STA511 Homework #3
Due Wednesday, October 28th 5:00PM

1. The following density on $[0, \infty)$ has both an infinite peak at 0 and a heavy tail:

$$f(x) = \frac{2}{\pi(1+x)\sqrt{x^2+2x}} \text{ for } x > 0.$$

   Consider as a possible candidate for dominating curve $c_\theta g_\theta(x)$ where

$$c_\theta g_\theta(x) = \begin{cases} \frac{2}{\pi\sqrt{2x}} & 0 \le x \le \theta \\ \frac{2}{\pi x^2} & x > \theta \end{cases} \tag{1}$$

   where c_θ is a constant depending upon only θ and $\theta > 0$ is a design parameter.

   (a) First show $f(x) \le c_\theta g_\theta(x)$.

   (b) Show that c_θ is minimal for $\theta = 2^{1/3}$ and find the optimal value of c_θ.

   (c) Perform the generalized rejection to obtain 500 observations from $f(x)$. For the results construct a histogram of the accepted observations with the pdf f superimposed.

2. Here we generate Laplace random variables using the Cauchy distribution as the dominating density. The Laplace distribution has pdf $f(x)$ defined by,

$$f(x) = \frac{\theta}{2} e^{-\theta|x|}, \text{ for } \theta > 0 \text{ and for } -\infty < x < \infty.$$

   (a) Assume $\theta = 1$ for the Laplace distribution and for our dominating density we will use a Cauchy distribution with pdf given by

$$g(x) = \frac{\mu}{\pi(\mu^2 + x^2)}, \text{ with } \mu > 0.$$

   Find the optimal value of μ and the optimal rejection constant c such that $c = sup \frac{f(x)}{g(x)}$.

   (b) Give an algorithm for generating random variables from the Cauchy distribution with the optimal parameter value for μ using Uniform(0,1) random variables.

   (c) Generate 1000 observations from the Laplace distribution ($\theta = 1$) using a generalized rejection algorithm. For the results construct a histogram of the accepted observations with the *pdf* of the Laplace distribution superimposed.

3. The beta distribution with parameters $\alpha > 0$ and $\beta > 0$ has a continuous density given by

$$f(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1}, \text{ for } 0 < x < 1,$$

   where $B(\alpha, \beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$ is the so-called complete beta function and $\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} \exp(-y) dy$ is the so-called gamma function.

   (a) Make plots in R of the target density when $(\alpha, \beta) = (0.5, 0.5), (1, 0.5), (0.5, 1), (1, 1), (2, 2), (5, 10)$. Note `gamma` is the gamma function in R.

(b) In this and the following questions, let $\alpha > 1$ and $\beta > 1$. Show that $f(x)$ is increasing for $x \leq (\alpha - 1)/(\alpha + \beta - 2)$ and decreasing for $x \geq (\alpha - 1)/(\alpha + \beta - 2)$.

(c) What do you suggest as a dominating density to use in the rejection sampling scheme?

(d) Implement the accept/rejection algorithm in R for one of the parameter settings in (a) and compare with results produced by the `rbeta` function in R. What is the acceptance percentage under each parameter setting?