Numerical Solution of Differential Equations
Assignment 1

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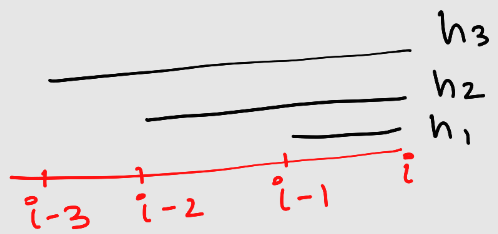
Question 1

Acknowledgement:

- ChatGPT was used in the code and makefile for all questions.
- GitHub Copilot was used in the code for all questions.

① (a) $\frac{\partial^2 u}{\partial x^2}$ in a non-uniform grid



$$c_0\left[u_i = u_i\right]$$

$$c_1\left[u_{i-1} = u_i - u_i' h_1 + u_i'' \frac{h_1^2}{2} - u_i''' \frac{h_1^3}{3!} + u_i'''' \frac{h_1^4}{4!} + \dots\right]$$

$$c_2\left[u_{i-2} = u_i - u_i' h_2 + u_i'' \frac{h_2^2}{2} - u_i''' \frac{h_2^3}{3!} + u_i'''' \frac{h_2^4}{4!} + \dots\right]$$

$$c_3\left[u_{i-3} = u_i - u_i' h_3 + u_i'' \frac{h_3^2}{2} - u_i''' \frac{h_3^3}{3!} + u_i'''' \frac{h_3^4}{4!} + \dots\right]$$

$$\uparrow \quad \uparrow \quad \uparrow \quad \uparrow$$

$$0 \quad 0 \quad 1 \quad 0$$

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -h_1 & -h_2 & -h_3 \\ 0 & h_1^2/2 & h_2^2/2 & h_3^2/2 \\ 0 & -h_1^3/6 & -h_2^3/6 & -h_3^3/6 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{b}$$

Using python to generate inverse and solving: $x = A^{-1} b$

→ Inverse is printed in code "q1.py"

$$c_0 = \frac{2(h_1+h_2+h_3)}{h_1 h_2 h_3}$$

$$c_1 = \frac{-2(h_2+h_3)}{h_1^3 - h_1^2 h_2 - h_1^2 h_3 + h_1 h_2 h_3}$$

$$C_2 = \frac{2(h_1 + h_3)}{h_1 h_2^2 - h_1 h_2 h_3 - h_2^3 + h_2^2 h_3}$$

$$C_3 = \frac{-2(h_1 + h_2)}{h_1 h_2 h_3 - h_1 h_3^2 - h_2 h_3^2 + h_3^3}$$

$$\frac{\partial^2 u}{\partial x^2} = C_0 u_i + C_1 u_{i-1} + C_2 u_{i-2} + C_3 u_{i-3} + O(h_1^2, h_2^2, h_3^2)$$

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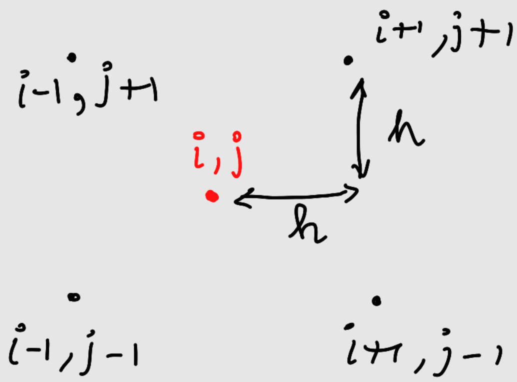
① (b)



