Problem 12.39 Show that

$$K_\mu K^\mu = \frac{1 - (u^2/c^2)\cos^2\theta}{1 - u^2/c^2} F^2,$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{F}$.

Problem 12.40 Show that the (ordinary) acceleration of a particle of mass m and charge q, moving at velocity $\mathbf{u}$ under the influence of electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$, is given by

$$\mathbf{a} = \frac{q}{m}\sqrt{1 - u^2/c^2}\left[\mathbf{E} + \mathbf{u} \times \mathbf{B} - \frac{1}{c^2}\mathbf{u}(\mathbf{u} \cdot \mathbf{E})\right].$$

[*Hint:* Use Eq. 12.73.]

12.3 Relativistic Electrodynamics

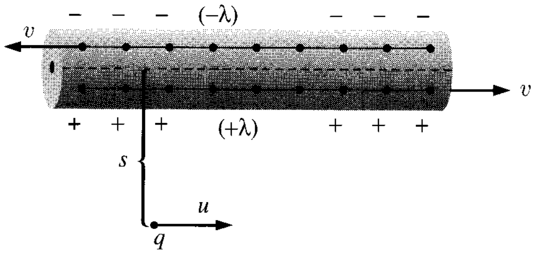
12.3.1 Magnetism as a Relativistic Phenomenon

Unlike Newtonian mechanics, classical electrodynamics is *already* consistent with special relativity. Maxwell's equations and the Lorentz force law can be applied legitimately in any inertial system. Of course, what one observer interprets as an electrical process another may regard as magnetic, but the actual particle motions they predict will be identical. To the extent that this did *not* work out for Lorentz and others, who studied the question in the late nineteenth century, the fault lay with the nonrelativistic mechanics they used, not with the electrodynamics. Having corrected Newtonian mechanics, we are now in a position to develop a complete and consistent formulation of relativistic electrodynamics. But I emphasize that we will not be changing the rules of electrodynamics in the slightest—rather, we will be *expressing* these rules in a notation that exposes and illuminates their relativistic character. As we go along, I shall pause now and then to rederive, using the Lorentz transformations, results obtained earlier by more laborious means. But the main purpose of this section is to provide you with a deeper understanding of the structure of electrodynamics—laws that had seemed arbitrary and unrelated before take on a kind of coherence and inevitability when approached from the point of view of relativity.

To begin with I'd like to show you why there *had* to be such a thing as magnetism, given electrostatics and relativity, and how, in particular, you can calculate the magnetic force between a current-carrying wire and a moving charge without ever invoking the laws of magnetism.[14] Suppose you had a string of positive charges moving along to the right at speed v. I'll assume the charges are close enough together so that we may regard them as a continuous line charge λ. Superimposed on this positive string is a negative one, $-\lambda$ proceeding to the left at the same speed v. We have, then, a net current to the right, of magnitude

$$I = 2\lambda v. \tag{12.75}$$

[14] This and several other arguments in this section are adapted from E. M. Purcell's *Electricity and Magnetism*. 2d ed. (New York: McGraw-Hill, 1985).



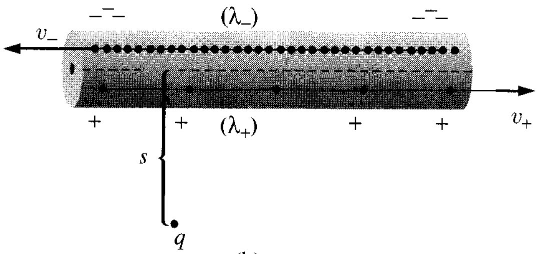


Figure 12.34

Meanwhile, a distance s away there is a point charge q traveling to the right at speed $u < v$ (Fig. 12.34a). Because the two line charges cancel, there is *no electrical force on* q in this system ($\mathcal{S}$).

However, let's examine the same situation from the point of view of system $\bar{\mathcal{S}}$, which moves to the right with speed u (Fig. 12.34b). In this reference frame q is at rest. By the Einstein velocity addition rule, the velocities of the positive and negative lines are now

$$v_{\pm} = \frac{v \mp u}{1 \mp vu/c^2}. \tag{12.76}$$

Because v_- is greater than v_+, the Lorentz contraction of the spacing between negative charges is more severe than that between positive charges; *in this frame,* therefore, *the wire carries a net negative charge!* In fact,

$$\lambda_{\pm} = \pm(\gamma_{\pm})\lambda_0, \tag{12.77}$$

where

$$\gamma_{\pm} = \frac{1}{\sqrt{1 - v_{\pm}^2/c^2}}, \tag{12.78}$$

and λ_0 is the charge density of the positive line in its own rest system. That's not the same as λ, of course—in $\mathcal{S}$ they're already moving at speed v, so

$$\lambda = \gamma\lambda_0, \tag{12.79}$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}. \tag{12.80}$$

It takes some algebra to put $\gamma_\pm$ into simple form:

$$\begin{aligned} \gamma_\pm &= \frac{1}{\sqrt{1 - \frac{1}{c^2}(v \mp u)^2(1 \mp vu/c^2)^{-2}}} = \frac{c^2 \mp uv}{\sqrt{(c^2 \mp uv)^2 - c^2(v \mp u)^2}} \\ &= \frac{c^2 \mp uv}{\sqrt{(c^2 - v^2)(c^2 - u^2)}} = \gamma\frac{1 \mp uv/c^2}{\sqrt{1 - u^2/c^2}}. \end{aligned} \tag{12.81}$$

Evidently, then, the net line charge in $\bar{\mathcal{S}}$ is

$$\lambda_{\text{tot}} = \lambda_+ + \lambda_- = \lambda_0(\gamma_+ - \gamma_-) = \frac{-2\lambda uv}{c^2\sqrt{1 - u^2/c^2}}. \tag{12.82}$$

Conclusion: As a result of unequal Lorentz contraction of the positive and negative lines, a current-carrying wire that is electrically neutral in one inertial system will be charged in another.

Now, a line charge λ_{tot} sets up an *electric* field

$$E = \frac{\lambda_{\text{tot}}}{2\pi\epsilon_0 s},$$

so *there is an electrical force on q in $\bar{\mathcal{S}}$*, to wit:

$$\bar{F} = qE = -\frac{\lambda v}{\pi\epsilon_0 c^2 s}\frac{qu}{\sqrt{1 - u^2/c^2}}. \tag{12.83}$$

But if there's a force on q in $\bar{\mathcal{S}}$, there must be one in $\mathcal{S}$; in fact, we can *calculate* it by using the transformation rules for forces. Since q is at rest $\bar{\mathcal{S}}$, and $\bar{F}$ is perpendicular to u, the force in $\mathcal{S}$ is given by Eq. 12.68:

$$F = \sqrt{1 - u^2/c^2}\,\bar{F} = -\frac{\lambda v}{\pi\epsilon_0 c^2}\frac{qu}{s}. \tag{12.84}$$

The charge is attracted toward the wire by a force that is purely electrical in $\bar{\mathcal{S}}$ (where the wire is charged, and q is at rest), but distinctly *non*electrical in $\mathcal{S}$ (where the wire is neutral). Taken together, then, electrostatics and relativity imply the existence of another force. This "other force" is, of course, *magnetic.* In fact, we can cast Eq. 12.84 into more familiar form by using $c^2 = (\epsilon_0\mu_0)^{-1}$ and expressing λv in terms of the current (Eq. 12.75):

$$F = -qu\left(\frac{\mu_0 I}{2\pi s}\right). \tag{12.85}$$

The term in parentheses is the magnetic field of a long, straight wire, and the force is precisely what we would have obtained by using the Lorentz force law in system $\mathcal{S}$.

