Functional Analysis Note

Shuo Wang

Dec. 2023

Contents

1 Introduction

This is the lecture note of Functional Analysis course in semeste A in 2023/24. The lecturer of this course is Prof. Laurent Mertz.

The origin of functional analysis started from the study of function whose inputs are functions. Mathematician used ”functional” to represent this special function: the function of functions.

2 Normed linear spaces and Banach Spaces

Definition (linear space)**.** X is a vector space over a scaler field (or) if X is a set equipped with two operations:

(i) (addition) a map from $X \times X$ to X

(ii) (scalar multiplication) a map from $\times X$ to X

Remark. A vector space X is a module over a special ring: field, so X is a commutative group with scalar multiplication.

Property. (i) $\forall x \in X$, $0_x = 0_X$

(ii) $\forall x \in X$, $(-1_)x = -x$

Example. (i) set of polynomials on or set of multivariate polynomials on ^{N}

(ii) space $\mathcal{C}(^N,)$ of continuous functions taking values in

(iii) set of sequences on : $\{\{a_n\}_{n\geq 1} :$ where $a_n \in, \forall n \geq 1\}$

(iv) $(\Omega, , \mu)$ where $\Omega \subseteq^N$ is open bounded, $= (\Omega)$ is the Borel algebra, and μ is a measure, then $X = \{f \in (\Omega,) : \int f \mu\}$

Definition (subspace)**.** Let $Y \subseteq X$ be a subset of a linear sapce X, then Y is a subspace if

$$\forall x, y \in Y, \forall \lambda \in, \quad x + \lambda y \in Y$$

Example. (i) $Y = \{0_X\}$ is the smallest subspace

(ii) if Y_1 and Y_2 are two subspaces, then

$$Y_1 + Y_2 \{y_1 + y_2 : y_1 \in Y_1 \text{ and } y_2 \in Y_2\}$$

is a subspace of X

(iii) if $\{Y_\theta : \theta \in\}$ is a collection of subspaces of X, then $\bigcap\limits_{\theta\in} Y_\theta$ is a subspace of X

Definition (linear span)**.** Let $S \subseteq X$, then

$$\text{span}(S) = \text{ set of linear combinations of elements in } S$$

then $z \in \text{span}(S)$

$\iff \exists N \in, \exists \lambda_1, \dots, \lambda_N \in, \exists x_1, \dots, x_N \in X, \quad z = \sum\limits_{k=1}^{N} \lambda_k x_k$. Clearly, span$(S)$ is the smallest linear space containing S, and

$$mathrmspan(S) = \left\{ \sum_{i=1}^{N} \lambda_1 x_i : \lambda_i \in, x_i \in S, N \geq 1 \right\}$$

Definition (linear maps)**.** Let X_1 and X_2 be two $-$linear spaces, $T : X_1 \to X_2$ is a linear map if

$$\forall x, y \in X_1, \forall \lambda \in, \quad T(x + \lambda y) = T(x) + \lambda T(y)$$

we say that X_1 and X_2 are isomorphic if there exists a bijective linear map between X_1 and X_2.

Property. (i) if $Y_1 \subseteq X_1$ is a subspace of X_1, then $T(Y_1)$ is a subspace of X_2

(ii) if $Y_2 \in X_2$ is a subspace of X_2, then $T^{-1}(Y_2)$ is a subspace of X_1

Definition (convex set)**.** Let $=$, X is a linear space. Let $S \subseteq X$, we say that S is convex if

$$\forall t \in [0, 1], \forall x, y \in S, \quad (1 - t)x + ty \in S$$

Example. $X =^2, =$

(i) $\{(x, y) \in R^2 : (x - a)^2 + (y - b)^2 \leq 1\}$

(ii) $\{(x, y) \in: 0 \leq y \leq \min\{x, -x + 2\}\}$

Definition (convex combination)**.** z is a convex combination of $x_1, \dots, x_n$ if it can be written as ${}_{k=1}^{N} t_k x_k$, where $t_k \in [0, 1]$ and ${}_{k=1}^{N} t_k = 1$

Property. (i) $S_\theta : \theta \in$ is a collection of convex sets in X, then $\bigcap\limits_{\theta \in} S_\theta$ is a convex set in X

(ii) if $T : X_1 \to X_2$ is a linear map and S is a convex subset of X_1, then $T(S)$ is a convex subset of X_2

(iii) if $R \in X_2$ is convex, then $T^{-1}(R)$ is convex

Definition (convex hull)**.** Let $S \in X$ be a subset of the linear sapce X, then the smallest convex set containing S which is also the intersection of all convex sets containing S can be characterized as

$$\mathrm{CH}(S) = {}_{j=1}^{N} t_j x_j : x_j \in S, N \geq 1, t_j \in [0, 1], \quad {}_{j=1}^{N} t_j = 1$$

Definition (normed linear space)**.** X is a normed linear space if X is a linear space over $=$ or and equipped with a norm $N : X \to_+$

(i) $\forall x \in X, N(X) \geq 0$
$\forall x \in X, N(x) = 0 \iff x = 0$

(ii) $\forall x \in X, \alpha \in, N(\alpha x) = \alpha N(x)$

(iii) $\forall x, y \in X, N(x + y) \leq N(x) + N(y)$

From now on, $N(x)$ is denoted by x.

Lemma 2.1 (triangle inequality). $\forall x, y \in X, x - y \leq x - y$

Proof. Using the third property of the norm. $\square$

Now we can define a metric on X

Definition (metric). A map from $X \times X$ to $_+$ is called a metric if

(i) $(x, y) \geq 0$ and $(x, y) = 0 \iff x = y$

(ii) $(x, y) = (y, x)$

(iii) $(x, z) \leq (x, y) + (y, z)$

It can be readily checked that the map $(x, y) \mapsto x - y$ is a metric. We can use norm to construct metric, use metric to construct topology, and use topology to construct convergence.

Definition (equivalence of norms). Two norms $_1$ and $_2$ are equivalent if

$$\exists\, 0 \leq c, \forall x \in X, \quad 1cx_1 \leq x_2 \leq cx_1$$

they give rise to the same topology

Definition (Banach space). Let $(X,)$ be a , $x_{kk\geq 1}$ is a Cauchy sequence if

$$\forall \epsilon \geq 0, \exists N_\epsilon \in, \forall p, q \geq N_\epsilon \quad x_p - x_q \leq \epsilon$$

we say X is complete is every Cauchy sequence converges in X. A Banach space is defined as a complete

Example. (i) $\ell = \left\{ \{a_j\}_{j\geq 1} : a_j \in, \quad \sup_j |a_j| < \infty \right\}$ is complete

(ii) $\ell^p = \left\{ \{a_j\}_{j\geqslant 1}, a_j \in \mathbb{K}, \sum_j |a_j|^p < \infty \right\}$ is complete

(iii) (M, d) metric space, $\mathcal{C}_0(M) = \{f : M \to \mathbb{C}, f$ is continuous and f has compact support$\}$ $\|f\|_\infty = \sup\{|f(x)| : x \in M\}$ is not complete (whereas M is compact)

we show that $\ell^\infty = \left\{ \{a_j\}_{j\geqslant 1}, a_j \in \mathbb{K}, \sup_j < \infty \right\}$ is complete

Proof. Let $x^n_{n\geqslant 1}$ be a Cauche sequence in ℓ^∞. For each $n \geqslant 1$, $x_n = \left\{x^n_j\right\}_{j\geqslant 1}$.
$\forall \epsilon > 1, \exists N_\epsilon \in \mathbb{N}, \forall p, q \geqslant N_\epsilon, \quad \|x^p - x^q\| \leqslant \epsilon$ which means $\sup_j \left|x^p_j - x^q_j\right| \leqslant \epsilon$.

Step I: identify a candidate x
Step II: show that $x^n \to x$

Claim. With j fixed, for each $j \geqslant 1$, $\left\{x_j^n\right\}_{n\geqslant 1}$ is a Cauchy sequence in $\mathbb{K}$, so for each $j \geqslant 1, x_j^n \to x_j$ as $n \to \infty$. The candidate is $x = \{x_j\}_{j\geqslant 1}$.
check that $x \in \ell^\infty$
for each j, $|x_j| = \lim_{n\to\infty} \left|x_j^n\right| \leqslant \overline{\lim_{n\to\infty}} \sup_j \left|x_j^n\right| \leqslant C_0$ which holds by thefact that Cauchy sequence is bounded.
Step II Since x^n is Cauchy, $\forall\epsilon > 0, \exists N_\epsilon \in \mathbb{N}, \forall p, q \geqslant N_\epsilon,\quad \|x^p - x^q\| = \sup_j \left|x_j^p - x_j^q\right| \leqslant \epsilon$, so for each $j \geqslant 1$, $\left|x_j^p - x_j^q\right| \leqslant \epsilon$. Take $q \to \infty$,

$$\forall\epsilon > 0, \exists N_\epsilon \in \mathbb{N}, \forall p \geqslant N_\epsilon, \quad \left|x_j^p - x_j\right| \leqslant \epsilon, \text{ for each } j \geqslant 1$$

which implies

$$\forall\epsilon \geqslant 0, \exists N_\epsilon \in \mathbb{N}, \forall p \geqslant N_\epsilon, \quad \sup_j \left|x_j^p - x_j^q\right| \leqslant \epsilon$$

That implies $\|x_n - x\| \to 0$

∎

we show that $\ell^p = \left\{\{a_j\}_{j\geqslant 1}, a_j \in \mathbb{K}, \sum_{j=1}^{\infty} |a_j|^p < \infty\right\}$ is complete

Proof. Let $\{x^n\}_{n\geqslant 1}$ be a Cauchy sequence in ℓ^p,

$$\forall\epsilon > 0, \exists N_\epsilon \in \mathbb{N}, \forall p, q \geqslant N_\epsilon, \quad \left(\sum_{j=1}^{\infty} \left|x_j^p - x_j^q\right|^p\right)^{1/p} < \infty$$

This implies for each $j \geqslant 1$

$$\forall\epsilon \geqslant 0, \exists N_\epsilon \in \mathbb{N}, \forall p, q \geqslant N_\epsilon, \quad \left|x_j^p - x_j^q\right| \leqslant \epsilon$$

Again $\left\{x_j^n\right\}_{n\geqslant 1}$ is a Cauchy sequence in $\mathbb{K}$, we choose $x = \{x_j\}_{j\geqslant 1}$ as the candidate.
Step II: For $K \in \mathbb{N}$,

$$\sum_{k=1}^{K} |x_k|^p = \lim_{n\to\infty} \sum_{k=1}^{K} |x_k^n|^p \geqslant \overline{\lim_{n\to\infty}} \|x^n\|^p \leqslant C_0$$

where C_0 is a constant not dependent on K, so $\sum_{k=1}^{\infty} < \infty$.
From Cauchy property, we deduce that

$$\forall\epsilon > 0, \exists N_\epsilon \in \mathbb{N}, \forall p, q \geqslant N_\epsilon, \forall K, \quad \left(\sum_{k=1}^{K} \left|x_j^p - x_J^q\right|^p\right)^{1/p} \leqslant \epsilon$$

which implies

$$\forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N}, \forall p \geqslant N_\epsilon, \forall K, \quad \left(\sum_{k=1}^{K} \left| x_j^p - x_j^q \right|^p \right)^{1/p} \leqslant \epsilon$$

which implies

$$\forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N}, \forall p \geqslant N_\epsilon, \quad \left(\sum_{k=1}^{\infty} \left| x_j^p - x_j \right|^p \right)^{1/p} \leqslant \infty$$