Only cross points as symmetrical nodes will cancel out

Using Taylor's series: $u = u_{i,j}$

$$C_0 \left[u_{i-1,j+1} = u - h u_x + h u_y + \frac{h^2}{2}[u_{xx} + u_{yy} + 2u_{xy}] + \frac{h^3}{6}[-u_{xxx} + 3u_{xxy} - 3u_{xyy} + u_{yyy}] + \cdots \right]$$

$$C_1 \left[u_{i-1,j-1} = u - h u_x - h u_y + \frac{h^2}{2}[u_{xx} + u_{yy} + 2u_{xy}] + \frac{h^3}{6}[-u_{xxx} - 3u_{xxy} - 3u_{xyy} - u_{yyy}] + \cdots \right]$$

$$C_2 \left[u_{i,j} = u \right]$$

$$C_3 \left[u_{i+1,j-1} = u + h u_x - h u_y + \frac{h^2}{2}[u_{xx} + u_{yy} - 2u_{xy}] + \frac{h^3}{6}[u_{xxx} - 3u_{xxy} + 3u_{xyy} - u_{yyy}] + \cdots \right]$$

$$\left[u_{i+1,j+1} = u + hu_x + hu_y + \frac{h^2}{2}\left[u_{xx} + u_{yy} + 2u_{xy}\right] + \frac{h^3}{6}\left[u_{xxx} + 3u_{xxy} + 3u_{xyy} + u_{yyy}\right] + \dots\right]$$

C_4

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -h & -h & 0 & h & h \\ h & -h & 0 & -h & h \\ h^2 & h^2 & 0 & h^2 & h^2 \\ -h^2 & h^2 & 0 & -h^2 & h^2 \end{bmatrix}$$

$$x = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

→ Inverting using code & solving:

$$c_0 = c_4 = \frac{1}{4h^2} \qquad c_2 = 0$$

$$c_1 = c_3 = \frac{-1}{4h^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1,j+1} - u_{i+1,j-1} + u_{i-1,j-1} - u_{i-1,j+1}}{4h^2}$$

