Campo scalare $\varphi(x)$ (vale il principio di sovrapposizione)

variazione $\begin{cases} x'^\mu = x^\mu + \delta x^\mu \\ \varphi'(x') = \varphi(x) + \delta'\varphi \end{cases}$, $\quad \delta'\varphi \doteq \varphi'(x') - \varphi(x)$

$$\delta'\varphi = \varphi'(x') - \varphi(x)$$

$$= \varphi'(x+\delta x) - \varphi(x)$$

$\delta' = \delta + \delta x^\mu \partial_\mu$

$$= \underbrace{\varphi'(x)} + \partial_\mu \varphi'(x)\,\delta x^\mu \underbrace{-\varphi(x)}$$

$\delta\varphi \doteq \varphi'(x) - \varphi(x)$

$\partial_\mu \varphi'(x) = \partial_\mu \varphi(x) + \underbrace{o(\partial^2\varphi(x))}$ ordine successivo

$$= \delta\varphi + \partial_\mu\varphi\,\delta x^\mu$$

Teorema: se un sistema è invariante per trasformazioni continue a $n$ parametri, allora esistono $n$ 4-correnti conservate

$$\underbrace{\delta x^\mu}_{\text{infinitesimo}} = \sum_{a=1}^{n} \omega_a \underbrace{\bar{\delta}_a x^\mu}_{\text{finito}}, \quad \omega_a \in \mathbb{R}$$

$$\lim_{\omega_a\to 0} \delta x = \lim_{\omega_a \to 0} \delta\varphi = 0$$

$$\delta\varphi = \sum_{a=1}^{n} \omega_a \bar{\delta}_a \varphi$$

invarianza $\iff \delta' S = 0$

$$S \doteq \int d^4x\, \mathcal{L} \quad \Longrightarrow \quad \delta' S = \int d^4x\, \delta'\mathcal{L} + \int \mathcal{L}\, \delta'(d^4x) = 0$$

$$\int \mathcal{L}\, \delta'(d^4x) = \int \mathcal{L}\, \delta(d^4x)$$

$\delta' x^\mu = \delta x$

$$\delta'\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\varphi}\delta'\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta'(\partial_\mu\varphi) + \partial_\mu\mathcal{L}\,\delta x^\mu$$

abbiamo già dimostrato che

$\delta(\partial_\mu\varphi) = \partial_\mu(\delta\varphi)$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\partial_\mu(\delta'\varphi) = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta'\varphi\right) - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right)\delta'\varphi$$

$$\delta' \mathcal{L} = \left( \frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \delta' \varphi + \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta' \varphi \right) + \partial_\mu \mathcal{L} \, \delta x^\mu$$

$$d^4x' = |J| \, d^4x , \qquad J = \left[ \frac{\partial x'^\mu}{\partial x^\nu} \right] \qquad \text{ma} \qquad \frac{\partial x'^\mu}{\partial x^\nu} = \frac{\partial x^\mu}{\partial x^\nu} + \frac{\partial}{\partial x^\nu} (\delta x^\mu)$$

$$|J| \doteq \det(J)$$

$$= g^\mu{}_\nu + \partial_\nu (\delta x^\mu)$$

$$= \delta^\mu_\nu + \delta(\partial_\nu x^\mu)$$

Siccome

$$2D: \begin{vmatrix} 1+\partial_0 \delta x^0 & 1+\partial_1 \delta x^0 \\ 1+\partial_0 \delta x^1 & 1+\partial_1 \delta x^1 \end{vmatrix} = (1+\partial_0 \delta x^0)(1+\partial_1 \delta x^1) - (1+\partial_1 \delta x^0)(1+\partial_0 \delta x^1)$$

$$= 1 + \partial_0 \delta x^0 + \partial_1 \delta x^1 + o(\partial^2 \delta x)$$

$$\Longrightarrow \quad |J| = 1 + \partial_\mu \delta x^\mu$$

$$\delta' S = \int d^4x \left( \left( \frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \delta' \varphi + \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta' \varphi \right) + \overbrace{\partial_\mu \mathcal{L} \, \delta x^\mu + \mathcal{L} \, \partial_\mu (\delta x^\mu)}^{\partial_\mu(\mathcal{L} \delta x^\mu)} \right)$$

$$= \int d^4x \left( \underbrace{\left( \frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right)}_{=0 \quad \text{eqq } E\text{-}L} \delta' \varphi + \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta' \varphi + \mathcal{L} \, \delta x^\mu \right) \right) = 0$$

