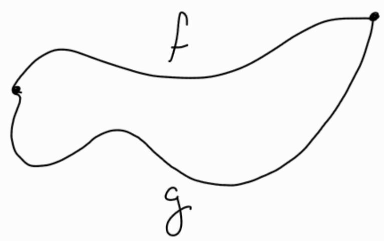
3 1 $\Rightarrow$ 2

$$X \text{ sempl conn} \Rightarrow X \text{ cpa} + \left(f,g : I \to X, \begin{cases} f(0)=g(0)=x_0 \\ f(1)=g(1)=x_1 \end{cases} \Rightarrow f \sim g \text{ rel } \{0,1\} \right)$$

$$X \text{ sempl conn} :\Rightarrow X \text{ cpa} + \pi_1(X)=\{\varepsilon\}$$



$$\pi_1(X,x) = \{ [\gamma] \mid \gamma : I \to X,\ \gamma(0)=\gamma(1)=x \}$$

$$f \sim g \text{ rel } \{0,1\} \Rightarrow \exists F : I \times I \to X \ \Bigg| \ \begin{cases} F(s,0)=f(s) \\ F(s,1)=g(s) \\ F(0,t)=f(0)=g(0) \\ F(1,t)=f(1)=g(1) \end{cases} \quad \forall t \in I$$

$$f \cdot i(g) : I \to X$$

$$(f \cdot i(g))(0) = (f \cdot i(g))(1) = x_0 \Rightarrow f \cdot i(g) \in \pi_1(X,x_0) \Rightarrow f \cdot i(g) \sim \varepsilon_{x_0}$$

$$\Rightarrow f \cdot i(g) \sim \varepsilon_{x_0} \Rightarrow f \cdot i(g) \cdot g = f \cdot \varepsilon_{x_1} = f \sim \varepsilon_{x_0} \cdot g = g \Rightarrow f \sim g \text{ rel } \{0,1\}$$

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2 $\Rightarrow$ 1

$$f \sim g \text{ rel } \{0,1\},\ \forall f,g : I \to X$$

sia $\gamma : I \to X$ laccio in x $\rightsquigarrow$ $\gamma(k) \in X,\ k \in (0,1)$

$$\begin{cases} f := \gamma|_{[0,k]} \\ g := i(\gamma)|_{[k,1]} \end{cases} \Rightarrow f \cdot i(g) = \gamma \quad \text{ma siccome} \quad \begin{cases} f(0)=g(0)=x \\ f(1)=g(1)=\gamma(k) \end{cases}$$

abbiamo che $f \sim g \text{ rel } \{0,1\} \Rightarrow \left. \begin{array}{l} f \cdot i(g) \sim \varepsilon_x \\ f \cdot i(g) = \gamma \end{array} \right\} \Rightarrow \pi_1(X) = \{\varepsilon\}$

---

③ $\Longrightarrow$ ①

X cpa + $f \sim \varepsilon$, $\forall f : S^1 \to X$, $f \in C^1(S^1)$

$$\begin{tikzcd}
I \arrow[r, "\tilde{f}"] \arrow[d, "\pi"'] & X \\
I/_{\{0,1\}} \cong S^1 \arrow[ur, "f"'] &
\end{tikzcd}$$

$\tilde{f} = f \circ \pi \sim \varepsilon \circ \pi \implies \tilde{f} : I \to X$

$\tilde{f}(0) = \tilde{f}(1)$ laccio

$\tilde{f} \sim \varepsilon \implies \pi_1(X) = \{\varepsilon\}$

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① $\Longrightarrow$ ③

passaggi inversi di ③ $\Longrightarrow$ ①

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④ $\Longleftrightarrow$ ③

Prop 1.3.6 : $f : S^1 \to X$ cont

$f \sim \varepsilon \iff f = g|_{\partial D^2 = S^1}$