Applying the Exponentic Mean to the Reals

A search for an algorithm is undertaken to find a 'reasonable' application of the exponentic mean to a multiset of real numbers.

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Exponentic Mean

Background

(Two years ago I gave a math seminar talk on some original research defining the *Exponentic Mean*, so the first part of this presentation is a review of that talk.)

The definition for the *Exponentic Mean* was derived by analogy from the definitions for the *Arithmetic Mean* and the *Geometric Mean*:

The ***Arithmetic Mean*** of $\boldsymbol{n}$ numbers is defined as:

$$\frac{a_1 + a_2 + \cdots + a_n}{n} = \left(\underset{\substack{\uparrow \\ \textit{rank 1}}}{\sum_{1}^{n}} a_i \right) \underset{\substack{\uparrow \\ \textit{rank 2}}}{\times} \left(\frac{1}{n} \right)$$

The ***Geometric Mean*** of $\boldsymbol{n}$ numbers is defined as:

$$\sqrt[n]{a_1 \times a_2 \times \cdots \times a_n} = \left(\underset{\substack{\uparrow \\ \textit{rank 2}}}{\prod_{1}^{n}} a_i \right) \underset{\substack{\uparrow \\ \textit{rank 3}}}{\uparrow} \left(\frac{1}{n} \right)$$

Analogous to the above,

the ***Exponentic Mean*** of $\boldsymbol{n}$ numbers is defined as:

$$\left({a_n}^{{a_{n-1}}^{\cdot^{\cdot^{\cdot^{a_1}}}}} \right) \uparrow\uparrow \frac{1}{n} = \left(\underset{\substack{\uparrow \\ \textit{rank 3}}}{\overset{1}{\underset{n}{E}}} a_i \right) \underset{\substack{\uparrow \\ \textit{rank 4}}}{\uparrow\uparrow} \left(\frac{1}{n} \right)$$

The analogy is based on the progression in the ranking of the operators, and applying *1/n* for normalization.

The ranking of the operators are taken from their position in the *Hyperoperation Sequence.*

Hyperoperation Sequence

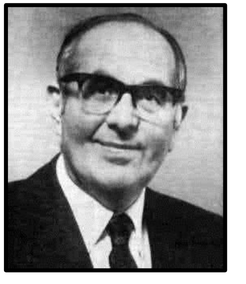
The hyperoperation sequence was first formally presented by Albert A. Bennett in 1915 in the *Annals of Mathematics.*

Rank

0	$'$	**Incrementation:**	a'	$\equiv$	$a+1$	(the basis operation of incrementing by 1)
1	$+$	**Addition:**	$a+b$	$\equiv$	$((a')'\cdots)'$	($a+b$ denotes incrementing a, b times.)
2	x	**Multiplication:**	$a\times b$	$\equiv$	$a+a+\cdots+a$	($a\text{x}b$ denotes adding a, b times.)
3	$\uparrow$	**Exponentiation:**	$a\uparrow b$	$\equiv$	$a\times a\times\cdots\times a$	($a\uparrow b$ denotes multiplying a, b times.)
4	$\uparrow\uparrow$	**Tetration:**	$a\uparrow\uparrow b$	$\equiv$	$a\uparrow a\uparrow\cdots\uparrow a$	($a\uparrow\uparrow b$ denotes exponentiating a, b times.)

5	$\uparrow\uparrow\uparrow$	**Pentation**
6	$\uparrow\uparrow\uparrow\uparrow$	**Hexation**
7	$\uparrow\uparrow\uparrow\uparrow\uparrow$	**Septation**
8	$\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow$	**Octation**
9	$\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow$	**Nonation**
10	$\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow$	**Decation**
$\vdots$		

In 1947, R. L. Goodstein coined terms for the operations beyond exponentiation.



Reuben Louis Goodstein
1912-1985
British Mathematician

Starting with tetration, the prefix of these terms are the Greek numbers for the *rank* of the operation, and the Latin suffix “*ation*” means “action” or “process.”

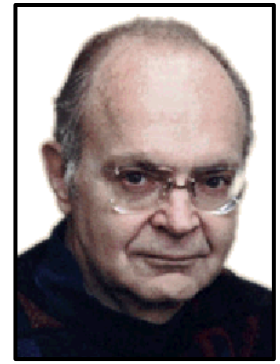
Hyperoperation Sequence

Hyperoperation Sequence

0	′	**Incrementation**
1	+	**Addition**
2	x	**Multiplication**
3	↑	**Exponentiation**
4	↑↑	**Tetration**
5	↑↑↑	**Pentation**
6	↑↑↑↑	**Hexation**
7	↑↑↑↑↑	**Septation**
8	↑↑↑↑↑↑	**Octation**
9	↑↑↑↑↑↑↑	**Nonation**
10	↑↑↑↑↑↑↑↑	**Decation**
⋮	⋮	⋮

Rather than using long strings of up-arrows, Donald Knuth has given a better notation:

$\uparrow^r$ denotes **r** repetitions of the up-arrow operator.



Donald Knuth
1938 –
U.S. Computer Scientist

A recursive definition for the hyperoperations is then more easily expressed as:

$$a\uparrow^r b = \begin{cases} 1 & \text{if } b = 0 \\ a^b & \text{if } r = 1 \\ a\uparrow^{r-1}(a\uparrow^r(b-1)) & \text{otherwise,} \quad a, b \in \mathbb{Z}_{\geq 0} \end{cases}$$

The ***rank*** of a hyper operator, is the no. of recursive levels required to completely express the given operation in terms of the basis operation, incrementation.

There are other recursive definitions that realize the hyperoperations, the earliest of which is Ackermann's Function (1928):

$$A(m,n) = \begin{cases} n+1 & \text{if } m = 0 \\ A(m-1, 1) & \text{if } m > 0 \text{ and } n = 0 \\ A(m-1, A(m, n-1)) & \text{if } m > 0 \text{ and } n > 0. \end{cases}$$

Derivation by Analogy

Definition: In mathematics, *Derivation by Analogy* is an informal method of reasoning (as opposed to the formal method of a proof) for deriving extended formulas or concepts.

For example, the formula $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

for the distance between two points in $\mathcal{E}^2$ extends by analogy to the distance between two points in $\mathcal{E}^3$ as:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

$\cdots$ and so forth over $\mathcal{E}^n$.

