Acta Appl Math
https://doi.org/10.1007/s10440-018-00232-2

A Solution-Generating Method in Einstein-Scalar Gravity

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Received: 3 October 2018 / Accepted: 10 December 2018

Abstract We present a method to generate static, spherically symmetric, solutions of Einstein gravity in $d+2$ dimensions minimally coupled to a real scalar field with a self-interacting potential. The solutions can be fully parametrised by a single function, whose behaviour encodes all the information about the local and global behaviour of the spacetime. We give several explicit applications of our solution-generating method that describe black holes, naked singularities and solitonic configurations.

Keywords Einstein-scalar gravity · Exact solutions · Black holes

1 Introduction

Einstein-scalar gravity provides simple but solid models and plays a central role in several branches of physics, including particle physics, cosmology, and black-hole physics. For instance, scalar field dark matter and dark energy models could explain the asymmetry in the amount of baryonic and non-baryonic matter; a single scalar field could drive cosmological inflation, as supported by the recent Planck data. Moreover, the existence of black holes with scalar hair and their astrophysical tests motivate the research of Einstein-scalar gravity and its extensions, as well as applications of the AdS/CFT correspondence.

In this work we consider $(d+2)$-dimensional Einstein gravity minimally coupled to a real scalar field with a non-zero self-interacting scalar potential. In four dimensions, for a vanishing scalar potential there exists a simple analytic solution, first found by Fisher [1], then by Buchdhal [2] and then again by Janis, Newman and Winicour [3], and Wyman [4]. However, for non-vanishing potentials, even in simple cases, analytic solutions are hard

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Published online: 27 December 2018

to find. For some examples of exact solutions see *e.g.* Refs. [5–12]. Besides the purely mathematical interest, studying these gravitational configuration is important because in realistic scenarios, the self-interacting scalar potential is unknown *a priori* and should be determined by observational data. Hence, it is important and useful to analyse and classify all admissible potentials.

In asymptotically flat spacetimes, the existence of black holes sourced by scalar fields is restricted by no-scalar-hair theorems which in their old form relate the existence of hairy black holes to the non-convexity of the potential [13–16] and in their recent form to the violation of the positive energy theorem [17, 18] with some notable exceptions [19, 20].

In this paper, we present a solution-generating method, slightly different from that introduced in Ref. [21] which has been applied to derive a large number of exact, static, asymptotically flat or anti-de Sitter (AdS) black-hole and black-brane solutions [22–26]. This previous method starts by fixing the scalar profile and then gives integral formulas for the other metric functions and the scalar potential. However, by construction, each solution is an *ad hoc* solution, and the spacetime may not be asymptotically flat or AdS. Alternative formulations of this inverse problem method and similar integral formulas can be found in Refs. [7, 12, 27]. In Refs. [7, 12] the authors presuppose acceptable forms of the radial metric function while the authors of Ref. [27] assume a monotonic scalar field profile with a sufficiently fast fall-off and they generate solutions considering suitable metric functions. The method presented here does not allow for arbitrary scalar profiles but only for those compatible with asymptotically flat spacetimes.

An interesting application of the method is the derivation of solutions representing solitonic stars and black holes sourced by a scalar field with a sine-Gordon profile [28]. It is remarkable that sine-Gordon solitons, which play a key role in several contexts of nonlinear physics [29–31] may also act as a gravitational scalar source in general relativity. In two-dimensional gravity, there exists a relationship between the sine-Gordon dynamics and the black-hole metric degrees of freedom [32, 33].

The structure of this paper is as follows. In Sect. 2 we present a solution-generating method in $d+2$ dimensions and we give general formulas for the scalar profile, the metric functions and the scalar potential in terms of a single function. In Sect. 3 we classify the solutions and we discuss the presence of singularities and event horizons and their relationship with no-hair theorems. In Sect. 4 we give the curvature invariants and some examples in four dimensions. Finally, in Sect. 5 we draw our conclusions.

Throughout this work we adopt $c = 16\pi G = 1$ units.

2 General Solving Method for Einstein-Scalar Gravity in $d+2$ Dimensions

Minimally coupled Einstein-scalar gravity in $d+2$ dimensions (with $d \geqslant 2$) is described by the action

$$S = \int \mathrm{d}^{d+2}x\,\sqrt{-g}\big(\mathcal{R} + 2\epsilon(\partial\phi)^2 - V(\phi)\big), \tag{1}$$

where $\mathcal{R}$ is the Ricci scalar, $\epsilon = \pm 1$ is the sign of the kinetic term, and $V(\phi)$ is the self-interacting scalar potential of the real scalar field ϕ. In this work we only consider a canonical (negative) kinetic term, *i.e.* no phantom scalar fields.

