(11) X sp top, $f,g: X \to \mathbb{S}^n$ continue $\mid f(x) \neq -g(x),\ \forall x \in X \implies f \sim g$

$$f \sim g \implies \exists F : X \times I \to \mathbb{S}^n \mid \begin{cases} F(x,0) = f(x) \\ F(x,1) = g(x) \end{cases} \quad \forall x \in X$$

$\mathbb{S}^n \subset \mathbb{R}^{n+1}$ convesso $\implies$ Esempio 1.1.1 $\qquad A$ convesso $\iff (1-t)x + ty \in A$ $\forall t \in [0,1],\ \forall x,y \in A$

Consideriamo

$$T^2 := \|(1-t)f(x) + t\,g(x)\|^2 = (1-t)^2\|f(x)\|^2 + t^2\|g(x)\|^2 + 2t(1-t)\,f(x)\cdot g(x) = 0$$

$$f(x) \neq 0 \wedge g(x) \neq 0 \quad \because \quad f(x), g(x) \in \mathbb{S}^n,\ \mathbb{S}^n \not\ni 0$$

$$\to \quad T^2 = 0 \iff (1-t)^2 = 0 \wedge t^2 = 0 \quad \text{impossibile} \implies T^2 \neq 0$$

Possiamo dunque considerare l'omotopia:

$$F : X \times I \to \mathbb{S}^n \subset \mathbb{R}^{n+1}$$
$$(x,t) \longmapsto \frac{(1-t)f(x) + t\,g(x)}{\|(1-t)f(x) + t\,g(x)\|}$$

$$\|F(x,t)\| = \frac{\|(1-t)f(x) + t\,g(x)\|}{\|(1-t)f(x) + t\,g(x)\|} = 1 \implies F(x,t) \in \mathbb{S}^n$$

$$f(x), g(x) \in \mathbb{S}^n \implies \|f(x)\| = \|g(x)\| = 1,\ \forall x \in X \implies \begin{cases} F(x,0) = \dfrac{f(x)}{\|f(x)\|} = f(x) \\[2ex] F(x,1) = \dfrac{g(x)}{\|g(x)\|} = g(x) \end{cases} \quad \forall x \in X$$