STA 511 Homework #3

Suruchi Jaikumar Ahuja

October 22 2015

1. The density on [0,∞) is given by

$$f(x) = \frac{2}{\pi(1+x)\sqrt{x^2+2x}}$$

The dominating curve is given by

$$c_\theta g_\theta(x) = \begin{cases} \frac{2}{\pi\sqrt{(2x)}} & 0 \leq x \leq \theta \\ \frac{2}{\pi x^2} & x \geq \theta \end{cases}$$

(a) To prove $0 \leq x \leq \theta$

$$\frac{f(x)}{c_\theta g_\theta} = \frac{\frac{2}{\pi(1+x)\sqrt{x^2+2x}}}{\frac{2}{\pi\sqrt{2x}}}$$

Since the numerators are same , they can be ignored and the remaining equation is now considered,which leaves us with;

$$\frac{f(x)}{c_\theta g_\theta} = \frac{\sqrt{2x}}{(1+x)\sqrt{x^2+2x}}$$

For every value of $x > 0$ in $\frac{f(x)}{c_\theta g_\theta}$; the denominator is greater than the numerator,

So $\frac{f(x)}{c_\theta g_\theta} \leq 1$

$\Rightarrow f(x) \leq c_\theta g_\theta$

Now to prove $x > \theta$

$$\frac{f(x)}{c_\theta g_\theta} = \frac{\dfrac{2}{\pi(1+x)\sqrt{x^2+2x}}}{\dfrac{2}{\pi x^2}}$$

Since the numerators are same , they can be ignored and the remaining equation is now considered,which leaves us with;

$$\frac{f(x)}{c_\theta g_\theta} = \frac{x}{(1+x)\sqrt{x^2+2x}}$$

For every value of $x > 0$ in $\dfrac{f(x)}{c_\theta g_\theta}$; the denominator is greater than the numerator,

So $\dfrac{f(x)}{c_\theta g_\theta} \leq 1$

$\Rightarrow f(x) \leq c_\theta g_\theta$

Hence the $f(x) \leq c_\theta g_\theta$ stands true for both conditions

(b) To show that c_θ is minimal for $\theta = 2^{\frac{1}{3}}$

$$\Rightarrow \int_0^\theta \frac{2}{\pi\sqrt{2x}} + \int_\theta^\infty \frac{2}{\pi x^2}$$

$$\rightarrow \left.\frac{4\sqrt{x}}{\pi}\right|_0^\theta - \left.\frac{2}{\pi x}\right|_\theta^\infty$$

$$\rightarrow \frac{4\theta^{\frac{1}{2}}}{\pi} + \frac{2}{\pi\theta}$$

$$\rightarrow \frac{dc(\theta)}{d\theta} = \frac{2}{\pi\sqrt{2\theta}} - \frac{2}{\pi\theta^2}$$

On simplifying the above equation and substituting $\theta = 2^{\frac{1}{3}}$, we get

$$\Rightarrow \frac{2}{\pi}[2^{-\frac{2}{3}} - 2^{-\frac{2}{3}}] \rightarrow 0$$

(c) Here $\theta = 2^{1/3}$ and the dominating curve is given by

$$c_\theta g_\theta(x) = \begin{cases} \frac{2}{\pi\sqrt{(2x)}} & 0 \leq x \leq \theta \\ \frac{2}{\pi x^2} & x \geq \theta \end{cases}$$

So now on integrating the $c_\theta g_\theta$, we get c = $\dfrac{(3 * 2^{2/3})}{\pi}$
Now the generalized rejection method is performed on the obtained 500 observations from f(x)
A histogram of the accepted observations with the pdf f superimposed is constructed (Refer to Figure 1).