Without any initial assumptions on the asymptotic behaviour of the spacetime, we consider a static and spherically symmetric ansatz for the spacetime metric

$$\mathrm{d}s^2 = -U(r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{U(r)} + R^2(r)\,\mathrm{d}\Omega_d^2, \tag{2}$$

where $\mathrm{d}\Omega_d^2$ is the line element of the d-dimensional sphere, and a radial scalar profile $\phi = \phi(r)$. The resulting field equations are

$$\frac{R''}{R} = -\frac{2}{d}\phi'^2, \tag{3}$$

$$\left(U R^d \phi'\right)' = \frac{1}{4} R^d \frac{\partial V}{\partial \phi}, \tag{4}$$

$$\left(U R^d\right)'' = d(d-1)R^{d-2} - \frac{d+2}{d} R^d V, \tag{5}$$

$$\left(U R^{d-1} R'\right)' = (d-1)R^{d-2} - \frac{1}{d} R^d V, \tag{6}$$

where the prime denotes derivation with respect to r. These four equations are not independent and Eq. (5) follows from the others. For a given scalar field profile $\phi(r)$, Eq. (3) is a case of the Riccati equation that can be written in the standard form[1] with the substitution $R(r) = \exp \int^r \mathrm{d}r'\, y(r')$.

Once the solution to the Riccati equation is known, and consequently R, the other metric function U and the scalar potential V can be computed using a solution-generating method as the one introduced in Ref. [21]. In that work, the asymptotic behaviour of the spacetime metric was fixed *a posteriori* and scalar profiles were chosen *ad hoc* to have relatively simple solutions to the Riccati equation. Here we slightly modify such a method and we show that the whole solution of the field equations can be entirely parametrised by a single function by requiring asymptotic flatness and the reality of the scalar field. Notice that once a particular solution to the Riccati equation is known, the general solution can be obtained by quadrature. Although in some cases, this general solution might be of physical interest, quite generically, it could generate a non-asymptotically flat spacetime, and/or could make the computations particularly involved, see *e.g.* the first example in Sect. 4.7.

To simplify the calculations, it is useful to introduce an auxiliary dimensionless coordinate $x \equiv \pm r_0/r$, with $r_0 > 0$ an arbitrary length scale, which sets the units to the mass of the solution and to the strength of the scalar field. By rescaling the radial coordinate r one can always use $r_0 = 1$ units, however, in view of physical applications, we prefer to use dimensionful coordinates. The overall sign is a matter of taste: the plus (minus) sign maps $r = 0$ into $x = \infty$ ($x = -\infty$) and $r = \infty$ into $x = 0^+$ ($x = 0^-$). For definiteness we choose the plus sign.

The general solution of Eq. (3) can be entirely parametrised in terms of a single function $P(x)$ and written as

$$\phi(x) = \sqrt{\frac{d}{2}} \int \mathrm{d}x \sqrt{-\frac{\ddot{P}}{P}}, \qquad R(x) = \frac{r_0}{x} P(x), \tag{7}$$

[1] The Riccati equation is an ordinary differential equation in the form $y' + p_1 y + p_2 y^2 = q$ where $p_1, p_2 \neq 0$ and $q \neq 0$ are functions of the independent variable.

where the dot denotes derivation with respect to x. Bronnikov [34] showed that Eqs. (3) to (6) do not admit wormhole, horn and flux tube type solutions. This means that depending on the possible zeros of $P(x)$, the range of x can be restricted without loss of generality to $0 \leqslant x < \infty$ or $0 \leqslant x \leqslant x_0$—see below Sect. 3.1.

By combining Eqs. (5) and (6) and using Eq. (7), the solution for the metric function U as a function of x can also be parametrised in terms of $P(x)$ and it is given by

$$U(x) = \frac{r_0^2 P^2}{x^2}\left[c_2 - \frac{2(d-1)}{r_0^2}\int \frac{\mathrm{d}x\, x^d}{P^{d+2}}\int \frac{\mathrm{d}x'\, P^{d-2}}{x'^d} + \frac{c_1}{r_0^{d+1}}\int \frac{\mathrm{d}x\, x^d}{P^{d+2}}\right], \tag{8}$$

where the integration constants c_1 and c_2 are determined by imposing boundary conditions on the asymptotic behaviour of the spacetime metric.