That $x^n \to x$ in ℓ^p $\square$

3 Zorn's lemma

Definition (relation)**.** X is a nonempry set, a relation R on X is a subset $\mathcal{G}_R \subset X \times X$

$$xRy \iff (x,y) \in \mathcal{G}_R$$

Example. Ω is a set, $X = \mathcal{P}(\Omega)$ is the power set of X. $\forall A, B \in \mathcal{P}(\Omega)$, $ARB \iff A \in B$

$$\mathcal{G}_R = \{(A,B) \in \mathcal{P}_\Omega \times \mathcal{P}_\Omega, A \in B\}$$

Definition (partial order)**.** we say that R is a partial order on X if it satisfies the following properties:

(i) (reflexive): $\forall x \in X, xRx$

(ii) (anti-symmetric): $\forall x, y \in X, xRy$ and $yRx \Rightarrow x - y$

(iii) (transitive): $\forall x, y, z \in X, xRy$ and $yRz \Rightarrow xRz$

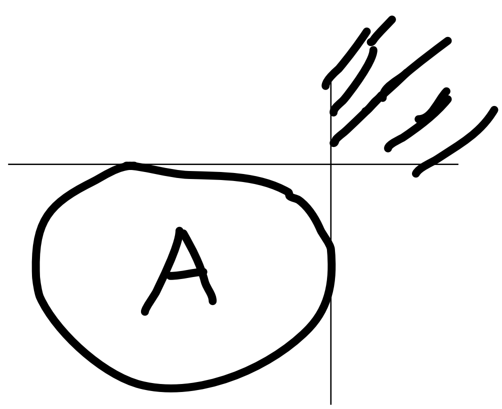
R becomes a totally ordered relation if in addition to a),b), and c)

(i) $\forall x, y \in X, xRy$ or yRx

Example. partial order in $\mathbb{R}^2$: Let $x = (x_1, x_2) \in \mathbb{R}^2, y = (y_1, y_2) \in \mathbb{R}^2$. We write xRy if $x_1 \leqslant y_1$ and $x_2 \leqslant y_2$ and use the notation $(x_1, x_2) < (y_1, y_2)$

Remark. We cannot compare two points $(x_1, x_2)\, and\, (y_1, y_2)$ when $x_1 > x_2$ or $y_1 > y_2$

Definition. Let X be with a partial order $<$. Consider $A \subseteq X$ and $z \in X$. We say that z is an upper bound for A if $\forall a \in A, a < z$

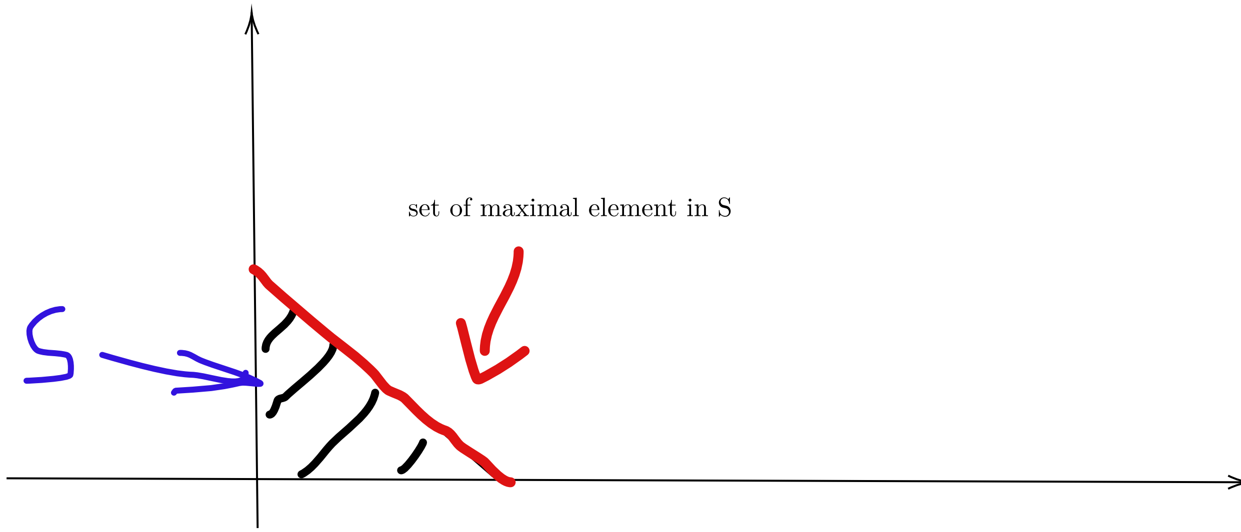


Example.
Any point in the shaded area is an upper bound for **A**

Definition (maximal element)**.** $A \subseteq X, z \in X$, we say that z is a maximal element if $\forall y \in A, z < y, \Rightarrow z = y$

Example.

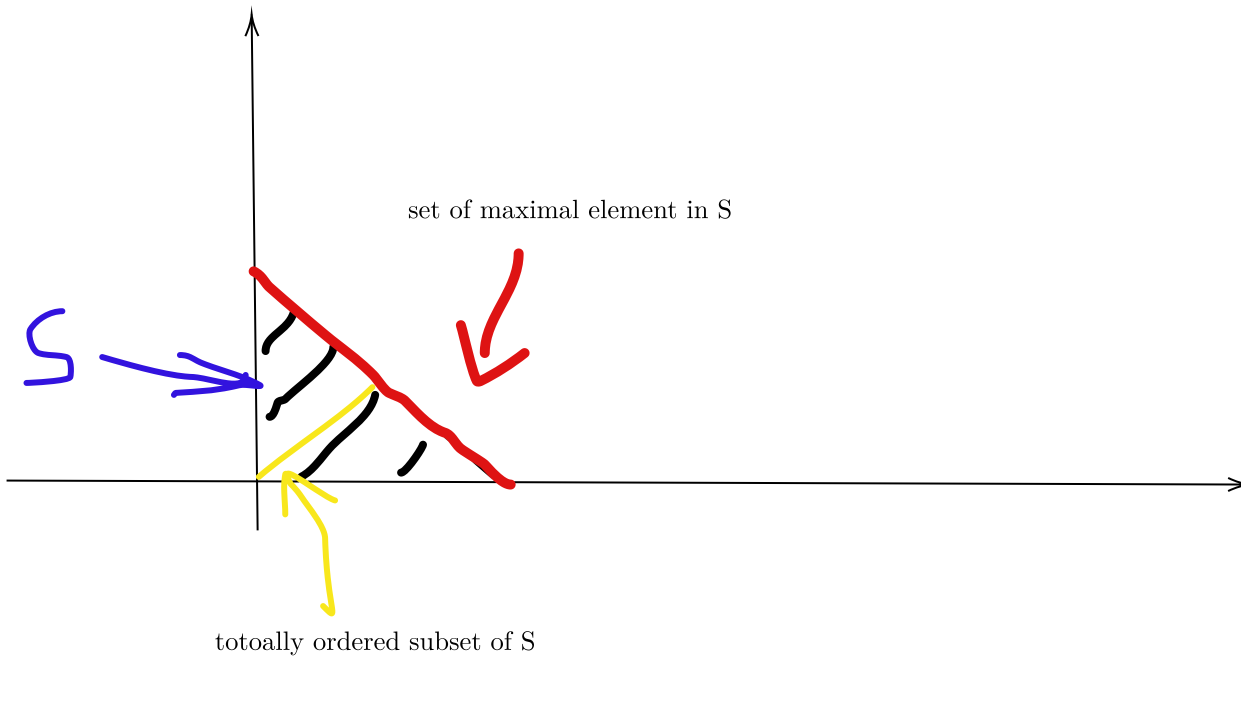
$$\mathcal{S} = \{(x,y) \in [0,\infty) \times [0,\infty) : x + y \leqslant 1\}$$



Remark. if R is a partial order on X, $\mathcal{G}_R \subseteq X \times X$. Let $Y \subseteq X$, $\forall x, y \in Y, xRy$ or $yRx, (x,y)$ or $(y,x) \in \mathcal{G}_R$, that means Y is totally ordered w.r.t. R

Example.

$$\mathcal{S} = \{(x,y) \in [0,\infty) \times [0,\infty) : x + y \leqslant 1\}$$



Lemma 3.1 (zorn's lemma)**.** Let $(X, <)$ be a nonempty partial order set, if any totally ordered subset Y of X has an upper bound, then X has a maximal element (at least one)

Remark. Zorn's lemma is equivalent to the **Axiom of Choice**, which can be stated as follows:
For any set X composed of nonempty sets, $\forall A \in X, \exists f : X \mapsto \bigcup\limits_{A \in X} A$ such that $f(A) \in A$

Example. $X = \{\{1,2\}, \{0\}, \{3,4\}\}$ when X is finite. $\bigcup\limits_{A \in X} A = \{0,1,2,3,4\}$. We define $f(\{1,2\}) = 1, f(\{0\}) = 0, f(\{3,4\}) = 3$.

Counterexample:

$$X = \left\{(x, y) \in \mathbb{R}^2 : x = y\right\}$$

it is not true that any totally ordered subset of X has an upper bound.
Now X be a vector space, we want to prove that X has a basis.

Definition. $\{X_\theta\}_{\theta \in \mathrm{I}}$ is a basis of X if

(i) a): $\forall N, \forall \theta_1, \ldots, \theta_N \in \mathrm{I}(\theta_i \neq \theta_j) \Rightarrow \{x_{\theta_1}, \ldots, x_{\theta_N}\}$ is linearly independent.

(ii) b): $\operatorname{span}\{x_\theta\}_{\theta \in \mathrm{I}} = X$

Theorem 3.2. Let X be a nonempty vector space, $\exists$ a basis for X.

