Page Rank Algorithm using Power Iteration

In [ ]:
```
import numpy as np
import matplotlib.pyplot as plt
```

In [ ]:
```
def rayleigh_quo(A,x):
    x = x/np.linalg.norm(x, ord=2)
    return np.dot(x, np.dot(A,x))
```

Markov Transition Matrix

In [ ]:
```
M = np.array([
    [0  ,0  ,1  ,0  ,0  ,0  ,0  ,0  ,0  ,0  ],
    [0.5,0  ,0  ,0  ,0  ,0  ,0  ,0  ,0.5,0  ],
    [0  ,0.5,0  ,0  ,0  ,0  ,0.5,0  ,0  ,0  ],
    [0  ,0  ,0  ,0  ,0  ,0  ,0  ,0  ,0  ,0  ],
    [0  ,0  ,0  ,1  ,0  ,0  ,0  ,0  ,0  ,0  ],
    [0  ,0  ,0  ,0  ,0.5,0  ,0  ,0.5,0  ,0  ],
    [0  ,0  ,0  ,0  ,0  ,0.5,0  ,0  ,0  ,0  ],
    [0  ,0.5,0  ,0  ,0.5,0  ,0  ,0  ,0  ,1  ],
    [0.5,0  ,0  ,0  ,0  ,0  ,0.5,0.5,0  ,0  ],
    [0  ,0  ,0  ,0  ,0  ,0.5,0  ,0  ,0.5,0  ]
])
```

Power Iteration

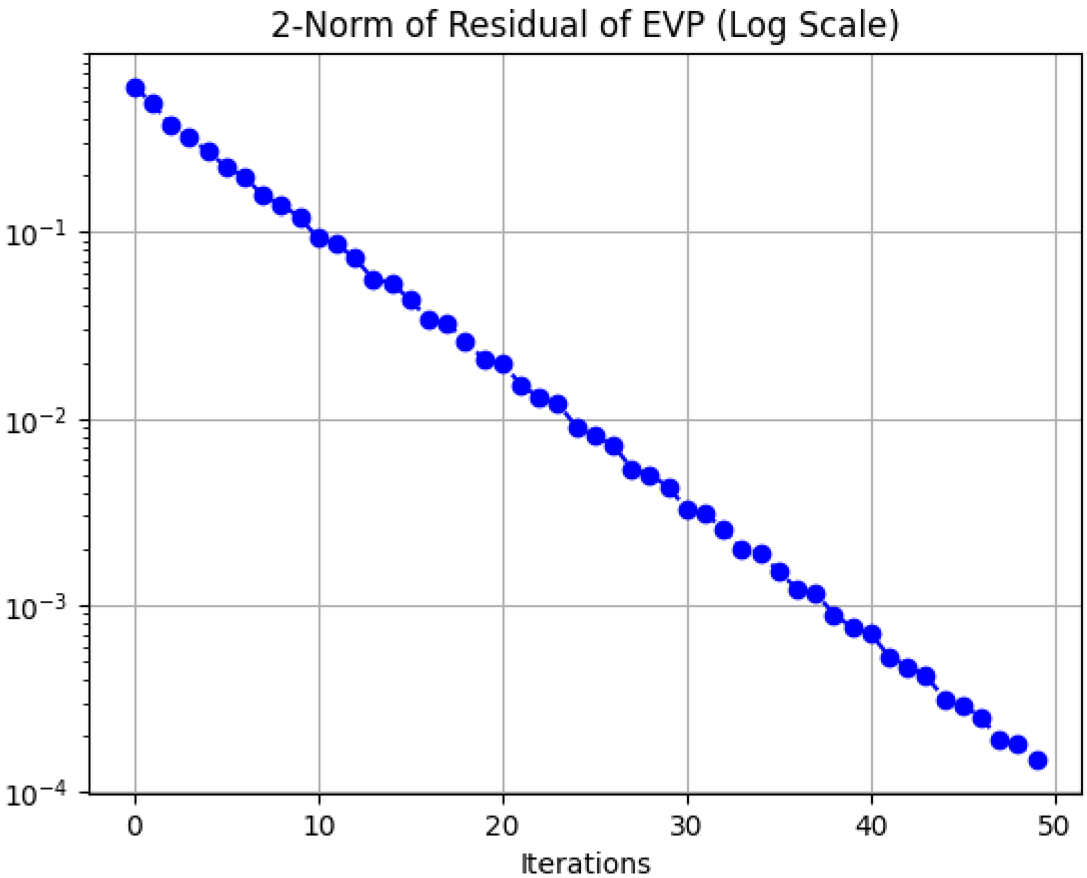
In [ ]:
```
max_iter = 50
m, n = M.shape
v = np.random.rand(n)
v = v/np.linalg.norm(v)

evp_residual = []
resPOW_list = []
rayleigh_quo_list = []

for i in range(max_iter):
    v_1 = np.dot(M,v)
    v_1 = v_1/np.linalg.norm(v_1)

    ray_quo_value = rayleigh_quo(M,v_1)
    evp_res_norm = np.dot(M,v_1) - ray_quo_value*v_1
    evp_residual.append(np.linalg.norm(evp_res_norm))

    resPOW_list.append(np.linalg.norm(v_1-v))
    rayleigh_quo_list.append(ray_quo_value)

    v = v_1
```