Finally, the scalar potential is

$$V[\phi(x)] = \frac{d^2(d-1)}{d+2}\frac{x^2}{r_0^2 P^2} - \frac{d}{d+2}\frac{x^{d+2}}{r_0^d P^d}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 \frac{\mathrm{d}}{\mathrm{d}x}\frac{UP^2}{x^2}\right). \tag{9}$$

The reality of the scalar field requires that the quantity in the square root in Eq. (7) must be positive. The strategy for finding solutions is therefore the following: one first chooses the function $P(x)$ such that P is positive and the ratio $\ddot{P}/P$ is negative at least in the range of the coordinate x corresponding to $0 \leqslant R < \infty$, and then determines the scalar profile $\phi(x)$, the metric functions $R(x)$ and $U(x)$, and the scalar potential $V(\phi)$ with Eqs. (7) to (9). In Sect. 4 we give some examples of application of this method.

3 Classification of the Solution

The solution (7) to (9) is parametrised by the single function $P(x)$. The information about the geometry and the causal structure of the spacetime is therefore completely encoded in the local and global behaviour of $P(x)$.

3.1 Positivity and Monotonicity Conditions on the Function $P(x)$

The radius of the d-sphere $R(x)$ must be an analytic, positive and monotonically decreasing function of x, *i.e.* $\dot{R} < 0$, as well as $P(x)$. The condition of asymptotic flatness requires that $P(0) = 1$ and $P(x)$ to be an analytic function near $x = 0$, *i.e.* $P(x \approx 0) = \sum_{n=0}^{N} a_n x^n$. Together with the reality condition for the scalar field, we get the following positivity, monotonicity and concavity conditions on the function $P(x)$:

$$P > 0, \quad \dot{P}/x - P/x^2 < 0, \quad \ddot{P} < 0. \tag{10}$$

The coordinate x must cover the whole $0 \leqslant R < \infty$ region so that the spacetime can be generated in different ways depending on the (possible) zeros of the function $P(x)$. We have two possible cases:

I. $P(x)$ has a zero at a finite value x_0, *i.e.* $P(x_0) = 0$, and hence the solution exists in the range $0 \leqslant x \leqslant x_0$. The r-origin corresponds to $x = x_0$, and the r-asymptotic region corresponds to $x = 0$.

II. $P(x)$ approaches a non-zero constant as $x \to \infty$, *i.e.* $P(x) \to \text{const} \neq 0$ and hence the solution exists in the whole range $0 \leqslant x < \infty$. The r-origin corresponds to $x = \infty$, while the r-asymptotic region corresponds again to $x = 0$.

3.2 Geometry of the Solutions

The value of c_2 in Eq. (8) is determined by imposing the asymptotic flatness of the solutions (7) to (9), *i.e.*,

$$U(r) \to 1, \quad R(r) \to r, \quad \phi(r) \to 0, \tag{11}$$

as $r \to \infty$. On the other hand, the integration constant c_1 in Eq. (8) determines the subleading terms in the $r = \infty$ expansion of the metric function U. Solutions that approach asymptotically the Schwarzschild geometry are of particular interest, *i.e.* those for which $U(r) \to 1 - M/(8\pi r)$, where M is the solution gravitational mass.

At the origin, $r \to 0$, there is no *a priori* specific boundary condition. Depending on the presences of zeros of U and curvature singularities, the solution describes the geometry of one of the following objects:

Naked singularities. When a curvature singularity is present and is not shielded by an event horizon, *i.e.* the metric function U has no zeros. This is always possible in case I, while in case II, this is possible when the condition B below holds.

Black holes. When a curvature singularity is shielded by an event horizon, *i.e.* the metric function U has a zero at some radius $r_h > 0$. This is always possible in case I, while in case II, this is possible when condition B below holds.

Regular solutions. When the spacetime is everywhere regular (star-like). This is only possible in case II when the condition A below holds.

3.3 Presence of Horizons and Curvature Singularities

The existence of an event horizon requires the existence of at least one zero of the metric function $U(x)$ at $x = x_h$ with $0 \leqslant x_h \leqslant x_0$ in case I or $0 \leqslant x_h < \infty$ in case II.

An important piece of information comes from the behaviour of $P(x)$ and its derivatives at $x = x_0$ as it determines the presence or absence of a curvature singularity. In case I (respectively case II), the integral form (8) of the metric function evaluated in $x = x_0$ (respectively $x = \infty$) becomes singular and, quite generically, the spacetime will develop a curvature singularity. For instance, in four dimensions, from Eqs. (16) and (17) one easily sees that $x = x_0$ or $x = \infty$ are generically curvature singularities unless $\ddot{P}$ goes sufficiently fast to zero. We have correspondingly two cases, describing respectively a regular point and a curvature singularity at $x = x_0$ or $x = \infty$:

A. $\ddot{P} \to 0$ keeping $x^4 U \ddot{P}/P$ and V finite;
B. $\ddot{P}$ stays finite or $\ddot{P} \to 0$ but $x^4 U \ddot{P}/P$ and/or V diverge.