Proof. A simple corollary application of Zorn's lemma. $\square$

4 Hahn-Banach theorem

Let X be an n.v.s., $\mathbb{K} = \mathbb{R}$

Theorem 4.1. Consider $Y \subseteq X$ to be a subspace, $l : Y \mapsto \mathbb{R}$ a linear map. Assume $\exists p : X \mapsto \mathbb{R}$ such that:

(i) $\forall \alpha > 0, \forall x \in X, \quad P(\alpha x) = \alpha p(x)$

(ii) $\forall x, y \in X, \quad p(x+y) \leqslant p(x) + p(y)$

if $\forall x \in X, l(x) \leqslant p(x)$, then $\exists L : X \mapsto \mathbb{R}$

(i) L is linear

(ii) $L\,|_Y = l$

(iii) $\forall x \in X, L(x) \leqslant p(x)$

Proof. Suppose first that X is spanned by Y and an element $x_0 \notin Y$, that is, suppose that

$$X = \{x = m + \alpha x_0 : m \in Y, \alpha \in \mathbb{R}\}$$

Since $x_0 \notin Y$, the above representation of $x \in X$ in the form $x = m + \alpha x_0$ is unique. It follows that, if, for any real number c, we set

$$L(x) = L(m + \alpha x_0) = l(m) + \alpha c$$

then L is a real linear functional on X which is an extension of l. We have to choose c such that $L(x) \leqslant p(x)$, that is, $l(m) + \alpha c \leqslant p(m + \alpha c)$. The condition is equivalent to the following two conditions:

$$l(m/\alpha) + c \leqslant p(x_0 + m/\alpha) \text{ for } \alpha > 0$$

$$l(m/(-\alpha)) - c \leqslant p(-x_0 + m/(-\alpha)) \text{ for } \alpha \leqslant 0$$

To satisfy these conditions, we shall choose c such that

$$l(m') - p(m' - x_0) \leqslant c \leqslant p(m'' + x_0) - l(m'') \text{ for all } m', m'' \in Y$$

uch a choice of c is possible since

$$\begin{aligned} l(m') + l(m'') = l(m' + m'') &\leqslant p(m' + m'') = p(m' - x_0 + m'' + x_0) \\ &\leqslant p(m' - x_0) + p(m'' + x_0) \end{aligned}$$

we have only to choose c between the two numbers

$$\sup_{m' \in Y} l(m') - p(m' - x_0) \text{ and } \inf_{m'' \in Y} p(m'' + x_0) - l(m'')$$

Consider now the family of all real linear extensions g of l for which the inequalities $g(x) \leqslant p(x)$ holds for all x in the domain of g. We make this family into a partiality ordered family by defining $g < h$ to mean that h is an extension of g. Then the Zorn's lemma ensures the existence of a maximal linear extension g of l for which the inequality $g(x) \leqslant p(x)$ holds for all x in the domain of g. We have to prove that the domain $D(g)$ of g coincides with X itself. If it does not, we obtain, taking $D(g)$ as Y and g as l, a proper extension L of g which satisfies $L(x) < p(x)$ for all x in the domain of L, contrary to the maximality of the linear extension g. $\square$

5 Dual of a n.l.s.

Let X be an **n.l.s**. and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Definition. l is contiunous if $\forall \{x_n\}_{n\geqslant 0}$ converging sequence in X such that $x = \lim\limits_{n\to\infty} x_n \in X$, then $\{l(x_n)\}_{n\geqslant 0}$ converging in $\mathbb{K}$

Remark. When X is finite dimensional, any linear forms on X is continuous.

Definition. l is bounded if $\exists C_0 > 0$ such that $\forall x \in X, \quad |l(x)| \leqslant C_0 \|x\|$

Proposition. $l : X \to \mathbb{K}$ linear form , then

$$l \text{ is continuous } \Leftrightarrow l \text{ is bounded}$$

Proof. $(\Longleftarrow)$ Assume l is bounded. Consider $\{x_n\}_{n\geqslant 0}, x_n \longrightarrow x$ in X, we want to show that $l(x_n) \longrightarrow l(x)$ in $\mathbb{K}$. Now $\exists C_0 > 0$ such that

$$|l(x_n) - l(x)| = |l(x_n - x)| \leqslant C_0 \|x_n - x\|$$

which shows that l is continuous.
$(\Longrightarrow)$ We prove by contradiction: we assume

$$\forall n \in N, \exists x_n \in X, \quad |l(x_n)| > n \|x_n\|$$

Define $z_n \triangleq \dfrac{x_n}{n\|x_n\|}$, then $\|z_n\| = \frac{1}{n}$ so $z_n \to 0$ in X, $l(z_n) = \dfrac{l(x_n)}{n\|x_n\|} > 1$, so $l(z_n) \nrightarrow l(0) = 0$ □

Definition (dual of **n.l.s.**). X' to denote the set of continuous linear forms on X, which is also the set of bounded linear forms on X

Property. X' is a n.l.s., $l \in X'$,

$$\begin{aligned} \|l\| &= \sup_{x\neq 0} \frac{|l(x)|}{\|x\|} \\ &= \sup_{x\neq 0} \left| L(\frac{x}{\|x\|}) \right| \\ &= \sup_{\|y\|=1} |l(y)| \end{aligned} \tag{1}$$

$\forall x \in X, |l(x)| \leqslant \|l\| \|x\|$

Exercise. $\|l\|$ is a norm on X'

Proposition. $l \in X', N = \{z \in X, l(z) = 0\}$

(i) N is a closed linear space

(ii) $l \neq 0, \exists x \in X, \forall y \in X, \exists \alpha \in \mathbb{K}, \exists \mu \in N$ such that $\quad y = \alpha x + \mu$

equivalently, $\exists x \in X, X = \mathbb{K}\langle x \rangle + N$

Proof. (i) Suppose that $x_n \to \hat{x} \in X$, then $l(\hat{x}) = l(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} l(x_n) = 0$, so $\hat{x} \in N$

(ii) $l \neq 0$. $\exists x \in X,$ such that $l(x) \neq 0$. Let $y \in X$, we want to find $\alpha \in \mathbb{K}, \mu \in N$, such that $y = \alpha x + \mu$
Step I: $l(y) = l(\alpha x + \mu) = \alpha l(x) + l(\mu)$, then the choice for α, given $y \in X$, is $\alpha \triangleq \frac{l(y)}{l(x)}$.
Step II: $y = \dfrac{l(y)}{l(x)} x + \left(y - \dfrac{l(y)}{l(x)} x \right)$, so we choose $\mu = y - \dfrac{l(y)}{l(x)} x$, It is easy to check the uniqueness of the represnetation of y in this form

Now $X = \mathbb{K}\langle x \rangle \oplus N$. If X is finite dimensional, say $\dim X = n$, then it is easy to find that N is of codimension 1 $\square$

Proposition. X' is always complete, (even if X is not complete)

Proof. $\{l_n\}_{n\geqslant 0}$ Cauchy sequence in X', then

$$\forall \epsilon > 0, \exists N_\epsilon \text{ such that } \forall p, q \geqslant N_\epsilon, \quad \|l_p - l_q\| \leqslant \epsilon$$

$\Leftrightarrow$

$$\forall \epsilon > 0, \exists N_\epsilon \text{ such that } \forall p, q \geqslant N_\epsilon, \quad \sup_{x \in X, x \neq 0} |l_p(x) - l_q(x)| \leqslant \epsilon \|x\| \tag{2}$$

we want to find $l \in X', l_n \to l$ in X' as $n \to \infty$
Step I: Fix an $x \in X$, $(\rightarrow)$ for (2):
$\{l_n(x)\}_{n\geqslant 0}$ is a Cauchy sequence in $\mathbb{K}$ since $\mathbb{K}$ is complete. There exists a limit $l(x)$. We can easily check that l is a linear mapping. Now we know that for each n, $\forall x \in X, \quad |l_n(x)| \leqslant \|l_n\| \|x\|$ since $\{l_n\}_{n\geqslant 0}$ is a Cauchy sequence, $\{\|l_n\|\}_{n\geqslant 0}$ is bounded. We deduce that

$$\forall x \in X, \quad |l(x)| \leqslant \overline{\lim_{n\to\infty}} \|l_n\| \|x\|$$

Step II: Use (2),

$$\forall \epsilon > 0, \exists N_\epsilon \text{ such that } \forall p, q \geqslant N_\epsilon, \forall x \in X, \quad |l_p(x) - l_q(x)| \leqslant \epsilon \|x\|$$

$(\Rightarrow)$

$$\forall \epsilon > 0, \exists N_\epsilon \text{ such that } \forall p \geqslant N_\epsilon, \forall x \in X, \quad |l_p(x) - l(x)| \leqslant \epsilon \|x\|$$

This is exactly:

$$\forall \epsilon > 0, \exists N_\epsilon \text{ such that } \forall p \geqslant N_\epsilon, \quad \|l_p - l_q\| \leqslant \epsilon$$

so $l_n \to l$ as $n \to \infty$ in X' $\square$

6 extension of bounded linear forms

$\mathbb{K} = \mathbb{R}$, X is a vector space, $Y \subseteq X$ is a subspace. $l : Y \mapsto \mathbb{R}$ is a linear form on Y. Define $p(x) = c_0 \|x\|$, and using Hahn-Banach Theorem to extend l on X: $\exists L : X \mapsto \mathbb{R}$ a linear form such that

(i) $\forall y \in Y, L(y) = l(y), \text{i.e.} L\mid_Y = l$

(ii) $\forall x \in X, |L(x)| \leqslant p(x)$

(iii) $\|L\| = \|l\|$

Proposition. Given $\{x_1, \dots, x_n\}$ linearly independent in X and $\alpha_1, \dots, \alpha_N \in \mathbb{R}$, $\exists L \in X'$ such that $L(x_j) = \alpha_j, 1 \leqslant j \leqslant N$

Proof. $Y \triangleq \text{span}\{x_1, \dots, x_n\}$,$\exists z \in Y, z = \sum\limits_{j=1}^{N} \beta_j x_j$ where $\beta_1, \dots, \beta_N \in \mathbb{R}$. Define l on Y:

$$l\left(\sum_{j=1}^{N} \beta_j x_j\right) = \sum_{j=1}^{N} \beta_j l(x_j) = \sum_{j=1}^{N} \beta_j \alpha_j$$

Claim. l is bounded

we want to find $c_0 > 0$ such that $\forall y \in Y, |l(y)| \leqslant c_0 \|y\|$ Take $z \in Y$,

$$\begin{aligned} |l(z)| &= \left| l(\sum_{j=1}^{N} \beta_j x_j) \right| \\ &= \left| \sum_{j=1}^{N} \beta_j \alpha_j \right| \leqslant \max_{j=1,\dots,N} |\alpha_j| \sum_{j=1}^{N} |\beta_j| \end{aligned}$$

since Y is finite dimensional, $\|z\|_1 \leqslant C_1 \|z\|_X$ because in finite dimensional vector space, all norms are equivalent. So $|l(z)| \leqslant C_0 \|z\|$.
Now we apply Hahn-Banach Theorem: writing $p(x) = C_0 \|x\|$, so L is indeed well-defined on X. □

Corollary. if $Y \subseteq X$ is a finite dimensional subspace, $\exists$ closed subspace M of X, such that $X = Y \oplus M$

Proof. since Y is finite dimension, $Y = \text{span}\{x_1, \dots, x_N\}$. Using the above proposition, we can define N bounded linear forms $L_j(x_i) = \delta_{i,j}$.
Define $M_j = \{x \in X : L_j(x) = 0\}$ and $M = \bigcap\limits_{j=1}^{N} M_j$

Claim. M is a closed subspace of X

Next take $x \in X$, we want to decompose x as $\sum\limits_{j=1}^{N} application_j x_j + \mu$ where $\alpha_j \in \mathbb{R}, \mu \in M$. Fix $1 \leqslant k \leqslant N$, $L_k(x) = L_k(\sum\limits_{j=1}^{N} \alpha_j x_j + \mu) = \sum\limits_{j=1}^{N} \alpha_j L_k(x_j) = \alpha_k$. Now we can define $\alpha_k \triangleq L_k(x)$, so $x = \sum\limits_{j=1}^{N} L_j(x)x_j + \left(x - \sum\limits_{j=1}^{N} L_j(x)x_j\right)$. It is easy to check that $x - \sum\limits_{j=1}^{N} L_j(x)x_j \in M$.

Finally, if $x \in X$, $x = \sum \alpha_j x_j + \mu = \sum \alpha_j' x_j + \mu'$, can we deduce $\alpha_j = \alpha_j'$ and $\mu = \mu'$? We study $Y \cap M$: Let $\xi \in Y \cap M$, $\exists \alpha_1, \dots, \alpha_N \in \mathbb{R}$ such that $\xi = \sum\limits_{j=1}^{N} \alpha_j x_j$. $\forall 1 \leqslant k \leqslant N, \quad L_k(\xi) = 0 = L_k(\sum\limits_{j=1}^{N} \alpha_j x_j) = \sum\limits_{j=1}^{N} \alpha_j L_k(x_j) = \alpha_k$, so $\xi = 0$.

