I corpi perfettamente rigidi non esistono in quanto le interazioni si propagano alla velocità della luce.

Considereremo le particelle come corpi rigidi (puntiformi)

Le particelle sono caratterizzate dalla loro carica (invariante)

invariante

$S = -mc \int ds$ aggiungiamo la parte dell'interazione $S = \int_a^b \Big(-mc\, ds \overbrace{- \frac{e}{c} A_\mu dx^\mu}^{\text{si trova questa forma sperimentalmente}}\Big)$

4-potenziale

$A^\mu = (\varphi, \bar{A})$

pot. scalare, pot. vettore

dunque

$ds = c\, d\tau = \frac{c}{\gamma} dt$

$$S = \int_a^b -mc\, ds - e\varphi\, dt + \frac{e}{c} \bar{A} \cdot d\bar{r} \qquad S = \int_a^b L\, dt = \int_a^b \left(\left(-mc^2\sqrt{1-\frac{v^2}{c^2}} - e\varphi + \frac{e}{c}\bar{A}\cdot\bar{v}\right) dt\right)$$

$$\boxed{L = \underbrace{-mc^2\sqrt{1-\frac{v^2}{c^2}}}_{\text{libera}} + \underbrace{\frac{e}{c}\bar{A}\cdot\bar{v} - e\varphi}_{\text{interazione}}}$$

[meccanica classica

$L = L(q, \dot{q}, t)$ $\qquad H = p\dot{q} - L$

$p = \frac{\partial L}{\partial \dot{q}}$]

$$\bar{P} = \frac{\partial L}{\partial \bar{v}} = \cancel{-} m c^{\cancel{2}} \cancel{\frac{1}{2}} \left(1 - \frac{v^2}{\cancel{c^2}}\right)^{-1/2} \left(\cancel{-}\frac{\cancel{2}\bar{v}}{c^2}\right) + \frac{e}{c}\bar{A} \implies \bar{P} = m\gamma\bar{v} + \frac{e}{c}\bar{A}$$

$\bar{P} = \bar{p} + \frac{e}{c}\bar{A}$

$$H = \bar{v}\cdot\bar{p} - L = \bar{v}\cdot\left(m\gamma\bar{v} + \frac{e}{c}\bar{A}\right) + mc^2\sqrt{1-\frac{v^2}{c^2}} - \frac{e}{c}\bar{A}\cdot\bar{v} + e\varphi$$

$$= m\gamma v^2 + \cancel{\frac{e}{c}\bar{v}\cdot\bar{A}} + \frac{mc^2}{\gamma} - \cancel{\frac{e}{c}\bar{A}\cdot\bar{v}} + e\varphi = \frac{mv^2 + mc^2\left(1-\frac{v^2}{c^2}\right)}{\sqrt{1-\frac{v^2}{c^2}}} - e\varphi = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} - e\varphi$$

$\implies H = mc^2\gamma - e\varphi$ solo il campo elettrico influenza l'energia

$$(H - e\varphi)^2 = \frac{m^2c^4}{1-\frac{v^2}{c^2}} = \overbrace{E^2 = p^2c^2 + m^2c^4}^{\text{eq mass-shell}} = \left(\bar{P} - \frac{e}{c}\bar{A}\right)^2 c^2 + m^2c^4$$

$$H=\sqrt{\left(\bar{P}-\frac{e}{c}\bar{A}\right)^2c^2+m^2c^4}+e\varphi$$

Per velocità piccole $\quad L=\frac{1}{2}mv^2+\frac{e}{c}\bar{A}\cdot\bar{v}-e\varphi$

$$H=\frac{1}{2m}\left(\bar{P}-\frac{e}{c}\bar{A}\right)^2+e\varphi$$

DIMOSTRA

Equazione del moto

Consideriamo una carica piccola di prova che non influenza il campo elettrico

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \bar{v}}\right)=\frac{\partial L}{\partial \bar{r}} \qquad L=-mc^2\sqrt{1-\frac{v^2}{c^2}}+\frac{e}{c}\bar{A}\cdot\bar{v}-e\varphi$$

$\bar{v}=\bar{v}(t)$ non dipende dalle coordinate

$$\frac{\partial L}{\partial \bar{r}}=\bar{\nabla}L=\frac{e}{c}\,\bar{\nabla}(\bar{A}\cdot\bar{v})-e\bar{\nabla}\varphi$$

siccome vale

$$\bar{\nabla}(\bar{a}\cdot\bar{b})=(\bar{a}\cdot\bar{\nabla})\bar{b}+(\bar{b}\cdot\bar{\nabla})\bar{a}+\bar{a}\wedge(\bar{\nabla}\wedge\bar{b})+\bar{b}\wedge(\bar{\nabla}\wedge\bar{a})$$