The presence of a curvature singularity at $R = 0$ is quite generic and in case I cannot be avoided. Consider for instance a power-law behaviour near x_0, $P \sim (x - x_0)^a$, $a > 0$. It follows that the Ricci scalar behaves near x_0 as $\mathcal{R} \sim 1/[P^2(x - x_0)]$. Similarly, with an exponential behaviour near x_0, $P \sim e^{1/(x-x_0)}$ the Ricci scalar behaves as $\mathcal{R} \sim 1/[P^2(x - x_0)^2]$.

The presence of a curvature singularity at $R = 0$ can be avoided only in case II. A sufficient condition for this to happen is that $\ddot{P}$ goes to zero exponentially as $x \to \infty$. To elucidate the way the mechanism works, consider an explicit example in four spacetime dimensions, characterised by $P = 2 - e^{-x}$, which has been studied in Ref. [28]. Near $x = \infty$, the metric function U behaves as

$$U = \frac{4r_0^2}{x^2}\left[c_2 - \frac{x^2}{16r_0^2} - \frac{11x}{24r_0^2} + \frac{\alpha_1}{r_0^2} + \frac{c_1}{4r_0^3}\left(\frac{x^3}{12} + \alpha_2\right)\right] + \mathcal{O}(e^{-x}), \tag{12}$$

where α_1 and α_2 are some numbers. The behaviour of the scalar potential at $x=\infty$ can be determined using Eq. (9) with $d=2$ and the previous equation, and one finds $V=\mathcal{O}(1)$. Finally, it is easy to verify that $\ddot{P}/P \sim -e^{-x}$. Thus, the two curvature invariants (16) and (17) are finite at $r=0$ as the power-law divergences are suppressed by the exponential.

Notice, however, that there might be cases in which the Ricci scalar is finite but there may still be a singularity, as in the Schwarzschild case. In such cases, one should also compute the Kretschmann scalar [35, 36] in order to infer about the presence of a curvature singularity.

3.4 Global Behaviour of the Potential and No-Hair Theorems

The global behaviour of the potential, such as bounded from below and convexity, is important for no-hair theorems. In the old formulation, no-scalar-hair theorems state that for $V \geqslant 0$, Eqs. (3) to (6) do not admit any hairy black-hole solutions and the only asymptotically flat solution is Schwarzschild. Recent versions of no-scalar-hair theorems relate the existence of black holes with the violation of the positive energy theorem. In both cases, sufficient conditions for the existence of black holes with scalar hair can be written in terms of constraints on the global form of the scalar potential. In view of Eq. (9), these global conditions on V can be in principle translated in terms of global conditions on the behaviour of the function P. However, this translation is in general not simple because it depends implicitly on the form of the profile of the scalar field.

Generically, in our picture, any black-hole solution must be parametrised by a function P in such a way that either the potential is unbounded from below, and/or it possesses a negative region such that the negative energy is not balanced by a positive contribution to violate the positive energy theorem. On the other hand, solitonic star-like solutions with regular centre and asymptotically flat typically respect the positive energy theorem. A nice example is the sine-Gordon solitonic star we describe in Sect. 4.5. The spacetime has an inner region of negative energy which is balanced by a positive energy contribution in the asymptotic region.

4 Application: Four-Dimensional Solutions

In this section we give examples of application of the method, discussing new and known four-dimensional solutions. In $d=2$, Eqs. (7) to (9) become

$$\phi(x) = \int \mathrm{d}x \sqrt{-\frac{1}{P}\frac{\mathrm{d}^2 P}{\mathrm{d}x^2}}, \qquad R(x) = \frac{r_0}{x} P(x), \tag{13}$$

$$U(x) = \frac{r_0^2 P^2}{x^2}\left[c_2 + \frac{2}{r_0^2}\int \frac{\mathrm{d}x\, x}{P^4} + \frac{c_1}{r_0^3}\int \frac{\mathrm{d}x\, x^2}{P^4}\right], \tag{14}$$

$$V\big[\phi(x)\big] = \frac{x^2}{r_0^2 P^2}\left[1 - \frac{x^2}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 \frac{\mathrm{d}}{\mathrm{d}x}\frac{U P^2}{x^2}\right)\right]. \tag{15}$$

For each example we give $P(x)$ and the respective range of the coordinate x; we express the scalar profile and the metric functions as functions of the radial coordinate r; when possible, we invert the scalar profile relation to get $r = r(\phi)$ and we express the scalar potential as a function of ϕ.