□

Theorem. $\forall x \in X, \quad \|x\| = \sup\limits_{\|l\|=1} |l(x)|$

Proof. On the one hand, fix an $x \in X$. Take $l \in X'$ such that $\|l\| = 1$, then $|l(x)| \leqslant \|l\| \, \|x\|$, so $\sup\limits_{\|l\|=1} |l(x)| \leqslant \|x\|$

On the other hand, again $x \in X$ is fixed, take $Y = \mathbb{K}x$. Define $\hat{l} : Y \mapsto \mathbb{R}, \hat{l}(tx) = t\|x\|^2$, then $\forall y \in Y, \quad \left|\hat{l}(y)\right| \leqslant \|y\| \, \|x\|$, so $\sup\limits_{y \in Y, \|y\| \leqslant 1} \left\|\hat{l}(y)\right\| \leqslant \|x\|$. Moreover, $\hat{l}(\frac{x}{\|x\|}) = \|x\|$, as a consequence, $\left\|\hat{l}\right\| = \|x\|$. Now use Hahn-Banach Theorem, $\exists l \in X', l \mid_Y = \hat{l}$ and $\|l\| = \left\|\hat{l}\right\|$ which is $\|x\|$. Introduce $\tilde{l} \triangleq \dfrac{l}{\|l\|}$, clearly $\tilde{l} = 1$ and $\tilde{l}(x) = \frac{l(x)}{\|x\|} = l(\frac{x}{\|x\|}) = \hat{l}(\frac{x}{\|x\|}) = \|x\|$ □

Theorem. $Y \subseteq X$ is a subspace, then $\forall x \in X, \quad \inf\limits_{y \in Y} \|x - y\| = \sup\limits_{\|l\|=1, l|_Y=0} |l(x)|$

Proof. On the one hand, consider $l \in X'$ such that $\|l\| = 1$ and $l \mid_Y = 0$. $\forall x \in X, \forall y \in Y$,

$$\begin{aligned} |l(x)| &= |l(x) - l(y)| = |l(x - y)| \\ &\leqslant \|l\| \, \|x - y\| \end{aligned}$$

since l is arbitrary,

$$\sup_{\|l\|=1, l|_Y=0} |l(x)| \leqslant \|x - y\|$$

since y is arbitrary,

$$\sup_{\|l\|=1, l|_Y} |l(x)| \leqslant \inf_{y \in Y} \|x - y\|$$

On the other hand, assume $x \notin Y$, define $d \triangleq \inf\limits_{y \in Y} \|x - y\| > 0$

$$Y_0 = \{\alpha x + y : y \in Y, \alpha \in \mathbb{R}\} \subseteq X$$

Remark. if $\alpha x + y = \alpha' x + y' \Rightarrow \alpha = \alpha'$ and $y = y'$

Define $l : Y \to \mathbb{R}$ linear forms such that $l(\alpha x + y) = \alpha l(x) + l(y) = \alpha l(x)$, we would like to have

$$\begin{aligned}|l(\alpha x + y)| \leqslant \|\alpha x + y\| &\Leftrightarrow |\alpha l(x)| \leqslant \|\alpha x + y\| \\ &\Leftrightarrow |l(x)| \leqslant \|x + y/\alpha\|\end{aligned}$$

This inequality will be satisfied if $|l(x)| \leqslant \inf\limits_{y\in Y} \|x - y\| = d$. Let $L(x) = d$, then $\forall y \in Y_0, |l(y)| \leqslant \|y\|$, so $\|y\| \leqslant 1$.
Finally apply Hahn-Banach Theorem: $\exists L : X \to \mathbb{R}$ which is a bounded linear form and $\|L\| = \|l\| \leqslant 1$, $L|_{Y_0} = l$, so $L|_Y \equiv 0$. Define $\hat{L} = \frac{L}{\|L\|}$, $\left\|\hat{L}\right\| = 1$ and $\hat{L}|_Y = 0$, $\hat{L}(x) = \frac{L(x)}{\|x\|} = \frac{L(x)}{\|l\|} = \frac{l(x)}{\|l\|} = \frac{d}{\|l\|} \geqslant d$. Here

$$\sup_{\|l\|=1, l|_Y=0} |l(x)| \geqslant \hat{L}(x) \geqslant d = \inf_{y\in Y} \|x - y\|$$

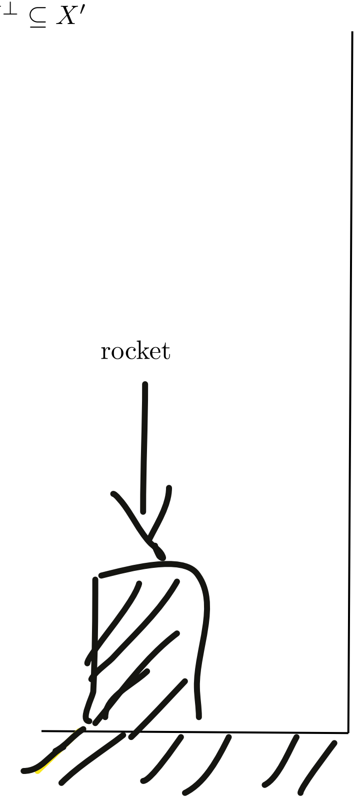
$\square$

Definition. $Y \subseteq X$ a subspace, $X' \supseteq Y^{\perp} \triangleq \{l \in X' : l|_Y = 0\}$

Theorem. $l \in X', \quad \|l\|_Y = \sup\limits_{y\in Y, \|y\|=1} |l(y)| = \inf\limits_{m\in Y^{\perp}} \|l - m\|_X$

7 application of Hahn-Banach Theorem in Optimal Control

Let X be a n.l.s. and $Y \subseteq X$ be a subspace. X' be the dual space of X and $Y^{\perp} \subseteq X'$



Question. $H > 0$ is the targeted height, and the motion is $1-$D moves alone the vertical axis. Let $x(t)$ be the position of the rocket at time $t \geqslant 0$ with initial values $x(0) = 0, \dot{x}(0) = 0$

We denote by M the mass of the rocket, then by **New's Law**:

$$M\ddot{x}(t) = -Mg + F(t)$$

where F is the applied force by engines. We want to send the rocket at a height H at time $T > 0$, so we need to find F such that

$$\mathcal{C}(T) = \int_0^T |F(s)| \, \mathrm{d}s$$

is minimal.
We are interested in the value and minimizer of $\inf \left\{ \mathcal{C}^F(T) : Fis \text{ such that } x^F(T) = H \right\}$.

Assume $M = 1$, then $\dot{x}(t) - \dot{x}(0) = \int_0^t (-g + F(s))\,\mathrm{d}s$, so $\dot{x}(t) = -gt + \int_0^t F(s)\mathrm{d}s$
Again,

$$x(t) - x(0) = -g\frac{t^2}{2} + \int_0^t \int_0^s F(r)\mathrm{d}r\mathrm{d}s$$

$$\int_0^t \int_0^s F(r)\mathrm{d}r\mathrm{d}s = \int_0^t \int_0^t \mathcal{X}_{\{r \leqslant s\}} F(r)\mathrm{d}s\mathrm{d}r = \int_0^t (t-r)F(r)\mathrm{d}r$$

$$x(t) = -g\frac{t^2}{2} + \int_0^t (t-r)F(r)\mathrm{d}r$$

We are interested in F such that $x(T) = H \Leftrightarrow \int_0^T (T-r)F(r)\mathrm{d}r = g\frac{T^2}{2} + H$. We are interested in $\inf\left\{\mathcal{C}^F(T) : F \text{ satisfies } \int_0^T (T-r)F(r)\mathrm{d}r = g\frac{T^2}{2} + H\right\}$. At the moment T is fixed, later we minimize over T.

Remark. $\forall g \in \mathcal{C}\,[0,T]\,, \int_0^T g(s)\mathrm{d}\mu(S) = \int_0^T g(s)F(s)\mathrm{d}s, \mu(\mathrm{d}s) = F(s)\mathrm{d}s$, then F is called the density of μ w.r.t. the Lebesgue measure, then $\int_0^T |F(s)|\,\mathrm{d}s = \|\mu\|_{TV}$. Now if μ is a general measure, $\|\mu\|_{TV} =\triangleq \sup\limits_{\Pi\text{partation of } [0,T]} \sum\limits_{A\in\Pi} |\mu(A)|$, recall that a partation of $[0,T]$ is a countable number of measurable subsets of $[0,T]$ and these subsets are pairwise disjoint, and their union is $[0,T]$.

Now we can reformulate the problem as

$$\inf_{\mu\in\mathcal{C}[0,T]'} \left\{\|\mu\|_{TV} : \mu(\omega) = H + g\frac{T^2}{2}\right\} \text{ where } \mu(\omega) = \int_0^T \omega(s)\mathrm{d}\mu(s) = \int_0^T (T-s)\mathrm{d}\mu(s)$$

Claim. the inf is $\frac{H}{T} + \frac{gT}{2}$ and the minimizer is $\mu^* = \left(\frac{H}{T} + \frac{gT}{2}\right)\delta_0$

Define $Y \triangleq \operatorname{span}(\omega) \subseteq \mathcal{C}\,[0,T]$ where $\omega(t) = T - t$, we also define $\hat{l}_0 : Y \to \mathbb{R}$ with $\hat{l}_0(\omega) = H + g\frac{T^2}{2}$. $\hat{l}_0$ is a bounded linear form on Y, so $\hat{l}_0 \in Y$. We can apply Hahn-Banach Theorem to extend $\hat{l}_0$ to $l_0 \in \mathcal{C}\,[0,T]'$. $\forall z \in Y, l_0(z) = \hat{l}_0(z)$. Now we reformulate the constraint $\mu(\omega) = H + g\frac{T^2}{2} \Leftrightarrow \mu(\omega) = l_0(\omega) \Leftrightarrow (\mu - l_0)(\omega) = 0$. We can reformulate the problem as

$$\begin{aligned}
&\inf_{\mu\in\mathcal{C}[0,T]'} \{\|l_0 - \mu - l_0\|_{TV} : (\mu - l_0)(\omega) = 0\} \\
&\Leftrightarrow \inf_{\hat{\mu}\in\mathcal{C}[0,T]'} \left\{\|\hat{\mu} - l_0\|_{TV} : \hat{\mu}(\omega) = 0 \text{ where } \hat{\mu} \triangleq l_0 - \mu \in Y^{\perp}\right\} \\
&= \sup_{y\in Y, \|y\|\leqslant 1} |l_0(y)| = \sup_{\alpha\in\mathbb{R}, \|\alpha\omega\|_\infty \leqslant 1} \left|\alpha(H + g\frac{T^2}{2})\right| \\
&\left(\text{ where } \|\omega\|_\infty = \max_{0\leqslant t\leqslant T} |\omega(t)| = T\right) \\
&= \left(\sup_{\alpha\in\mathbb{R}, |\alpha|\leqslant \frac{1}{T}} |\alpha|\right)\left(H + g\frac{T^2}{2}\right) \\
&= H + g\frac{T}{2}
\end{aligned}$$

