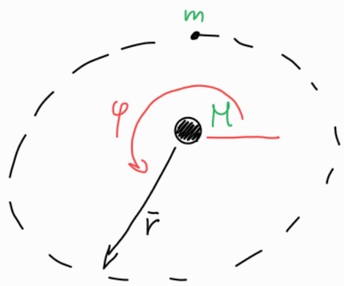
Moto di una particella in potenziale $V = V(r)$

momento angolare $\bar{L} = \bar{r} \wedge \bar{p}$ con $\bar{p} = \gamma m \bar{v}$



$$\begin{cases} M \gg m \\ \dot{V} = 0 \end{cases}$$

moto sul piano

$\Downarrow$

$\dot{L} = 0$

coordinate cilindriche
$$\bar{L} = \bar{r} \wedge \left(\dot{r}\hat{r} + r\dot{\varphi}\hat{\varphi}\right)\gamma m$$
$$= m\gamma r^2 \dot{\varphi}\hat{z}$$

$$\mathcal{E} = m\gamma c^2 + V(r)$$
$$= mc^2\left(1 - \frac{(\dot{r}\hat{r} + r\dot{\varphi}\hat{\varphi})^2}{c^2}\right)^{-1/2} + V(r)$$
$$= \frac{mc^2}{\sqrt{1 - \frac{\dot{r}^2 + r^2\dot{\varphi}^2}{c^2}}} + V(r)$$

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\varphi}\frac{d\varphi}{dt} \doteq r'\dot{\varphi}$$

$$\bar{L} = m\gamma r^2 \dot{\varphi}\hat{z} \Rightarrow \dot{\varphi} = \frac{L}{m\gamma r^2}$$
$$= \frac{L}{mr^2}\sqrt{1 - \frac{(r'^2 + r^2)\dot{\varphi}^2}{c^2}}$$

$$\mathcal{E} = \frac{mc^2}{\left(1 - \frac{(r'^2 + r^2)\dot{\varphi}^2}{c^2}\right)^{1/2}} + V(r)$$

$$\dot{\varphi}^2 = \frac{L^2}{m^2 r^4}\left(1 - \frac{(r'^2 + r^2)\dot{\varphi}^2}{c^2}\right)$$

$$\frac{m^2 r^4}{L^2}\dot{\varphi}^2 = 1 - \left(\frac{r'^2 + r^2}{c^2}\right)\dot{\varphi}^2$$

$$\left(\frac{m^2 r^4}{L^2} + \frac{r'^2 + r^2}{c^2}\right)\dot{\varphi}^2 = 1 \longrightarrow \frac{\dot{\varphi}^2}{c^2} = \frac{L^2}{m^2c^2r^4 + (r'^2 + r^2)L^2}$$

$$\mathcal{E} = mc^2\left(1 - \frac{(r'^2 + r^2)\dot{\varphi}^2}{c^2}\right)^{-1/2} + V(r)$$

$$= mc^2\left(1 - (r'^2 + r^2)\frac{L^2}{m^2c^2r^4 + (r'^2 + r^2)L^2}\right)^{-1/2} + V(r)$$

$$= mc^2 \left(\frac{m^2c^2r^4 + \cancel{(r'^2+r^2)L^2} - \cancel{(r'^2+r^2)L^2}}{m^2c^2r^4+(r'^2+r^2)L^2} \right) + V(r)$$

$$= mc^2 \sqrt{\frac{m^2c^2r^4+(r'^2+r^2)L^2}{m^2c^2r^4}} + V(r)$$

$$= mc^2 \sqrt{1 + \frac{(r'^2+r^2)L^2}{m^2c^2r^4}} + V(r)$$

$$(\mathcal{E}-V)^2 = m^2c^4 \left(1 + \frac{(r'^2+r^2)L^2}{m^2c^2r^4}\right)$$

$$(r'^2+r^2)\,\frac{c^2L^2}{r^4} = (\mathcal{E}-V)^2 - m^2c^4$$

$$\left(\frac{dr}{d\varphi}\right)^2 + r^2 = \frac{r^4}{c^2L^2}\left((\mathcal{E}-V)^2 - m^2c^4\right)$$

