⑤ X connesso, Y sp top, $X \sim Y \Rightarrow Y$ connesso

X, Y sp top, $\quad f: X \to Y, \quad f$ continua $\wedge$ X connesso $\Rightarrow f(X) \subseteq Y$ connesso

$X \sim Y \Rightarrow \exists\, f: X \to Y,\ g: Y \to X$ continue $\mid g \circ f \sim id_X \wedge f \circ g \sim id_Y$

$\Rightarrow \exists f: X \to Y$ continua

$\Rightarrow f(X) \subseteq Y$ connesso $\because$ X connesso

$$f \circ g \sim id_Y \Rightarrow \exists F: Y \times I \to Y \;\Big|\; \begin{cases} F(y,0) = (f \circ g)(y) \\ F(y,1) = id_Y(y) = y \end{cases}$$

Sia $h: Y \to \{0,1\} \subset \mathbb{N}$

$h \circ f: X \to \{0,1\}$ costante $\Leftrightarrow$ X connesso, $\quad h \circ f = \varepsilon$

$(h \circ f \circ g)(y) = (h \circ f \circ g)(y')$, $\forall y, y' \in Y$ $\quad \because \quad h \circ f \circ g = \varepsilon \circ g = \varepsilon'$ costante $\because$ ε costante

Siccome $\quad F_y: I \to Y$, $F_y(0) = (f \circ g)(y)$, $F_y(1) = y$ è un arco tra $(f \circ g)(y)$ e y in Y, allora questi sono connessi e h è costante tra i due punti, i.e. $(h \circ f \circ g)(y) = h(y)$

Analogo per $(h \circ f \circ g)(y') = h(y')$

A questo punto $(h \circ f \circ g)(y) = (h \circ f \circ g)(y')$, $\forall y, y' \in Y$

$$\Downarrow$$

$$h(y) = h(y'),\ \forall y, y' \in Y$$

$$\Updownarrow$$

$$h: Y \to \{0,1\} \text{ costante}$$

$$\Updownarrow$$

$$Y \text{ connesso}$$