George Pólya
1888-1985
Jewish Hungarian Mathematician

In his two volumes on *Mathematics and Plausible Reasoning*, George Pólya states that *analogy* is a sort of *similarity*, but at a more definitive level:

"Two systems are analogous, if they agree in clearly definable relations of their respective parts."

For example, *Recurrence Relations* (also called *Difference Equations*) are the discrete analog of *Ordinary Linear Differential Equations*, which are continuous.

The broad terms used in these definitions are open to interpretation on a case-by-case basis, showing why *Derivation by Analogy* is an informal method. One must be aware of critical exceptions and differences.

(One can consider the *homomorphisms* of abstract algebra as more formally defined analogies.)

Derivation by Analogy

Here is a simple visual display of the analogy in progressing from arithmetic mean to geometric mean to exponentic mean for the following four numbers: 5 3 2 3

When asking for the arithmetic mean of the four above numbers, we are asking for the value of four x's whose sum is the same as the sum of those four numbers:

$$x + x + x + x = 13 \Rightarrow 4x = 13$$

$$5 + 3 + 2 + 3 = 13 \qquad x = 13/4 = 3.25$$

When asking for the geometric mean of those four numbers, we are asking for the value of four x's whose product is the same as the product of those four numbers:

$$x \times x \times x \times x = 90 \Rightarrow x^4 = 90$$

$$5 \times 3 \times 2 \times 3 = 90 \qquad x = \sqrt[4]{90} = 3.08\cdots$$

When asking for the exponentic mean of the four numbers, we are asking for the value of four x's whose exponentiation is the same as the exponentiation of those four numbers:

$$x^{x^{x^{x}}} \simeq 8.75 \text{ x } 10^{4585} \Rightarrow x{\uparrow\uparrow}4 \simeq 8.75\text{e}4585$$

$$x \simeq 8.75\text{e}4585{\uparrow\uparrow}¼ = 2.51\cdots$$

$$5^{3^{2^{3}}} \simeq 8.75 \text{ x } 10^{4585}$$

Exponentic Mean Example

Arithmetic Mean: $(5+3+2+3)\times\frac{1}{4} \;=\; 13\times\frac{1}{4} \;=\; 3.25$

Geometric Mean: $(5\times3\times2\times3)\uparrow\frac{1}{4} \;=\; 90\uparrow\frac{1}{4} \;=\; 3.08\cdots$

Exponentic Mean: $(5\uparrow3\uparrow2\uparrow3)\uparrow\uparrow\frac{1}{4} \;\approx\; 8.75\times10^{4585}\uparrow\uparrow\frac{1}{4} \;=\; 2.51\cdots$

Question: In what significant way does the exponentic mean differ from the other two means?

Answer: The exponentiation operator is not commutative, whereas addition and multiplication operators are commutative.

Thus, as defined above, the order of the operands for the exponentic mean can yield different values.

For ***n*** operands there may be up to ***n!*** different results.

To see how the noncommutativity of the exponentiation operator complicates a definition for the exponentic mean, consider computing the exponentic mean for a set of just 3 integers: {2, 3, 5}.

Since set elements are unordered, which of the 3! orderings should be chosen to compute the exponentic mean?

$$\left(2^{3^{5}}\right)\uparrow\uparrow ⅓ \quad \left(2^{5^{3}}\right)\uparrow\uparrow ⅓ \quad \left(3^{2^{5}}\right)\uparrow\uparrow ⅓ \quad \left(3^{5^{2}}\right)\uparrow\uparrow ⅓ \quad \left(5^{3^{2}}\right)\uparrow\uparrow ⅓ \quad \left(5^{2^{3}}\right)\uparrow\uparrow ⅓$$

Exponentic Mean Example

The following 6 different exponentic means for the set {2, 3, 5} are arranged in ascending order:

$$(5\uparrow2\uparrow3)\uparrow\uparrow\tfrac{1}{3} = 3.906e05\uparrow\uparrow\tfrac{1}{3} = 2.649370\cdots$$
$$(5\uparrow3\uparrow2)\uparrow\uparrow\tfrac{1}{3} = 1.953e06\uparrow\uparrow\tfrac{1}{3} = 2.699197\cdots$$
$$(3\uparrow5\uparrow2)\uparrow\uparrow\tfrac{1}{3} = 8.472e11\uparrow\uparrow\tfrac{1}{3} = 2.967930\cdots$$
$$(3\uparrow2\uparrow5)\uparrow\uparrow\tfrac{1}{3} = 1.853e15\uparrow\uparrow\tfrac{1}{3} = 3.070583\cdots$$
$$(2\uparrow5\uparrow3)\uparrow\uparrow\tfrac{1}{3} = 4.253e37\uparrow\uparrow\tfrac{1}{3} = 3.440132\cdots$$
$$(2\uparrow3\uparrow5)\uparrow\uparrow\tfrac{1}{3} = 1.413e73\uparrow\uparrow\tfrac{1}{3} = 3.706743\cdots$$

As the magnitude of the base ranges from 10^5 to 10^{73} the exponentic mean increases by only a little over 1.

What would be some ways of defining the exponentic mean(s) for a multiset of numbers?

1) Define the exponentic mean as the minimum over the permutations of the multiset of numbers.
2) Define the exponentic mean as the maximum over the permutations of the multiset of numbers.
3) Just accept that there will be no one single number, but up to n! exponentic means.
4) Do something hoakey like computing the arithmetic or geometric mean of the exponentic means.
5) Develop a different definition for the exponentic mean.

It was at this point in the presentation two years ago that I dropped the exponentic mean and went on to some other related topics.

Then, just a few months ago, the following idea came to mind:

An Iterative Definition

Consider a set of just two numbers, say {2, 3}.

There are only two ways to compute the exponentic mean:

$$(2\uparrow 3)\uparrow\uparrow\tfrac{1}{2} \;=\; 8\uparrow\uparrow\tfrac{1}{2} = 2.3884234844993385\cdots$$

$$(3\uparrow 2)\uparrow\uparrow\tfrac{1}{2} \;=\; 9\uparrow\uparrow\tfrac{1}{2} = 2.4509539280155797\cdots$$

Question: If there is to be only one result for their exponentic mean, what do these two results suggest that you do?