4.1 Curvature Invariants

The equations of motion give for the Ricci tensor $\mathcal{R}_{ij} = 2\partial_i\phi\partial_j\phi + \frac{1}{2}g_{ij}V(\phi)$. Using the static, spherically symmetric solution (13) to (15), we get the curvature invariants

$$\mathcal{R} = 2\left(V - \frac{x^4 U \ddot{P}}{r_0^2 P}\right), \tag{16}$$

$$\mathcal{R}_{ij}\mathcal{R}^{ij} = V^2 - \frac{2x^4 U \ddot{P}}{r_0^2 P}V + \left(\frac{2x^4 U \ddot{P}}{r_0^2 P}\right)^2. \tag{17}$$

The Kretschmann curvature invariant simplified for the solution (13) to (15) is

$$\begin{aligned}\mathcal{R}_{ijkl}\mathcal{R}^{ijkl} &= \frac{4x^2}{r_0^4 P^3}\big[U(P - x\dot{P})^2 - 1\big]\big(r_0^2 V P + 2x^4 U \ddot{P}\big) \\ &\quad + \frac{12x^4}{r_0^4 P^4}\big[U(P - x\dot{P})^2 - 1\big]^2 + V^2 + \frac{8x^8 U^2 \ddot{P}^2}{r_0^4 P^2}.\end{aligned} \tag{18}$$

4.2 Harmonic Solutions

Harmonic solutions are generated by the choice

$$P(x) = \cos x, \quad 0 \leqslant x \leqslant \frac{\pi}{2}. \tag{19}$$

The scalar profile and the metric function R are given by Eq. (13)

$$\phi(r) = \frac{r_0}{r}, \qquad R(r) = r\cos\frac{r_0}{r}. \tag{20}$$

The other metric function $U(r)$ is given by Eq. (14), which with the condition of asymptotic flatness ($c_1 = 0$ and $c_2 = 1/3r_0^2$) gives

$$\begin{aligned}U(r) = \frac{r^2}{3r_0^2}\cos^2\frac{r_0}{r}\Bigg[&1 + 4\log\left(\cos\frac{r_0}{r}\right) + \left(\frac{2r_0}{r}\tan\frac{r_0}{r} - 1\right)\sec^2\frac{r_0}{r} \\ &+ \frac{4r_0}{r}\tan\frac{r_0}{r}\Bigg].\end{aligned} \tag{21}$$

The $r = \infty$ asymptotic behaviour is $U(r) \approx 1 + r_0^4/9r^4 + \mathcal{O}(1/r^6)$. Moreover, $U(r) \geqslant 1$ for $r > 2r_0/\pi$, *i.e.* the solution does not possess a horizon. The Ricci scalar $\mathcal{R}$ diverges at $r = 2r_0/\pi$, then this solution represents a naked singularity.

The scalar potential is

$$\begin{aligned}V(\phi) = \frac{1}{3r_0^2}\big[&2\phi\sin 2\phi\big(8\phi^2 - 12\log(\cos\phi) - 9\big) + 3 - 8\phi^3\tan\phi - 10\phi^2 \\ &- 4\big(2\phi^2 + 3\big)\log(\cos\phi) \\ &+ \cos 2\phi\big(28\phi^2 + 4\big(4\phi^2 - 3\big)\log(\cos\phi) - 3\big)\big].\end{aligned} \tag{22}$$

The asymptotic behaviour of the scalar potential is $V(\phi \approx 0) = -4\phi^6/3r_0^2 + \mathcal{O}(\phi^8)$.

4.3 Arcsine Solutions

Arcsine scalar profiles can be obtained by taking

$$P(x) = 1 - x^2, \quad 0 \leqslant x \leqslant 1. \tag{23}$$

Equation (13) gives

$$\phi(r) = \sqrt{2} \arcsin \frac{r_0}{r}, \qquad R(r) = r - \frac{r_0^2}{r}, \tag{24}$$

with $r_0 \leqslant r < \infty$ and $0 \leqslant \phi \leqslant \pi/\sqrt{2}$. The scalar profile behaves asymptotically as $\phi \sim r_0/r$.

