Consideriamo la lagrangiana di Proca per il campo magnetico

$$\mathcal{L}_P = -\frac{1}{16\pi} F^{\mu\nu}F_{\mu\nu} + \underbrace{\mu^2 A^\mu A_\mu}_{\text{eventuale termine di massa del fotone}} \qquad (c=1 \text{ qui})$$

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\right) - \frac{\partial \mathcal{L}}{\partial \varphi} = 0 \longrightarrow \partial_\mu F^{\mu\nu} + \mu^2 A^\nu = \frac{4\pi}{c} j^\nu$$

però questa non è invariante di gauge

$$A^\mu \to A^\mu + \partial^\mu f \implies A^\mu A_\mu \to \underbrace{A^\mu A_\mu - A^\mu \partial_\mu f - A_\mu \partial^\mu f + \partial^\mu f \partial_\mu f}_{\text{questi non si semplificano}}$$

$$\partial_\nu \cancel{\partial_\mu F^{\mu\nu}}^{\,0} + \mu^2 \partial_\nu A^\nu = \frac{4\pi}{c} \cancel{\partial_\nu j^\nu}^{\,0} \implies \mu^2 \partial_\nu A^\nu = 0 \implies \partial_\nu A^\nu = 0 \quad \because \partial_\nu j^\nu = 0$$

gauge di Lorenz

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \underbrace{\partial_\mu \partial^\mu}_{\square} A^\nu - \partial^\nu \cancel{(\partial_\mu A^\mu)} = \frac{4\pi}{c} j^\nu \xRightarrow{j=0} \square A^\nu = 0$$

eq di d'Alembert

Tramite gli stessi passaggi:

$$\partial_\mu F^{\mu\nu} + \mu^2 A^\nu = \frac{4\pi}{c} j^\nu \xRightarrow{\partial_\nu A^\nu = 0} \square A^\nu + \mu^2 A^\nu = \frac{4\pi}{c} j^\nu$$

$$\left[\begin{array}{c} j=0 \\ (\square + \mu^2) A^\nu = 0 \\ \text{eq Klein-Gordon} \end{array}\right]$$

<u>Esempio</u>: caso statico + sorgente di carica singola

$$q(r=0) \longrightarrow \varphi \sim r^{-1}$$

segni negativi $\because \square \doteq \partial_t^2 - \nabla^2$

$$(\square + \mu^2)\varphi^\nu = \frac{4\pi}{c} j^\nu \longrightarrow \nabla^2 \varphi - \mu^2 \varphi = -\frac{4\pi}{c} q \delta^3(\bar{r})$$

ipotesi: $\varphi = q \dfrac{e^{-\mu r}}{r}$

$$\nabla^2 \varphi = \bar{\nabla} \cdot \left(\bar{\nabla}\left(q \frac{e^{-\mu r}}{r}\right)\right) = \bar{\nabla}\left(\frac{q}{r} \bar{\nabla}(e^{-\mu r}) + q e^{-\mu r} \bar{\nabla}\left(\frac{1}{r}\right)\right)$$

$$= \frac{q}{r} \nabla^2(e^{-\mu r}) + q \bar{\nabla}\left(\frac{1}{r}\right) \cdot \bar{\nabla}(e^{-\mu r}) + q \bar{\nabla}(e^{-\mu r}) \cdot \bar{\nabla}\left(\frac{1}{r}\right) + q e^{-\mu r} \nabla^2\left(\frac{1}{r}\right)$$

$$\bar{\nabla}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\frac{\bar{r}}{r} = -\frac{\hat{r}}{r^2}, \qquad \bar{\nabla}(e^{-\mu r}) = -\mu e^{-\mu r} \frac{\bar{r}}{r} = -\mu e^{-\mu r} \hat{r}$$

$$\bar{\nabla}\left(\frac{1}{r}\right)\cdot\bar{\nabla}\left(e^{-\mu r}\right)=\mu\,\frac{e^{-\mu r}}{r^2}$$

$\bar{\nabla}\cdot\bar{r}=3$

$$\nabla^2\left(e^{-\mu r}\right)=\bar{\nabla}\cdot\left(\bar{\nabla}\left(e^{-\mu r}\right)\right)=\mu^2 e^{-\mu r}\underbrace{\hat{r}\cdot\hat{r}}_{1}-\mu e^{-\mu r}\bar{r}\cdot\left(-\frac{\hat{r}}{r^2}\right)-3\mu\,\frac{e^{-\mu r}}{r}$$

$\bar{r}\cdot\hat{r}=r$

$$=\mu^2 e^{-\mu r}+\mu\,\frac{e^{-\mu r}}{r}-3\mu\,\frac{e^{-\mu r}}{r}=\mu^2 e^{-\mu r}-2\mu\,\frac{e^{-\mu r}}{r}$$

$$\nabla^2\left(\frac{1}{r}\right)=-4\pi\,\delta^3(\bar{r})$$

Ora sostituiamo in:

$$\nabla^2\varphi-\mu^2\varphi=-\frac{4\pi}{c}\,q\,\delta^3(\bar{r})\qquad \text{dove}\qquad \varphi=q\,\frac{e^{-\mu r}}{r}$$

$$\frac{q}{r}\,\nabla^2\left(e^{-\mu r}\right)+2q\,\bar{\nabla}\left(\frac{1}{r}\right)\cdot\bar{\nabla}\left(e^{-\mu r}\right)+q\,e^{-\mu r}\,\nabla^2\left(\frac{1}{r}\right)-\mu^2 q\,\frac{e^{-\mu r}}{r}=-4\pi\,q\,\delta^3(\bar{r})$$

$$\frac{q}{r}\left(\cancel{\mu^2 e^{-\mu r}}-\cancel{2\mu\,\frac{e^{-\mu r}}{r}}\right)+\cancel{2q\mu\,\frac{e^{-\mu r}}{r^2}}+q\,e^{-\mu r}\left(-4\pi\,\delta^3(\bar{r})\right)-\cancel{\mu^2 q\,\frac{e^{-\mu r}}{r}}=-4\pi\,q\,\delta^3(\bar{r})$$

$$-4\pi q\,e^{\mu r}\delta^3(\bar{r})=-\frac{4\pi}{c}\,q\,\delta^3(\bar{r})\qquad\longrightarrow\qquad\begin{cases}\bar{r}=\bar{0}\implies e^{-\mu r}=1\\ \bar{r}\neq 0\implies \delta(\bar{r})=0\end{cases}$$

$$\implies\quad \varphi=q\,\frac{e^{-\mu r}}{r}$$

potenziale di Yukawa

$$\bar{E}=-\bar{\nabla}\varphi=e^{-\mu r}\left(\frac{\mu}{r}+\frac{1}{r^2}\right)\hat{r}$$