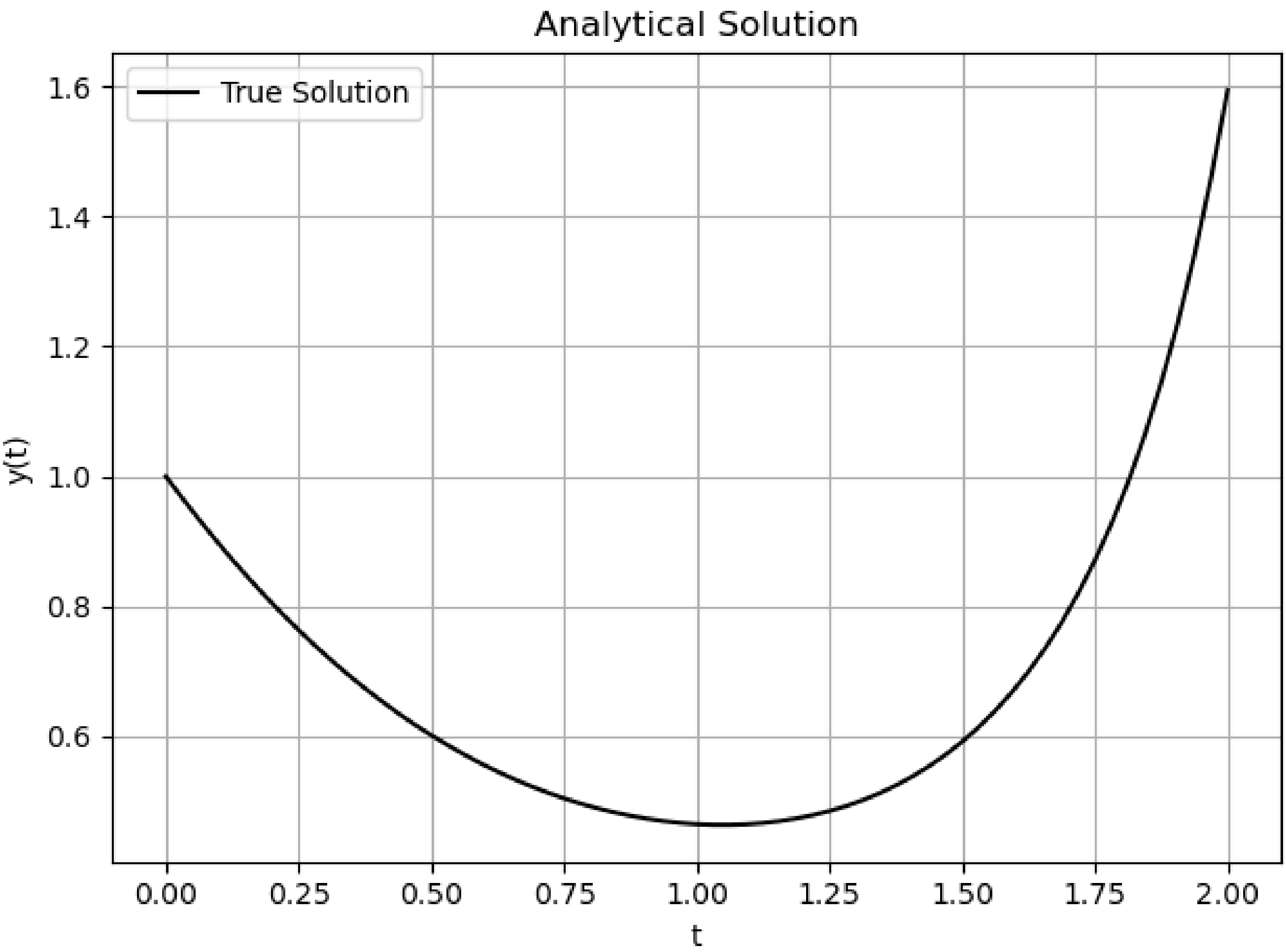
Question 2

The given initial value problem over the interval $t = 0$ to $t = 20$ is:

$$\frac{dy}{dt} = yt^2 - 1.1y$$