Answer: Replace the original elements in the set with these two results and iterate – hoping that convergence will occur!

$$(2.3884234844993385\cdots \uparrow 2.4509539280155797\cdots)\uparrow\uparrow\tfrac{1}{2} \;= 8.4476022332573859\cdots \uparrow\uparrow\tfrac{1}{2} = 2.4174327152212283\cdots$$

$$(2.4509539280155797\cdots \uparrow 2.3884234844993385\cdots)\uparrow\uparrow\tfrac{1}{2} \;= 8.5093661233470889\cdots \uparrow\uparrow\tfrac{1}{2} = 2.4213004031177277\cdots$$

Already, these two results are converging closer to some value between the two first results.

Iteratively, the exponentic mean of $\{a_0, b_0\}$ is defined as:

$$\mu_e(a_0, b_0) \;=\; \lim_{n\to\infty}(a_n) \;=\; \lim_{n\to\infty}(b_n)$$

$$\text{where} \quad a_n \;=\; (a_{n-1} \uparrow b_{n-1})\uparrow\uparrow \tfrac{1}{2}$$

$$b_n \;=\; (b_{n-1} \uparrow a_{n-1})\uparrow\uparrow \tfrac{1}{2}$$

On a computer, iteration continues until $|\, a_n - b_n \,| \le \varepsilon$ where epsilon governs the desired accuracy within the constraints of the machine and computer language.

An Iterative Definition

Here is the output from a C++ program that shows $\mu_e(2, 3)$ converging to 15 places of accuracy within 12 iterations.

$$\mu_e(2, 3) \;=\; 2.41936532184492\cdots$$

Note 1:

$2.41936532184492\cdots \uparrow 2.41936532184492\cdots$
$= 8.47840162427882\cdots$

This is a little less than half way between 8 and 9.

Note 2:

The value for the exponentic mean continues the trend set by the geometric mean in producing decreasing values compared to the arithmetic mean:

$$\mu_a(2, 3) \;=\; 2.5$$
$$\mu_g(2, 3) \;=\; 2.44948974278317\cdots$$
$$\mu_e(2, 3) \;=\; 2.41936532184492\cdots$$

```
P:\Research\ExponenticMean\BinaryTetraticRoot.exe
Enter two number to determine their exponentic mean:
x = 2
y = 3
x^y = 8
y^x = 9
8^^(1/2) = 2.3884234844993384                              1
9^^(1/2) = 2.4509539280155797
continue ? y
x^y = 8.4476022332573847
y^x = 8.5093661233470877
8.4476022332573847^^(1/2) = 2.4174327152212283             2
8.5093661233470877^^(1/2) = 2.4213004031177277
continue ? y
x^y = 8.4764919026362221
y^x = 8.4803119775270672
8.4764919026362221^^(1/2) = 2.4192457183509459             3
8.4803119775270672^^(1/2) = 2.4194849348054763
continue ? y
x^y = 8.4782834897152169
y^x = 8.4785197612586495
8.4782834897152169^^(1/2) = 2.4193579240967895             4
8.4785197612586495^^(1/2) = 2.4193727196292678
continue ? y
x^y = 8.4783943175885882
y^x = 8.4784089309783684
8.4783943175885882^^(1/2) = 2.4193648642933114             5
8.4784089309783684^^(1/2) = 2.4193657793966707
continue ? y
x^y = 8.4784011723599555
y^x = 8.478402076197793
8.4784011723599555^^(1/2) = 2.4193652935453609             6
8.4784020761977930^^(1/2) = 2.4193653501444836
continue ? y
x^y = 8.4784015963276841
y^x = 8.4784016522300345
8.4784015963276841^^(1/2) = 2.4193653200945948             7
8.4784016522300345^^(1/2) = 2.4193653235952493
continue ? y
x^y = 8.4784016225500804
y^x = 8.4784016260076395
8.4784016225500804^^(1/2) = 2.4193653217366644             8
8.4784016260076395^^(1/2) = 2.4193653219531797
continue ? y
x^y = 8.478401624171935
y^x = 8.4784016243857848
8.4784016241719350^^(1/2) = 2.4193653218382263             9
8.4784016243857848^^(1/2) = 2.4193653218516178
continue ? y
x^y = 8.4784016242722466
y^x = 8.4784016242854733
8.4784016242722466^^(1/2) = 2.4193653218445079             10
8.4784016242854733^^(1/2) = 2.4193653218453361
continue ? y
x^y = 8.4784016242784509
y^x = 8.478401624279269
8.4784016242784509^^(1/2) = 2.4193653218448965             11
8.4784016242792690^^(1/2) = 2.4193653218449476
continue ? y
x^y = 8.4784016242788347
y^x = 8.4784016242788852
8.4784016242788347^^(1/2) = 2.4193653218449205             12
8.4784016242788852^^(1/2) = 2.4193653218449236
continue ? y
x^y = 8.4784016242788584
y^x = 8.4784016242788615
8.4784016242788584^^(1/2) = 2.4193653218449218
8.4784016242788615^^(1/2) = 2.4193653218449223
continue ?
```

An Iterative Definition

How could the iterative definition of the exponentic mean of two numbers be extended to three numbers?

With two numbers we initially start the process with only two different results, and iterate to one converging value:

$$(2\uparrow 3)\uparrow\uparrow \tfrac{1}{2} = 8\uparrow\uparrow \tfrac{1}{2} = 2.3884234844993385\cdots$$

$$(3\uparrow 2)\uparrow\uparrow \tfrac{1}{2} = 9\uparrow\uparrow \tfrac{1}{2} = 2.4509539280155797\cdots$$

With three numbers we initially start the process with six different results:

$$(5\uparrow 2\uparrow 3)\uparrow\uparrow \tfrac{1}{3} = 3.906e05\uparrow\uparrow \tfrac{1}{3} = 2.649370\ \cdots$$

$$(5\uparrow 3\uparrow 2)\uparrow\uparrow \tfrac{1}{3} = 1.953e06\uparrow\uparrow \tfrac{1}{3} = 2.699197\ \cdots$$

$$(3\uparrow 5\uparrow 2)\uparrow\uparrow \tfrac{1}{3} = 8.472e11\uparrow\uparrow \tfrac{1}{3} = 2.967930\ \cdots$$

$$(3\uparrow 2\uparrow 5)\uparrow\uparrow \tfrac{1}{3} = 1.853e15\uparrow\uparrow \tfrac{1}{3} = 3.070583\ \cdots$$

$$(2\uparrow 5\uparrow 3)\uparrow\uparrow \tfrac{1}{3} = 4.253e37\uparrow\uparrow \tfrac{1}{3} = 3.440132\ \cdots$$

$$(2\uparrow 3\uparrow 5)\uparrow\uparrow \tfrac{1}{3} = 1.413e73\uparrow\uparrow \tfrac{1}{3} = 3.706743\ \cdots$$

Can we expect convergence to one value for this case?