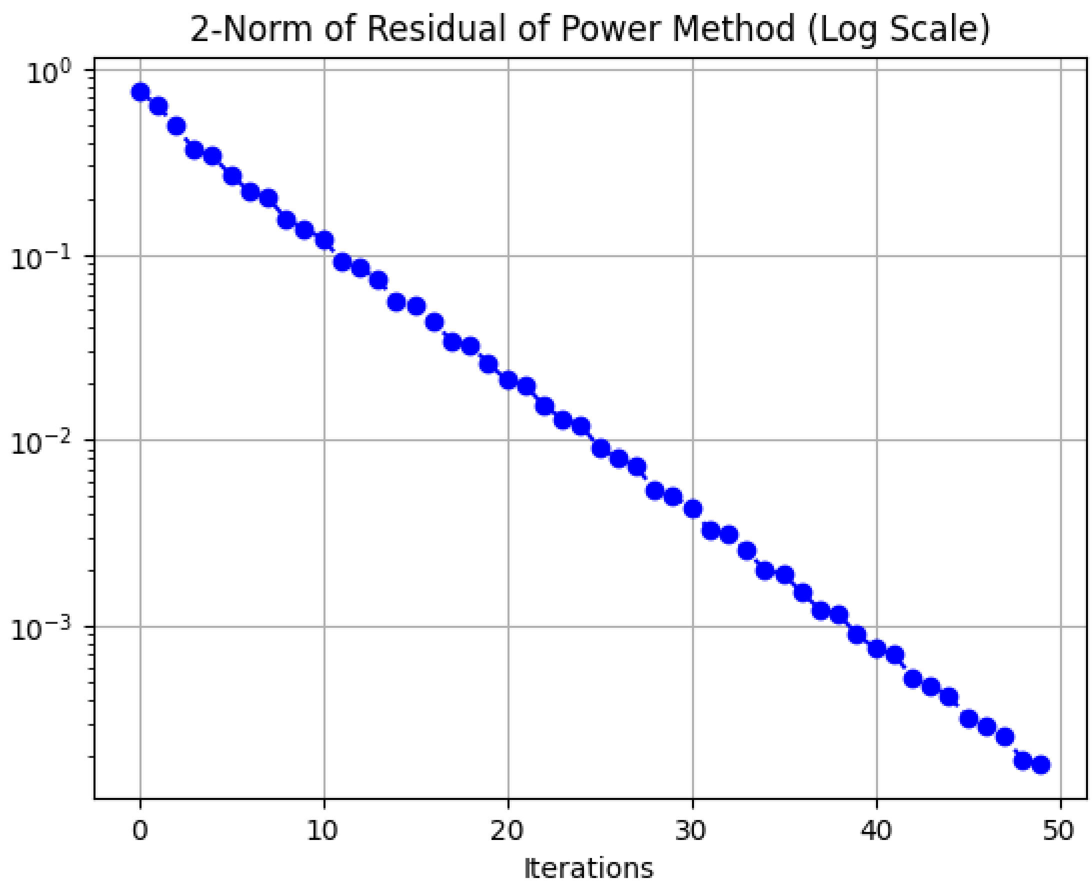
Plots

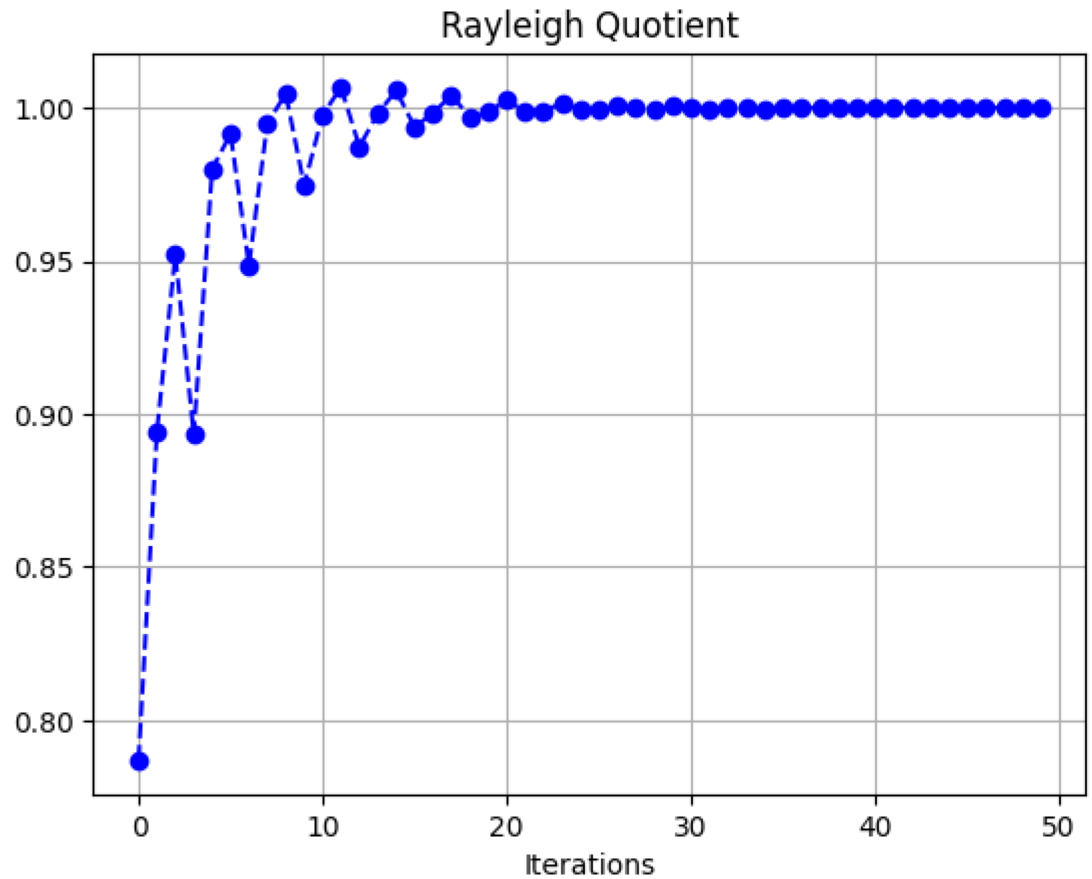
In [ ]:

```
resEVP_np = np.array(evp_residual)
resEV_np = np.array(resPOW_list)
rayleigh_np = np.array(rayleigh_quo_list)

iterations = np.arange(max_iter)

print(f"Highest page rank is for page {np.argmax(v)+1} with value {rayleigh_np[n
print(f"Lowest page rank is for page {np.argmin(v)+1} with value {rayleigh_np[np

plt.plot(iterations, resEVP_np, 'o--', color='blue')
plt.yscale('log')
plt.xlabel('Iterations')
plt.title('2-Norm of Residual of EVP (Log Scale)')
plt.grid()
plt.show()

plt.plot(iterations, resEV_np, 'o--', color='blue')
plt.yscale('log')
plt.xlabel('Iterations')
plt.title('2-Norm of Residual of Power Method (Log Scale)')
plt.grid()
plt.show()

plt.plot(iterations, rayleigh_np, 'o--', color='blue')
plt.xlabel('Iterations')
plt.title('Rayleigh Quotient')
plt.grid()
plt.show()
```

```
Highest page rank is for page 8 with value 0.9949383826470746
Lowest page rank is for page 4 with value 0.8933567553539793
```