12.3.2 How the Fields Transform

We have learned, in various special cases, that one observer's electric field is another's magnetic field. It would be nice to know the *general* transformation rules for electromagnetic fields: Given the fields in $\mathcal{S}$, what are the fields in $\bar{\mathcal{S}}$? Your first guess might be that $\mathbf{E}$ is the spatial part of one 4-vector and $\mathbf{B}$ the spatial part of another. If so, your intuition is wrong—it's more complicated than that. Let me begin by making explicit an assumption that was already used implicitly in Sect. 12.3.1: *Charge is invariant.* Like mass, but unlike energy, the charge of a particle is a fixed number, independent of how fast it happens to be moving. We shall assume also that the transformation rules are the same no matter how the fields were produced—electric fields generated by changing magnetic fields transform the same way as those set up by stationary charges. Were this not the case we'd have to abandon the field formulation altogether, for it is the essence of a field theory that the fields at a given point tell you *all there is to know,* electromagnetically, about that point; you do *not* have to append extra information regarding their source.

With this in mind, consider the *simplest possible* electric field: the uniform field in the region between the plates of a large parallel-plate capacitor (Fig. 12.35a). Say the capacitor is at rest in $\mathcal{S}_0$ and carries surface charges $\pm\sigma_0$. Then

$$\mathbf{E}_0 = \frac{\sigma_0}{\epsilon_0}\,\hat{\mathbf{y}}. \tag{12.86}$$

But what if we examine this same capacitor from system $\mathcal{S}$, moving to the right at speed v_0 (Fig. 12.35b)? In this system the plates are moving to the left, but the field still takes the form

$$\mathbf{E} = \frac{\sigma}{\epsilon_0}\,\hat{\mathbf{y}}; \tag{12.87}$$

the only difference is the value of the surface charge σ. [Wait a minute! *Is* that the only difference? The formula $E = \sigma/\epsilon_0$ for a parallel plate capacitor came from Gauss's law, and whereas Gauss's law is perfectly valid for moving charges, this particular application also relies on symmetry. Are we sure that the field is still perpendicular to the plates? What if the field of a moving plane *tilts,* say, in the direction of motion, as in Fig. 12.35c? Well, *even if it did* (it *doesn't*), the field between the plates, being the superposition of the $+\sigma$ field and the $-\sigma$ field, would nevertheless run perpendicular to the plates. For the $-\sigma$ field would aim as indicated in Fig. 12.35c (changing the sign of the charges reverses the direction of the field), and the vector sum kills off the parallel components.]

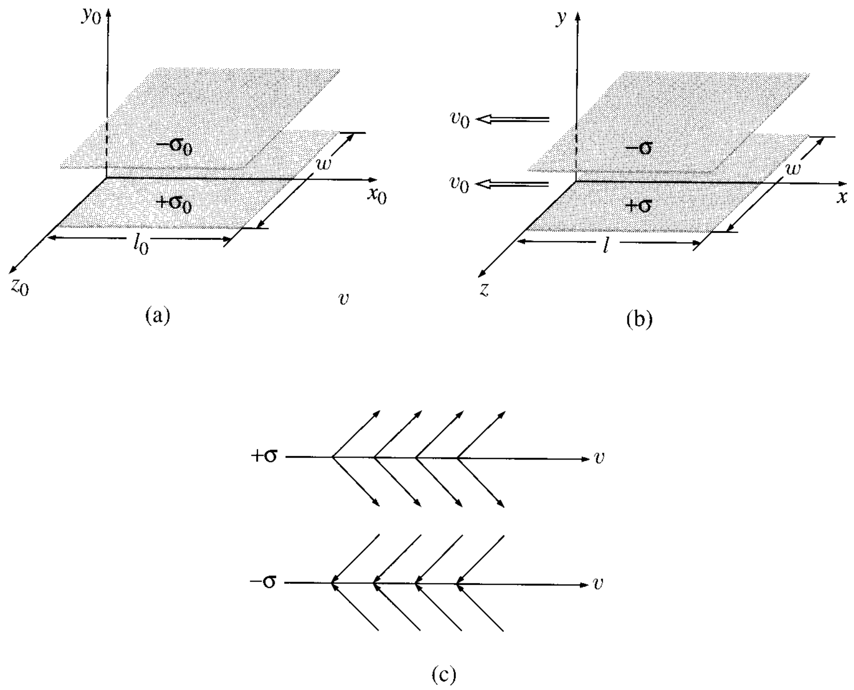


Figure 12.35

Now, the total charge on each plate is invariant, and the *width* (w) is unchanged, but the *length* (l) is Lorentz-contracted by a factor

$$\frac{1}{\gamma_0} = \sqrt{1 - v_0^2/c^2}, \tag{12.88}$$

so the charge per unit area is *increased* by a factor γ_0:

$$\sigma = \gamma_0 \sigma_0. \tag{12.89}$$

Accordingly,

$$\mathbf{E}^{\perp} = \gamma_0 \mathbf{E}_0{}^{\perp}. \tag{12.90}$$

I have put in the superscript $\perp$ to make it clear that this rule pertains to components of $\mathbf{E}$ that are *perpendicular* to the direction of motion of $\mathcal{S}$. To get the rule for *parallel* components, consider the capacitor lined up with the yz plane (Fig. 12.36). This time it is the plate separation (d) that is Lorentz-contracted, whereas l and w (and hence also σ) are the same in both frames. Since the field does not depend on d, it follows that

$$E^{\parallel} = E_0{}^{\parallel}. \tag{12.91}$$

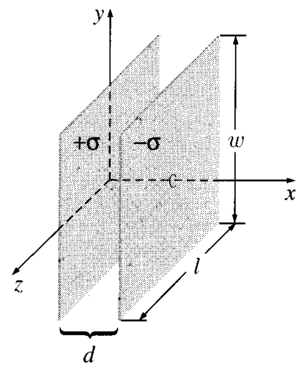


Figure 12.36

Example 12.13

Electric field of a point charge in uniform motion. A point charge q is at rest at the origin in system $\mathcal{S}_0$. *Question*: What is the electric field of this same charge in system $\mathcal{S}$, which moves to the right at speed v_0 relative to $\mathcal{S}_0$?

Solution: In $\mathcal{S}_0$ the field is

$$\mathbf{E}_0 = \frac{1}{4\pi\epsilon_0}\frac{q}{r_0^2}\hat{\mathbf{r}}_0,$$

or

$$\left\{\begin{array}{rcl} E_{x0} &=& \dfrac{1}{4\pi\epsilon_0}\dfrac{qx_0}{(x_0^2+y_0^2+z_0^2)^{3/2}}, \\[2ex] E_{y0} &=& \dfrac{1}{4\pi\epsilon_0}\dfrac{qy_0}{(x_0^2+y_0^2+z_0^2)^{3/2}}, \\[2ex] E_{z0} &=& \dfrac{1}{4\pi\epsilon_0}\dfrac{qz_0}{(x_0^2+y_0^2+z_0^2)^{3/2}}. \end{array}\right.$$

From the transformation rules (Eqs. 12.90 and 12.91), we have

$$\left\{\begin{array}{rcccl} E_x &=& E_{x0} &=& \dfrac{1}{4\pi\epsilon_0}\dfrac{qx_0}{(x_0^2+y_0^2+z_0^2)^{3/2}}, \\[2ex] E_y &=& \gamma_0 E_{y0} &=& \dfrac{1}{4\pi\epsilon_0}\dfrac{\gamma_0 q y_0}{(x_0^2+y_0^2+z_0^2)^{3/2}}, \\[2ex] E_z &=& \gamma_0 E_{z0} &=& \dfrac{1}{4\pi\epsilon_0}\dfrac{\gamma_0 q z_0}{(x_0^2+y_0^2+z_0^2)^{3/2}}. \end{array}\right.$$