$$\Longrightarrow \quad \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta' \varphi + \mathcal{L} \, \delta x^\mu \right) = 0$$

ha la forma di un'eq di continuità

Sostituiamo le trasformazioni:

$$\begin{cases} \delta x^\mu = \sum_{a=1}^{n} \omega_a \, \bar{\delta}_a x^\mu \\ \delta \varphi = \sum_{a=1}^{n} \omega_a \, \bar{\delta}_a \varphi \end{cases}$$

$$\sum_a \partial_\mu\left(\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\,\delta_a\varphi + \mathcal{L}\,\delta_a x^\mu\right)\omega_a\right)=0$$

siccome $\partial_\mu \omega_a = 0$ abbiamo che

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\,\bar{\delta}_a\varphi + \mathcal{L}\,\bar{\delta}_a x^\mu\right)=0, \quad \forall a\in[1,n]$$

dove ci sono $n$ correnti di Noether conservate:

$$j_a^\mu \doteq \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\,\bar{\delta}_a\varphi + \mathcal{L}\,\bar{\delta}_a x^\mu$$

cariche di Noether: $Q_a \doteq \int_\sigma d\sigma_\mu\, j_a^\mu$ sono conservate

se $d\sigma_\mu = (d^3x, 0, 0, 0)$ $\quad Q_a = \int d^3x\, j_a^0 \Rightarrow \frac{dQ_a}{dt}=0$

$$\int d^3x\,\partial_\mu j^\mu = \frac{\partial}{\partial t}\int d^3x\, j^0 + \int d^3x\,\bar{\nabla}\cdot\bar{J} = 0 \quad \because\ \partial_\mu j^\mu = 0$$

$$\Longrightarrow \quad \frac{\partial Q}{\partial t} = -\int d\bar{\sigma}\cdot\bar{J}$$

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Esempio: traslazioni $\qquad x'^\mu = x^\mu + \varepsilon^\mu$ $\quad$ 4 cost. infinitesime

$$\delta x^\mu = \varepsilon^\mu = g^\mu{}_\nu \varepsilon^\nu$$
$$= \varepsilon^\nu \underbrace{\bar{\delta}_\nu x^\mu}_{g^\mu{}_\nu}$$

$\varphi'(x') = \varphi(x)$

non $\varphi'(x) = \varphi(x)$ perché abbiamo solo ridefinito il nome delle coordinate

Per il campo abbiamo solo una variazione funzionale (cambia la forma ma conserva il suo valore):

$$\delta\varphi(x) = \varphi'(x) - \varphi(x)$$
$$= \cancel{\delta'\varphi} - \delta x^\nu \partial_\nu\varphi$$
$$= -\delta x^\nu \partial_\nu \varphi$$

ma $\delta\varphi = \varepsilon^\nu \bar{\delta}_\nu \varphi$

$$\Downarrow$$

$$\bar{\delta}_\nu\varphi = -\partial_\nu\varphi$$

$= -\varepsilon^\nu \partial_\nu \varphi$

Traslazione : $d^4x' = d^4x$ (volume invariato)

$\nu$ indice delle correnti corrisponde a quello delle coordinate

$$(j_\alpha)^\mu \equiv (j_\nu)^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \bar{\delta}_\nu \varphi + \mathcal{L}\, \bar{\delta}_\nu x^\mu$$

$$= -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\nu \varphi + \mathcal{L}\, g^\mu{}_\nu \doteq -T^\mu{}_\nu$$

$$T^{\mu\nu} \doteq \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial^\nu \varphi - \mathcal{L} g^{\mu\nu} \implies \partial_\mu T^{\mu\nu} = 0$$

$d\sigma_\mu (d^3x, 0, 0, 0)$

$$P^\nu(t) = \int_\sigma d\sigma_\mu T^{\mu\nu} \doteq \int d^3x\, T^{0\nu}$$

$\dot{\varphi} = \frac{\partial \varphi}{\partial t}$

$$T^{00} = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \dot{\varphi} - \mathcal{L} \overbrace{g^{00}}^{1} = \mathcal{H} \quad \text{densità di energia}$$

invarianza traslazione temporale $\implies$ conservazione energia

$$T^{0i} = \frac{\partial \mathcal{L}}{\partial \varphi_i'} \partial^i \varphi - \cancel{\mathcal{L} g^{0i}} = p^i$$

invarianza traslazione spaziale $\implies$ conservazione impulso/momento

$$\begin{cases} T^{i0} & \text{flusso di densità di energia} \\ T^{ij} & \text{flusso di componente } j \text{ lungo } i \text{ di } \bar{p} \end{cases}$$

tensore di stress