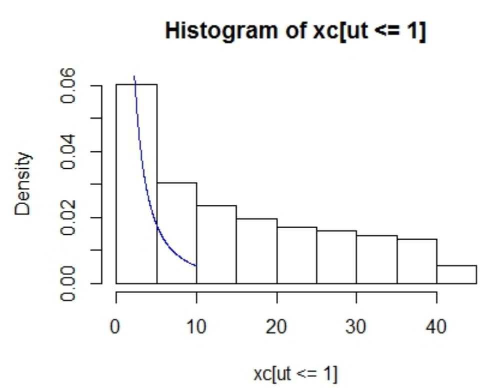

Figure 1: Histogram of accepted observations with the pdf f superimposed on it

2. The Laplace distribution has pdf f(x) given by

$$f(x) = \frac{\theta}{2}e^{-\theta|x|} for \theta > 0 and for -\infty < x < \infty$$

(a) Here we assume
$\theta = 1$
$\mu = 3$

$$g(x) = \frac{\mu}{\pi(\mu^2 + x^2)}$$

To find the optimal rejection constant c we use the formula

$$\Rightarrow c = sup \frac{f(x)}{g(x)}$$

$$\rightarrow c = \frac{\frac{\theta}{2}e^{-\theta|x|}}{\frac{\mu}{\pi(\mu^2 + x^2)}}$$

$$\rightarrow c = \frac{\frac{1}{2}e^{-}|x|}{\frac{3}{\pi}(9 + x^2)}$$

$$\rightarrow c = \frac{\pi}{6}e^{-|x|}(9 + x^2)$$

```
x = seq(-10,10,length=1000)
y = pi/6*exp(-abs(x))*(9+x^2)
plot(x,y)
```

(b) Algorithm for generating random variables from the Cauchy distribution with the optimal parameter value for $\mu = 3$ using Uniform(0,1) random variables.
Consider Y = F (x)
$\rightarrow$ x $=F^{-1}$(Y)

$$F = \frac{1}{\pi}\arctan(x) + \frac{1}{2}$$

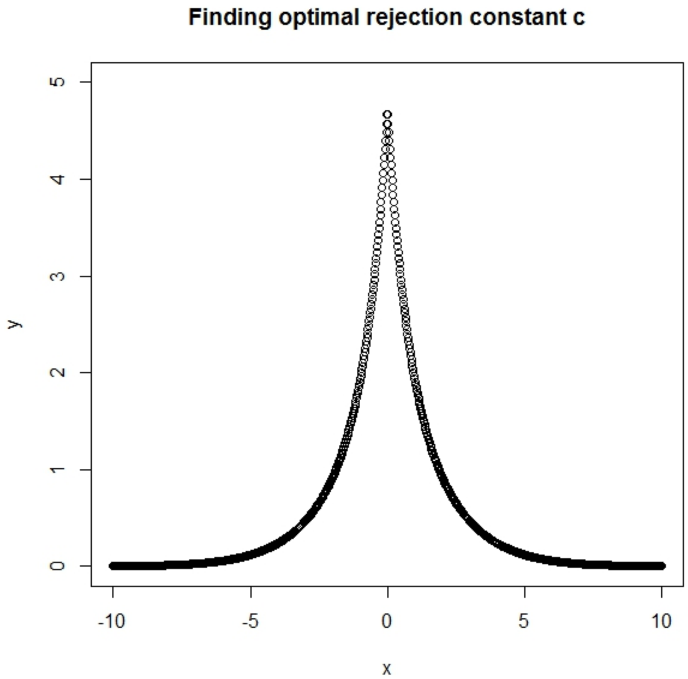

Figure 2: Optimal rejection constant c-Maximum value

$$x = \tan(\pi(y - \frac{1}{2}))$$

For F(x;μ,c)

$$\rightarrow \frac{1}{\pi} \arctan(\frac{x-\mu}{c}) - \frac{1}{2}$$

Step 1: In this case the first step is to generate U from Uniform(0,1)
Step 2: First we have

$$g(x) = \frac{\mu}{\pi(\mu^2 + x^2)}$$

$$G(x) = \int_{-\infty}^{x} \frac{\mu}{\pi(\mu^2 + z^2)} dz$$

$$\rightarrow \frac{1}{\mu\pi}\int_{-\infty}^{x}\frac{1}{1+\dfrac{z^2}{\mu^2}}dz$$