Intuitively, I believe the answer is *yes*, however an *intractable algorithm* occurs if we blindly forge ahead:

$$\underbrace{\mu_e(2,3,5)}_{\text{3 values}} = \underbrace{\mu_e(2.64\cdots, 2.69\cdots, 2.96\cdots, 3.07\cdots, 3.44\cdots, 3.70\cdots)}_{3! = 6 \text{ values}} = \underbrace{\mu_e(\cdots)}_{6! = 720 \text{ values}} = \underbrace{\mu_e(\cdots)}_{720! = 2.6 \times 10^{1746} \text{ values}} = \cdots$$

There hasn't been enough nanoseconds (10^{26}) since the big bang to even make a dent in this iterative process!

(An *intractable algorithm* theoretically halts with an answer, but there is not enough time or matter in this world to make this practical on a human scale.)

An Iterative Definition

Here is a formal definition for the exponentic mean that leads to an intractable algorithm:

Definition 1 Define the exponentic mean of a multiset S_0 of positive reals $x_{i,0}$, $|S_0| \geq 2$, to be $x_{i,n} \in S_n$ where $x_{i,n}$ is the n^{th} tetratic root of one of the $n!$ permutations of the elements in an exponential tower: $(x_{j,n-1} \uparrow \cdots \uparrow x_{k,n-1}) \uparrow\uparrow (1/n)$, subject to the constraints that $x_{i,n} \in (\, min(x_{j,n}), max(x_{j,n})\,)$ and $(max(x_{j,n}) - min(x_{j,n})\,) \leq \varepsilon$

As mentioned previously, for $n \geq 3$ this definition leads to an intractable implementation because $|S_n| = |S_{n-1}|!$.

Only for $n = 2$ is this definition algorithmically tractable, since under the factorial function, 2 factorial maps back into itself.

So, what "reasonable" definition could be given for the exponentic mean of a multiset S of size n ?

$$\mu_e(S) = \left(\overset{n}{\underset{i=1}{E}} a_i \right) \uparrow\uparrow 1/n \;\; =/\approx \;\; ?$$

"Reasonableness" should require a definition whose algorithmic implementation is tractable.

Since we already have a reasonable definition for computing the exponentic mean of 2 numbers, could we find a definition for 3 or more numbers, that only involves working with them 2 at a time, and yet involve all possible combinations of them as we work our way down the towers of all possible permutations? For example, for S = {2, 3, 4}:

$$2^{3^{4}} \qquad 2^{4^{3}} \qquad 3^{2^{4}} \qquad 3^{4^{2}} \qquad 4^{3^{2}} \qquad 4^{2^{3}}$$

An Iterative Definition

Because the exponentic mean for 2 numbers has been defined so that $\mu_e(a, b) = \mu_e(b, a)$, the exponentic means of the towers above the three bases, 2, 3, and 4, are the same according to their respective circled colors:

Example for $S = \{2, 3, 4\}$:

$$2^{3^4} \quad 2^{4^3} \qquad 3^{2^4} \quad 3^{4^2} \qquad 4^{3^2} \quad 4^{2^3}$$

$$2^{\mu_e(3,4)} \qquad 3^{\mu_e(2,4)} \qquad 4^{\mu_e(2,3)}$$

Thus the 6 permutations reduce to 3 different values as shown above:

Question: Now, what does this suggest that we do next?

Answer 1: Replace the original 3 elements in set $S_0 = \{2, 3, 4\}$ with these values, and iterate, until convergence among the 3 values occurs:

$$S_0 = \{2, 3, 4\} \;\rightarrow\; S_1 = \{\, 2^{\mu e(3,4)}, 3^{\mu e(2,4)}, 4^{\mu e(2,3)} \,\} \;\rightarrow \cdots$$

But, convergence does not occur!

Divergence to infinity occurs.

An Iterative Definition

Because the exponentic mean for 2 numbers has been defined so that $\mu_e(a, b) = \mu_e(b, a)$, the exponentic means of the towers above the three bases, 2, 3, and 4, are the same according to their respective circled color:

Example for S = {2, 3, 4}:

$$2^{3^{4}} \quad 2^{4^{3}} \qquad 3^{2^{4}} \quad 3^{4^{2}} \qquad 4^{3^{2}} \quad 4^{2^{3}}$$

$$\mu_e(2, \mu_e(3,4)) \qquad \mu_e(3, \mu_e(2,4)) \qquad \mu_e(4, \mu_e(2,3))$$

Thus the 6 permutations reduce to 3 different values as shown above:

Question: Now, what does this suggest that we do next?

Answer 2: Replace the original 3 elements in set S_0 = {2, 3, 4} with the exponentic mean of the base and its exponent, and iterate, until convergence among the 3 values occurs:

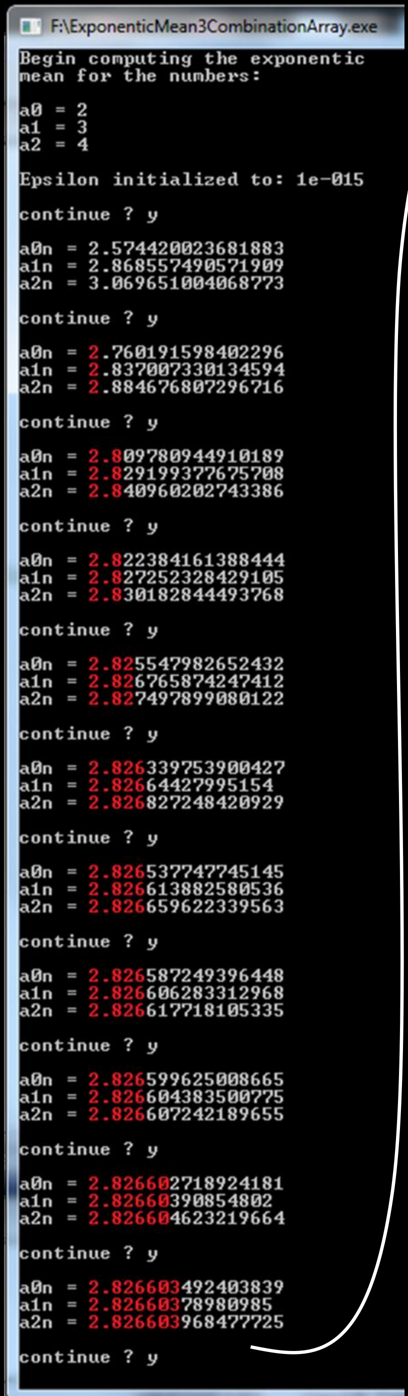
$$S_0 = \{2, 3, 4\} \;\rightarrow\; S_1 = \{\, \mu_e(2, \mu_e(3,4)),\ \mu_e(3, \mu_e(2,4)),\ \mu_e(4, \mu_e(2,3)) \,\} \;\rightarrow \cdots$$