2-Norm of Residual of Power Method (Log Scale)
10^0
10^{-1}
10^{-2}
10^{-3}
0
10
20
30
40
50
Iterations


Rayleigh Quotient
1.00
0.95
0.90
0.85
0.80
0
10
20
30
40
50
Iterations

Question 4:

④ (a) $\underset{\sim}{A}\,\underset{\sim}{u_i} = \lambda_i \underset{\sim}{B}\,\underset{\sim}{u_i}$ Eqn (1)

$\underset{\sim}{B}$ is normal, so: $\underset{\sim}{B} = \underset{\sim}{Q}\,\underset{\sim}{D}\,\underset{\sim}{Q}^T$

$\underset{\sim}{D} = \underset{\sim}{D}^{1/2}\,\underset{\sim}{D}^{1/2}$; $\underset{\sim}{Q}\,\underset{\sim}{Q}^T = \underset{\sim}{Q}^T\underset{\sim}{Q} = \underset{\sim}{I}$

$\underset{\sim}{D}^{1/2}$ is also diag. & invertible

$$\underset{\sim}{A}\,\underset{\sim}{u_i} = \lambda_i \underset{\sim}{Q}\,\underset{\sim}{D}\,\underset{\sim}{Q}^T u_i$$

$$\underset{\sim}{A}\,\underset{\sim}{Q}\,\underset{\sim}{Q}^T \underset{\sim}{u_i} = \underset{\sim}{A}\,\underset{\sim}{Q}\,\underset{\sim}{D}\,\underset{\sim}{Q}^T \underset{\sim}{u_i}$$

$$\left[\underset{\sim}{Q}^T \underset{\sim}{A}\,\underset{\sim}{Q}\,\underset{\sim}{D}^{-1/2}\right] \underset{\sim}{D}^{1/2} \underset{\sim}{Q}^T \underset{\sim}{u_i} = \lambda_i \underset{\sim}{D}\,\underset{\sim}{Q}^T \underset{\sim}{u_i}$$

$$\left[\underset{\sim}{D}^{-1/2} \underset{\sim}{Q}^T \underset{\sim}{A}\,\underset{\sim}{Q}\,\underset{\sim}{D}^{-1/2}\right] \left[\underset{\sim}{D}^{1/2} \underset{\sim}{Q}^T \underset{\sim}{u_i}\right] = \lambda_i \left[\underset{\sim}{D}^{1/2} \underset{\sim}{Q}^T \underset{\sim}{u_i}\right]$$

is of the form:

$\underset{\sim}{H}\,\underset{\sim}{v_i} = \lambda_i \underset{\sim}{v_i}$; $\underset{\sim}{v_i} = \underset{\sim}{D}^{1/2} \underset{\sim}{Q}^T \underset{\sim}{u_i}$

Eqn (2)

$$\underset{\sim}{H} = \underset{\sim}{D}^{-1/2} \underset{\sim}{Q}^T \underset{\sim}{A} \underset{\sim}{Q} \underset{\sim}{D}^{-1/2} = \underset{\sim}{H}^T$$

→ $\underset{\sim}{H}$ is symmetric.

4 (b)

→ Finding λ_i closest to μ ($\mu = 2 \neq \lambda_i$) and the corresponding eigenvector is equivalent to finding $\underset{\sim}{v}_i$ corresponding to same λ_i from eqn (2). $\underset{\sim}{u}_i$ can be obtained from $\underset{\sim}{v}_i$.

→ We can use shift inverse iteration as we have a guess μ for λ_i.

→ $\underset{\sim}{B}$ has to be diagonalized once.

Algo: Start with random $\underset{\sim}{v}^{(0)}$ unit vector.

for $k = 1, 2, \ldots$

$$\text{Solve } (\underset{\sim}{H} - \mu \underset{\sim}{I}) \underset{\sim}{w} = \underset{\sim}{v}^{(k-1)}$$

$$\underset{\sim}{v}^{(k)} = \underset{\sim}{w} / \|\underset{\sim}{w}\|_2$$

Find approx. λ using RQ if needed.