where $y(0) = 1$

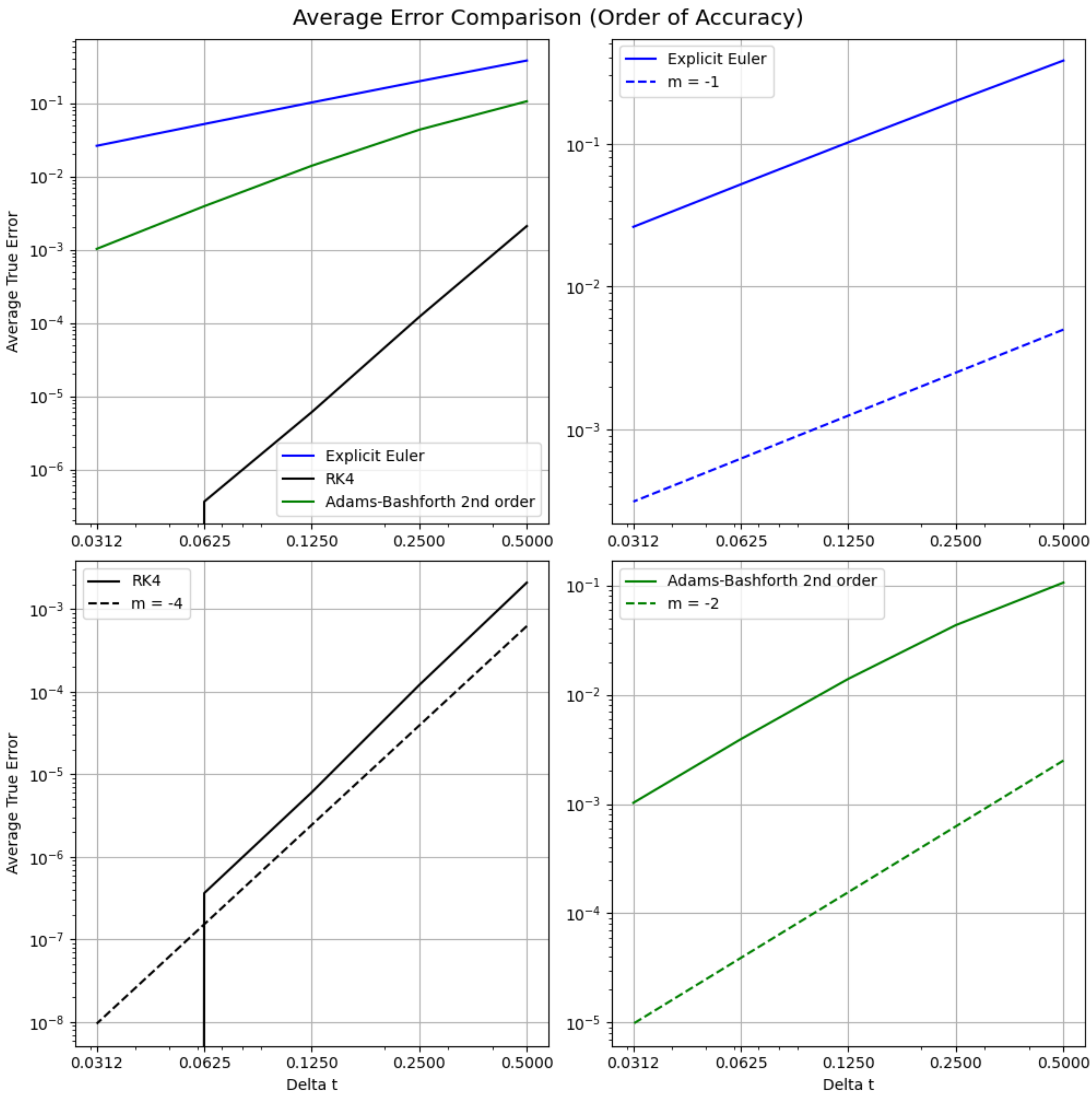
The equation can be solved analytically by variable separable method.