Now what about the minimizer? if μ^* is a minimizer with $\|\mu^*\| = \frac{H}{T} + g\frac{T}{2}$, we know the properties of the minimizer which come from the constraint in the minimizer problem that

$$\begin{aligned} \mu^*(\omega) = H + \frac{T^2 g}{2} = T\left(\frac{H}{T} + g\frac{T}{2}\right) = \|\omega\|_\infty \|\mu^*\|_{TV} \\ (\text{ where } \omega(t) = T - t) \\ \Rightarrow \mu^* = \beta\delta_0 \text{ where } \beta = \frac{H}{T} + \frac{Tg}{2} \end{aligned}$$

Last step is to minimize over T: $f(T) \triangleq \frac{H}{T} + \frac{Tg}{2}$, then what is $T^* > 0$ such that $f'(T^*) = 0$. The answer is $T^* = \sqrt{\frac{2H}{g}}$ and $f(T^*) = \sqrt{2gH}$

8 Reflexive Spaces

X n.l.s. and X' the dual space of X.

$$\|l\| \triangleq \sup_{\|x\|=1} |l(x)| = \sup_{\|x\|\leqslant 1} |l(x)|$$

for the second equality, ($\leqslant$) is obvious, we have to check ($\geqslant$): let $x \in X$ be arbitrary nonzero element, $l(\frac{x}{\|x\|}) \leqslant \sup_{\|\bar{x}\|=1} l(\bar{x}) = \|l\|$, so $l(x) \leqslant \|l\| \, \|x\|$, which is $\sup_{\|x\|\leqslant 1} |l(x)| \leqslant \|l\|$

Now $X'' =$ set of bounded linear forms on X', and $\|L\| = \sup_{\|l\|=1} |L(l)| = \sup_{\|l\|\leqslant 1} |L(l)|$

Let $x \in X$, and $L_x : X' \to \mathbb{R}$ such that $l \mapsto l(x)$.

Claim. $L_x \in X''$

Proof. It is easy to see L_x is linear. $\sup_{\|l\|=1} |L_x(l)| = \sup_{\|l\|=1} l(x) = \|x\|$ □

So we have a mapping: $\mathcal{L} : X \to X''$ such that $x \mapsto L_x$, and $\|\mathcal{L}_x\| = \|L_x\| = \|x\|$. $\mathcal{L}$ is an isometry, so $\mathcal{L} \subseteq X''$

Question. Can we find an element M of X'' such that $M \notin \mathcal{L}(x)$

Definition (reflexive sapce)**.** we say that X is reflexive under the condition $\mathcal{L}(X) = X''$

Example. (i) if X is finite dimension, X is reflexive

(ii) $X = \mathcal{C}\,[-1,1]$ with $\|\|_\infty$, then this is a counterexample.

Proof. we know that

$$\forall l \in X', \quad \|l\| = \sup_{\|L\|=1} |L(l)| = L_0(l) \text{ for some } L_0 \in X'' \text{ where } \|L_0\| = 1 \quad (3)$$

Suppose $X = X''$, then

$$\forall L \in \mathcal{C}\,[-1,1]'' \,, \exists x_L \in \mathcal{C}\,[-1,1] \,, \forall l \in \mathcal{C}\,[-1,1]' \,, \quad L(l) = l(x_L) \quad (4)$$

Next step, put (4) in 3: we apply (4) to L_0:

$$\exists x_0 \in \mathcal{C}\,[-1,1] \,, \forall l \in \mathcal{C}\,[-1,1]' \,, \|l\| = L_0(l) = l(x_0) \quad (5)$$

Because $\mathcal{L} : X \to X''$ is an isometry, $\|L_0\| = \|x\| = 1$. Define $l \in \mathcal{C}\,[-1,1]'$ as follows:

$$\forall g \in \mathcal{C}\,[-1,1] \,, \quad l(g) \triangleq \int_{-1}^{0} g(t)\mathrm{d}t - \int_{0}^{1} g(t)\mathrm{d}t$$

Claim. $\|l\| = 2$

$$
\begin{aligned}
|l(g)| &\leqslant \left|\int_{-1}^{0} g(t)\mathrm{d}t\right| + \left|\int_{0}^{1} g(t)\mathrm{d}t\right| \\
&\leqslant \int_{-1}^{0} |g(t)|\,\mathrm{d}t + \int_{0}^{1} |g(t)|\,\mathrm{d}t \\
&\leqslant 2\,\|g\|_{\infty}
\end{aligned}
$$

so $\|l\| \leqslant 2$. Now introduce $g_\epsilon(t) \triangleq \mathcal{X}_{[-1,-\epsilon)}(t) - \mathcal{X}_{(\epsilon,1]}(t) + \frac{t}{\epsilon}\mathcal{X}_{[-\epsilon,\epsilon]}(t)$, then $l(g_\epsilon) = 2(1-\epsilon)$ and $\|g_\epsilon\| = 1$.

$$
\|l\| = \sup_{\|g\|=1} |l(g)| \geqslant |l(g_\epsilon)| = 2(1-\epsilon)
$$

so $\|l\| = 2$. Now can we find a function $x \in \mathcal{C}\,[-1,1]$ such that $l(x) = 2$ and $\|x\| = 1$? In fact, $\forall y \in \mathcal{C}\,[-1,1]$, if $\|y\| \leqslant 1$, then $|l(y)| < 2$, here 2 is not reached by l. $\forall y \in \mathcal{C}\,[-1,1]\,, l(y) \neq \|l\|$, this contradicts the function in (5). As a consequence, $\mathcal{C}\,[-1,1]$ is not reflexive.

□

Theorem. Let X be a n.l.s., $\mathbb{K} = \mathbb{R}$. Assume X' is separable and $D = \{l_n : n \geqslant 1\} \subseteq X'$ is a countable and dense subset, then X is separable.

Proof. Since X' is separable. For each $n \in \mathbb{N}$, $\|l_n\| = \sup\limits_{\|z\|=1} |l_n(z)|$. $\exists z_n \in X$ such that $\|z_n\| = 1$ and $|l_n(z_n)| \geqslant \frac{1}{2}\,\|l_n\|$.

Remark. we can replace $|l_n(z_n)|$ by $l_n(z_n)$, since we can use $\hat{z_n} = \pm z_n$

we work on $\{z_n : n \in \mathbb{N}\}$:

Claim. $\overline{\operatorname{span}\{z_n : n \in \mathbb{N}\}} = X$

Define $Y \triangleq \overline{\operatorname{span}\{z_n : n \in \mathbb{N}\}}$ and assume $Y \neq X$, then $\exists x \in X, x \notin Y$. Define $Z_x \triangleq \{\alpha x + y : \alpha \in \mathbb{R}, y \in Y\}$, we then construct $l : Z_x \to \mathbb{R}$ such that $l(x) \neq 0$ and $l|_Y = 0$. Now use Hahn-Banach Theorem: $\exists \hat{l} : X \to \mathbb{R}$ a bounded linear form on X and $\left\|\hat{l}\right\| = \|l\|$, $\forall z \in Z_x, \hat{l}(z) = l(z)$, in particular, $\hat{l}(x) \neq 0$ and $\hat{l}|_Y = 0$. $\forall n \in \mathbb{N}, \hat{l}(z_n) = 0$, we can, up to normalization, assume $\left\|\hat{l}\right\| = 1$. Next we use that D is dense: $\forall \epsilon > 0, \exists n$ such that $\left\|l_n - \hat{l}\right\| \leqslant \epsilon$

$$
\begin{aligned}
\|l_n\| &= \left\|\hat{l} + l_n - \hat{l}\right\| \\
&\geqslant \left\|\hat{l}\right\| - \left\|l_n - \hat{l}\right\| \geqslant 1 - \epsilon
\end{aligned}
$$

and

$$\begin{aligned} \|l_n\| &\leqslant 2l_n(z_n) = 2\left(l_n(z_n) - \hat{l}(z_n)\right) \\ &2\left\|l_n - \hat{l}\right\| \|z_n\| \geqslant 2\epsilon \end{aligned}$$

which is impossible. So we obtained that $\overline{\operatorname{span}\{z_n : n \in \mathbb{N}\}} = X$.

$$\begin{aligned} X &= \overline{\operatorname{span}\left\{\sum_{j=1}^{M} \alpha_j z_{n_j} : M \geqslant 1, \alpha_1, \ldots, \alpha_M \in \mathbb{R}\right\}} \\ &= \overline{\operatorname{span}\left\{\sum_{j=1}^{M} \alpha_j z_{n_j} : M \geqslant 1, \alpha_1, \ldots, \alpha_M \in \mathbb{Q}\right\}} \end{aligned}$$

the last set under the overline is countable. As a consequence, X is separable.

□

Exercise. Show that $\mathcal{C}[-1,1]$ is not reflexive.

Solution 1. Suppose $\mathcal{C}[-1,1] = \mathcal{C}[-1,1]''$, we know that $\mathcal{C}[-1,1]$ is separable, so $\mathcal{C}[-1,1]''$ is separable. Using the above theorem, we know that $\mathcal{C}[-1,1]'$ is separable, which is not true and is checked as follows: We can define a family $\{l_t\}_{t\in[-1,1]}$ which is not countable, such that $\forall t \neq t', \quad \|l_t - l_{t'}\| = 2$. Then $\forall t \neq t', B(l_t, \frac{1}{2}) \cap B(l_{t'}, \frac{1}{2}) = \varnothing$. If $D \subseteq \mathcal{C}[-1,1]'$ is dense, then $\forall t, D \cap B(l_t, \frac{1}{2}) \neq \varnothing$. Then D cannot be countable.

9 Hilbert Space

X linear space, $\mathbb{K} = \mathbb{R}$, a mapping $\langle , \rangle : X \times X \to \mathbb{K}$ is called a inner product if $\forall x, y, z \in X, \alpha \in \mathbb{K}$:

(i) $\langle \alpha x + y, z \rangle = \alpha \langle x, z \rangle + \langle y, z \rangle$

(ii) $\langle x, y \rangle = \overline{\langle y, x \rangle}$

(iii) $\langle x, x \rangle \geqslant 0$ and $\langle x, x \rangle = 0 \iff x = 0$

Definition. $\|x\| \triangleq \sqrt{\langle x, x \rangle}$

Theorem (Schwarz Inequality)**.** $\forall x, y \in X, \quad |\langle x, y \rangle| \leqslant \|x\| \|y\|$

Remark. (i) $|\langle x, y \rangle| = \|x\| \|y\| \iff \exists \alpha \in \mathbb{K}, x = \alpha y$

(ii) $\|x\| = \sup\limits_{y \in X, \|y\|=1} |\langle x, y \rangle|$
On the one hand, $\forall y \in X$ such that $\|y\| = 1, \quad |\langle x, y \rangle| \leqslant \|x\| \Rightarrow \sup\limits_{y \in X, \|y\|=1} |\langle x, y \rangle| \leqslant \|x\|$. On the other hand, $\|x\| = |x, y^*|$ for a specific y^*. Take $y^* = \frac{x}{\|x\|}$, then $|\langle x, y^* \rangle| = \left| \left\langle x, \frac{x}{\|x\|} \right\rangle \right| = \frac{1}{\|x\|} |\langle x, x \rangle| = \|x\|$

