Ripasso relatività ristretta

4-velocità $\quad u^\mu = \left(\gamma, \frac{\gamma}{c}\bar{v}\right)$ $\qquad$ 4-posizione $\quad x^\mu$

una legge scritta in 4-vettori è invariante per trasf. tra SRI

Definiamo la 4-accelerazione:

$u^\mu \doteq \frac{dx^\mu}{ds}$ ← scalare (invariante)

$$\boxed{w^\mu = \frac{du^\mu}{ds}}$$

tempo

$$ds = c\,d\tau = \frac{c}{\gamma}dt$$

$$\frac{d}{dt}v^2 = \frac{d}{dt}(\bar{v}\cdot\bar{v})$$

$$w^0 = \frac{du^0}{ds} = \frac{d\gamma}{ds} = \frac{\gamma}{c}\frac{d\gamma}{dt} = \frac{\gamma}{c}\frac{d}{dt}\left(1-\frac{v^2}{c^2}\right)^{-1/2} = \frac{\gamma}{c}\left(-\cancel{\frac{1}{2}}\right)\left(1-\frac{v^2}{c^2}\right)^{-3/2}\left(-\cancel{2}\frac{\bar{v}}{c^2}\frac{d\bar{v}}{dt}\right) = \frac{\gamma}{c}\gamma^3\frac{\bar{v}}{c^2}\overbrace{\frac{d\bar{v}}{dt}}^{\bar{a}} = \frac{\gamma^4}{c^3}\bar{v}\cdot\bar{a}$$

$$w^i = \frac{du^i}{ds} = \frac{\gamma}{c}\frac{du^i}{dt} = \frac{\gamma}{c^2}\frac{d}{dt}(\gamma v^i) = \frac{\gamma}{c^2}\left(\frac{d\gamma}{dt}v^i + \gamma\frac{dv^i}{dt}\right) = \frac{\gamma}{c^2}\left(\gamma^3\frac{\bar{v}}{c^2}\cdot\bar{a}\,v^i + \gamma a^i\right) = \left(\frac{\gamma^4}{c^4}\bar{v}\cdot\bar{a}\,v^i + \frac{\gamma^2}{c^2}a^i\right)$$

$$\boxed{w^\mu = \left(\frac{\gamma^4}{c^3}\bar{v}\cdot\bar{a},\ \frac{\gamma^2}{c^2}\left(\gamma^2\left(\frac{\bar{v}\cdot\bar{a}}{c}\right)\bar{v} + \bar{a}\right)\right)}$$

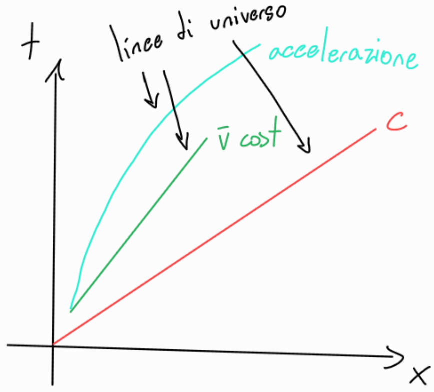
termine lungo $\bar{v}$

Proprietà

$$u^\mu u_\mu = 1 \Rightarrow \frac{d}{ds}(u^\mu u_\mu) = 0 \Rightarrow w^\mu u_\mu + u^\mu w_\mu = 2w^\mu u_\mu = 0$$

$$\Rightarrow \boxed{w^\mu u_\mu = 0} \qquad \left(\text{non necessario che } \bar{v}\cdot\bar{a} = \bar{0}\right)$$

SR particella: $\quad \bar{v} = \bar{0} \Rightarrow w^\mu = \left(0, \frac{\gamma^2}{c^2}\bar{a}\right)$ $\quad$ questo indica che non è un SRI



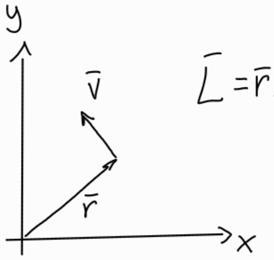
Tensori

$$\underbrace{T^{\mu\nu}}_{\text{i tensori si trasformano come}} = \underbrace{A^\mu B^\nu}_{\text{prodotto di 4-vettori}} \rightarrow \quad (V')^\mu = \Lambda^\mu{}_\nu V^\nu$$

$$\Downarrow$$

$$(T')^{\mu\nu} = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}T^{\rho\sigma}$$