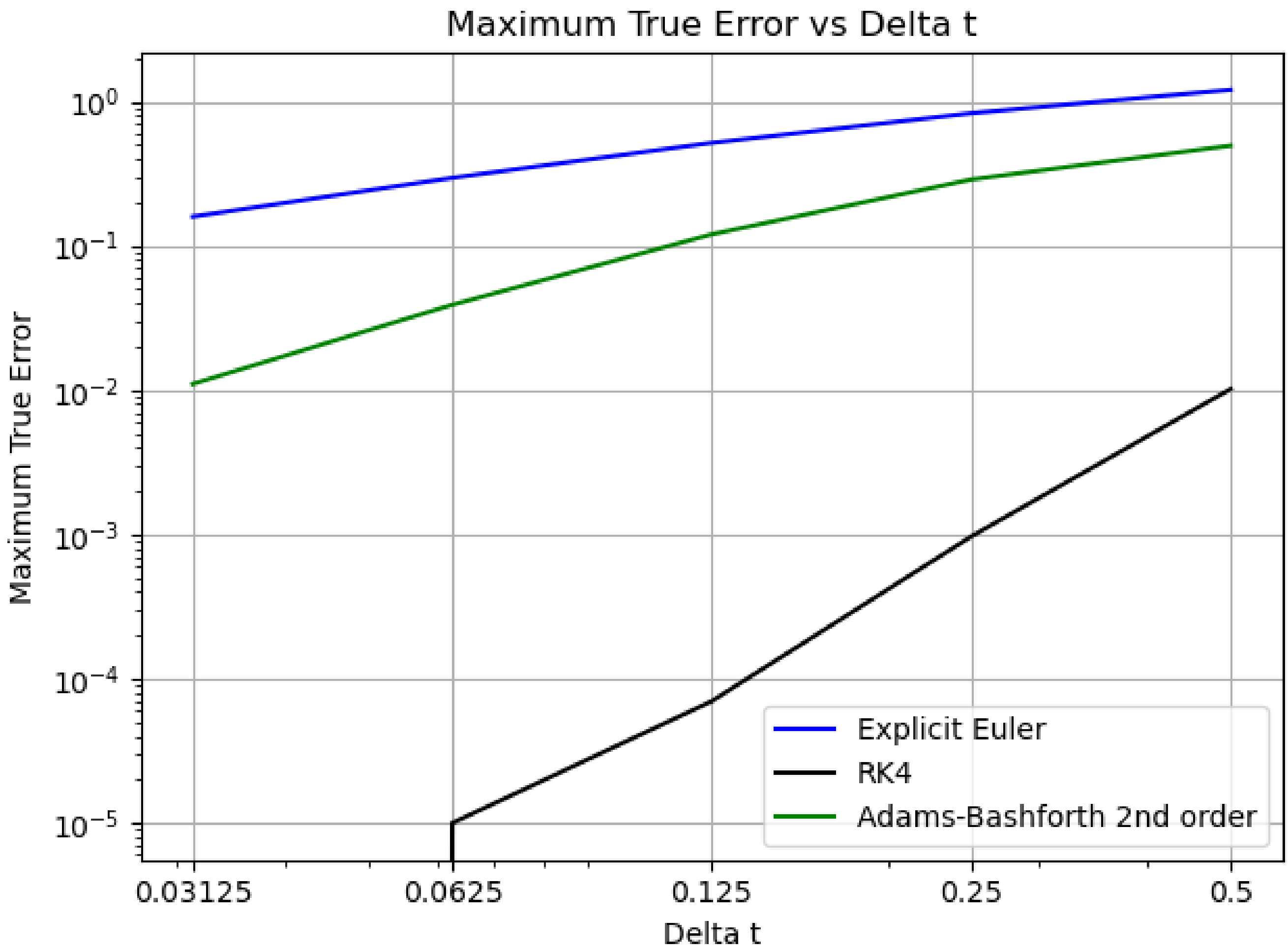
The analytical solution is given below.