$$G(x) = \mu = \frac{1}{\pi}\tan^{-1}(\frac{x}{\mu}) + \frac{\pi}{2}$$

$$\rightarrow \mu = \frac{1}{\pi}\tan^{-1}(\frac{x}{\mu}) + \frac{\pi}{2}$$

$$\Rightarrow x = \mu\tan[\pi(\mu - \frac{1}{2})]$$

(c) Using a generalized rejection algorithm, 1000 observations are generated from the Laplace distribution($\theta = 1$)
A histogram of the accepted observations with the pdf of the Laplace distribution was constructed.(Refer to Figure 3)
R Code

```
x=seq(-10,10,length=1000)
c <- (pi/6)*exp(-abs(x))*(9+x^2)#optimal rejection constant
xc <- rcauchy(1000,0,1) # generating 1000 observations
n <- runif(1000,0,1)

fun <- function(x)
{
  s<- (6/(pi*(9+x^2)*exp(-abs(x))))
}
t <- c*sapply(xc, fun)
t1 <- n*t

hist(xc[t1<=1],prob=T,ylim=c(0,0.5))

laplace <- function(x)
{
  f<- (1/2)*(exp(-abs(x)))
}
```

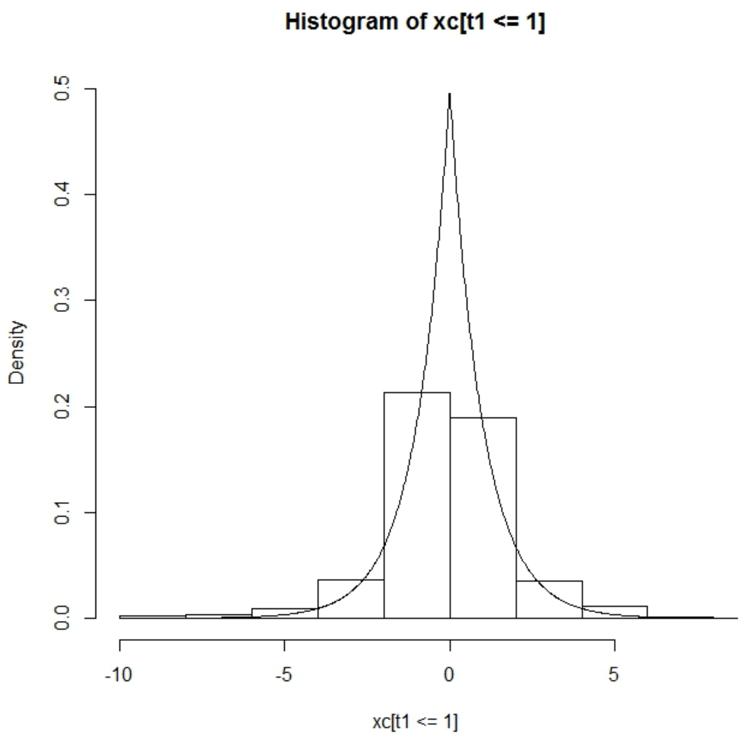

Figure 3: Histogram of xc[t1$\leq$ 1]

```
lines(x,laplace(x))
sum(t1<=1)/1000
```

OUTPUT:

The acceptance percentage is 84.8 percent

3. The beta distribution wih parameters $\alpha > 0$ and $\beta > 0$ has a continuous density given by

$$f(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1} for 0 < x < 1$$

   (a) To make plots in R of the target density when (α, β)= (0.5, 0.5),(1, 0.5),(0.5, 1),(1, 1),(2, 2),(5, 10)