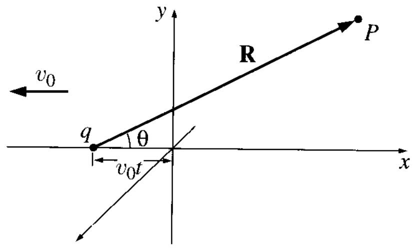


Figure 12.37

These are still expressed in terms of the $\mathcal{S}_0$ coordinates (x_0, y_0, z_0) of the field point (P); I'd prefer to write them in terms of the $\mathcal{S}$ coordinates of P. From the Lorentz transformations (or, actually, the inverse transformations),

$$\left\{\begin{array}{lcccl} x_0 &=& \gamma_0(x + v_0 t) &=& \gamma_0 R_x, \\ y_0 &=& y &=& R_y, \\ z_0 &=& z &=& R_z, \end{array}\right.$$

where $\mathbf{R}$ is the vector from q to P (Fig. 12.37). Thus

$$\begin{aligned} \mathbf{E} &= \frac{1}{4\pi\epsilon_0}\frac{\gamma_0 q\mathbf{R}}{(\gamma_0^2 R^2\cos^2\theta + R^2\sin^2\theta)^{3/2}} \\ &= \frac{1}{4\pi\epsilon_0}\frac{q(1 - v_0^2/c^2)}{[1 - (v_0^2/c^2)\sin^2\theta]^{3/2}}\frac{\hat{\mathbf{R}}}{R^2}. \end{aligned} \tag{12.92}$$

This, then, is the field of a charge in uniform motion; we got the same result in Chapter 10 using the retarded potentials (Eq. 10.68). The present derivation is far more efficient, and sheds some light on the remarkable fact that the field points away from the instantaneous (as opposed to the retarded) position of the charge: E_x gets a factor of γ_0 from the Lorentz transformation of the *coordinates;* E_y and E_z pick up theirs from the transformation of the *field.* It's the balancing of these two γ_0's that leaves $\mathbf{E}$ parallel to $\mathbf{R}$.

But Eqs. 12.90 and 12.91 are not the most general transformation laws, for we began with a system $\mathcal{S}_0$ in which the charges were at rest and where, consequently, there was no magnetic field. To derive the *general* rule we must start out in a system with both electric and magnetic fields. For this purpose $\mathcal{S}$ itself will serve nicely. In addition to the electric field

$$E_y = \frac{\sigma}{\epsilon_0}, \tag{12.93}$$

there is a *magnetic* field due to the surface currents (Fig. 12.35b):

$$\mathbf{K}_\pm = \mp\sigma v_0\,\hat{\mathbf{x}}. \tag{12.94}$$

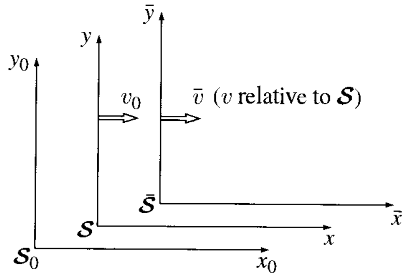


Figure 12.38

By the right-hand rule, this field points in the negative z direction; its magnitude is given by Ampère's law:

$$B_z = -\mu_0 \sigma v_0. \tag{12.95}$$

In a *third* system, $\bar{\mathcal{S}}$, traveling to the right with speed v relative to $\mathcal{S}$ (Fig. 12.38), the fields would be

$$\bar{E}_y = \frac{\bar{\sigma}}{\epsilon_0}, \quad \bar{B}_z = -\mu_0 \bar{\sigma} \bar{v}, \tag{12.96}$$

where $\bar{v}$ is the velocity of $\bar{\mathcal{S}}$ relative to $\mathcal{S}_0$:

$$\bar{v} = \frac{v + v_0}{1 + v v_0/c^2}, \quad \bar{\gamma} = \frac{1}{\sqrt{1 - \bar{v}^2/c^2}}, \tag{12.97}$$

and

$$\bar{\sigma} = \bar{\gamma} \sigma_0. \tag{12.98}$$

It remains only to express $\bar{\mathbf{E}}$ and $\bar{\mathbf{B}}$ (Eq. 12.96), in terms of $\mathbf{E}$ and $\mathbf{B}$ (Eqs. 12.93 and 12.95). In view of Eqs. 12.89 and 12.98, we have

$$\bar{E}_y = \left(\frac{\bar{\gamma}}{\gamma_0}\right) \frac{\sigma}{\epsilon_0}, \quad \bar{B}_z = -\left(\frac{\bar{\gamma}}{\gamma_0}\right) \mu_0 \sigma \bar{v}. \tag{12.99}$$

With a little algebra, you will find that

$$\frac{\bar{\gamma}}{\gamma_0} = \frac{\sqrt{1 - v_0^2/c^2}}{\sqrt{1 - \bar{v}^2/c^2}} = \frac{1 + v v_0/c^2}{\sqrt{1 - v^2/c^2}} = \gamma\left(1 + \frac{v v_0}{c^2}\right), \tag{12.100}$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}, \tag{12.101}$$

as always. Thus,

$$\bar{E}_y = \gamma\left(1 + \frac{v v_0}{c^2}\right) \frac{\sigma}{\epsilon_0} = \gamma\left(E_y - \frac{v}{c^2 \epsilon_0 \mu_0} B_z\right),$$

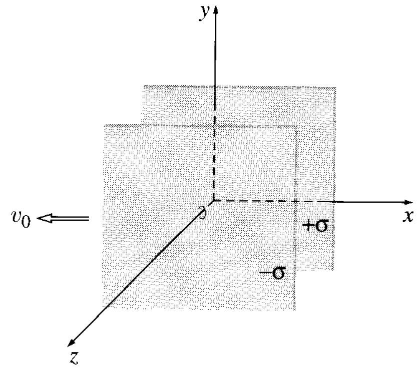


Figure 12.39

whereas

$$\bar{B}_z = -\gamma\left(1 + \frac{vv_0}{c^2}\right)\mu_0\sigma\left(\frac{v + v_0}{1 + vv_0/c^2}\right) = \gamma(B_z - \mu_0\epsilon_0 v E_y).$$

Or, since $\mu_0\epsilon_0 = 1/c^2$,

$$\left.\begin{aligned} \bar{E}_y &= \gamma(E_y - vB_z), \\ \bar{B}_z &= \gamma\left(B_z - \frac{v}{c^2}E_y\right). \end{aligned}\right\} \tag{12.102}$$

This tells us how E_y and B_z transform—to do E_z and B_y we simply align the same capacitor parallel to the xy plane instead of the xz plane (Fig. 12.39). The fields in $\mathcal{S}$ are then

$$E_z = \frac{\sigma}{\epsilon_0}, \quad B_y = \mu_0\sigma v_0.$$

(Use the right-hand rule to get the sign of B_y.) The rest of the argument is identical—everywhere we had E_y before, read E_z, and everywhere we had B_z, read $-B_y$:

$$\left.\begin{aligned} \bar{E}_z &= \gamma(E_z + vB_y), \\ \bar{B}_y &= \gamma\left(B_y + \frac{v}{c^2}E_z\right). \end{aligned}\right\} \tag{12.103}$$

As for the x components, we have already seen (by orienting the capacitor parallel to the yz plane) that

$$\bar{E}_x = E_x. \tag{12.104}$$

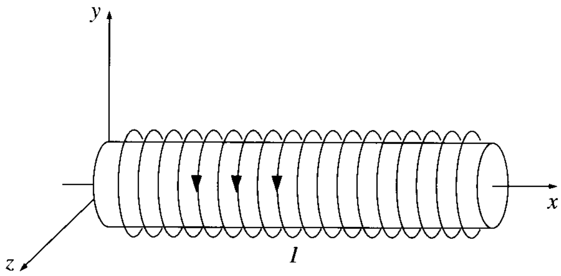


Figure 12.40

Since in this case there is no accompanying magnetic field, we cannot deduce the transformation rule for B_x. But another configuration will do the job: Imagine a long *solenoid* aligned parallel to the x axis (Fig. 12.40) and at rest in $\mathcal{S}$. The magnetic field within the coil is

$$B_x = \mu_0 n I, \tag{12.105}$$

where n is the number of turns per unit length, and I is the current. In system $\bar{\mathcal{S}}$, the length contracts, so n *increases:*

$$\bar{n} = \gamma n. \tag{12.106}$$

On the other hand, time *dilates:* The $\mathcal{S}$ clock, which rides along with the solenoid, runs slow, so the current (charge *per unit time*) in $\bar{\mathcal{S}}$ is given by

$$\bar{I} = \frac{1}{\gamma} I. \tag{12.107}$$

The two factors of γ exactly cancel, and we conclude that

$$\bar{B}_x = B_x.$$

Like **E**, the component of **B** *parallel* to the motion is unchanged.