The other metric function $U(r)$ is

$$U(r) = \left(r - \frac{r_0^2}{r}\right)^2 \left[\frac{c_1 r(3r^4 + 8r^2 r_0^2 - 3r_0^4) + 16r^6}{48 r_0^2 (r^2 - r_0^2)^3} - \frac{c_1}{32 r_0^3} \log \frac{r + r_0}{r - r_0} - \frac{1}{3r_0^2}\right]. \tag{25}$$

The metric function behaves as $U(r) = 1 + c_1/3r + \mathcal{O}(1/r^3)$ as $r \to \infty$. This example satisfies the condition I, therefore we always have a black hole solution whenever the metric function $U(r)$ has a zero. The transcendental equation $U(r) = 0$ has always an acceptable solution for $c_1/r_0 < -2$.

Finally, the scalar potential is

$$\begin{aligned} V(\phi) = \frac{\sec^4 \frac{\phi}{\sqrt{2}}}{48 r_0^3} \Bigg[& 2 \sin^6 \frac{\phi}{\sqrt{2}} \left(5 \sin^2 \frac{\phi}{\sqrt{2}} - 6\right) \left(3c_1 \operatorname{arctanh}\left(\sin \frac{\phi}{\sqrt{2}}\right) + 16 r_0\right) \\ & - 30 c_1 \sin^7 \frac{\phi}{\sqrt{2}} + 26 c_1 \sin^5 \frac{\phi}{\sqrt{2}} - 3c_1 \sin \frac{\phi}{\sqrt{2}} (\cos \sqrt{2}\phi + 5) \\ & - 3c_1 (\cos \sqrt{2}\phi + 2) \log\left(1 - \frac{2}{\csc \frac{\phi}{\sqrt{2}} + 1}\right)\Bigg]. \end{aligned} \tag{26}$$

In this case, the $r = \infty$ asymptotic behaviour of the scalar potential is $V(\phi \approx 0) = \sqrt{2} c_1 \phi^5 / 15 r_0^3 + \mathcal{O}(\phi^6)$. This solution has been previously discussed in Sect. 4.1 and Eq. (B.1) of Ref. [12].

4.4 Logarithmic Solutions

Logarithmic scalar profiles can be obtained using

$$P(x) = (1 - x)^w, \quad 0 \leqslant x \leqslant 1, \ 0 \leqslant w \leqslant 1. \tag{27}$$

The scalar profile and the radial metric function are given by

$$\phi(r) = -\sqrt{w(1 - w)} \log\left(1 - \frac{r_0}{r}\right), \qquad R(r) = r \left(1 - \frac{r_0}{r}\right)^w, \tag{28}$$

while the metric function U and the scalar potential can be found, *e.g.* in Refs. [25, 37].

A slightly different solution can be obtained with

$$P(x) = \sqrt{1 - x^2}, \quad 0 \leqslant x \leqslant 1. \tag{29}$$

In this case, the scalar profile and the radial metric function are

$$\phi = -\frac{1}{2}\log\frac{r-r_0}{r+r_0}, \qquad R(r) = r\sqrt{1-\frac{r_0^2}{r^2}}. \tag{30}$$

Notice that the scalar field vanishes at infinity and diverges to infinity at r_0.

Asymptotic flatness implies $c_2 = -1/r_0^2$ and the metric function U reads

$$U(r) = \frac{r^2}{2r_0^2}\left(1-\frac{r_0^2}{r^2}\right)\left[\frac{c_1 r + 2r_0^2}{r^2 - r_0^2} + \frac{c_1}{2r_0}\log\frac{r-r_0}{r+r_0}\right]. \tag{31}$$

This example satisfies the condition I and it describes a black hole whenever $c_1/r_0 < -2$.

The scalar potential is

$$V(\phi) = \frac{c_1}{r_0^3}\left(2\phi + \phi\cosh 2\phi - \frac{3}{2}\sinh 2\phi\right). \tag{32}$$

Notice that the potential is everywhere negative and is unbounded below. This example has been also discussed in Ref. [7], appendix A. Notice also that the scalar potential is identical to the one obtained with $w = 1/2$ in Eq. (27), although the metric function is different, cfr. Ref. [25].

4.5 Sine-Gordon Solutions

A star-like regular solution sourced by a scalar profile having the form of a sine-Gordon soliton can be generated choosing

$$P(x) = 2 - e^{-x}, \quad 0 \leqslant x < \infty. \tag{33}$$

From Eq. (13) we get

$$\phi(r) = 2\arcsin\frac{e^{-r_0/2r}}{\sqrt{2}}, \qquad R(r) = r\left(2 - e^{-r_0/r}\right). \tag{34}$$

The scalar profile is identical to the solitons of the sine-Gordon theory, while U and V can be given in an analytic and closed form in terms of non-elementary functions and we do not report them here.