Refer to Figure 4 for the plots of target density

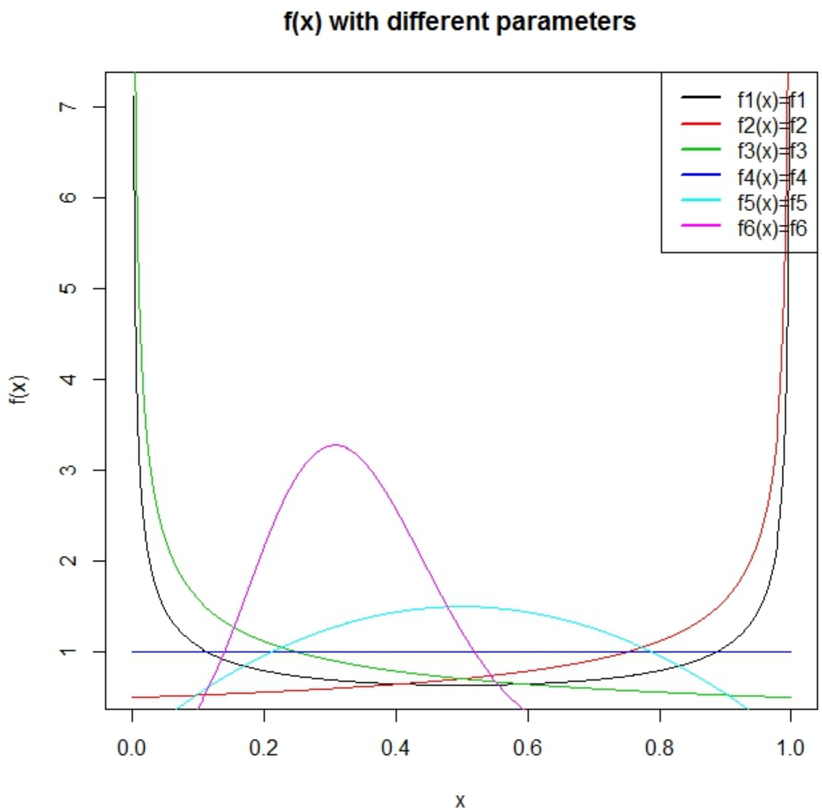

Figure 4: Plots to show target density when (α, β)= (0.5, 0.5),(1, 0.5),(0.5, 1),(1, 1),(2, 2),(5, 10)

R Code :

```
x = seq(0.1,1,300)
a1=0.5; a2=1; a3=0.5; a4=1; a5=2; a6=5;
b1=0.5; b2=0.5; b3=1 ;b4=1; b5=2; b6=10

f=function(x){
  gamma(a1+b1)/(gamma(a1)*gamma(b1))*x^(a1-1)*(1-x)^(b1-1)
  }
f2=function(x){
  gamma(a2+b2)/(gamma(a2)*gamma(b2))*x^(a2-1)*(1-x)^(b2-1)
```

```
  }
f3=function(x){
  gamma(a3+b3)/(gamma(a3)*gamma(b3))*x^(a3-1)*(1-x)^(b3-1)
  }
f4=function(x){
  gamma(a4+b4)/(gamma(a4)*gamma(b4))*x^(a4-1)*(1-x)^(b4-1)
  }
f5=function(x){
  gamma(a5+b5)/(gamma(a5)*gamma(b5))*x^(a5-1)*(1-x)^(b5-1)
  }
f6=function(x){
  gamma(a6+b6)/(gamma(a6)*gamma(b6))*x^(a6-1)*(1-x)^(b6-1)
  }

plot(x, f(x), type="l", col=1)
lines(x,f2(x),type="l",col=2)
lines(x,f3(x),type="l",col=3)
lines(x,f4(x),type="l",col=4)
lines(x,f5(x),type="l",col=5)
lines(x,f6(x),type="l",col=6)
legend("topright",lwd =c(2,2,2,2,2,2),
legend = c("f(x)=f","f2(x)=f2","f3(x)=f3","f4(x)=f4","f5(x)=f5","f6(x)=f6"),
col=c(1,2,3,4,5,6))
```

(b) Let $\alpha > 1$ and $\beta > 1$

$$f(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} for 0 < x < 1$$

On differentiating the f(x) equation ,

$$\rightarrow \frac{df}{dx} = \frac{1}{B(\alpha,\beta)} x^{\alpha-2}(1-x)^{\beta-1} - \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-2}$$