<u>Pseudo tensori</u>: si trasformano come $(\tilde{T}')^{\mu\nu} = \det(\Lambda)\,\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}\tilde{T}^{\rho\sigma}$



$\bar{L} = \bar{r}\wedge\bar{v}$

per inversione:

$\bar{r} \to -\bar{r}$

$\bar{v} \to -\bar{v}$

$\bar{L} \to \bar{L}$

in 4D ci sono dei tensori che si comportano così

Esistono anche gli pseudoscalari (cambiano segno per trasf)

Derivazione e Integrazione

$$\underbrace{\partial_\mu = \frac{\partial}{\partial x^\mu}}_{\text{covariante}} \doteq \left(\frac{\partial}{\partial x^0}, \bar{\nabla}\right) \qquad d\varphi = \underbrace{\frac{\partial\varphi}{\partial x^\mu}}_{\text{covariante}} \overbrace{dx^\mu}^{\text{controvariante}}$$

$$\partial_\mu \varphi = \left(\frac{\partial\varphi}{\partial x^0}, \bar{\nabla}\varphi\right) \qquad \text{scalare}$$

$$\partial_\mu V^\mu = \partial_0 V^0 + \partial_i V^i = \frac{1}{c}\frac{\partial V^0}{\partial t} + \bar{\nabla}\cdot\bar{V} \qquad \text{vettore}$$

$$\partial^\mu = (\partial_0, -\bar{\nabla})$$

$$\Box\varphi \doteq \overbrace{\partial^\mu}^{\text{controvariante}}\partial_\mu\varphi = \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \nabla^2\varphi \qquad \text{(ricorda l'eq. delle onde)}$$

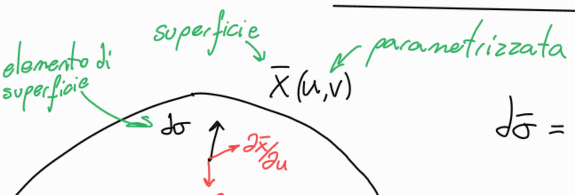
$d^4x = dx^0 dx^1 dx^2 dx^3 = c\,dt\,d^3x \doteq d\Omega$ invariante (scalare di Lorentz)

intuitivamente (tr Lorentz ≡ rotazione in 4D) $\Rightarrow$ $d\Omega$ non cambia

$$d^4x' = |J(x)|\,d^4x \qquad \text{con} \qquad J(x) = \det\left[\frac{\partial (x')^\mu}{\partial x^\nu}\right] = \det(\Lambda) = 1$$

dx^0 e d^3x non sono invarianti ma cambiano in modo opposto (dunque $d\Omega$ è a posto)

<u>3D</u>



$$d\bar{\sigma} = d\sigma\,\hat{n} = \left(\frac{\partial\bar{x}}{\partial u} \wedge \frac{\partial\bar{x}}{\partial v}\right) du\,dv$$

$$d\sigma^i = \varepsilon^{ijk} \frac{\partial x^j}{\partial u} \frac{\partial x^k}{\partial v} du\, dv$$

$$(u,v) \xrightarrow{\text{scegliamo}} (x,y) \implies d\sigma = (dy\,dz,\ dx\,dz,\ dx\,dy)$$

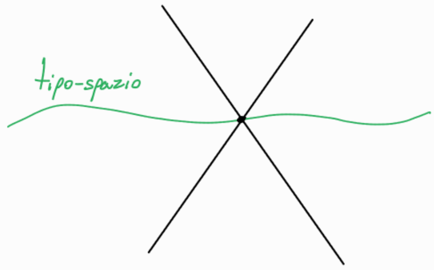
4D

$$d\sigma_\mu = \frac{1}{3!} \varepsilon_{\mu\nu\rho\sigma} \frac{\partial(x^\nu, x^\rho, x^\sigma)}{\partial(u,v,w)} du\, dv\, dw = \hat{n}_\mu\, d\sigma$$