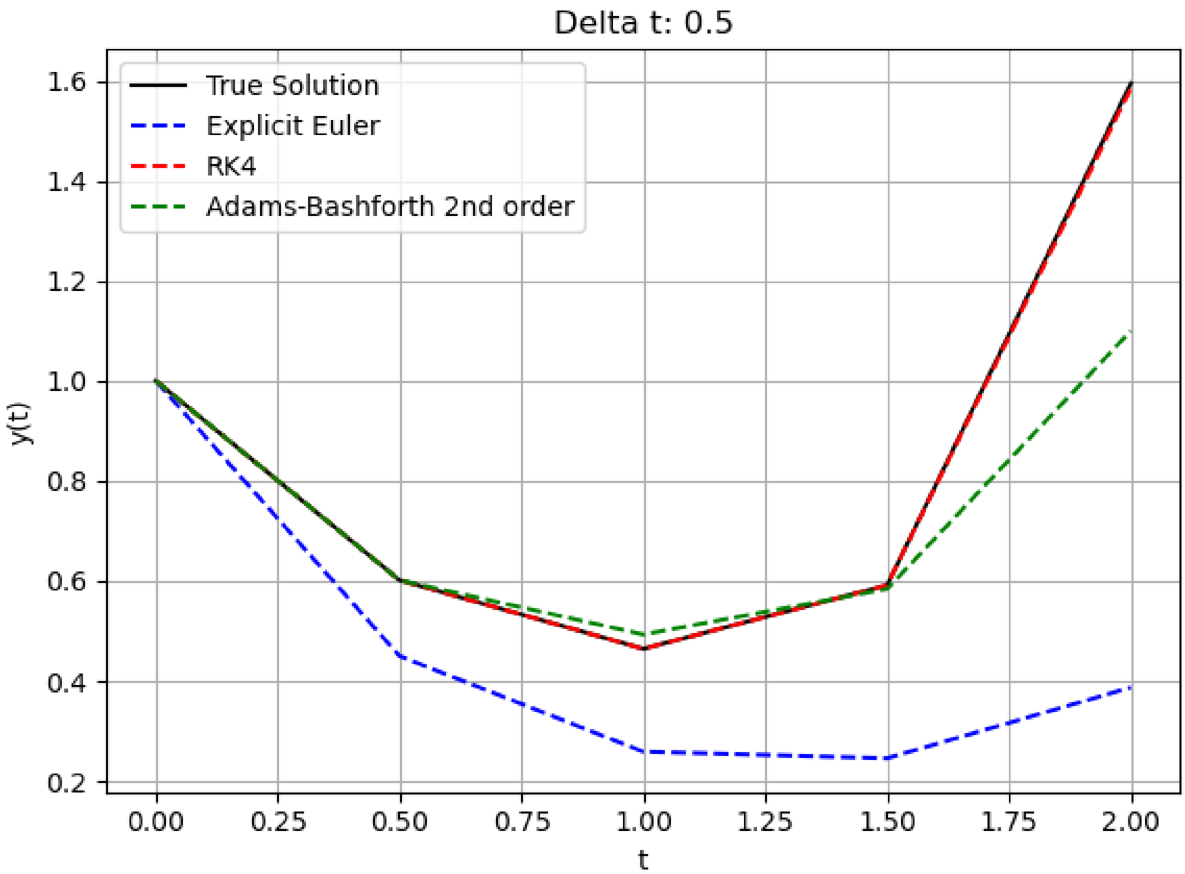
$$ln(y) = \frac{t^3}{3} - 1.1t$$

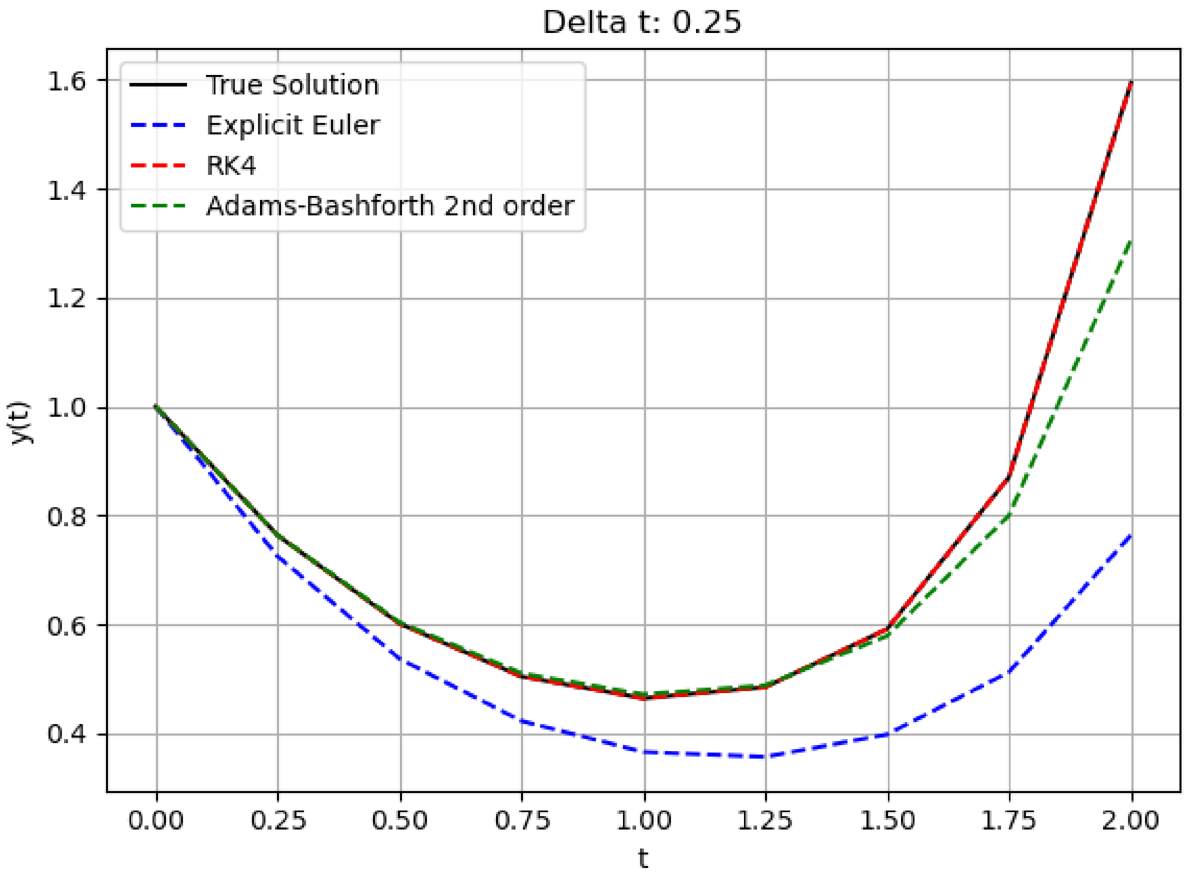
Plots:



