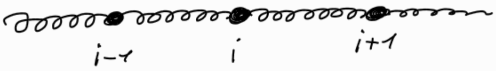
$L = T - V$ 1 particella

Esempio: molla $L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 \longrightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = \frac{d}{dt}(m\dot{x}) - kx = 0 \longrightarrow m\ddot{x} = kx$

Per tante particelle: $L = \sum_{i=1}^{n} T_i - V(x_i)$ comprende interazioni tra particelle

Per un sistema di N masse e molle



$$L_i = \frac{1}{2}\left(\sum_{i=1}^{N} m_i \dot{x}_i^2\right) - \frac{1}{2} k \left(x_{i+1}^2 - x_i^2\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0 \qquad i = 1, \dots, N$$

Per $N \to +\infty$: $(x \to \varphi$ funzione$)$

$$L = \int dx \left(\frac{1}{2} \eta \left(\frac{\partial \varphi}{\partial t}\right)^2 - \frac{1}{2} Y \left(\frac{\partial \varphi}{\partial x}\right)^2 \right)$$

densità lineare (η)

modulo di Young (tensione) (Y)

dunque $\varphi = \varphi(x,t)$ e' un campo

Consideriamo un campo $\varphi = \varphi(x,t) \longrightarrow L = L\left(\varphi, \frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial x}, x, t\right)$

densità di lagrangiana $\mathcal{L}$: $L = \int \mathcal{L}\, dx \longrightarrow S = \int L\, dt = \int \mathcal{L}\, dx\, dt$

Per una particella, il principio di minima azione da' $x^\mu(s)$ (linea di universo), in questo caso dara' $\varphi = \varphi(x,t)$

Consideriamo $\varphi(x,t,\alpha) = \varphi(x,t,0) + \alpha \overbrace{\xi(x,t)}^{\text{variazione come } \varphi' \to \varphi + \delta\varphi}$, $\quad \xi(x_1,t_1) = \xi(x_2,t_2) = 0$ variazione nulla al contorno

$\delta S = 0 \iff \left.\frac{dS}{d\alpha}\right|_{\alpha=0} = 0$ per le funzioni considerate possiamo commutare $\frac{d}{d\alpha}$ e gli integrali

$$\left.\frac{dS}{d\alpha}\right|_{\alpha=0} = \int_{(t_2,x_2)}^{(t_1,x_1)} dt\,dx \left(\frac{\partial \mathcal{L}}{\partial \varphi}\frac{\partial \varphi}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \varphi}{\partial t}\right)} \frac{\partial}{\partial \alpha}\left(\frac{\partial \varphi}{\partial t}\right) + \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \varphi}{\partial x}\right)} \frac{\partial}{\partial \alpha}\left(\frac{\partial \varphi}{\partial x}\right) \right)$$

x e t non dipendono da α $\left(\frac{\partial x}{\partial \alpha} = \frac{\partial t}{\partial \alpha} = 0\right)$

$$\int_{t_1}^{t_2} dt \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \varphi}{\partial t}\right)} \frac{\partial}{\partial \alpha}\left(\frac{\partial \varphi}{\partial t}\right) = \int_{t_1}^{t_2} dt \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \varphi}{\partial t}\right)} \frac{\partial}{\partial t}\left(\frac{\partial \varphi}{\partial \alpha}\right) \overset{\text{parti}}{=} \cancel{\left[\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \varphi}{\partial t}\right)} \underbrace{\frac{\partial \varphi}{\partial \alpha}}_{\xi}\right]_{t_2}^{t_2}} - \int_{t_1}^{t_2} dt \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \varphi}{\partial t}\right)}\right)\frac{\partial \varphi}{\partial \alpha}$$