4-vettore

$$(u,v,w) \to (x^0, \bar{x}) \implies d\sigma_\mu = (dx^1 dx^2 dx^3,\ dx^0 dx^2 dx^3,\ dx^0 dx^1 dx^3,\ dx^0 dx^1 dx^2)$$

<u>ipersuperficie tipo spazio</u>: $\underbrace{(x_1 - x_2)^2 < 0}_{}$, $x_1, x_2 \in \sigma$ (eventi: x_1, x_2)



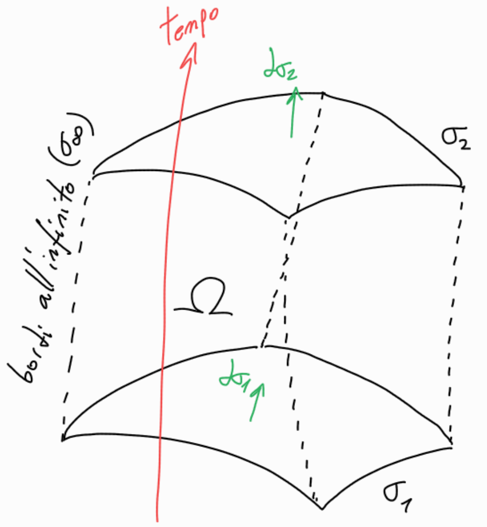
Teoremi utili

<u>Gauss</u>:

$$\int_\Omega \partial_\mu V^\mu(x)\, d^4x = \int_{\partial\Omega} V^\mu(x)\, d\sigma_\mu$$

caso particolare: $\partial_\mu V^\mu = 0$, Ω delimitato da σ_1, σ_2 tipo-spazio

$$\partial\Omega = (\sigma_1, \sigma_2, \sigma_\infty)$$



$$\partial_\mu V^\mu = 0 \implies \int_\Omega \partial_\mu V^\mu d^4x = 0$$

$$\implies 0 = \int_{\partial\Omega} V^\mu d\sigma_\mu = \int_{\sigma_2} V^\mu d\sigma_\mu - \int_{\sigma_1} V^\mu d\sigma_\mu + \int_{\sigma_\infty} V^\mu d\sigma_\mu$$

$$\lim_{\bar{x} \to \infty} V^\mu = 0^\mu \implies \int_{\sigma_\infty} V^\mu dx_\mu = 0$$

$I(\sigma_2)$ $\qquad$ $I(\sigma_1)$

$$\Longrightarrow \int_{\sigma_1} V^\mu d\sigma_\mu = \int_{\sigma_2} V^\mu d\sigma_\mu \qquad \text{vale per } \forall \sigma_i \text{ tipo-spazio}$$

Esempio: $d\sigma_\mu$ tutto lo spazio in un tempo fissato $\Rightarrow dx^0 = 0$

$$\Rightarrow d\sigma_\mu = (dx^1 dx^2 dx^3, 0, 0, 0) = (dV, 0)$$

$$I(\sigma) \doteq \int_\sigma V^\mu d\sigma_\mu = \int V^0 d\sigma^0 = \int V^0 dx\,dy\,dz = \text{cost} \Rightarrow V^0 \text{ si conserva nel tempo per } \partial_\mu V^\mu = 0$$

cambiamo σ con un altro σ (tipo-spazio, tempo fissato)

Eq Minkowski in caso di forze

$$\frac{dp^\mu}{ds} = F^\mu \qquad p^\mu = \left(\frac{\mathcal{E}}{c}, \bar{p}\right) \qquad ds = c\,d\tau = \frac{c}{\gamma} dt$$

$$\frac{dp^\mu}{ds} = \left(\frac{\gamma}{c^2} \underbrace{\frac{d\mathcal{E}}{dt}}_{W = \bar{F}\cdot\bar{v}\ \text{potenza}}, \frac{\gamma}{c} \underbrace{\frac{d\bar{p}}{dt}}_{\bar{F}}\right) \Longrightarrow F^\mu = \left(\frac{\gamma}{c^2} \bar{F}\cdot\bar{v}, \frac{\gamma}{c} \bar{F}\right)$$