Let's now collect together the complete set of transformation rules:

$$\boxed{\begin{array}{lll} \bar{E}_x = E_x, & \bar{E}_y = \gamma(E_y - vB_z), & \bar{E}_z = \gamma(E_z + vB_y), \\ \bar{B}_x = B_x, & \bar{B}_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right), & \bar{B}_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right). \end{array}} \tag{12.108}$$

Two special cases warrant particular attention:

1. If $\mathbf{B} = 0$ in $\mathcal{S}$, then

$$\bar{\mathbf{B}} = \gamma \frac{v}{c^2}(E_z\,\hat{\mathbf{y}} - E_y\,\hat{\mathbf{z}}) = \frac{v}{c^2}(\bar{E}_z\,\hat{\mathbf{y}} - \bar{E}_y\,\hat{\mathbf{z}}),$$

or, since $\mathbf{v} = v\,\hat{\mathbf{x}}$,

$$\boxed{\bar{\mathbf{B}} = -\frac{1}{c^2}(\mathbf{v} \times \bar{\mathbf{E}}).} \tag{12.109}$$

2. If $\mathbf{E} = 0$ in $\mathcal{S}$, then

$$\bar{\mathbf{E}} = -\gamma v(B_z\,\hat{\mathbf{y}} - B_y\,\hat{\mathbf{z}}) = -v(\bar{B}_z\,\hat{\mathbf{y}} - \bar{B}_y\,\hat{\mathbf{z}}),$$

or

$$\boxed{\bar{\mathbf{E}} = \mathbf{v} \times \bar{\mathbf{B}}.} \tag{12.110}$$

In other words, if either $\mathbf{E}$ or $\mathbf{B}$ is zero (at a particular point) in *one* system, then in any other system the fields (at that point) are very simply related by Eq. 12.109 or Eq. 12.110.

Example 12.14

Magnetic field of a point charge in uniform motion. Find the *magnetic* field of a point charge q moving at constant velocity $\mathbf{v}$.

Solution: In the particle's *rest* frame ($\mathcal{S}_0$) the magnetic field is zero (everywhere), so in a system $\mathcal{S}$ moving to the right at speed v,

$$\mathbf{B} = -\frac{1}{c^2}(\mathbf{v} \times \mathbf{E}).$$

We calculated the *electric* field in Ex. 12.13. The magnetic field, then, is

$$\mathbf{B} = \frac{\mu_0}{4\pi} \frac{qv(1 - v^2/c^2)\sin\theta}{[1 - (v^2/c^2)\sin^2\theta]^{3/2}} \frac{\hat{\boldsymbol{\phi}}}{R^2}, \tag{12.111}$$

where $\hat{\boldsymbol{\phi}}$ aims counterclockwise as you face the oncoming charge. Incidentally, in the nonrelativistic limit ($v^2 \ll c^2$), Eq. 12.111 reduces to

$$\mathbf{B} = \frac{\mu_0}{4\pi} q \frac{\mathbf{v} \times \mathbf{R}}{R^2},$$

which is exactly what you would get by naïve application of the Biot-Savart law to a point charge (Eq. 5.40).

Problem 12.41 Why can't the electric field in Fig. 12.35b have a z component? After all, the *magnetic* field does.

Problem 12.42 A parallel-plate capacitor, at rest in $\mathcal{S}_0$ and tilted at a 45° angle to the x_0 axis, carries charge densities $\pm\sigma_0$ on the two plates (Fig. 12.41). System $\mathcal{S}$ is moving to the right at speed v relative to $\mathcal{S}_0$.

(a) Find $\mathbf{E}_0$, the field in $\mathcal{S}_0$.

(b) Find $\mathbf{E}$, the field in $\mathcal{S}$.

(c) What angle do the plates make with the x axis?

(d) Is the field perpendicular to the plates in $\mathcal{S}$?

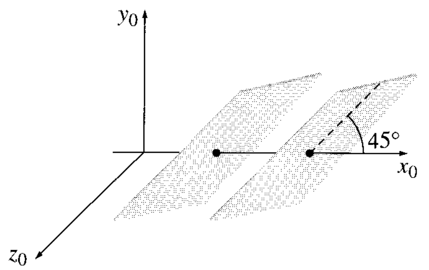


Figure 12.41

Problem 12.43

(a) Check that Gauss's law, $\int \mathbf{E} \cdot d\mathbf{a} = (1/\epsilon_0)Q_{\text{enc}}$, is obeyed by the field of a point charge in uniform motion, by integrating over a sphere of radius R centered on the charge.

(b) Find the Poynting vector for a point charge in uniform motion. (Say the charge is going in the z direction at speed v, and calculate $\mathbf{S}$ at the instant q passes the origin.)

Problem 12.44

(a) Charge q_A is at rest at the origin in system $\mathcal{S}$; charge q_B flies by at speed v on a trajectory parallel to the x axis, but at $y = d$. What is the electromagnetic force on q_B as it crosses the y axis?

(b) Now study the same problem from system $\bar{\mathcal{S}}$, which moves to the right with speed v. What is the force on q_B when q_A passes the $\bar{y}$ axis? [Do it two ways: (i) by using your answer to (a) and transforming the force; (ii) by computing the fields in $\bar{\mathcal{S}}$ and using the Lorentz force law.]

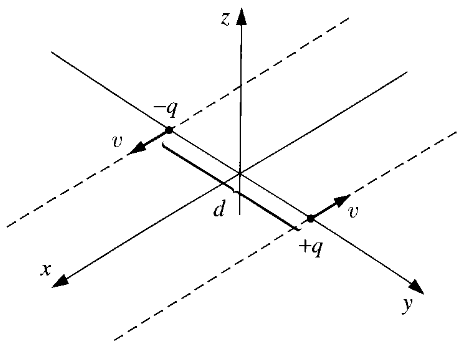


Figure 12.42

Problem 12.45 Two charges $\pm q$, are on parallel trajectories a distance d apart, moving with equal speeds v in opposite directions. We're interested in the force on $+q$ due to $-q$ at the instant they cross (Fig. 12.42). Fill in the following table, doing all the consistency checks you can think of as you go along.

	System A (Fig. 12.42)	System B ($+q$ at rest)	System C ($-q$ at rest)
E at $+q$ due to $-q$:			
B at $+q$ due to $-q$:			
F on $+q$ due to $-q$:			

Problem 12.46

(a) Show that $(\mathbf{E} \cdot \mathbf{B})$ is relativistically invariant.

(b) Show that $(E^2 - c^2 B^2)$ is relativistically invariant.

(c) Suppose that in one inertial system $\mathbf{B} = 0$ but $\mathbf{E} \neq 0$ (at some point P). Is it possible to find another system in which the *electric* field is zero at P?

Problem 12.47 An electromagnetic plane wave of (angular) frequency ω is traveling in the x direction through the vacuum. It is polarized in the y direction, and the amplitude of the electric field is E_0.

(a) Write down the electric and magnetic fields, $\mathbf{E}(x, y, z, t)$ and $\mathbf{B}(x, y, z, t)$. [Be sure to define any auxiliary quantities you introduce, in terms of ω, E_0, and the constants of nature.]

(b) This same wave is observed from an inertial system $\bar{S}$ moving in the x direction with speed v relative to the original system S. Find the electric and magnetic fields in $\bar{S}$, and express them in terms of the $\bar{S}$ coordinates: $\bar{\mathbf{E}}(\bar{x}, \bar{y}, \bar{z}, \bar{t})$ and $\bar{\mathbf{B}}(\bar{x}, \bar{y}, \bar{z}, \bar{t})$. [Again, be sure to define any auxiliary quantities you introduce.]