Average Error Comparison (Order of Accuracy)
Average True Error
Explicit Euler
RK4
Adams-Bashforth 2nd order
Explicit Euler
m = -1
RK4
m = -4
Adams-Bashforth 2nd order
m = -2
Delta t
0.0312
0.0625
0.1250
0.2500
0.5000

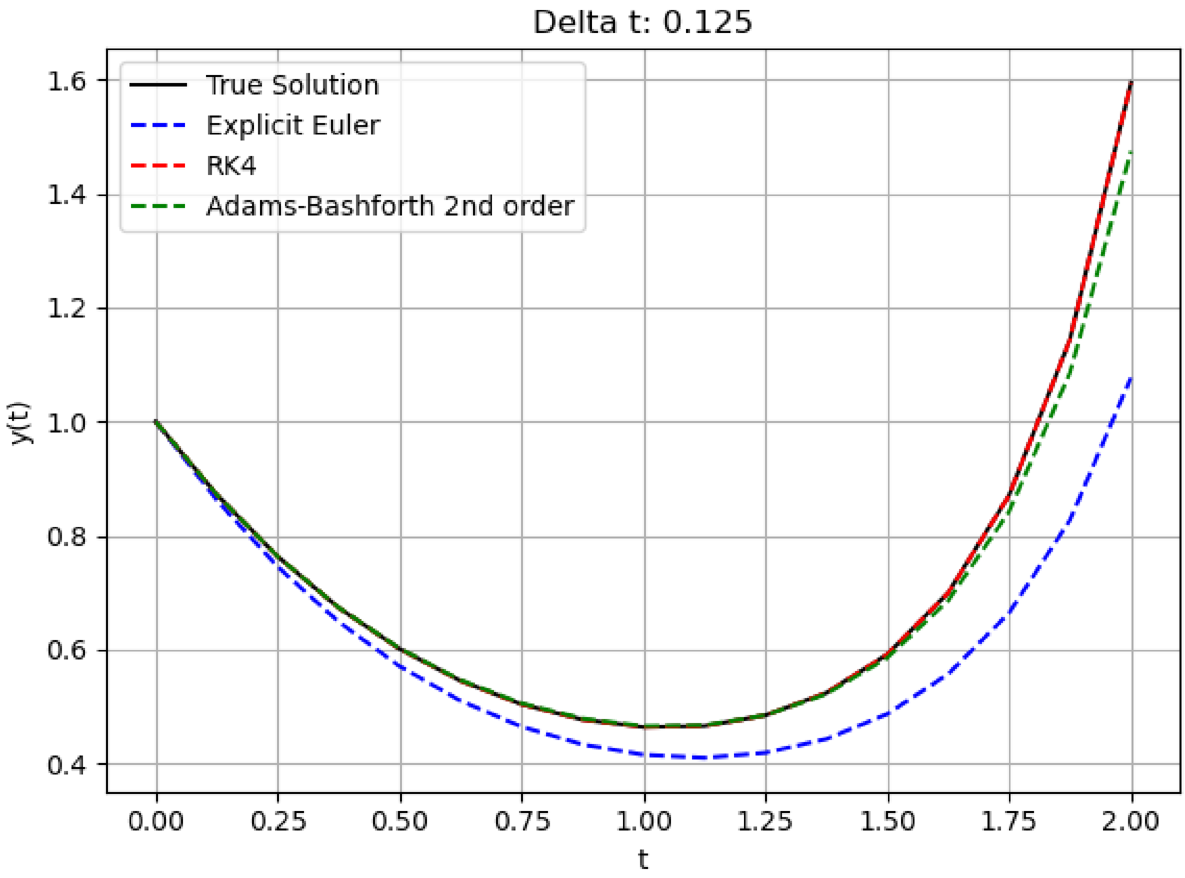

Maximum True Error vs Delta t
Maximum True Error
Delta t
Explicit Euler
RK4
Adams-Bashforth 2nd order
0.03125
0.0625
0.125
0.25
0.5