$\xi \to \xi|_{t_1} = \xi|_{t_2} = 0$

analogamente

$$\int_{x_1}^{x_2} dx \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \varphi}{\partial x}\right)} \frac{\partial}{\partial \alpha}\left(\frac{\partial \varphi}{\partial x}\right) = -\int_{x_1}^{x_2} dx \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \varphi}{\partial x}\right)}\right) \frac{\partial \varphi}{\partial \alpha}$$

sostituendo

ξ variazione arbitraria

$$\left.\frac{dS}{d\alpha}\right|_{\alpha=0} = \int_{(t_2,x_2)}^{(t_1,x_1)} dt\,dx \left(\frac{\partial \mathcal{L}}{\partial \varphi} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \varphi}{\partial t}\right)}\right) - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \varphi}{\partial x}\right)}\right)\right) \overbrace{\frac{\partial \varphi}{\partial \alpha}}^{\xi} = 0$$

$$\Longrightarrow \quad \frac{\partial \mathcal{L}}{\partial \varphi} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \varphi}{\partial t}\right)}\right) - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \varphi}{\partial x}\right)}\right) = 0$$

eqq Eulero-Lagrange
per $\varphi = \varphi(x,t)$

Queste eqq valgono per somme di campi, a causa delle proprietà delle derivate.
Nella lagrangiana possono anche esserci termini di interazione.

Per $\mathcal{L} = \mathcal{L}(\varphi_r, \partial_\mu \varphi_r, x^\mu)$
$r = 1, \dots, N$

usiamo solo campi scalari φ_r perché per campi vettoriali ogni componente è un campo scalare indipendente dalle altre

$$L = \int d^3x\, \mathcal{L} \qquad S = \int dt\, L = \int d^4x\, \mathcal{L}$$

$$\varphi_r(x^\mu, \alpha) = \varphi_r(x^\mu, 0) + \alpha\, \xi_r(x^\mu), \qquad \left.\xi_r\right|_{\partial\Omega} = 0$$

$$\left.\frac{dS}{d\alpha}\right|_{\alpha=0} = \int \left(\frac{\partial \mathcal{L}}{\partial \varphi_r} \frac{\partial \varphi_r}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_r)} \frac{\partial}{\partial \alpha}(\partial_\mu \varphi_r)\right) d^4x$$

non c'è la sommatoria in r perché i vari campi non sono accoppiati

$$= \int \left(\frac{\partial \mathcal{L}}{\partial \varphi_r} \frac{\partial \varphi_r}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_r)} \partial_\mu\left(\frac{\partial \varphi_r}{\partial \alpha}\right)\right) d^4x$$

$dx^0 dx^1 dx^2 = d\sigma^3$
$dx^1 dx^2 dx^3 = d\sigma^0$

stiamo integrando solo uno dei dx

parti

$$\int_\Omega \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_r)} \partial_\mu\left(\frac{\partial \varphi_r}{\partial \alpha}\right) d^4x = \cancel{\int_{\partial\Omega} d\sigma_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_r)} \underbrace{\frac{\partial \varphi_r}{\partial \alpha}}_{\left.\xi_r\right|_{\partial\Omega}=0}} - \int_\Omega d^4x\, \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_r)}\right) \frac{\partial \varphi_r}{\partial \alpha}$$

$$\left.\frac{dS}{d\alpha}\right|_{\alpha=0} = \int \left(\frac{\partial \mathcal{L}}{\partial \varphi_r} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_r)} \right) \right) \underbrace{\frac{\partial \varphi_r}{\partial \alpha}}_{\xi_r \text{ variazione arbitraria}} d^4x = 0$$

$$\Longrightarrow \quad \frac{\partial \mathcal{L}}{\partial \varphi_r} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_r)} \right) = 0$$

eqq. Eulero-Lagrange per N campi φ_r in 4D

La lagrangiana è definita a meno di una 4-divergenza:

$$\mathcal{L} \to \mathcal{L} + \partial_\nu F^\nu(\varphi_r, x^\mu) \qquad \because \int_\Omega \partial_\nu F^\nu d^4x = \int_{\partial\Omega} d\sigma^\nu F_\nu = 0$$

Le eqq di E-L sono covarianti solo se $\mathcal{L}$ è uno scalare di Lorentz