(c) What is the frequency $\bar{\omega}$ of the wave in $\bar{S}$? Interpret this result. What is the wavelength $\bar{\lambda}$ of the wave in $\bar{S}$? From $\bar{\omega}$ and $\bar{\lambda}$, determine the speed of the waves in $\bar{S}$. Is it what you expected?

(d) What is the ratio of the intensity in $\bar{\mathcal{S}}$ to the intensity in $\mathcal{S}$? As a youth, Einstein wondered what an electromagnetic wave would look like if you could run along beside it at the speed of light. What can you tell him about the amplitude, frequency, and intensity of the wave, as v approaches c?

12.3.3 The Field Tensor

As Eq. 12.108 indicates, **E** and **B** certainly do *not* transform like the spatial parts of the two 4-vectors—in fact, the components of **E** and **B** are stirred together when you go from one inertial system to another. What sort of an object is this, which has six components and transforms according to Eq. 12.108? *Answer*: It's an **antisymmetric, second-rank tensor**.

Remember that a 4-vector transforms by the rule

$$\bar{a}^{\mu} = \Lambda^{\mu}_{\nu} a^{\nu} \tag{12.112}$$

(summation over ν implied), where Λ is the Lorentz transformation matrix. If $\bar{\mathcal{S}}$ is moving in the x direction at speed v, Λ has the form

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{12.113}$$

and Λ^{μ}_{ν} is the entry in row μ, column ν. A (second-rank) tensor is an object with *two* indices, which transform with *two* factors of Λ (one for each index):

$$\bar{t}^{\mu\nu} = \Lambda^{\mu}_{\lambda}\Lambda^{\nu}_{\sigma} t^{\lambda\sigma}. \tag{12.114}$$

A tensor (in 4 dimensions) has $4 \times 4 = 16$ components, which we can display in a 4×4 array:

$$t^{\mu\nu} = \left\{ \begin{matrix} t^{00} & t^{01} & t^{02} & t^{03} \\ t^{10} & t^{11} & t^{12} & t^{13} \\ t^{20} & t^{21} & t^{22} & t^{23} \\ t^{30} & t^{31} & t^{32} & t^{33} \end{matrix} \right\}.$$

However, the 16 elements need not all be different. For instance, a *symmetric* tensor has the property

$$t^{\mu\nu} = t^{\nu\mu} \quad \textbf{(symmetric tensor)}. \tag{12.115}$$

In this case there are 10 distinct components; 6 of the 16 are repeats ($t^{01} = t^{10}$, $t^{02} = t^{20}$, $t^{03} = t^{30}$, $t^{12} = t^{21}$, $t^{13} = t^{31}$, $t^{23} = t^{32}$). Similarly, an *anti*symmetric tensor obeys

$$t^{\mu\nu} = -t^{\nu\mu} \quad \textbf{(antisymmetric tensor)}. \tag{12.116}$$

Such an object has just 6 distinct elements—of the original 16, six are repeats (the same ones as before, only this time with a minus sign) and four are zero (t^{00}, t^{11}, t^{22}, and t^{33}). Thus, the general antisymmetric tensor has the form

$$t^{\mu\nu} = \left\{ \begin{array}{cccc} 0 & t^{01} & t^{02} & t^{03} \\ -t^{01} & 0 & t^{12} & t^{13} \\ -t^{02} & -t^{12} & 0 & t^{23} \\ -t^{03} & -t^{13} & -t^{23} & 0 \end{array} \right\}.$$

Let's see how the transformation rule 12.114 works, for the six distinct components of an antisymmetric tensor. Starting with $\bar{t}^{01}$, we have

$$\bar{t}^{01} = \Lambda^0_\lambda \Lambda^1_\sigma t^{\lambda\sigma},$$

but according to Eq. 12.113, $\Lambda^0_\lambda = 0$ unless $\lambda = 0$ or 1, and $\Lambda^1_\sigma = 0$ unless $\sigma = 0$ or 1. So there are four terms in the sum:

$$\bar{t}^{01} = \Lambda^0_0\Lambda^1_0 t^{00} + \Lambda^0_0\Lambda^1_1 t^{01} + \Lambda^0_1\Lambda^1_0 t^{10} + \Lambda^0_1\Lambda^1_1 t^{11}.$$

On the other hand, $t^{00} = t^{11} = 0$, while $t^{01} = -t^{10}$, so

$$\bar{t}^{01} = (\Lambda^0_0\Lambda^1_1 - \Lambda^0_1\Lambda^1_0)t^{01} = (\gamma^2 - (\gamma\beta)^2)t^{01} = t^{01}.$$

I'll let you work out the others—the complete set of transformation rules is

$$\left. \begin{array}{lll} \bar{t}^{01} = t^{01}, & \bar{t}^{02} = \gamma(t^{02} - \beta t^{12}), & \bar{t}^{03} = \gamma(t^{03} + \beta t^{31}), \\ \bar{t}^{23} = t^{23}, & \bar{t}^{31} = \gamma(t^{31} + \beta t^{03}), & \bar{t}^{12} = \gamma(t^{12} - \beta t^{02}). \end{array} \right\} \tag{12.117}$$

These are precisely the rules we derived on physical grounds for the electromagnetic fields (Eq. 12.108)—in fact, we can construct the **field tensor** $F^{\mu\nu}$ by direct comparison:[15]

$$F^{01} \equiv \frac{E_x}{c}, \quad F^{02} \equiv \frac{E_y}{c}, \quad F^{03} \equiv \frac{E_z}{c}, \quad F^{12} \equiv B_z, \quad F^{31} \equiv B_y, \quad F^{23} \equiv B_x.$$

Written as an array,

$$F^{\mu\nu} = \left\{ \begin{array}{cccc} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{array} \right\}. \tag{12.118}$$

Thus relativity completes and perfects the job begun by Oersted, combining the electric and magnetic fields into a single entity, $F^{\mu\nu}$.

If you followed that argument with exquisite care, you may have noticed that there was a *different* way of imbedding **E** and **B** in an antisymmetric tensor: instead of comparing

[15]Some authors prefer the convention $F^{01} \equiv E_x$, $F^{12} \equiv cB_z$, and so on, and some use the opposite signs. Accordingly, most of the equations from here on will look a little different, depending on the text.

the first line of Eq. 12.108 with the first line of Eq. 12.117, and the second with the second, we could relate the first line of Eq. 12.108 to the *second* line of Eq. 12.117, and vice versa. This leads to **dual tensor,** $G^{\mu\nu}$:

$$G^{\mu\nu} = \left\{ \begin{array}{cccc} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{array} \right\}. \tag{12.119}$$

$G^{\mu\nu}$ can be obtained directly from $F^{\mu\nu}$ by the substitution $\mathbf{E}/c \to \mathbf{B}$, $\mathbf{B} \to -\mathbf{E}/c$. Notice that this operation leaves Eq. 12.108 unchanged—that's why both tensors generate the correct transformation rules for **E** and **B.**

Problem 12.48 Work out the remaining five parts to Eq. 12.117.

Problem 12.49 Prove that the symmetry (or antisymmetry) of a tensor is preserved by Lorentz transformation (that is: if $t^{\mu\nu}$ is symmetric, show that $\bar{t}^{\mu\nu}$ is also symmetric, and likewise for antisymmetric).

Problem 12.50 Recall that a *covariant* 4-vector is obtained from a *contravariant* one by changing the sign of the zeroth component. The same goes for tensors: When you "lower an index" to make it covariant, you change the sign if that index is zero. Compute the tensor invariants

$$F^{\mu\nu}F_{\mu\nu}, \quad G^{\mu\nu}G_{\mu\nu}, \quad \text{and} \quad F^{\mu\nu}G_{\mu\nu},$$

in terms of **E** and **B.** Compare Prob. 12.46.

Problem 12.51 A straight wire along the z axis carries a charge density λ traveling in the $+z$ direction at speed v. Construct the field tensor and the dual tensor at the point $(x, 0, 0)$.