Theorem. $\forall x, y \in X, \|x + y\| \leqslant \|x\| + \|y\|$

Theorem (parallelogram identity)**.** $\|x + y\|^2 + \|x - y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$

Definition (orthogonality)**.** We say that $x, y \in X$ are orthogonal if $\langle x, y \rangle = 0$

Definition (Hilbert space)**.** X is a Hilbert sapce if X is a linear space which is complete w.r.t. the norm induced by the inner product

Remark. (i) $\{x_n\}_{n \geqslant 0}, \|x_n - x\| \to 0$ as $n \to \infty$, then $\forall y \in X, \quad \langle x_n, y \rangle \to \langle x, y \rangle$ as $n \to \infty$

(ii) $X, \langle , \rangle$. Assume X is not complete with the norm induced by the inner product. We can construct the completion $\bar{X}$ of X as the set of limits of converging sequences in X and define $\langle \langle x, y \rangle \rangle \triangleq \lim\limits_{n \to \infty} \langle x_n, y_n \rangle$ if $X \ni x_n \to x \in \bar{X}$ and $X \ni y_n \to y \in \bar{X}$

Example. (i) $X = \mathcal{C}[a, b]$ with $\langle f, g \rangle = \int_a^b f(x) g(x) \mathrm{d}x$ is not complete

(ii) $\ell^2 = \left\{ \{a_j\}_{j \geqslant 1} : a_j \in \mathbb{C}, \sum\limits_{j \geqslant 1} |a_j|^2 < \infty \right\}$ and $\langle a, b \rangle = \sum\limits_{j=1}^{\infty} a_j \overline{b_j}$ is complete

(iii) $D \subseteq \mathbb{R}^2$, $\langle f, g\rangle = \int_D f(x)g(x)\mathrm{d}x$ and $L^2(D) = \left\{ f : D \to \mathbb{R} : \int_D |f(x)|^2 \,\mathrm{d}x < \infty \right\}$ with the relation $f = \tilde{f}$ if $f = \tilde{f}$a.e., this space is complete. And when $D = (a, b)$, iii) is the completion of i).

10 Riesz Representation THeorem

A little observation: $x \in X$ and if we define $l_x : X \to \mathbb{K}$ such that $y \mapsto \langle y, x \rangle$, it is readily checked that $l_x \in X'$ and $\|l_x\| = \|x\|$

Theorem (riesz representation theorem)**.** If X is a Hilbert space and $l \in X'$, then $\exists x_l \in X$ such that $\forall y \in X, \quad l(y) = \langle y, x_l \rangle$

Proof. The Uniqueness of x_l is clear, since $\langle y, z \rangle = 0$ for all $y \in X$ implies $z = 0$. To prove its existence, consider the null space $N = N(l) = \{y \in X : l(y) = 0\}$. Since l is continuous and linear, N is a closed linear subspace. The theorem is trivial in the case $N = X$; we take in this case $x_l = 0$. Suppose $N \neq X$. Then there exists a x_0 whhich belongs to $N^{\perp}$. Define

$$x_l = \left(l(\bar{x}_0) / \|x_0\|^2 \right) x_0$$

We will show that this x_l meets the condition of the Theorem. First, if $y \in N$, then $l(y) = \langle y, x_l \rangle$ since both sides vanish. Next, if y is of the form $y = \alpha x_0$, then we have

$$\langle y, x_l \rangle = \langle \alpha x_0, x_l \rangle = \left\langle \alpha x_0, \frac{l(\bar{x}_0)}{\|x_0\|^2} x_0 \right\rangle = \alpha l(x_0) = f(\alpha x_0) = l(y)$$

Since $f(x)$ and $\langle y, x_l \rangle$ are both linear in y, the equality $l(y) = \langle y, x_l \rangle, y \in X$, is proved if we have proved that X is spanned by N and x_0. To show the last assertion, we write, remembering that $l(x_l) \neq 0$,

$$y = \left(y - \frac{l(y)}{l(x_l)} x_l \right) + \frac{l(y)}{l(x_l)} x_l$$

Since the first term is an element of N, we have thus proved the representation $l(y) = \langle y, x_l \rangle$.
Therefore, we have

$$\|f\| = \sup_{\|y\| \leqslant 1} |l(y)| = \sup_{\|y\| \leqslant 1} |\langle y, x_l \rangle| \leqslant \sup_{\|y\| \leqslant 1} \|y\| \|x_l\| = \|x_l\|$$

and also $\|\| = \sup_{\|y\| \leqslant 1} |l(y)| \geqslant |l(x_l / \|x_l\|)| = \left\langle \frac{x_l}{\|x_l\|}, x_l \right\rangle = \|x_l\|$. Hence we have proved the equality $\|l\| = \|x_l\|$

□

11 Lax-Milgram Theorem

Theorem (Lax-Milgram Theorem)**.** If X is a Hilbert space and $B : X \times X \to \mathbb{K}$ is such that

(i) (sesqui-linearity): $\forall x \in X, \quad B$ is linear in the first component and skew-linear in the second component

(ii) (boundedness): $c_1 > 0, \forall x, y \in X, \quad |B(x,y)| \leqslant c_1 \|x\| \|y\|$

(iii) (positivity): $c_0 > 0, \forall x \in X, \quad |B(x,x)| \geqslant c_0 \|x\|^2$

then $\forall l \in X', \exists x_l \in X$ such that $\forall y \in X, l(y) = B(y,x)$

CLearly, this theorem is a stronger version and a generalization of Riesz Representation.

11.1 variational Problems

$D \sqsubseteq \mathbb{R}^N$ is open and connected, $f : D \to R \in \mathcal{C}(D)$ and $g : \partial D \to \mathbb{R} \in \mathcal{C}(\partial D)$. We now consider the Poisson equation: $u \in \mathcal{C}^2(D) \cap \mathcal{C}(\bar{D})$

$$\begin{cases} -\nabla^2 u = f \text{ in } D \\ u = g \text{ on } \partial D \end{cases}$$

we want to solve this type of equation using appropriate Hilbert space:

Theorem. If $a : X \times X \to \mathbb{R}$ is sesqui-linear, bounded, and positive, $b \in X'$, then $\exists! x \in X$ such that

$$\inf_{x \in X} \left\{ \frac{1}{2} a(x,x) - b(x) \right\} = \frac{1}{2} a(x^*, x^*) - b(x^*)$$

Proof. $F(x) \triangleq \frac{1}{2} a(x,x) - b(x)$ and $\alpha \triangleq \inf_{x \in X} F(x) \leqslant F(0) = 0 < \infty$

Claim. $\alpha > -\infty$

since $a(x,x) \leqslant c_1 \|x\|^2$ and $b(x) \leqslant c_2 \|x\|$, we see $F(x) \geqslant \frac{1}{2} c_1 \|x\|^2 - c_2 \|x\|$, we see from the right hand side is a quadratic function in $\|x\|$ that $\alpha > -\infty$. Now there eixsts $u_n \in X$ such that $F(u_n) \to \alpha$ as $n \to \infty$

Claim. u_n is Cauchy

$$
\begin{aligned}
F(u_n) + F(u_m) &= \frac{1}{2}a(u_n, u_n) - b(u_n) + \frac{1}{2}a(u_m, u_m) - b(u_m) \\
&= \frac{1}{2}\left(a(u_n, u_n) + a(u_m, u_m)\right) - 2b(\frac{u_n + u_m}{2}) \\
&= 2\left(\frac{1}{2}a(\frac{u_n + u_m}{2}, \frac{u_n + u_m}{2}) - b(\frac{u_n + u_m}{2})\right) - 2a(\frac{u_n - u_m}{2}, \frac{u_n - u_m}{2}) \\
&= 2F(\frac{u_n + u_m}{2}) + 2a(\frac{u_n - u_m}{2}, \frac{u_n - u_m}{2}) \\
&\geqslant 2\alpha + 2a(\frac{u_n - u_m}{2}, \frac{u_n - u_m}{2})
\end{aligned}
$$

$\implies \|u_n - u_m\|^2 \leqslant \frac{4}{c_1}\|u_n - u_m\|^2$ Fix $\epsilon > 0, \exists N_\epsilon$, such that $\forall n, m \geqslant N_\epsilon$, $\quad \frac{4}{c_1}\left(F(u_n) + F(u_m) - 2\alpha\right) \leqslant \epsilon^2$. So $\{u_n\}$ is Cauchy and we call $u = \lim_{n\to\infty} u_n$

Claim. $F(u) = \alpha$

This is easy since F is continuous and $F(u) = F\left(\lim_{n\to\infty} u_n\right) = \lim_{n\to\infty} F(u_n) = \alpha$, we thus have proved the existence.
Uniqueness: property of a minimizer: suppose $\hat{u}$ is a minimizer, then $\forall x \in X, a(\hat{u, x}) = b(x)$. Indeed $\varphi(t) = F(\hat{u} + tx)$, $\varphi(0)$ minimizes at $t = 0$ and φ is differentiable, then $\varphi'(0) = 0$ which is equivalent to the minimizer property of $\hat{u}$. Assume u_0 and u_1 are two minimizers, then $\forall x \in X, a(u_0, x) = a(u_1, x) = b(x)$, then $a(u_0 - u_1, x) = 0 \forall x \in X$ and $a(u_0 - u_1, u_0 - u_1) = 0$. Positivity of a shows the uniqueness. $\square$

Theorem (equivalence of the two problems)**.**

$$
\begin{aligned}
\inf_{x\in X} F(x) = F(x^*) \\
\Leftarrow \\
\forall y \in X, a(x^*, y) = b(y)
\end{aligned}
$$

The sufficiency has already been proved. For the necessity, take $x \in X$, then

$$
\begin{aligned}
F(x) &= F(x^* + x - x^*) \\
&= \frac{1}{2}a(x^* + x - x^*, x^* + x - x^*) - b(x^* + x - x^*) \\
&= \frac{1}{2}\left(a(x^*, x^*) + 2a(x^*, x - x^*) + a(x - x^*, x - x^*)\right) - b(x^*) - b(x - x^*) \\
&= F(x^*) + \left(a(x^*, x - x^*) - b(x - x^*)\right) + \frac{1}{2}a(x - x^*, x - x^*)
\end{aligned}
$$

So $\forall x \in X, F(x) \geqslant F(x^*)$

12 Sobolev Space

D is an open connected subset of $\mathbb{R}^N$, $\mathcal{C}_0^\infty(D) = \{\varphi \in \mathcal{C}^\infty(D) : \operatorname{supp}(\varphi) \sqsubseteq D\}$. $\forall 1 \leqslant j \leqslant N, \forall \varphi, \psi \in \mathcal{C}_0^\infty(D), \mathbb{Q} \int_D \frac{\partial \varphi}{\partial x_j} \psi \mathrm{d}x = -\int_D \varphi \frac{\partial \psi}{\partial x_j} x$ Now consider $u, w \in L^2(D), \left\{h : D \to \mathbb{R} : \int_D |h(x)|^2 \mathrm{d}x < \infty\right\}$ Assume $\forall \varphi \in \mathcal{C}_0^\infty(D), \int_D \frac{\partial \varphi}{\partial x_j} u \mathrm{d}x = -\int_D \varphi w \mathrm{d}x$, we say that u ahs a generalized derivative w.r.t x_j and w is denoted by $\frac{\partial u}{\partial x_j}$.