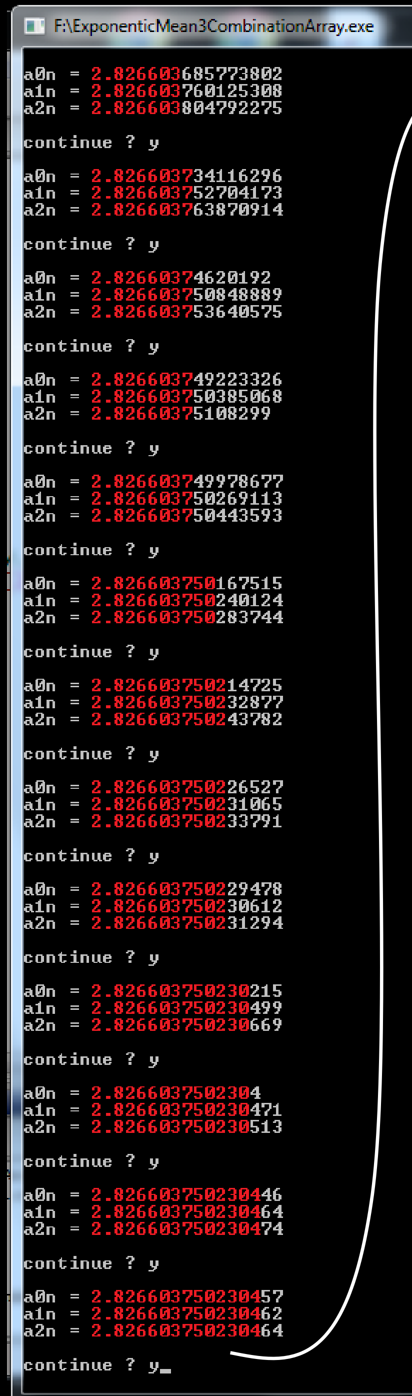
An Iterative Definition

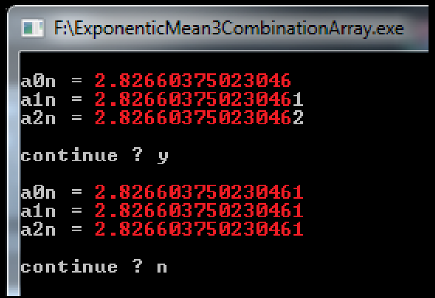
```
Fcn muE3(a_i, n)

    b_i = a_i   i=1··n

    while max(abs(b_i  -b_j)>ε)
                  i≠j
      b′_i = b_i   i=1··n

      b_1 = muE2(b′_1, muE2(b′_2, b′_3)
      b_2 = muE2(b′_2, muE2(b′_1, b′_3)
      b_3 = muE2(b′_3, muE2(b′_1, b′_2)

    endWhile
endFcn
```

An iterative algorithm for the above pseudo-code was written in C++, with output converging to the exponentic mean of {2, 3, 4}.

Using type **`long double`** and an epsilon of 10^{-15}, the three convergents converged to the same 16 digits of overall accuracy.







An Iterative Definition

These are screen shots from the C++ program that show $\mu_e(2, 3, 4)$ converging to 16 places of accuracy in 26 iterations for the 3 elements.

$$\mu_e(2, 3, 4) = 2.826603750230461\cdots$$

Again, note that the value for the exponentic mean continues the trend set by the geometric mean in producing decreasing values compared to the arithmetic mean:

$$\begin{aligned} \mu_a(2, 3, 4) &= 3 \\ \mu_g(2, 3, 4) &= 2.884499140614816\cdots \\ \mu_e(2, 3, 4) &= 2.826603750230461\cdots \end{aligned}$$

An Iterative Definition

Extending the iterative definition for the exponentic mean from 2 numbers to 3 numbers to 4 numbers is explored next as an aid to generalizing the definition.

Let $S = \{ 2, 3, 4, 5 \}$. The 4! = 24 permutations of their exponential towers are shown below:

Their respective towers are the 3!=6 permutations of the remaining elements: $\mathbf{S \backslash \{a_i\}}$

Since the towers of height 3 in any row are of the same 3 elements, their exponentic means are the same:

The base for each row comes from the 4 elements in S:

$2^{3^{4^{5}}}$ $\quad 2^{3^{5^{4}}}$ $\quad 2^{4^{3^{5}}}$ $\quad 2^{4^{5^{3}}}$ $\quad 2^{5^{3^{4}}}$ $\quad 2^{5^{4^{3}}}$ $\Rightarrow$ $\mu e_3(3, 4, 5)$

$3^{2^{4^{5}}}$ $\quad 3^{2^{5^{4}}}$ $\quad 3^{4^{2^{5}}}$ $\quad 3^{4^{5^{2}}}$ $\quad 3^{5^{4^{2}}}$ $\quad 3^{5^{2^{4}}}$ $\Rightarrow$ $\mu e_3(2, 4, 5)$

$4^{2^{3^{5}}}$ $\quad 4^{2^{5^{3}}}$ $\quad 4^{3^{2^{5}}}$ $\quad 4^{3^{5^{2}}}$ $\quad 4^{5^{2^{3}}}$ $\quad 4^{5^{3^{2}}}$ $\Rightarrow$ $\mu e_3(2, 3, 5)$

$5^{2^{3^{4}}}$ $\quad 5^{2^{4^{3}}}$ $\quad 5^{3^{2^{4}}}$ $\quad 5^{3^{4^{2}}}$ $\quad 5^{4^{2^{3}}}$ $\quad 5^{4^{3^{2}}}$ $\Rightarrow$ $\mu e_3(2, 3, 4)$

Thus we can iterate in turn over these towers of heights 2, 3, and 4 as convergence occurs for each tower height.