$$\bar{\nabla}(\bar{A}\cdot\bar{v})=(\bar{v}\cdot\bar{\nabla})\bar{A}+\underbrace{(\bar{A}\cdot\bar{\nabla})\bar{v}}_{0}+\bar{v}\wedge(\bar{\nabla}\wedge\bar{A})+\bar{A}\wedge\underbrace{(\bar{\nabla}\wedge\bar{v})}_{0}=(\bar{v}\cdot\bar{\nabla})\bar{A}+\bar{v}\wedge(\bar{\nabla}\wedge\bar{A})$$

$$\frac{\partial L}{\partial \bar{r}}=\frac{e}{c}(\bar{v}\cdot\bar{\nabla})\bar{A}+\frac{e}{c}\bar{v}\wedge(\bar{\nabla}\wedge\bar{A})-e\bar{\nabla}\varphi$$

$$\frac{d\bar{A}}{dt}=\frac{\partial\bar{A}}{\partial t}+(\bar{v}\cdot\bar{\nabla})\bar{A}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \bar{v}}\right)=\frac{d}{dt}\left(\bar{p}+\frac{e}{c}\bar{A}\right)=\frac{d\bar{p}}{dt}+\frac{e}{c}\frac{d\bar{A}}{dt}$$

$$\cancel{\frac{e}{c}(\bar{v}\cdot\bar{\nabla})\bar{A}}+\frac{e}{c}\bar{v}\wedge(\bar{\nabla}\wedge\bar{A})-e\bar{\nabla}\varphi=\frac{d\bar{p}}{dt}+\frac{e}{c}\frac{\partial\bar{A}}{\partial t}+\cancel{\frac{e}{c}(\bar{v}\cdot\bar{\nabla})\bar{A}}$$

$$\underbrace{\frac{d\bar{p}}{dt}}_{\bar{F}} = \frac{e}{c}\left(\bar{v}\wedge(\bar{\nabla}\wedge\bar{A}) - \frac{\partial A}{\partial t}\right) - e\bar{\nabla}\varphi$$

$$\begin{cases} \bar{E} = -\frac{1}{c}\frac{\partial A}{\partial t} - \bar{\nabla}\varphi \\ \bar{B} = \bar{\nabla}\wedge\bar{A} \end{cases}$$

dovrebbe essere $\bar{H}$ ma siamo nel vuoto

$$\bar{F} = \frac{e}{c}\bar{v}\wedge\bar{B} + e\bar{E}$$

forza di Lorentz

$$\frac{d\mathcal{E}}{dt} = \frac{d}{dt}(mc^2\gamma) = mc^2\frac{d}{dt}\left(\left(1-\frac{v^2}{c^2}\right)^{-1/2}\right) = m\cancel{c^2}\left(-\frac{1}{2}\right)\overbrace{\left(1-\frac{v^2}{c^2}\right)^{-3/2}}^{\gamma^3} 2\left(-\frac{1}{\cancel{c^2}}\frac{d\bar{v}}{dt}\right) = m\gamma^3\,\bar{v}\cdot\frac{d\bar{v}}{dt}$$

$$\frac{d\bar{p}}{dt} = \bar{F} = \frac{d}{dt}(m\gamma\bar{v}) = m\gamma^3\left(\bar{v}\cdot\frac{d\bar{v}}{dt}\right)\frac{\bar{v}}{c^2} + m\gamma\frac{d\bar{v}}{dt} = m\gamma^3\left(\left(\bar{v}\cdot\frac{d\bar{v}}{dt}\right)\frac{\bar{v}}{c^2} + \left(1-\frac{v^2}{c^2}\right)\frac{d\bar{v}}{dt}\right)$$

$$\underbrace{\frac{d\bar{p}}{dt}\cdot\bar{v}}_{\text{potenza}} = m\gamma^3\left(\cancel{\bar{v}\cdot\frac{d\bar{v}}{dt}\frac{v^2}{c^2}} + \bar{v}\cdot\frac{d\bar{v}}{dt} - \cancel{\frac{v^2}{c^2}\bar{v}\cdot\frac{d\bar{v}}{dt}}\right) = m\gamma^3\,\bar{v}\cdot\frac{d\bar{v}}{dt} = \frac{d\mathcal{E}}{dt} \Longrightarrow \frac{d\mathcal{E}}{dt} = \frac{d\bar{p}}{dt}\cdot\bar{v}$$

potenza come in meccanica classica

$\bar{v}\cdot\bar{v}\wedge\bar{B} = 0$

$$\frac{d\mathcal{E}}{dt} = \bar{v}\cdot\bar{F} = \bar{v}\cdot e\left(\bar{E} + \bar{v}\wedge\bar{B}\right) = e\,\bar{v}\cdot\bar{E}$$

il campo magnetico non influisce sulla potenza perché perpendicolare al movimento della particella