Convergence of shift inverse:

If $(\lambda_J, \underset{\sim}{q}_J)$ are the target pair with λ_K as second closest eigen value to μ:

(i) $\underset{\sim}{q}_J^T \underset{\sim}{v}^{(0)} \neq 0$

(ii) $|\mu - \lambda_J| < |\mu - \lambda_i| \ \forall (i \neq J)$

(iii) $\mu \neq \lambda_J$ (given)

(iv) $\lambda_i \neq \lambda_J \quad \forall (i \neq J)$

④ (d)

If $\lambda_i = \lambda_{min}$ is the target, then inverse iteration without any shift can be used as it is the power iteration on $\underset{\sim}{A}^{-1}$.

For shift inverse:

$$\left\| \underset{\sim}{v}^{(k)} - (\pm \underset{\sim}{q}_{min}) \right\|_2 = O\left[\left| \frac{\mu - \lambda_{min}}{\mu - \lambda_g} \right|^k \right]$$

g is second closest to μ.

For inverse:

$$\left\| \underset{\sim}{v}^{(k)} - (\pm \underset{\sim}{q}_{min}) \right\|_2 = O\left[\left| \frac{\lambda_{min}}{\lambda_g} \right|^k \right]$$

The convergence is clearly faster than shift inverse.

(4) (c)

→ Computationally intensive step is solving the system $(\underset{\sim}{H} - \mu \underset{\sim}{I})\underset{\sim}{w} = \underset{\sim}{v}^{(k-1)}$.

→ LU based methods offer $O(m^3)$ flops as cost of computation.

Question 6 :

(6) (a)

$\underset{\sim}{x}_i$ is eigenvector with ϵ_i as exact eigenvalue.

$\underset{\sim}{V}_{(0)}^{n} = \{ \tilde{x}_i^{(0)} \}_{i=1}^{n} \longrightarrow$ first n smallest eigen vector

$\underset{\sim}{t}_i = \underset{\sim}{x}_i - \tilde{x}_i^{(0)}$

Corrector Eqn:

$$(\underset{\sim}{A} - \epsilon_i \underset{\sim}{I})\underset{\sim}{t}_i = (\epsilon_i \underset{\sim}{I} - \underset{\sim}{A})\tilde{x}_i^{(0)}$$

$$\underset{\sim}{A}\underset{\sim}{t}_i - \epsilon_i \underset{\sim}{t}_i = \epsilon_i \tilde{x}_i^{(0)} - \underset{\sim}{A}\tilde{x}_i^{(0)}$$

$$\underset{\sim}{A}\left[\underset{\sim}{t}_i + \tilde{x}_i^{(0)}\right] = \epsilon_i\left[\underset{\sim}{t}_i + \tilde{x}_i^{(0)}\right]$$

eigenvector corresponding to ϵ_i, so it is $\underset{\sim}{x}_i$.

$\underset{\sim}{A}\underset{\sim}{x}_i = \epsilon_i \underset{\sim}{x}_i$ with $\underset{\sim}{x}_i = \underset{\sim}{t}_i + \tilde{x}_i^{(0)}$

$$\Rightarrow \boxed{\underset{\sim}{t}_i = \underset{\sim}{x}_i - \tilde{x}_i^{(0)}}$$

→ Corrector eqn cannot be solved with knowledge of ϵ_i as $(\underset{\sim}{A} - \epsilon_i \underset{\sim}{I})$ is singular.

⑥ (b)

→ $\tilde{\epsilon}_i^{(0)}$ is best estimated using the Rayleigh Quo.

$$\tilde{\epsilon}_i^{(0)} = \tilde{x}_i^{(0)^T} \underset{\sim}{A}\, \tilde{x}_i^{(0)}$$

→ Since $\underset{\sim}{V}_{(0)}^{n}$ is orthogonal, each vector approximating its own eigenvector, $\tilde{\epsilon}_i^{(0)}$ thus obtained will also be close to corresponding eigenvalues.