An Iterative Definition

To set the limits for this algorithm, let $S = \{ a_1 , a_2 , \cdots , a_k \} \quad a_i \in \mathbb{R}_{>1}$ be a multiset of cardinality k.

Note 1:

Multisets allow $a_i = a_j = a_k = \cdots$ for $i \neq j \neq k \neq \cdots$ whereas sets do not allow this.

This allows us to compute the exponentic mean for a numeric list that contains repeated numbers.

For example: { 5, 3, 3, 2, 4, 4, 4 }

Note 2:

Currently we will restrict the elements a_i to be real and greater than 1.

There are several reasons for doing this:

a) There is no convergence when an element equals 1 since whenever 1 is at the base of a tower of exponents, the overall tower <u>always</u> evaluates to 1:

$$1^{a^{b^{c^{\cdot^{\cdot^{\cdot}}}}}} = 1$$

b) I have yet to explore the behavior of convergence in the algorithm for values of a_i in critical regions such as (0, 1); and complex numbers would play a role in (-∞, 0).

The next slide shows the progression in the iterative definitions for the exponentic mean of elements in S of cardinality k, denoted $\mu e_k(S)$, for the first few values of k:

An Iterative Definition

k = 2: $S = \{ a_1 , a_2 \} \quad a_i \in \mathbb{R} > 1$

$$\mu e_2(S) = \begin{cases} a_{1_0} = a_1 \\ a_{2_0} = a_2 \\ \\ a_{1_n} = (a_{1_{n-1}} \uparrow a_{2_{n-1}}) \uparrow\uparrow (½) & n = 1, 2, \cdots \\ a_{2_n} = (a_{2_{n-1}} \uparrow a_{1_{n-1}}) \uparrow\uparrow (½) & \text{and } \lim\limits_{n\to\infty} a_{1_n} = \lim\limits_{n\to\infty} a_{2_n} \end{cases}$$

Observing these definitions,
we generalize for cardinality *k*:
$S = \{ a_1 , a_2 , \cdots a_k \} \quad a_i \in \mathbb{R} > 1$

$$\mu e_k(S) = \begin{cases} \mu e_2(a_1 , a_2) & k = 2 \\ \\ a_{m_0} = a_m \qquad m = 1 \cdot\cdot k & k > 2 \\ a_{m_n} = \mu e_2 (a_{m_{n-1}} , \mu e_{k-1} (S \setminus a_{m_{n-1}})) & m = 1 \cdot\cdot k, \\ & n = 1, 2, \cdots \end{cases}$$

k = 3: $S = \{ a_1 , a_2 , a_3 \} \quad a_i \in \mathbb{R} > 1$

$$\mu e_3(S) = \begin{cases} a_{1_0} = a_1 \\ a_{2_0} = a_2 \\ a_{3_0} = a_3 \\ \\ a_{1_n} = \mu e_2(a_{1_{n-1}} , \mu e_2(S \setminus a_{1_{n-1}})) & n = 1, 2, \cdots \\ a_{2_n} = \mu e_2(a_{2_{n-1}} , \mu e_2(S \setminus a_{2_{n-1}})) & \text{and } \lim\limits_{n\to\infty} a_{1_n} = \lim\limits_{n\to\infty} a_{2_n} = \lim\limits_{n\to\infty} a_{3_n} \\ a_{3_n} = \mu e_2(a_{3_{n-1}} , \mu e_2(S \setminus a_{3_{n-1}})) \end{cases}$$

k = 4: $S = \{ a_1 , a_2 , a_3 , a_4 \} \quad a_i \in \mathbb{R} > 1$

$$\mu e_4(S) = \begin{cases} a_{1_0} = a_1 \\ a_{2_0} = a_2 \\ a_{3_0} = a_3 \\ a_{4_0} = a_4 \\ \\ a_{1_n} = \mu e_2(a_{1_{n-1}} , \mu e_3(S \setminus a_{1_{n-1}})) & n = 1, 2, \cdots \\ a_{2_n} = \mu e_2(a_{2_{n-1}} , \mu e_3(S \setminus a_{2_{n-1}})) & \text{and } \lim\limits_{n\to\infty} a_{1_n} = \lim\limits_{n\to\infty} a_{2_n} = \lim\limits_{n\to\infty} a_{3_n} = \lim\limits_{n\to\infty} a_{4_n} \\ a_{3_n} = \mu e_2(a_{3_{n-1}} , \mu e_3(S \setminus a_{3_{n-1}})) \\ a_{4_n} = \mu e_2(a_{4_{n-1}} , \mu e_3(S \setminus a_{4_{n-1}})) \end{cases}$$

A Recursive Definition

Definition 2 Cardinality $k > 1$: $S = \{ a_1, a_2, \cdots a_k \}$ $a_i \in \mathbb{R} > 1$

$$\mu e_k(S) = \begin{cases} \mu e_2(a_1, a_2) & k = 2 \longleftarrow \text{Boundary Condition} \\ a_{m_0} = a_m \quad m = 1 \cdot\cdot k & k > 2 \\ a_{m_n} = \mu e_2(a_{m_{n-1}}, \mu e_{k-1}(S \setminus a_{m_{n-1}})) & m = 1 \cdot\cdot k;\ n = 1, 2, \cdots;\ \max\limits_{1 \le i \ne j \le k}(|a_{m_i} - a_{m_j}|) \le \varepsilon \end{cases}$$

Recursive Definition

In the process of generalizing the iterative definitions, we have also condensed it to a recursive definition.

One might object to this approach by saying that nowhere in the definition does the k^{th} tetratic root appear as it does in the original definition for the exponentic mean:

$$\left({a_k}^{{a_{k-1}}^{\cdot^{\cdot^{\cdot^{a_1}}}}} \right) \uparrow\uparrow \frac{1}{k} = \left(\mathop{\boldsymbol{E}}_{k}^{1} a_i \right) \uparrow\uparrow \left(\frac{1}{k} \right)$$

Instead, the tetratic root ½ is solely used on pairs of arguments as we recursively work our way down the towers of permutations.

