$$L_i = -\frac{m_i c^2}{\gamma_i} - e_i\varphi + \frac{e_i}{c}\bar{A}\cdot\bar{v}_i$$

[nel caso classico: $L = \sum_i \frac{1}{2} m_i v_i^2 - \sum_i\sum_j \frac{e_i e_j}{r_{ij}}$]

$$\varphi = \int \frac{\rho_{rit}}{R}\,dV \qquad \bar{A} = \frac{1}{c}\int \frac{\bar{J}_{rit}}{R}\,dV$$

con $\beta \ll 1$ (ma non limite classico)

Espandiamo in $\frac{R}{c}$ $(\ll 1)$:

$$f\left(t-\frac{R}{c}\right) = f(t) - \frac{R}{c}f'(t) + \frac{1}{2}\frac{R^2}{c^2}f''(t) + o\left(\frac{R^3}{c^3}\right)$$

$$\varphi \simeq \int \frac{\rho}{R}\,dV - \frac{R}{c}\frac{\partial}{\partial t}\int \frac{\rho}{R}\,dV + \frac{1}{2}\frac{R^2}{c^2}\frac{\partial^2}{\partial t^2}\int\frac{\rho}{R}\,dV$$

$$= \int \frac{\rho}{R}\,dV - \frac{1}{c}\cancel{\frac{\partial}{\partial t}\underbrace{\int \rho\,dV}_{Q_{TOT} = cost}} + \frac{1}{2c^2}\frac{\partial^2}{\partial t^2}\int R\rho\,dV$$

$\frac{\partial R}{\partial t} \sim 0$ ∵ $R$ cambia poco

$\rho = e\,\delta(\bar{r})$

$$= \frac{e}{R} + \frac{e}{2c^2}\frac{\partial^2 R}{\partial t^2}$$

espandiamo al primo ordine perché c'è già un $\frac{1}{c^2}$ nella lagrangiana

$$\bar{A} = \frac{1}{c}\int \frac{\bar{J}_{rit}}{R}\,dV = \frac{1}{c}\int\frac{\rho\bar{v}}{R}\,dV = \frac{e}{cR}\bar{v}$$

Specifichiamo un gauge:

$$\begin{cases} \varphi' = \varphi - \frac{1}{c}\frac{\partial f}{\partial t} \\ \bar{A}' = \bar{A} + \bar{\nabla} f \end{cases} \xrightarrow{f = \frac{e}{2c}\frac{\partial R}{\partial t}} \begin{cases} \varphi' = \frac{e}{R} + \cancel{\frac{e}{2c^2}\frac{\partial^2 R}{\partial t^2}} - \cancel{\frac{e}{2c^2}\frac{\partial^2 R}{\partial t^2}} = \frac{e}{R} \\ \bar{A}' = \frac{e}{cR}\bar{v} + \frac{e}{2c}\bar{\nabla}\frac{\partial R}{\partial t} \end{cases}$$

R posizione carica

$$\bar{\nabla}\frac{\partial R}{\partial R} = \frac{\partial}{\partial t}\bar{\nabla}R = \frac{\partial}{\partial t}\hat{n} = \frac{\partial}{\partial t}\left(\frac{\bar{R}}{R}\right) = \frac{\dot{\bar{R}}}{R} - \frac{\dot{R}}{R^2}\bar{R} = -\frac{\bar{v}}{R} + \frac{\hat{n}\cdot\bar{v}}{R^2}\bar{R}$$

$$\frac{\partial}{\partial t}(R^2) = 2\bar{R}\cdot\dot{\bar{R}} = -2\bar{R}\cdot\bar{v} \Rightarrow \dot{R} = -\frac{\bar{R}\cdot\bar{v}}{R}$$

$$\bar{A} = \frac{e}{cR}\bar{v} + \frac{e}{2c}\frac{\partial}{\partial t}\bar{\nabla}R$$

$$= \frac{e}{cR}\bar{v} + \frac{e}{2c}\left(-\frac{\bar{v}}{R} + \frac{\hat{n}\cdot\bar{v}}{R^2}\bar{R}\right)$$

$$\hat{n} = \frac{\bar{R}}{R}$$

$$= \frac{e}{cR}\bar{v} - \frac{e}{2cR}\bar{v} + \frac{e}{2cR}(\hat{n}\cdot\bar{v})\,\hat{n}$$

$$= \frac{e}{2cR}\left(\bar{v} + (\hat{n}\cdot\bar{v})\,\hat{n}\right)$$

Espandiamo $-\frac{mc^2}{\gamma_i} = -mc^2\sqrt{1-\frac{v_i^2}{c^2}}$

$$\sqrt{1-x} = 1 - \frac{x}{2} - \frac{x^2}{8} + o(x^3)$$

costante (inutile per lagrangiana)

termine aggiuntivo dalla relatività

$$-mc^2\sqrt{1-\frac{v_i^2}{c^2}} \simeq -mc^2\left(1 - \frac{1}{2}\frac{v_i^2}{c^2} - \frac{1}{8}\frac{v_i^4}{c^4}\right) = \cancel{-mc^2} + \underbrace{\frac{m}{2}v_i^2}_{T\ \text{standard}} + \overbrace{\frac{m}{8}\frac{v_i^4}{c^2}}$$

$$L = -\frac{m_i c^2}{\gamma_i} - e_i\varphi' + \frac{e_i}{c}\bar{A}'\cdot\bar{v}_i$$

$$\simeq \frac{m_i v_i^2}{2} + \frac{m_i v_i^4}{8c^2} - e_i\sum_j \frac{e_j}{R_{ij}} + \frac{e_i}{2c^2}\sum_j \frac{e_j}{R_{ij}}\left(\bar{v}_i\cdot\bar{v}_j + (\bar{v}_i\cdot\hat{n}_{ij})(v_j\cdot\hat{n}_{ij})\right)$$

Questa è la lagrangiana della particella i causata da tutte le altre (j)

$$\boxed{\begin{aligned} L &= \sum_i L_i \\ &= \sum_i \left(\frac{m_i v_i^2}{2} + \frac{m_i v_i^4}{8c^2}\right) + \sum_{i>j}\left(\frac{e_i e_j}{R_{ij}} + \frac{e_i e_j}{2c^2 R_{ij}}\left(\bar{v}_i\cdot\bar{v}_j + (\bar{v}_i\cdot\hat{n}_{ij})(v_j\cdot\hat{n}_{ij})\right)\right)\end{aligned}}$$

Lagrangiana di Darwin

Questa lagrangiana gestisce le interazioni tra particelle senza introdurre i potenziali (almeno formalmente/matematicamente)