Approx. corrector eqn:

$$\left(\underset{\sim}{D} - \tilde{\epsilon}_i^{(0)} \underset{\sim}{I}\right) \tilde{t}_i = \left(\tilde{\epsilon}_i^{(0)} \underset{\sim}{I} - \underset{\sim}{A}\right) \tilde{x}_i^{(0)}$$

$\underset{\sim}{D} \rightarrow \mathrm{diag}(\underset{\sim}{A})$ | $\tilde{\epsilon}_i^{(0)} = \tilde{x}_i^{(0)^T} \underset{\sim}{A}\, \tilde{x}_i^{(0)}$

$\left(\underset{\sim}{D} - \tilde{\epsilon}_i^{(0)} \underset{\sim}{I}\right)$ is easily invertible.

So,

$$\underset{\sim}{t}_i = \left[\underset{\sim}{D} - \tilde{\epsilon}_i^{(0)}\underset{\sim}{I}\right]^{-1}\left[\tilde{\epsilon}_i^{(0)}\underset{\sim}{I} - \underset{\sim}{A}\right]\tilde{x}_i^{(0)}$$

6 (c)

→ $\tilde{t}_i^{(K)}$ approximates $\underset{\sim}{t}_i^{(K)}$ which is the correction vector that relates i^{th} guess to i^{th} eigenvector.

→ $\tilde{t}_i^{(K)}$ is a good approx. to $\underset{\sim}{t}_i^{(K)}$ as $\underset{\sim}{D} = \text{diag}(\underset{\sim}{A})$ and $\underset{\sim}{A}$ is diagonally dominant.

$\underset{\sim}{V}_{(K)}^{2n}$ has vectors $\{\tilde{x}_i^{(K)}, \tilde{t}_i^{(K)}\}$. Lin. comb. of $\tilde{x}_i^{(K)}$ & $\tilde{t}_i^{(K)}$ will take us closer to the actual eigenvectors.

$\underset{\sim}{V}_{(k)}^{n}$ does not have the correction vectors, so it is not as good as $\underset{\sim}{V}_{(k)}^{2n}$ even though it has information about the eigenvectors.

6 (d) Define:

$$\underset{\sim}{V} = \underset{\sim}{V}_{(0)}^{2n} \quad \Big| \quad \underset{\sim}{r}_i = \underset{\sim}{A}\, \tilde{\underset{\sim}{x}}_i^{(1)} - \tilde{\epsilon}_i^{(1)} \tilde{\underset{\sim}{x}}_i^{(1)}$$

Galerkin Condition: $\underset{\sim}{V}^T \underset{\sim}{r}_i = \underset{\sim}{0}$

$\rightarrow$ $\tilde{\underset{\sim}{x}}_i^{(1)}$ is a linear comb. of columns of $\underset{\sim}{V}$.

$$\tilde{\underset{\sim}{x}}_i^{(1)} = \underset{\sim}{V}\, \underset{\sim}{y}_i$$ for some $\underset{\sim}{y}_i$

$$\underset{\sim}{V}^T\left[\underset{\sim}{A}\,\underset{\sim}{V}\,\underset{\sim}{y_i} - \tilde{\epsilon}_i^{(1)}\,\underset{\sim}{V}\,\underset{\sim}{y_i}\right] = \underset{\sim}{0}$$

$$\overset{\underset{\sim}{A}'}{\underset{\sim}{V}^T\underset{\sim}{A}}\;\overset{\underset{\sim}{u_i}}{\left(\underset{\sim}{V}\,\underset{\sim}{y_i}\right)} = \tilde{\epsilon}_i^{(1)}\;\overset{\underset{\sim}{B}}{\underset{\sim}{V}^T}\;\overset{\underset{\sim}{u_i}}{\left(\underset{\sim}{V}\,\underset{\sim}{y_i}\right)}$$

This is of the Gen. EVP form. This problem can be solved to find the eigen pair corresponding to the n smallest eigen values.

- Simultaneous iteration on $(\underset{\sim}{A}')^{-1}$ seems feasible, $\underset{\sim}{A}'$ is not guaranteed to be non-singular & it may not be optimal.

- Power iteration on $\underset{\sim}{A}''$ which where: $\underset{\sim}{A}'' = -(\underset{\sim}{A}' - g\underset{\sim}{I})$
  g is very large.