12.3.4 Electrodynamics in Tensor Notation

Now that we know how to represent the fields in relativistic notation, it is time to reformulate the laws of electrodynamics (Maxwell's equations and the Lorentz force law) in that language. To begin with, we must determine how the *sources* of the fields, ρ and **J**, transform. Imagine a cloud of charge drifting by; we concentrate on an infinitesimal volume V, which contains charge Q moving at velocity **u** (Fig. 12.43). The charge density is

$$\rho = \frac{Q}{V},$$

and the current density[16] is

$$\mathbf{J} = \rho\mathbf{u}.$$

[16] I'm assuming all the charge in V is of one sign, and it all goes at the same speed. If not, you have to treat the constituents separately: $\mathbf{J} = \rho_+\mathbf{u}_- + \rho_-\mathbf{u}_-$. But the argument is the same.

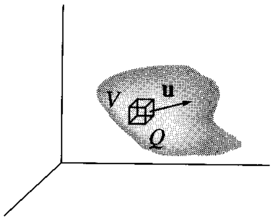


Figure 12.43

I would like to express these quantities in terms of the **proper charge density** ρ_0, the density *in the rest system of the charge:*

$$\rho_0 = \frac{Q}{V_0},$$

where V_0 is the rest volume of the chunk. Because one dimension (the one along the direction of motion) is Lorentz-contracted,

$$V = \sqrt{1 - u^2/c^2}\, V_0, \tag{12.120}$$

and hence

$$\rho = \rho_0 \frac{1}{\sqrt{1 - u^2/c^2}}, \quad \mathbf{J} = \rho_0 \frac{\mathbf{u}}{\sqrt{1 - u^2/c^2}}. \tag{12.121}$$

Comparing this with Eqs. 12.40 and 12.42, we recognize here the components of *proper velocity,* multiplied by the invariant ρ_0. Evidently charge density and current density go together to make a 4-vector:

$$J^\mu = \rho_0 \eta^\mu, \tag{12.122}$$

whose components are

$$\boxed{J^\mu = (c\rho, J_x, J_y, J_z,).} \tag{12.123}$$

We'll call it the **current density 4-vector.**

The continuity equation (Eq. 5.29),

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t},$$

expressing the local conservation of charge, takes on a nice compact form when written in terms of J^μ. For

$$\nabla \cdot \mathbf{J} = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} = \sum_{i=1}^{3} \frac{\partial J^i}{\partial x^i},$$

while

$$\frac{\partial \rho}{\partial t} = \frac{1}{c}\frac{\partial J^0}{\partial t} = \frac{\partial J^0}{\partial x^0}. \tag{12.124}$$

while

$$\frac{\partial \rho}{\partial t} = \frac{1}{c}\frac{\partial J^0}{\partial t} = \frac{\partial J^0}{\partial x^0}. \tag{12.124}$$

Thus, bringing $\partial\rho/\partial t$ over to the left side, we have:

$$\boxed{\frac{\partial J^\mu}{\partial x^\mu} = 0,} \tag{12.125}$$

with summation over μ implied. Incidentally, $\partial J^\mu/\partial x^\mu$ is the four-dimensional *divergence* of J^μ, so the continuity equation states that the current density 4-vector is divergenceless.

As for Maxwell's equations, they can be written

$$\boxed{\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu, \quad \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0,} \tag{12.126}$$

with summation over ν implied. Each of these stands for four equations—one for every value of μ. If $\mu = 0$, the first equation reads

$$\begin{aligned}
\frac{\partial F^{0\nu}}{\partial x^\nu} &= \frac{\partial F^{00}}{\partial x^0} + \frac{\partial F^{01}}{\partial x^1} + \frac{\partial F^{02}}{\partial x^2} + \frac{\partial F^{03}}{\partial x^3} \\
&= \frac{1}{c}\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) = \frac{1}{c}(\nabla \cdot \mathbf{E}) \\
&= \mu_0 J^0 = \mu_0 c\rho,
\end{aligned}$$

or

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho.$$

This, of course, is Gauss's law. If $\mu = 1$, we have

$$\begin{aligned}
\frac{\partial F^{1\nu}}{\partial x^\nu} &= \frac{\partial F^{10}}{\partial x^0} + \frac{\partial F^{11}}{\partial x^1} + \frac{\partial F^{12}}{\partial x^2} + \frac{\partial F^{13}}{\partial x^3} \\
&= -\frac{1}{c^2}\frac{\partial E_x}{\partial t} + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \left(-\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{B}\right)_x \\
&= \mu_0 J^1 = \mu_0 J_x.
\end{aligned}$$

Combining this with the corresponding results for $\mu = 2$ and $\mu = 3$ gives

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t},$$

which is Ampère's law with Maxwell's correction.

Meanwhile, the second equation in 12.126, with $\mu = 0$, becomes

$$\begin{aligned}\frac{\partial G^{0\nu}}{\partial x^\nu} &= \frac{\partial G^{00}}{\partial x^0} + \frac{\partial G^{01}}{\partial x^1} + \frac{\partial G^{02}}{\partial x^2} + \frac{\partial G^{03}}{\partial x^3} \\ &= \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = \nabla \cdot \mathbf{B} = 0\end{aligned}$$

(the third of Maxwell's equations), whereas $\mu = 1$ yields

$$\begin{aligned}\frac{\partial G^{1\nu}}{\partial x^\nu} &= \frac{\partial G^{10}}{\partial x^0} + \frac{\partial G^{11}}{\partial x^1} + \frac{\partial G^{12}}{\partial x^2} + \frac{\partial G^{13}}{\partial x^3} \\ &= -\frac{1}{c}\frac{\partial B_x}{\partial t} - \frac{1}{c}\frac{\partial E_z}{\partial y} + \frac{1}{c}\frac{\partial E_y}{\partial z} = -\frac{1}{c}\left(\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E}\right)_x = 0.\end{aligned}$$

So, combining this with the corresponding results for $\mu = 2$ and $\mu = 3$,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

which is Faraday's law. In relativistic notation, then, Maxwell's four rather cumbersome equations reduce to two delightfully simple ones.

In terms of $F^{\mu\nu}$ and the proper velocity η^μ, the *Minkowski* force on a charge q is given by

$$\boxed{K^\mu = q\eta_\nu F^{\mu\nu}.} \tag{12.127}$$

For if $\mu = 1$, we have

$$\begin{aligned}K^1 &= q\eta_\nu F^{1\nu} = q(-\eta^0 F^{10} + \eta^1 F^{11} + \eta^2 F^{12} + \eta^3 F^{13}) \\ &= q\left[\frac{-c}{\sqrt{1-u^2/c^2}}\left(\frac{-E_x}{c}\right) + \frac{u_y}{\sqrt{1-u^2/c^2}}(B_z) + \frac{u_z}{\sqrt{1-u^2/c^2}}(-B_y)\right] \\ &= \frac{q}{\sqrt{1-u^2/c^2}}[\mathbf{E} + (\mathbf{u} \times \mathbf{B})]_x,\end{aligned}$$

with a similar formula for $\mu = 2$ and $\mu = 3$. Thus,

$$\mathbf{K} = \frac{q}{\sqrt{1-u^2/c^2}}[\mathbf{E} + (\mathbf{u} \times \mathbf{B})], \tag{12.128}$$

and therefore, referring back to Eq. 12.70,

$$\mathbf{F} = q[\mathbf{E} + (\mathbf{u} \times \mathbf{B})],$$

which is the Lorentz force law. Equation 12.127, then, represents the Lorentz force law in relativistic notation. I'll leave for you the interpretation of the zeroth component (Prob. 12.54).

Problem 12.52 Obtain the continuity equation (12.125) directly from Maxwell's equations (12.126).