Observation: Assume w_1 and $w_2 \in L^2(D)$ such that $\forall \varphi \in \mathcal{C}_0^\infty(D), \quad \int_D \frac{\partial \varphi}{\partial x_j} u \mathrm{d}x = -\int_D \varphi w_i \mathrm{d}x, i = 1, 2 \Rightarrow \forall \varphi \in \mathcal{C}_0^\infty(D), \int_D \varphi(w_1 - w_2) \mathrm{d}x = 0$. Since $\mathcal{C}_0^\infty(D)$ is dense in $L^2(D)$, we see that $\|w_1 - w_2\|^2 = \int_D |w_1(x) - w_2(X)|^2 \mathrm{d}x = 0$, then $w_1 = w_2$a.e. in D.

Example. $D = (-1, 1), u(x) = |x|$. The generalized derivative is

$$u'(x) = \begin{cases} -1 \text{ if } x < 0 \\ \alpha \text{ if } x = 0 \\ 1 \text{ if } x > 0 \end{cases}$$

Definition. $H^1 \triangleq \left\{u \in L^2(D) : \forall 1 \leqslant j \leqslant N, \exists w_j \in L^2(D), \forall \varphi \in \mathcal{C}_0^\infty(D), \quad \int_D \frac{\partial \varphi}{\partial x_j} u \mathrm{d}x = -\int_D \varphi w_j \mathrm{d}x\right\}$.

Define an inner product $\langle u, v \rangle_{H^1(D)} \triangleq \langle u, v \rangle_{L^2(D)} + \sum_{j=1}^N \langle u_j, v_j \rangle_{L^2(D)}$

Claim. $\langle u, v \rangle_{H^1(D)}$ is an inner product.

Theorem. $H^1(D)$ is a Hilbert space

Definition. $H_0^1(D) \triangleq \overline{\mathcal{C}_0^\infty(D)}^{H^1(D)}$

This is indeed a closed linear subspace of $H^1(D)$, so $H_0^1(D)$ is a Hilbert space.

Proposition. $H_0^1(D) = \left\{u \in H^1(D) : u = 0 \text{ on } \partial D\right\}$

12.1 Dirichlet Problem

(DP1)

$$\begin{cases} -\nabla^2 u = f \text{ in } D \\ u = g \text{ in } \partial D \end{cases}$$

Let $w \in \mathcal{C}_0^\infty(D), \int_D -\nabla^2 u \mathrm{d}x = \int_D f w \mathrm{d}x \Leftrightarrow \int_D \nabla u \cdot \nabla w \mathrm{d}x = \int_D f w \mathrm{d}x$. Assume u_g is smooth such that $u_g|_{\partial D} = g$. Let $\hat{u} = u - u_g$, so

$$\begin{cases} -\nabla^2 \hat{u} = \hat{f} \text{ in } D \\ \hat{u} = 0 \text{ in } \partial D \end{cases}$$

where $\hat{f} = f - \nabla^2 u_g$
$(DP2)$ Find $\hat{u} \in H_0^1(D)$ such that $\forall w \in H_0^1(D), a(\hat{u}, w) = b(w)$ where $a(\varphi, \psi) \triangleq \int_D \nabla^2 \varphi \cdot \nabla^2 \psi \mathrm{d}x$ and $b(\psi) \triangleq \int_D \hat{f} \psi \mathrm{d}x$

13 Weak formulation of PDEs

let $D \subseteq \mathbb{R}^N$ be an open domain, find $\bar{u} : \bar{D} \to \mathbb{R}$ which is the solution of

$$\begin{cases} -\sum\limits_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x x_j}(x) \right) + \lambda(x)u(x) = f(x) \text{ for } x \in D \\ u(x) = 0 \text{ for } x \in \partial D \end{cases}$$

a_{ij}, λ, f are given, $f \in L^2(D)$.

(i) we assume they are continuous on $\bar{D}$

(ii) $\exists c_0 > 0, \forall x \in D, \forall \xi, \quad \sum\limits_{ij=1}^{N} a_{ij}(x)\xi_i\xi_j \geqslant c_0 \|x\|^2$

(iii) $\exists \lambda_0 > 0, \forall x \in D, \quad \lambda(x) \geqslant \lambda_0$

We want to define a solution to the Dirichlete problem in the weak sense: Weak formulation of DP,

$$a_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}, \lambda(x) \equiv 1$$

If DP has a solution such that $u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N}, \nabla^2(u) \in L^2(D)$, we can multiply the equation

$$-\nabla^2(u) + u = f \text{ in } D$$

by arbitrary $v \in \mathcal{C}_0^\infty(D)$

$$\int_D \left(-\nabla^2 u + u\right) v \mathrm{d}x = \int_D f v \mathrm{d}x$$

Using Green's identity: $\forall v \in \mathcal{C}_0^\infty(D)$,

$$\int_D (-\nabla^2 u) v \mathrm{d}x = \int_D \nabla u \cdot \nabla v \mathrm{d}x$$

$$\int_D \nabla u \cdot \nabla v \mathrm{d}x + \int_D u v \mathrm{d}x = \int_D f v \mathrm{d}x \tag{6}$$

We want to work in $H_0^1(D)$ where $H_0^1(D) = \overline{\mathcal{C}_0^\infty(D)}^{H^1(D)}$:

$$H^1(D) = \left\{ \varphi \in L^2(D) : \text{ generalized partial derivatives } \frac{\partial \varphi}{\partial x_1}, \ldots, \frac{\partial \varphi}{\partial x_N} \in L^2(D) \right\}$$

$$\langle \varphi, \psi \rangle = \int_D \varphi\psi + \int_D \nabla\varphi \cdot \nabla\psi$$

Since $H_0^1(D) \subseteq H^1(D)$ is a subspace, then $H_0^1(D)$ is equipped with the same inner product as $H^1(D)$:

$$\|\varphi\|_{H^1(D)} = \left(\int_D \varphi(x)^2 \mathrm{d}x + \int_D \|\nabla\varphi(x)\|^2 \mathrm{d}x\right)^{1/2}$$

$H^1(D), H_0^1(D)$ are Hilbert spaces.
$l : H_0^1(D) \to \mathbb{R}$ such that $v \mapsto \int_D fv$

Claim. l is a bounded linear form on $H_0^1(D)$

Proof. Let $v \in H_0^1(D)$, $|l(v)| \leqslant \left|\int_D fv\right| \leqslant \left(\int_D f^2\right)^{1/2} \left(\int_D v^2\right)^{1/2} = \|f\|_{L^2(D)} \|v\|_{L^2(D)}$. Since $\|v\|_{L^2(D)} \leqslant \|v\|_{H_0^1(D)}$, we see $|l(v)| \leqslant \|f\|_{L^2(D)} \|v\|_{H_0^1(D)}$, then $\|l\|_{H_0^1(D)'} \leqslant \|f\|_{L^2(D)}$. □

From Riesz Representation theorem: $\exists! u_l \in H_0^1(D), \forall v \in H_0^1(D), \quad \langle u_l, v\rangle_{H^1(D)} = l(v)$

Claim. u_l is a weak solution of the Dirichlet Problem with $a_{ij}(x) = \begin{cases} 1 \text{ where } i = j \\ 0 \text{ where } i \neq j \end{cases}$ and $\lambda(x) \equiv 1$

Explicitly, $\forall v \in H_0^1(D)$,

$$\int_D \nabla u_l \cdot \nabla v + u_l v \mathrm{d}x = \int_D fv \mathrm{d}x$$

Further regularity: $\exists h \in L^2(D)$ such that $\forall v \in \mathcal{C}_0^\infty(D)$

$$\int_D \nabla u_l \cdot \nabla v = -\int_D hv$$

Then h is denoted by $\nabla^2 u_l$

$$\int_D \left(-\nabla^2 u_l + u_l - f\right) v \mathrm{d}x = 0$$

$\to$

$$-\nabla^2 u_l + u_l = f \text{ in } L^2(D)$$
$$-\nabla^2 u_l(x) + u_l(x) = f(x) \text{a.e. in } D$$

For the general case: assume all the terms below are $L^2(D)$:

$$-\sum_{ij=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}(x)\right) + \lambda(x)u(x) = f(x)$$

Multiplying on both sides by $v \in \mathcal{C}_0^\infty(D)$ and integrate:

$$\int_D A(x)\nabla u(x) \cdot \nabla v(x)\mathrm{d}x + \int_D \lambda(x)u(x)v(x)\mathrm{d}x = \int_D f(x)v(x)\mathrm{d}x \tag{7}$$

where $A(x) = \{a_{ij}(x)\}_{ij}$, we must have $\forall v \in H_0^1(D)$:

$$\int_D A(x)\nabla u(x) \cdot \nabla v(x)\mathrm{d}x + \int_D \lambda(x)u(x)v(x)\mathrm{d}x = \int_D f(x)v(x)\mathrm{d}x \tag{8}$$

Now define $a : H_0^1(D) \times H_0^1(D) \to \mathbb{R}$ such that $(\varphi, \psi) \mapsto \int_D A(x)\nabla u(x) \cdot \nabla v(x)\mathrm{d}x + \int_D \lambda(x)u(x)v(x)\mathrm{d}x$

(i) (symmetry): suppose for every x, $A(x)$ is a positive definite matrix: $\forall \xi \in \mathbb{R}^N, \quad \xi^T A(x)\xi \geqslant c_0 \|\xi\|^2$, where c_0 does not depend on x

(ii) (linearity): trivial

(iii) $a(\varphi,\varphi) \geqslant 0, \quad a(\varphi,\varphi) = 0, \Rightarrow \varphi = 0$ in $H_0^1(D)$

So a becomes an inner product and we define $\|\varphi\|_a = \sqrt{a(\varphi,\varphi)}$. Now consider $l : H_0^1(D) \to \mathbb{R}$ such that $v \mapsto \int_D fv$, for any function $v \in H_0^1(D)$, $|l(v)| \leqslant \|f\|_{L^2(D)} \|v\|_{L^2(D)}$ where $\|v\|_{H_0^1(D)} \leqslant \|v\|_{H_0^1(D)} \leqslant c\|v\|_a$ In this context, we can apply Riesz Representation Theorem.

Now remove the assumption that a is symmetric, we need to use Lax-Milgram Theorem. Then we need to assume:

(i) $\exists \bar{\lambda} > 0$ such that $\lambda(x) \leqslant \bar{\lambda}$

(ii) $\exists \bar{a_{ij}} > 0$, such that $a_{ij}(x) \leqslant \bar{a_{ij}}$

under these two assumptions, $\exists c > 0, \quad |a(\varphi,\psi)| \leqslant c\|\varphi\|\|\psi\|$

By Lax-Milgram theorem we get

$$\forall v \in H_0^1(D), \quad a(u_l, v) = l(v) \leftrightarrow$$
$$\int_D A(x)\nabla u_l(x) \cdot \nabla v(x) + \lambda(x)u_l(x)v(x)\mathrm{d}x = \int_D f(x)v(x)\mathrm{d}x$$

The boundary condition for u_l is satisfied by definition. Assume that $\nabla(A(x)\nabla u_l(x))$ exists in a genelized sense, $\int_D A(x)\nabla u_l(x) \cdot \nabla v(x) = -\int_D \nabla(A(x)\nabla u_l(x))v(x)\mathrm{d}x$ Therefore, $\forall v \in H_0^1(D)$

$$\int_D \left(-\nabla(A(x)\nabla u_l(x)) + \lambda(x)u_l(x) - f(x)\right)v(x)\mathrm{d}x = 0$$

$\Rightarrow$

$$-\nabla\left(A(x)\nabla u_l(x)\right) + \lambda(x)u_l(x) = f(x) \text{ in } L^2(D)$$

Remark. It is indeed possible to show that we have more regularity than $H^1(D)$ on u the solution given by Riesz or Lax-Milgram theorems.