The rebuttal of this objection is that arithmetic means and geometric means for $k > 2$ can <u>analogously</u> be computed using <u>only</u> their respective normalizing operations of division by 2 and square roots.

In other words, keeping the above recursive definition the same, but changing the boundary function to return *(a+b)/2* or *sqrt(a*b)*, will respectively return the arithmetic mean or the geometric mean of a multiset S of any size $k \ge 2$.

This not only rebuts the objection, it also experimentally establishes the veracity of this newly derived algorithm for computing the tetratic mean.

A Recursive Definition

Here is the C++ function that serves as the boundary condition, returning the exponentic mean of its 2 arguments:

```
long double muExp2(long double a, long double b)
{   long double an, bn, anm1, bnm1, epsilon = 1.e-17;

    an = a;
    bn = b;
    while(abs(an-bn)>epsilon)
    { anm1 = an;
      bnm1 = bn;

      an = bisect(pow(anm1, bnm1), 2);
      bn = bisect(pow(bnm1, anm1), 2);
    }
    return an;              // exponentic mean of {a, b}

    // return (a+b)/2;      // arithmetic mean of {a, b}

    // return sqrt(a*b);    // geometric  mean of {a, b}
}
```

If instead of returning the exponentic mean of the two arguments, a and b, their arithmetic mean is returned, or their geometric mean is returned, then the respective mean for the overall multiset of cardinality k is computed with no code changes needed in the recursive definition.

```
F:\ExponenticMean3CombinationArray.exe
Begin computing the arithmetic
mean for the numbers:

a0 = 2
a1 = 3
a2 = 4

Epsilon initialized to: 1e-016

continue ? y

a0n = 2.75
a1n = 3
a2n = 3.25

continue ? y

a0n = 2.9375
a1n = 3
a2n = 3.0625

continue ? y

a0n = 2.984375
a1n = 3
a2n = 3.015625

continue ? y

a0n = 2.99609375
a1n = 3
a2n = 3.00390625

continue ? y

a0n = 2.9990234375
a1n = 3
a2n = 3.0009765625

continue ? y

a0n = 2.999755859375
a1n = 3
a2n = 3.000244140625

continue ? y

a0n = 2.99993896484375
a1n = 3
a2n = 3.00006103515625

continue ? y

a0n = 2.999984741210938
a1n = 3
a2n = 3.000015258789062

continue ? y

a0n = 2.999996185302734
a1n = 3
a2n = 3.000003814697266

continue ? y

a0n = 2.999999046325684
a1n = 3
a2n = 3.000000953674316

continue ? y

a0n = 2.999999761581421
a1n = 3
a2n = 3.000000238418579

continue ? y

a0n = 2.999999940395355
a1n = 3
a2n = 3.000000059604645
```

```
F:\ExponenticMean3CombinationArray.exe
continue ? y

a0n = 2.999999985098839
a1n = 3
a2n = 3.000000014901161

continue ? y

a0n = 2.99999999627471
a1n = 3
a2n = 3.00000000372529

continue ? y

a0n = 2.999999999068677
a1n = 3
a2n = 3.000000000931323

continue ? y

a0n = 2.999999999767169
a1n = 3
a2n = 3.000000000232831

continue ? y

a0n = 2.999999999941792
a1n = 3
a2n = 3.000000000058208

continue ? y

a0n = 2.999999999985448
a1n = 3
a2n = 3.000000000014552

continue ? y

a0n = 2.999999999996362
a1n = 3
a2n = 3.000000000003638

continue ? y

a0n = 2.999999999999091
a1n = 3
a2n = 3.000000000000909

continue ? y

a0n = 2.999999999999773
a1n = 3
a2n = 3.000000000000227

continue ? y

a0n = 2.999999999999943
a1n = 3
a2n = 3.000000000000057

continue ? y

a0n = 2.999999999999986
a1n = 3
a2n = 3.000000000000014

continue ? y

a0n = 2.999999999999996
a1n = 3
a2n = 3.000000000000004

continue ? y

a0n = 2.999999999999999
a1n = 3
a2n = 3.000000000000001

continue ? y
```

```
F:\ExponenticMean3CombinationArray.exe

a0n = 3
a1n = 3
a2n = 3

continue ? n
```

With the boundary function changed to return (a+b)/2, the arithmetic mean of { 2, 3, 4 } was returned in 26 iterations.

When the boundary function was changed to return sqrt(a*b), the geometric mean of { 2, 3, 4 } was also returned in 26 iterations:

```
F:\ExponenticMean3CombinationArray.exe
Begin computing the geometric
mean for the numbers:

a0 = 2
a1 = 3
a2 = 4

Epsilon initialized to: 1e-016

continue ? y

a0n = 2.632148025904985
a1n = 2.912950630243941
a2n = 3.130169160146575

continue ? y

a0n = 2.8192292183656
a1n = 2.891585853984704
a2n = 2.944047159392526
```

```
continue ? y

a0n = 2.884499140614816
a1n = 2.884499140614817
a2n = 2.884499140614818

continue ? y

a0n = 2.884499140614817
a1n = 2.884499140614817
a2n = 2.884499140614817

continue ? n
```

Timing Analysis

To derive the run-time order of execution for this algorithm, the unit-of-cost is chosen to be the constant time, b, to determine the exponentic mean for 2 numbers at the boundary:

$$S = \{ a_1 , a_2 , \cdots\ a_k \} \quad a_i \in \mathbb{R} > 1 , \quad k > 1$$

T(2)

$$\mu e_k(S) = \begin{cases} \mu e_2 (a_1, a_2) & k = 2 \\ a_{m_0} = a_m \quad m = 1 \cdot\cdot\ k & k > 2 \\ a_{m_n} = \mu e_2 (a_{m_{n-1}} , \mu e_{k-1} (S \setminus a_{m_{n-1}})) & m = 1 \cdot\cdot\ k ; \quad n = 1, 2, \cdots \end{cases}$$

$$\max_{1 \le i \ne j \le k} (|a_{m_i} - a_{m_j}|) \le \varepsilon$$

T(k)

T(k-1) x k

The overall time is designated as $T(k)$ for determining the exponentic mean of k numbers.

The body of the recursive definition shows k number of calls over sets of size $k\text{-}1$.