Problem 12.53 Show that the second equation in (12.126) can be expressed in terms of the field tensor $F^{\mu\nu}$ as follows:

$$\frac{\partial F_{\mu\nu}}{\partial x^\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x^\mu} + \frac{\partial F_{\lambda\mu}}{\partial x^\nu} = 0. \tag{12.129}$$

Problem 12.54 Work out, and interpret physically, the $\mu = 0$ component of the electromagnetic force law, Eq. 12.127.

12.3.5 Relativistic Potentials

From Chapter 10 we know that the electric and magnetic fields can be expressed in terms of a scalar potential V and a vector potential $\mathbf{A}$:

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}. \tag{12.130}$$

As you might guess, V and $\mathbf{A}$ together constitute a 4-vector:

$$\boxed{A^\mu = (V/c, A_x, A_y, A_z).} \tag{12.131}$$

In terms of this **4-vector potential** the field tensor can be written

$$\boxed{F^{\mu\nu} = \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu}.} \tag{12.132}$$

(Observe that the differentiation is with respect to the *covariant* vectors x_μ and x_ν; remember, that changes the sign of the zeroth component: $x_0 = -x^0$. See Prob. 12.55.)

To check that Eq. 12.132 is equivalent to Eq. 12.130, let's evaluate a few terms explicitly. For $\mu = 0$, $\nu = 1$,

$$\begin{aligned} F^{01} &= \frac{\partial A^1}{\partial x_0} - \frac{\partial A^0}{\partial x_1} = -\frac{\partial A_x}{\partial (ct)} - \frac{1}{c}\frac{\partial V}{\partial x} \\ &= -\frac{1}{c}\left(\frac{\partial \mathbf{A}}{\partial t} + \nabla V\right)_x = \frac{E_x}{c}. \end{aligned}$$

That (and its companions with $\nu = 2$ and $\nu = 3$) is the first equation in 12.130. For $\mu = 1$, $\nu = 2$, we get

$$F^{12} = \frac{\partial A^2}{\partial x_1} - \frac{\partial A^1}{\partial x_2} = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = (\nabla \times \mathbf{A})_z = B_z,$$

which (together with the corresponding results for F^{13} and F^{23}) is the second equation in 12.130.

The potential formulation automatically takes care of the homogeneous Maxwell equation ($\partial G^{\mu\nu}/\partial x^\nu = 0$). As for the inhomogeneous equation ($\partial F^{\mu\nu}/\partial x^\nu = \mu_0 J^\mu$), that becomes

$$\frac{\partial}{\partial x_\mu}\left(\frac{\partial A^\nu}{\partial x^\nu}\right) - \frac{\partial}{\partial x_\nu}\left(\frac{\partial A^\mu}{\partial x^\nu}\right) = \mu_0 J^\mu. \tag{12.133}$$

This is an intractable equation as it stands. However, you will recall that the potentials are not uniquely determined by the fields—in fact, it's clear from Eq. 12.132 that you could add to A^μ the gradient of any scalar function λ:

$$A^\mu \longrightarrow A^{\mu\prime} = A^\mu + \frac{\partial \lambda}{\partial x_\mu}, \tag{12.134}$$

without changing $F^{\mu\nu}$. This is precisely the **gauge invariance** we noted in Chapter 11; we can exploit it to simplify Eq. 12.133. In particular, the Lorentz gauge condition (Eq. 10.12)

$$\nabla \cdot \mathbf{A} = -\frac{1}{c^2}\frac{\partial V}{\partial t}$$

becomes, in relativistic notation,

$$\frac{\partial A^\mu}{\partial x^\mu} = 0. \tag{12.135}$$

In the Lorentz gauge, therefore, Eq. 12.133 reduces to

$$\boxed{\Box^2 A^\mu = -\mu_0 J^\mu,} \tag{12.136}$$

where $\Box^2$ is the **d'Alembertian**,

$$\Box^2 \equiv \frac{\partial}{\partial x_\nu}\frac{\partial}{\partial x^\nu} = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}. \tag{12.137}$$

Equation 12.136 combines our previous results into a single 4-vector equation—it represents the most elegant (and the simplest) formulation of Maxwell's equations.[17]

[17] Incidentally, the *Coulomb* gauge is a *bad* one, from the point of view of relativity, because its defining condition, $\nabla\cdot\mathbf{A} = 0$, is destroyed by Lorentz transformation. To restore this condition, it is necessary to perform an appropriate gauge transformation every time you go to a new inertial system, in *addition* to the Lorentz transformation itself. In this sense A^μ is not a true 4-vector, in the Coulomb gauge.

Problem 12.55 You may have noticed that the **four-dimensional gradient** operator $\partial/\partial x^{\mu}$ functions like a *covariant* 4-vector—in fact, it is often written ∂_{μ}, for short. For instance, the continuity equation, $\partial_{\mu} J^{\mu} = 0$, has the form of an invariant product of two vectors. The corresponding *contravariant* gradient would be $\partial^{\mu} \equiv \partial x_{\mu}$. *Prove* that $\partial^{\mu}\phi$ is a (contravariant) 4-vector, if ϕ is a scalar function, by working out its transformation law, using the chain rule.

Problem 12.56 Show that the potential representation (Eq. 12.132) automatically satisfies $\partial G^{\mu\nu}/\partial x^{\nu} = 0$. [*Suggestion:* Use Prob. 12.53.]

More Problems on Chapter 12

Problem 12.57 Inertial system $\bar{S}$ moves at constant velocity $\mathbf{v} = \beta c(\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}})$ with respect to S. Their axes are parallel to one another, and their origins coincide at $t = \bar{t} = 0$, as usual. Find the Lorentz transformation matrix Λ (Eq. 12.25).

$$\left[Answer: \begin{pmatrix} \gamma & -\gamma\beta\cos\phi & -\gamma\beta\sin\phi & 0 \\ -\gamma\beta\cos\phi & (\gamma\cos^2\phi + \sin^2\phi) & (\gamma-1)\sin\phi\cos\phi & 0 \\ -\gamma\beta\sin\phi & (\gamma-1)\sin\phi\cos\phi & (\gamma\sin^2\phi + \cos^2\phi) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right]$$

Problem 12.58 Calculate the **threshold** (minimum) momentum the pion must have in order for the process $\pi + p \to K + \Sigma$ to occur. The proton p is initially at rest. Use $m_{\pi}c^2 = 150$, $m_K c^2 = 500$, $m_p c^2 = 900$, $m_{\Sigma}c^2 = 1200$ (all in MeV). [*Hint:* To formulate the threshold condition, examine the collision in the center-of-momentum frame (Prob. 12.30). *Answer:* 1133 MeV/c]

Problem 12.59 A particle of mass m collides elastically with an identical particle at rest. *Classically,* the outgoing trajectories always make an angle of 90°. Calculate this angle *relativistically,* in terms of ϕ, the scattering angle, and v, the speed, in the center-of-momentum frame. [*Answer:* $\tan^{-1}(2c^2/v^2\gamma\sin\phi)$]

Problem 12.60 Find x as a function of t for motion starting from rest at the origin under the influence of a constant *Minkowski* force in the x direction. Leave your answer in implicit form (t as a function of x). [*Answer:* $2Kt/mc = z\sqrt{1+z^2} + \ln(z + \sqrt{1+z^2})$, where $z \equiv \sqrt{2Kx/mc^2}$]

! **Problem 12.61** An electric dipole consists of two point charges ($\pm q$), each of mass m, fixed to the ends of a (massless) rod of length d. (Do *not* assume d is small.)

(a) Find the net self-force on the dipole when it undergoes hyperbolic motion (Eq. 12.62) along a line perpendicular to its axis. [*Hint:* Start by appropriately modifying Eq. 11.90.]

(b) Notice that this self-force is *constant* (t drops out), and points in the direction of motion—just right to *produce* hyperbolic motion. Thus it is possible for the dipole to undergo *self-sustaining accelerated motion* with no external force at all![18] [Where do you suppose the energy comes from?] Determine the self-sustaining force, F, in terms of m, q, and d. [*Answer:* $(2mc^2/d)\sqrt{(\mu_0 q^2/8\pi m d)^{2/3} - 1}$]

[18] F. H. J. Cornish, *Am. J. Phys.* **54**, 166 (1986).