Now taking all the terms common in one side and simlifying the above equation;

$$\rightarrow \frac{df}{dx} = \frac{1}{B(\alpha,\beta)} x^{\alpha-2}(1-x)^{\beta-2}[(1-x)(\alpha-1) - (\beta-1)x]$$

Using the second part of the equation and simplifying it further we get

[(1-x)(α -1)-(β-1)x]
= [(α - 1) -x(α -1) - x(β - 1)]
=[(α-1)-x[(α-1)+(β-1)]
=(α)-x[($\alpha + \beta$)-2]

Now substituting the [(1-x)(α -1)-(β-1)x] in $\dfrac{df}{dx}$;

$\rightarrow \dfrac{1}{B(\alpha,\beta)} x^{\alpha-2}(1-x)[(\alpha+\beta)-2][\dfrac{(\alpha-1)}{(\alpha+\beta-2)} - x]$

Considering only $[\dfrac{(\alpha-1)}{(\alpha+\beta-2)} - x]$

The equation can be re written as

$$x \leq \frac{(\alpha-1)}{((\alpha+\beta)-2)}$$

Then $\dfrac{df}{dx} > 0 \rightarrow$ f(x) is increasing.

Simillarly

$$x \geq \frac{(\alpha-1)}{((\alpha+\beta)-2)}$$

Then $\dfrac{df}{dx} < 0 \rightarrow$ f(x) is decreasing.

To prove this further we take $\alpha = 2$ and $\beta = 2$, and plot a graph(Refer to Figure 5)
f(x) is maximum at 0.5
R Code:

```
a <- 2
b<- 2
xn <- seq(0,1,length=500)
f <-xn^(a-1)*(1-xn)^(b-1)/beta(a,b)
plot(x=c(0,1),y=c(0,10),xlab="x",ylab="f(x)")
lines(xn,f)
```

(c) The dominating density that can be used in the rejection sampling technique can be of Uniform distribution.

(d) Implementing the acceptance and rejection algorithm for the parameter B(α, β)=(2,2)
(Refer to Figure 6)
R Code :

```
f <- function(x){
  fx <- x^(2-1)*(1-x)^(2-1)/beta(2,2)
}
num <- runif(500,0,1)
u <- runif(500,0,1)
t <- sapply(num,f)
ut <- u/t
hist(num[ut <=1], prob=T)
sum(ut<=1)/1000
```

Output: The acceptance percentage is 40.7 percent

Implementing the rbeta function(Refer to Figure 7)

```
r<-rbeta(1,2,2)
ut1<-u/r
hist(num[ut1 <=1], prob=T)
```