14 Uniform boundedness Principle

Definition (Baire Space)**.** X is a Baire space if $\forall \{E_n\}_{n\geqslant 1}$ family of open and dense subsets of X, $\bigcap\limits_{n\geqslant 1} E_n$ is dense in X

Theorem. if X is a complete metric space, then X is a Baire space.

Proof. Omitted. $\square$

Theorem (principle of uniform Boundedness)**.** X is a complete metric space, $f_\alpha : X \to \mathbb{R}$ is continuous for each $\alpha \in \mathrm{I}$. Assume $\forall x \in X, \exists 0 < M(x) < \infty$ such that $\sup\limits_{\alpha\in\mathrm{I}} |f_\alpha(x)| \leqslant M(x)$, then $\exists G$ nonempty open subset of X, $\exists c > 0$ such that $\sup\limits_{xinG} \sup\limits_{\alpha\in\mathrm{I}} |f_\alpha(x)| \leqslant c$

Proof. $n \in \mathbb{N}, F_n = \left\{ x \in X : \sup\limits_{\alpha\in\mathrm{I}} |f_\alpha(x)| \leqslant n \right\}$.

Claim. F_n is closed and $\bigcup\limits_{n\in\mathbb{N}}^{F_n} = X$

Let $\{x_p\}_{p\geqslant 1} \subseteq F_n$, so $\forall p \geqslant 1, \sup\limits_{\alpha\in\mathrm{I}} |f_\alpha(x_p)| \leqslant n$. $\forall p \geqslant 1, \forall \alpha \in \mathrm{I}, \quad |f_\alpha(x_p)| \leqslant n$. $\forall \alpha \in \mathrm{I}, \forall p \geqslant 1, |f_\alpha(x_p)| \leqslant n$. $\Rightarrow \forall \alpha \in \mathrm{I}, \quad f_\alpha(x) \leqslant n$. $\Rightarrow \sup\limits_{\alpha\in\mathrm{I}} |f_\alpha(x)| \leqslant n, \Rightarrow x \in F_n$. Let $x \in X$, we say that $\exists n_x \in \mathbb{N}, x \in F_{n_x}$, clearly, $x \in F_{n_{\lceil M(x) \rceil}}$.

Claim. $\exists n \in \mathbb{N}, \exists G$ nonempty open subset of X, $G \subseteq F_n$

we prove by contradiction: Assume $\forall n \in \mathbb{N}, \forall G$ nonempty open subset of X, $G \cap F_n^c \neq \varnothing$. We know $E_n \triangleq F_n^c$ is open and dense, so $\bigcap\limits_{n\geqslant 1} E_n$ is dense since X is aa Baire space. Then $\bigcap\limits_{n\geqslant 1} F_n^c = \left(\bigcup\limits_{n\geqslant 1} F_n \right)^c = X^c = \varnothing$, which is a contradiction. SO $\exists G$ nonempty open subset of X, $\exists c \in \mathbb{N}$ such that $\sup\limits_{x\in G} \sup\limits_{\alpha\in\mathrm{I}} |f_\alpha(x) \leqslant c|$ $\square$

15 weak convergence

Definition (weak convergence)**.** X is n.v.s., $\{x_n\}_{n\geqslant 1}$ is a sequence in X and $x \in X$. We say that x_n converges weakly towards x if

$$\forall l \in X', \quad l(x_n) \to l(x) \text{ in } \mathbb{K}$$

, we denote weak convergence by $x_n \xrightarrow{\text{w}} x$

Remark. (i) Suppose $x_n \to x$, then $x_n \xrightarrow{\text{w}} x$, that is, strong convergence implies weak convergence.

(ii) Weak convergence does not immplies strong convergence: $\ell^2 = \left\{ \{a_n\}_{n\geqslant 1} : \sum\limits_{n\geqslant 1} |a_n|^2 < \infty \right\}$

Consider $x^p = \{x_n^p\}_{n\geqslant 1}$, where $x_n^p = \delta_{n,p}$. Let $m \in (\ell^2)'$, we know that, $\exists b \in \ell^2$ such that $\forall a \in \ell^2, \quad m(a) = \sum\limits_{n\geqslant 1} a_n b_n$. In particular, $m(x^p) = b_p$ and $|m(x^p)| = |b_p| \to 0$ as $p \to \infty$ since $\sum\limits_{n\geqslant 1} |b_p|^2 < \infty$, so $x^p \xrightarrow{\text{w}} 0$, while $\|x^p\| = \left(\sum\limits_{n\geqslant 1} (x_n^p)^2 \right)^{1/2} = 1$, which implies x^p does not strongly converge to 0

Lemma. Let X be n.v.s. and $\{x_n\}_{n\geqslant 1}$ be a sequence, $x \in X$. If

(i) $\exists c > 0, \forall n \in \mathbb{N}, \quad \|x_n\| \leqslant c$

(ii) $\forall l \in A \subseteq X'$ where A is dense in X', $\quad l(x_n) \to l(x)$ as $n \to \infty$

then $x_n \xrightarrow{\text{w}} x$ as $n \to \infty$

Proof. We show that $\forall m \in X', \quad m(x_n) \to m(x)$ as $n \to \infty$.
Now fix $\epsilon > \epsilon, \exists l_\epsilon \in A$ such that $|m - l_\epsilon|_{X'} < \epsilon$.
Fix $\epsilon' > 0$

$$\begin{aligned} |m(x_n) - m(x)| = |m(x_n - x)| &= |m(x_n - x) - l_\epsilon(x_n - x) + l_\epsilon(x_n - x)| \\ &\leqslant |m(x_n - x) - l_\epsilon(x_n - x)| + |l_\epsilon(x_n - x)| \\ &\leqslant \|m - l_\epsilon\| \, \|x_n - x\| + \frac{\epsilon'}{2} \leqslant \epsilon(c + \|x\|) + \frac{\epsilon'}{2} \end{aligned}$$

□

Theorem. Let X be a Banach space, $f_\alpha : X \to \mathbb{R}, \alpha \in \mathrm{I}$ is such that

(i) continuous

(ii) $\forall x, y \in X, f_\alpha(x+y) \leqslant f_\alpha(x) + f_\alpha(y)$

(iii) $\forall \lambda \in \mathbb{R}, |f_\alpha(\lambda x)| = |\lambda| \, |f_\alpha(x)|$

and $\forall x \in X, \exists 0 > M(x) < \infty, \quad \sup\limits_{\alpha \in \mathrm{I}} |f_\alpha(x)| \leqslant M(x)$, then $\exists c > 0, \forall x \in X, \sup\limits_{\alpha \in \mathrm{I}} |f_\alpha(x)| \leqslant c \, \|x\|$

Proof. Apply principle of uniform boundedness, $\exists c > 0, \exists G$ nonempty open subset of X such that $\sup\limits_{x \in G} \sup\limits_{\alpha \in \mathrm{I}} |f_\alpha(x)| \leqslant c$. Since G is nonempty open set, $\exists z \in G, \exists r > 0$ such that $B(z,r) \subseteq G, \quad \sup\limits_{x \in B(z,r)} \sup\limits_{\alpha \in \mathrm{I}} |f_\alpha(x)| \leqslant c$. Now take $y \in B(0,r), \|y\| \leqslant r$

$$\begin{aligned} |f_\alpha(y)| = |f_\alpha(y+z-z)| &\leqslant |f_\alpha(y+z) + f_\alpha(-z)| \\ &\leqslant |f_\alpha(y+z)| + |f_\alpha(z)| \end{aligned}$$

Since both $z, y+z \in B(z,r), \quad |f_\alpha(y+z)| \leqslant c, |f_\alpha(z) \leqslant c|$, so $\sup\limits_{\alpha \in \mathrm{I}} |f_\alpha(y)| \leqslant 2c$. Let $x \in X$ arbitrary, $\frac{x}{\|x\|}\frac{r}{2} \in B(0,r)$, so $\sup\limits_{\alpha \in \mathrm{I}} \left| f_\alpha(\frac{rx}{2\|x\|}) \right| \leqslant 2c$. This means that $\forall \alpha \in \mathrm{I}, \quad \left| f_\alpha(\frac{rx}{2\|x\|}) \right| \leqslant 2c$. then $\forall \alpha \in \mathrm{I}, \quad \left| \frac{r}{2\|x\|} f_\alpha(x) \right| \leqslant 2c$, then $\forall \alpha \in \mathrm{I}, |f_\alpha(x)| \leqslant \frac{4c}{r} \|x\|$. So $\forall x \in X, \sup\limits_{\alpha \in \mathrm{I}} |f_\alpha(x)| \leqslant \frac{4c}{r} \|x\|$ □

Remark. Let X be a Banach space, $\{l_\alpha\}_{\alpha \in \mathrm{I}} \subseteq X', \forall x \in X, \exists M(x), \quad \sup\limits_{\alpha \in \mathrm{I}} |l_\alpha(x)| \leqslant M(x)$, then $\exists 0 < c < \infty, \quad \sup\limits_{\alpha \in \mathrm{I}} \|l_\alpha\| \leqslant c$.

Proof. Define $f_\alpha(x) \triangleq |l_\alpha(x)|$, then f_α satisfies the condition of the above theorem. □

Remark. Let X be a n.v.s., $\{x_\alpha\}_{\alpha \in \mathrm{I}} \subseteq X'$ such that $\forall l \in X', \exists c_l > 0, \sup\limits_{\alpha \in \mathrm{I}} |l(x_\alpha)| \leqslant c_l$. then $\exists c > 0, \sup\limits_{\alpha \in \mathrm{I}} \|x_\alpha\| \leqslant c$

Proof. See $L_\alpha = x_\alpha \in X''$, $\|L_\alpha\| = \|x_\alpha\|$. $\forall l \in X', \exists c_l > 0, \sup\limits_{\alpha \in \mathrm{I}} |L_\alpha(l)| \leqslant c_l$, $\Rightarrow \exists c > 0, \sup\limits_{\alpha \in \mathrm{I}} \|L_\alpha\| \leqslant c \leftrightarrow \sup\limits_{\alpha \in \mathrm{I}} \|x_\alpha\| \leqslant c$ □

Remark. Let X be n.v.s., then any weakly converging sequence is bounded.

Proof. $x_n \xrightarrow{\mathrm{w}} x$, then $\forall l \in X', l(x_n) \to l(x)$. Then $\exists c > 0, \sup\limits_{n \in \mathbb{N}} |l(x_n)| \leqslant c$. So by remark 2, $\exists 0 < K < \infty, \quad \sup\limits_{n \in \mathbb{N}} \|x_n\| \leqslant K$. □

Remark. if $x_n \xrightarrow{\mathrm{w}} x$, then $\|x\| \leqslant \liminf \|x_n\|$

Proof. $x \in X, \exists l \in X', \|x\| = l(x), \|l\| = 1$. Clearly, $l(x_n) \to l(x)$. So $\|x\| = |l(x)| = \lim_{n\to\infty} |l(x_n)| \leqslant \liminf \|x_n\|$ $\square$