$$\frac{1}{r^4}\left(\frac{dr}{d\varphi}\right)^2 + \frac{1}{r^2} = \frac{1}{c^2L^2}\left((\mathcal{E}-V)^2 - m^2c^4\right)$$

$$\begin{cases} u \doteq \dfrac{1}{r} \\ \dfrac{du}{d\varphi} = \dfrac{d}{d\varphi}\left(\dfrac{1}{r}\right) = -\dfrac{1}{r^2}\dfrac{dr}{d\varphi} \end{cases}$$

eq Binet

$$\left(\frac{du}{d\varphi}\right)^2 + u^2 = \frac{1}{L^2c^2}\left((\mathcal{E}-V)^2 - m^2c^4\right)$$

$$V = -\frac{\alpha}{r} = -\alpha u\,, \quad \alpha \in \mathbb{R}^+$$

$$\left(\frac{du}{d\varphi}\right)^2 + u^2 = \frac{1}{L^2c^2}\left((\mathcal{E}+\alpha u)^2 - m^2c^4\right)$$

$$\frac{d}{d\varphi}$$

$$\cancel{2}\,\cancel{\frac{du}{d\varphi}}\,\frac{d^2u}{d\varphi^2} + \cancel{2}u\,\cancel{\frac{du}{d\varphi}} = \frac{1}{L^2c^2}\,\cancel{2}(\mathcal{E}+\alpha u)\,\alpha\,\cancel{\frac{du}{d\varphi}}$$

$$\frac{d^2u}{d\varphi^2} + u = \frac{\alpha}{L^2c^2}(\mathcal{E}+\alpha u)$$

$$\frac{d^2u}{d\varphi^2} + \underbrace{\left(1 - \frac{\alpha^2}{L^2c^2}\right)}_{\doteq q^2} u = \frac{\alpha\mathcal{E}}{L^2c^2}$$

$$\begin{cases} q = \sqrt{1 - \dfrac{\alpha^2}{L^2c^2}} \\ p = \dfrac{q^2L^2c^2}{\alpha\mathcal{E}} \end{cases}$$

$$\frac{d^2u}{d\varphi^2} + q^2u = \frac{q^2}{p}$$

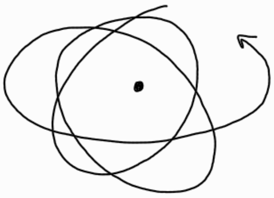
w

$$\frac{d^2u}{d\varphi^2}+q^2\left(u-\frac{1}{p}\right)=0$$

$$\frac{d^2w}{d\varphi^2}+q^2w=0 \implies w=A\cos\left(q(\varphi-\varphi_0)\right)=u-\frac{1}{p}=\frac{1}{r}-\frac{1}{p}$$

$$r=\frac{p}{1+\epsilon\cos\left(q(\varphi-\varphi_0)\right)}\quad,\quad \epsilon\doteq Ap \quad \text{eccentricità}$$

Il periodo non è 2π $\because$ $q(\varphi-\varphi_0)$: dopo 2π il raggio non è lo stesso



la precessione del perielio è un effetto relativistico

periodo $\varphi=\varphi_0+\overbrace{\frac{2\pi}{q}}^{T}$

$$\delta=\frac{2\pi}{q}-2\pi=2\pi\left(\frac{1}{q}-1\right)=2\pi\left(\left(1-\frac{\alpha^2}{L^2c^2}\right)^{-1/2}-1\right)\xrightarrow{\alpha\ll Lc}\frac{\pi\alpha^2}{L^2c^2}$$

$$q=\sqrt{1-\frac{\alpha^2}{L^2c^2}}\xrightarrow{c\to\infty}1 \quad\therefore\quad \text{no precessione}$$

gravitazionale
$\alpha=GM_1M_2$