For values of $c_1 \neq 0$ it describes either a naked singularity or a black hole, while for $c_1 = 0$ it describes a horizonless and everywhere regular solution [28]. This latter solitonic solution interpolates between an AdS and an asymptotically flat spacetime. The inner region has negative energy but it is balanced by a positive energy contribution in the asymptotic region, so that the total energy of the solution is positive.

Similar solutions can be obtained by the general class of models described by

$$P(x) = \left(\frac{c+d}{c+de^{-ax}}\right)^b, \tag{35}$$

where a, b, c and d are some numbers.

In the simpler case when $a = c = d = 1$ and $b = 1/4$, we obtain

$$\phi(r) = \sqrt{5}\arctan\sqrt{\frac{4e^{r_0/r}-1}{5}} - \arctan\sqrt{4e^{r_0/r}-1} + \phi_0, \tag{36}$$

$$R(r) = r\sqrt[4]{\frac{2}{e^{-r_0/r}+1}}, \tag{37}$$

$$U(r) = \frac{r^2 e^{-r_0/r}}{r_0^2\sqrt{2e^{-r_0/r}+2}}\left[e^{r_0/r}\left(\frac{r_0^2}{r^2}+2\right) - 2\left(\frac{r_0}{r}+1\right)\right], \tag{38}$$

$$\begin{aligned} V\big[\phi(r)\big] = {} & \frac{\sqrt{e^{-r_0/r}+1}}{2\sqrt{2}r_0^2(e^{r_0/r}+1)^3}\left[2\left(\frac{r_0^2}{r^2}+\frac{6r_0}{r}+12\right)\right. \\ & + e^{r_0/r}\left(-\frac{r_0^4}{r^4}+\frac{8r_0^2}{r^2}+\frac{48r_0}{r}+24\right) \\ & \left. - e^{2r_0/r}\left(-\frac{r_0^4}{r^4}-\frac{4r_0^3}{r^3}-\frac{14r_0^2}{r^2}-\frac{36r_0}{r}+24\right) - 24e^{3r_0/r}\right], \end{aligned} \tag{39}$$

where $\phi_0 = \pi/3 - \sqrt{5}\arctan\sqrt{3/5}$.

4.6 Another Regular Solution

The choice

$$P(x) = 1 + \log(1+x), \quad 0 \leqslant x < \infty, \tag{40}$$

gives the following scalar profile and radial metric function

$$\phi(r) = 2\sqrt{1+\log\left(1+\frac{r_0}{r}\right)} - 2, \qquad R(r) = r\left[1+\log\left(1+\frac{r_0}{r}\right)\right]. \tag{41}$$

The scalar field goes to zero as r_0/r as $r \to \infty$ but diverges as $r \to 0$. After choosing c_2 in order to have an asymptotically flat solution, the metric function U reads

$$\begin{aligned} U(r) = {} & \frac{1}{6e^3 Prr_0^3}\big[e^2\,\mathrm{Ei}(1)P^3r^3(2r_0-c_1) + 16e\,\mathrm{Ei}(2)P^3r^3(c_1-r_0) \\ & + P^3r^3\big(e^2(c_1-2r_0)\,\mathrm{Ei}(P) + 16e(r_0-c_1)\,\mathrm{Ei}(2P) + 27c_1\,\mathrm{Ei}(3P)\big) \\ & - 27c_1\,\mathrm{Ei}(3)P^3r^3 + e^3\big(2P^3r^3(c_1+4r_0) - P^2(r+r_0)\big(2r^2(c_1+3r_0) \\ & + 2rr_0(5c_1+4r_0) + 9c_1r_0^2\big) - Pr_0(r+r_0)\big(c_1(2r+3r_0)+2r(r+2r_0)\big) \\ & - 2r_0^2(c_1+2r)(r+r_0)\big)\big], \end{aligned} \tag{42}$$

where $\mathrm{Ei}(x)$ is the exponential integral function. For $c_1/r_0 < 0$ the solution is a black hole; for $c_1 = 0$ the solution is regular at the origin, $U(0) = 0$; for $c_1/r_0 > 0$ there is a naked singularity, and in particular, for $0 < c_1/r_0 < 2$, U has a local minimum while for $c_1/r_0 \geqslant 2$, $U \geqslant 1$.