The order of execution is then the solution to this recurrence relation:
$$T(k) = \begin{cases} b & k = 2 \\ k\ T(k\text{-}1) & k > 2 \end{cases}$$

The solution for this at $k = n$ is: $bn!/2$

Thus this algorithm is $\mathcal{O}(n!)$ in execution time, which should not be surprising in light of the $n!$ arrangements of exponential towers for n different numbers.

Research Summary

1) The definition for the *Exponentic Mean* was derived by analogy from the definitions for the *Arithmetic Mean* and the *Geometric Mean*:

Arithmetic Mean
$$\frac{a_1 + a_2 + \cdots + a_n}{n} = \left(\underbrace{\sum_1^n}_{\text{rank 1}} a_i\right) \underbrace{\times}_{\text{rank 2}} \left(\frac{1}{n}\right)$$

Geometric Mean
$$\sqrt[n]{a_1 \times a_2 \times \cdots \times a_n} = \left(\underbrace{\prod_1^n}_{\text{rank 2}} a_i\right) \underbrace{\uparrow}_{\text{rank 3}} \left(\frac{1}{n}\right)$$

Exponentic Mean
$$\left({a_n}^{{a_{n-1}}^{\cdot^{\cdot^{\cdot^{a_1}}}}}\right) \uparrow\uparrow \frac{1}{n} = \left(\underbrace{\mathop{E}_n^1}_{\text{rank 3}} a_i\right) \underbrace{\uparrow\uparrow}_{\text{rank 4}} \left(\frac{1}{n}\right)$$

2) But, because the exponentiation operator is noncommutative, how can we assign only one value as the exponentic mean to an unordered list of two or more numbers?

$$-\ e.g.\quad S = \{\,2,\,3\,\}$$
$$2^3 \neq 3^2$$
$$8\uparrow\uparrow ½ = 2.3884\cdots \neq 9\uparrow\uparrow ½ = 2.4509\cdots$$

3) The answer is to replace the original two elements in the list with their respective tetratic half roots, and iterate over these and subsequent values until convergence occurs to within some epsilon:

$$\mu e_2(a_0, b_0) = \lim_{n\to\infty}(a_n) = \lim_{n\to\infty}(b_n)$$

where
$$a_n = (a_{n-1} \uparrow b_{n-1}) \uparrow\uparrow \tfrac{1}{2}$$
$$b_n = (b_{n-1} \uparrow a_{n-1}) \uparrow\uparrow \tfrac{1}{2}$$

$$-\ e.g.\quad \mu e_2(\,2, 3\,) =$$
$$2.41936532184492\cdots$$
$$|a_{12} - b_{12}| \leq 10^{-14}$$

Research Summary

4) Defining the exponentic mean of two numbers serves as the boundary condition for the recursive definition covering multisets of 3 or more numbers:

$$S = \{ a_1 , a_2 , \cdots \ a_k \} \quad a_i \in \mathbb{R} > 1 , \quad k > 1$$

$$\mu e_k(S) = \begin{cases} \mu e_2 (a_1, a_2) & k = 2 \\ a_{m_0} = a_m \qquad m = 1 \cdots k & k > 2 \\ a_{m_n} = \mu e_2 (a_{m_{n-1}} , \mu e_{k-1} (S \setminus a_{m_{n-1}})) & m = 1 \cdots k \, ; \quad n = 1, 2, \cdots \end{cases}$$

$$\max_{1 \le i \neq j \le k} (|a_{m_i} - a_{m_j}|) \le \varepsilon$$

$\mu_{arithmetic}$ $\mu_{geometric}$

5) Replacing the boundary condition that computes the exponentic mean with other boundary conditions such as the arithmetic mean or the geometric mean, causes the algorithm to compute those respective means for numbers from multisets of cardinality greater than 2.

This demonstrates that the recursive definition part of the algorithm serves as a general purpose engine for pairwise-computing the various definitions of mathematical means (averages).

6) As presented here, the order of execution for this algorithm is ***O(n!)***. Although this order is quite large compared to polynomial-time algorithms, it is much faster than the brute-force *algorithm for solving the exponentic mean problem as originally posed. Also, speed-ups are possible for this algorithm since many of the same combinations of numbers are dealt with more than once.

*The brute-force approach would be $O(\sum_{i=1}^{n} k!^{i})$ where superscript i denotes repetitions of the factorial operator, and n is the number of iterations required for convergence.

Conclusion

Although no proof has been presented here that the derived algorithm computes the exponentic mean of a multiset, the fact that it was originally derived without regard to computing arithmetic and geometric means, and yet turned out to be "backwards compatible" in doing so, experimentally establishes its original intent.

Thus we may conclude that the algorithm is not only reasonable for computing the exponentic mean, but that it is most likely the 'correct' algorithm in the platonic sense.

Further Research

1) Extend the exponentic mean definition to include the reals less than 1, and complex numbers.

2) Improve the efficiency of the algorithm by omitting recurring over previously computed sets of numbers.

3) Apply the algorithm to computing combinations of the AGEM to see if it results in transcendental constants as does the AGM, further bolstering confidence in the definition for the exponentic mean.

For Example:

AGM

$$\boxed{\begin{aligned} a_{n+1} &= (a_n + b_n) \times \tfrac{1}{2} \\ b_{n+1} &= (a_n \times b_n) \uparrow \tfrac{1}{2} \end{aligned}}$$

$$AGM(1, \tfrac{\sqrt{2}}{2}) = 0.84721308\cdots = \frac{\Gamma^2(\frac{3}{4})}{\sqrt{\pi}}$$

$$AGM(1, \sqrt{2}) = 1.19814023 = \frac{2\sqrt{2\pi^3}}{\Gamma^2(\frac{1}{4})}$$

Further Research

4) Develop a formula for a weighted exponentic mean.

5) Prove that the exponentic mean of a multiset of reals > 1 is less than the geometric mean of the same multiset.

6) What further complications occur when extending the exponentic mean to higher ranking arithmetic operators? Is there any useful or theoretical applications for a countably infinite hierarchy of extensions?

Philosophical Questions to Ponder

Why wasn't tetration the subject of more research in previous centuries?

Why isn't there a button for tetration on our calculators –or– as a function in our computer languages?

Is there no natural phenomena that is modeled by repeated exponentiation?

Why is it that natural phenomena is modeled by our current set of operations / functions?

Is it because that's just the way the universe is?

– Or, is it because science has yet to discover more complex phenomena best modeled by formulas involving tetration.

To the extent that theory and practice inform each other, answers to these questions are always historically interesting.

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