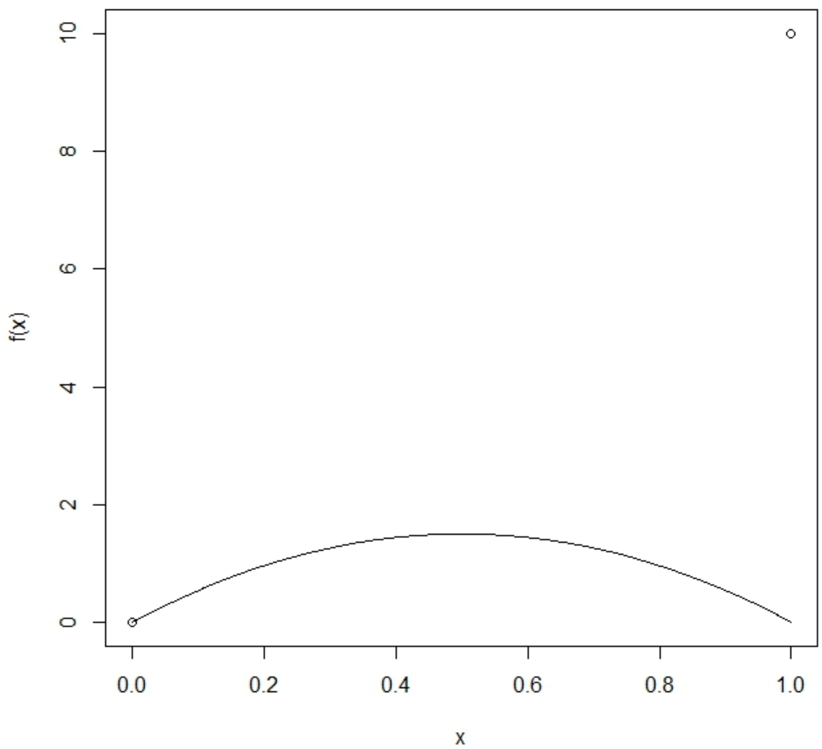

Figure 5: Plot to show f(x) is increasing for $x \leq \dfrac{(\alpha-1)}{((\alpha+\beta)-2)}$ and f(x) is decreasing for$x \geq \dfrac{(\alpha-1)}{((\alpha+\beta)-2)}$

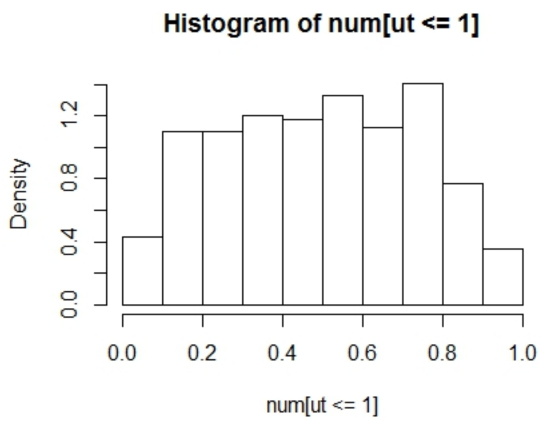

Figure 6: Acceptance - Rejection algorithm (α, β)=(2, 2)

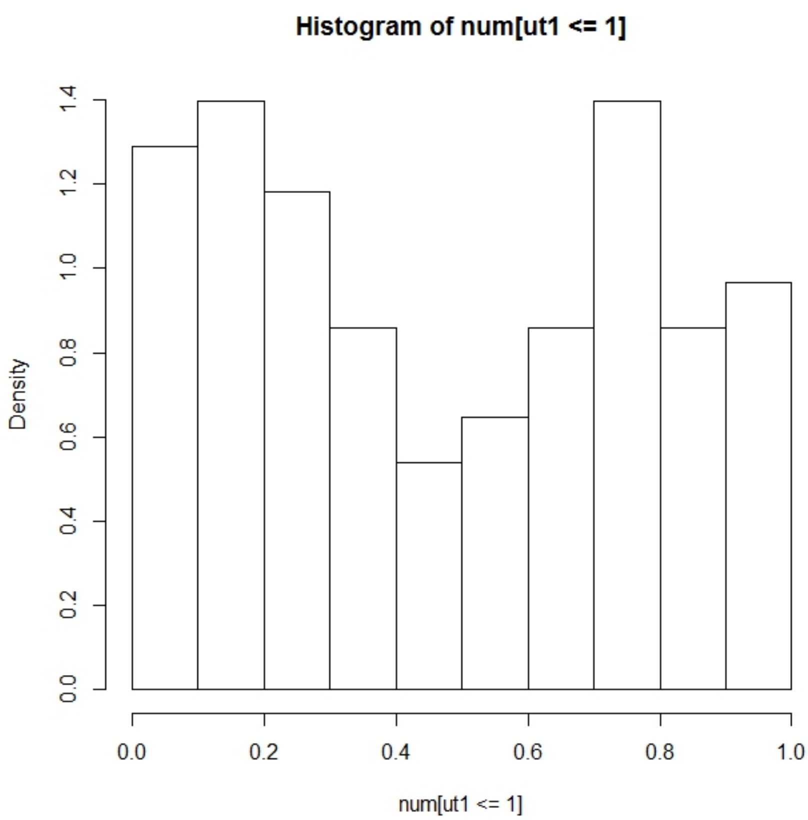

Figure 7: Rbeta Histogram (α, β)=(2, 2)