The potential can be given analytically but it is cumbersome, around $\phi = 0$ it behaves as follows,

$$V(\phi) \approx \frac{4c_1 - 2r_0}{15r_0^3}\phi^5 + \mathcal{O}\big(\phi^6\big). \tag{43}$$

4.7 Other Known Solutions

The black-hole solution with exponentially decaying scalar hair studied in Ref. [5], *i.e.* $\phi(r) = \phi_0 e^{-mr}$ with $\phi_0 > 0$ and $m > 0$, can in principle be obtained with the following choice

$$P(x) = \frac{x}{m}\left[K_0\left(\frac{\phi_0}{2}e^{-m/x}\right) + \left(\log\frac{\phi_0}{4} + \gamma\right)I_0\left(\frac{\phi_0}{2}e^{-m/x}\right)\right], \tag{44}$$

where I_0 and K_0 are the modified Bessel functions of the first and second kind, and γ is the Euler-Mascheroni constant. However, this choice of the generating function is not very natural and the series of P around $x = 0$ is not a polynomial, *i.e.* $P(x) = 1 + \mathcal{O}(x^2) + \mathcal{O}(\phi^2)$, as well as the technical difficulties with the integrals of powers of special functions. With some work, the corresponding expressions for R, U and V can be computed, but the potential is non-analytic.

The solutions discussed in Sects. 4.2 and 4.3 of Ref. [12]—which describes, respectively, a black hole with positive scalar potential at spatial infinity or a naked singularity with positive scalar potential near the centre, and a regular solution—can be obtained using

$$P(x) = 1 - \frac{b^2x^2}{4r_0(r_0 - bx)}, \quad b > 0,\ 0 < x < \frac{r_0}{b}\frac{2}{\sqrt{2}+1}, \tag{45}$$

$$P(x) = 1 - 2x^2 + \frac{4x^3}{3} - \frac{x^4}{3}, \quad 0 \leqslant x \leqslant 1. \tag{46}$$

The regular solution derived in Appendix B of Ref. [7] can be obtained with

$$P(x) = \sqrt{\frac{\tanh(c+x)}{\tanh c}}, \quad c > 0,\ 0 \leqslant x < \infty, \tag{47}$$

from which it follows

$$\phi(r) = \frac{\sqrt{3}\pi}{4} - \frac{1}{4}\log\frac{1-Z}{Z+1} - \frac{\sqrt{3}}{2}\arctan\frac{Z}{\sqrt{3}}, \tag{48}$$

$$R(r) = r\sqrt{\coth c \tanh\left(c + \frac{r_0}{r}\right)}, \tag{49}$$

where $Z^2 \equiv 2\cosh(2c + 2r_0/r) - 1$.

Asymptotic flatness requires $c_1 = 0$ and $c_2 = \tanh^2 c(c^2 - 2\log\sinh c)/r_0^2$,

$$U(r) = \tanh c \tanh\left(c + \frac{r_0}{r}\right)\left[1 - \frac{2r}{r_0}\coth\left(c + \frac{r_0}{r}\right) + \frac{2r^2}{r_0^2}\log\frac{\sinh(c + r_0/r)}{\sinh c}\right]. \tag{50}$$

Due to the form (48) of the scalar field, the scalar potential can be written in analytic form but cannot be written as function of ϕ, and we do not present it here. But it is smooth and finite in the whole range.

5 Summary and Outlook

In this paper we have presented a general method for constructing static, spherically symmetric, solutions of $(d+2)$-dimensional Einstein gravity minimally coupled to a real scalar field with a self-interacting potential. The whole solution is parametrised in terms of a single function, which encodes all the information about the local and global behaviour of the spacetime. Our method gives integral formulas for the scalar field profile, the metric functions and the scalar potential.

We have also given several applications of our solution-generating method by working out explicit solutions of Einstein-scalar gravity theory describing black holes, naked singularities and stars. A particularly interesting solution is sourced by a scalar field with the sine-Gordon solitonic profile.

There are two important points that we have not discussed in this paper and which deserve future investigation: the stability and the thermodynamical behaviour of the black-hole solutions in Sect. 4. We generically expect the black-hole solutions to be unstable, as they violate the no-scalar-hair theorems. In addition, specific results in Refs. [28, 38] on logarithmic and sine-Gordon solutions show that these solutions are linearly unstable.

Acknowledgements EF is partially supported by the Spanish MINECO under projects FPA2016-76005-C2-C-P and MDM-2014-0369 of ICCUB (Unidad de Excelencia "María de Maeztu"), and by AGAUR grant 2017-SGR-754.

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