Problem 12.62 An ideal magnetic dipole moment $\mathbf{m}$ is located at the origin of an inertial system $\bar{\mathcal{S}}$ that moves with speed v in the x direction with respect to inertial system $\mathcal{S}$. In $\bar{\mathcal{S}}$ the vector potential is

$$\bar{\mathbf{A}} = \frac{\mu_0}{4\pi}\frac{\bar{\mathbf{m}} \times \hat{\bar{\mathbf{r}}}}{\bar{r}^2},$$

(Eq. 5.83), and the electric potential $\bar{V}$ is zero.

(a) Find the scalar potential V in $\mathcal{S}$. [*Answer:* $(1/4\pi\epsilon_0)(\hat{\mathbf{R}} \cdot (\mathbf{v} \times \mathbf{m})/c^2R^2)(1 - v^2/c^2)/(1 - (v^2/c^2)\sin^2\theta)^{3/2}$]

(b) In the nonrelativistic limit, show that the scalar potential in $\mathcal{S}$ is that of an ideal *electric* dipole of magnitude

$$\mathbf{p} = \frac{\mathbf{v} \times \mathbf{m}}{c^2},$$

located at $\bar{\mathcal{O}}$.

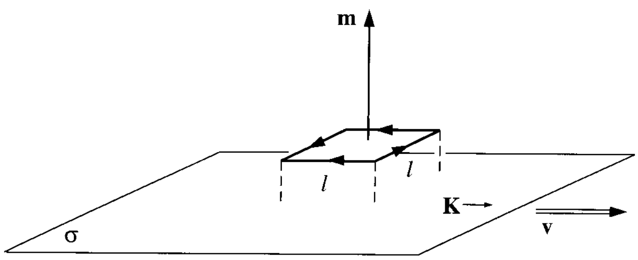


Figure 12.44

! **Problem 12.63** A stationary magnetic dipole, $\mathbf{m} = m\,\hat{\mathbf{z}}$, is situated above an infinite uniform surface current, $\mathbf{K} = K\,\hat{\mathbf{x}}$ (Fig. 12.44).

(a) Find the torque on the dipole, using Eq. 6.1.

(b) Suppose that the surface current consists of a uniform surface charge σ, moving at velocity $\mathbf{v} = v\,\hat{\mathbf{x}}$, so that $\mathbf{K} = \sigma\mathbf{v}$, and the magnetic dipole consists of a uniform line charge λ, circulating at speed v (same v) around a square loop of side l, as shown, so that $m = \lambda v l^2$. Examine the same configuration from the point of view of system $\bar{\mathcal{S}}$, moving in the x direction at speed v. In $\bar{\mathcal{S}}$ the surface charge is at *rest*, so it generates no magnetic field. Show that in this frame the current loop carries an *electric* dipole moment, and calculate the resulting torque, using Eq. 4.4.

Problem 12.64 In a certain inertial frame $\mathcal{S}$, the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ are neither parallel nor perpendicular, at a particular space-time point. Show that in a different inertial system $\bar{\mathcal{S}}$, moving relative to $\mathcal{S}$ with velocity $\mathbf{v}$ given by

$$\frac{\mathbf{v}}{1 + v^2/c^2} = \frac{\mathbf{E} \times \mathbf{B}}{B^2 + E^2/c^2},$$

the fields $\bar{\mathbf{E}}$ and $\bar{\mathbf{B}}$ are *parallel* at that point. Is there a frame in which the two are *perpendicular*?

Problem 12.65 Two charges $\pm q$ approach the origin at constant velocity from opposite directions along the x axis. They collide and stick together, forming a neutral particle at rest. Sketch the electric field before and shortly after the collision (remember that electromagnetic "news" travels at the speed of light). How would you interpret the field after the collision, physically?[19]

Problem 12.66 "Derive" the Lorentz force law, as follows: Let charge q be at rest in $\bar{S}$, so $\bar{\mathbf{F}} = q\bar{\mathbf{E}}$, and let $\bar{S}$ move with velocity $\mathbf{v} = v\,\hat{\mathbf{x}}$ with respect to S. Use the transformation rules (Eqs. 12.68 and 12.108) to rewrite $\bar{\mathbf{F}}$ in terms of $\mathbf{F}$, and $\bar{\mathbf{E}}$ in terms of $\mathbf{E}$ and $\mathbf{B}$. From these deduce the formula for $\mathbf{F}$ in terms of $\mathbf{E}$ and $\mathbf{B}$.

Problem 12.67 A charge q is released from rest at the origin, in the presence of a uniform electric field $\mathbf{E} = E_0\hat{\mathbf{z}}$ and a uniform magnetic field $\mathbf{B} = B_0\,\hat{\mathbf{x}}$. Determine the trajectory of the particle by transforming to a system in which $\mathbf{E} = 0$, finding the path in that system and then transforming back to the original system. Assume $E_0 < cB_0$. Compare your result with Ex. 5.2.

Problem 12.68

(a) Construct a tensor $D^{\mu\nu}$ (analogous to $F^{\mu\nu}$), out of $\mathbf{D}$ and $\mathbf{H}$. Use it to express Maxwell's equations inside matter in terms of the free current density J_f^{μ}. [*Answer:* $D^{01} \equiv cD_x$, $D^{12} \equiv H_z$, etc.; $\partial D^{\mu\nu}/\partial x^{\nu} = J_f^{\mu}$.]

(b) Construct the dual tensor $H^{\mu\nu}$ (analogous to $G^{\mu\nu}$). [*Answer:* $H^{01} \equiv H_x$, $H^{12} \equiv -cD_z$, etc.]

(c) Minkowski proposed the **relativistic constitutive relations** for linear media:

$$D^{\mu\nu}\eta_{\nu} = c^2\epsilon F^{\mu\nu}\eta_{\nu} \quad \text{and} \quad H^{\mu\nu}\eta_{\nu} = \frac{1}{\mu}G^{\mu\nu}\eta_{\nu},$$

where ϵ is the proper[20] permittivity, μ is the proper permeability, and η^{μ} is the 4-velocity of the material. Show that Minkowski's formulas reproduce Eqs. 4.32 and 6.31, when the material is at rest.

(d) Work out the formulas relating $\mathbf{D}$ and $\mathbf{H}$ to $\mathbf{E}$ and $\mathbf{B}$ for a medium moving with (ordinary) velocity $\mathbf{u}$.

! **Problem 12.69** Use the Larmor formula (Eq. 11.70) and special relativity to derive the Liénard formula (Eq. 11.73).

Problem 12.70 The natural relativistic generalization of the Abraham-Lorentz formula (Eq. 11.80) would seem to be

$$K^{\mu}_{\text{rad}} = \frac{\mu_0 q^2}{6\pi c}\frac{d\alpha^{\mu}}{d\tau}.$$

This is certainly a 4-vector, and it reduces to the Abraham-Lorentz formula in the non-relativistic limit $v \ll c$.

[19] See E. M. Purcell, *Electricity and Magnetism*, 2d ed. (New York: McGraw-Hill, 1985), Sect. 5.7 and Appendix B (in which Purcell obtains the Larmor formula by masterful analysis of a similar geometrical construction), and R. Y. Tsien, *Am. J. Phys.* **40**, 46 (1972).

[20] As always, "proper" means "in the rest frame of the material."

(a) Show, nevertheless, that this is not a possible Minkowski force. [*Hint:* See Prob. 12.38d.]

(b) Find a correction term that, when added to the right side, removes the objection you raised in (a), without affecting the 4-vector character of the formula or its nonrelativistic limit.[21]

Problem 12.71 Generalize the laws of relativistic electrodynamics (Eqs. 12.126 and 12.127) to include magnetic charge. [Refer to Sect. 7.3.4.]

[21]For interesting commentary on the relativistic radiation reaction, see F. Rohrlich, *Am. J. Phys.* **65**, 1051 (1997).