Delta t: 0.5
y(t)
t
True Solution
Explicit Euler
RK4
Adams-Bashforth 2nd order

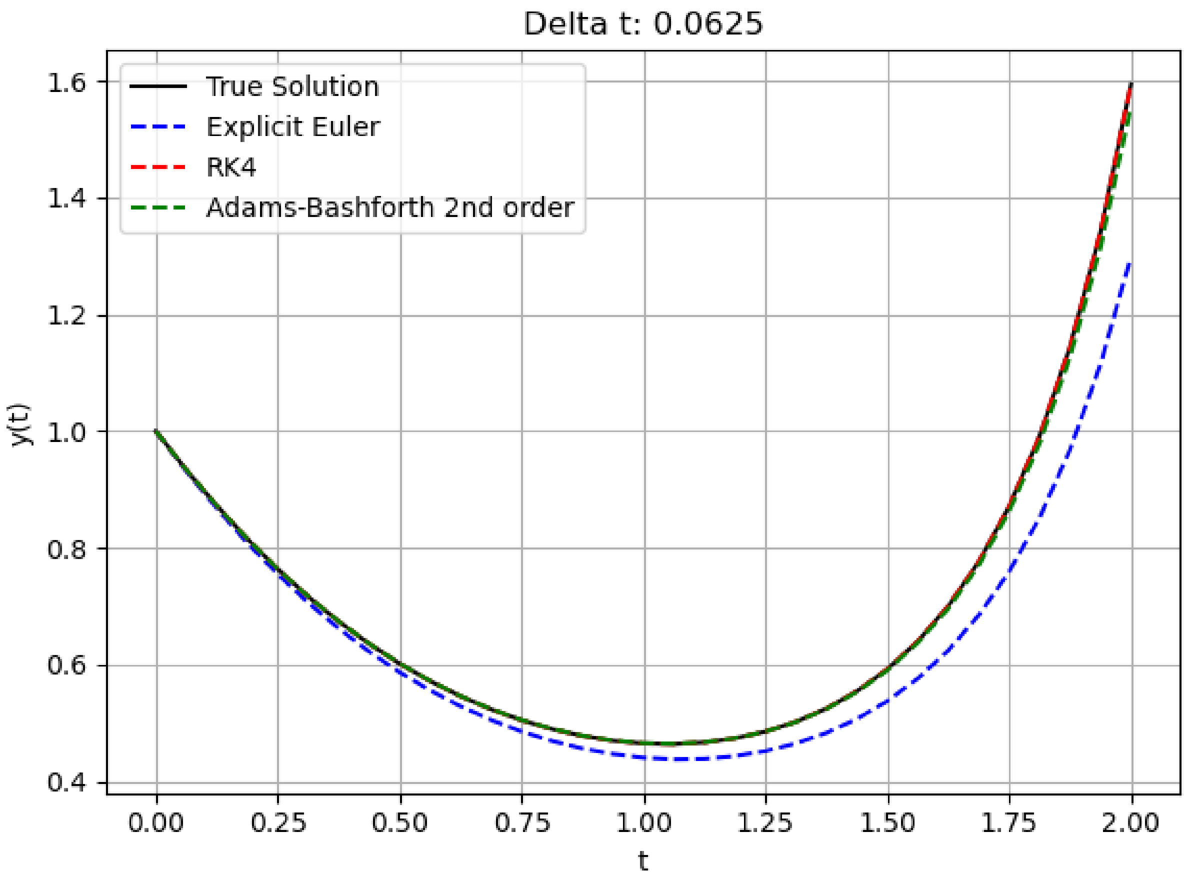


Discussion:

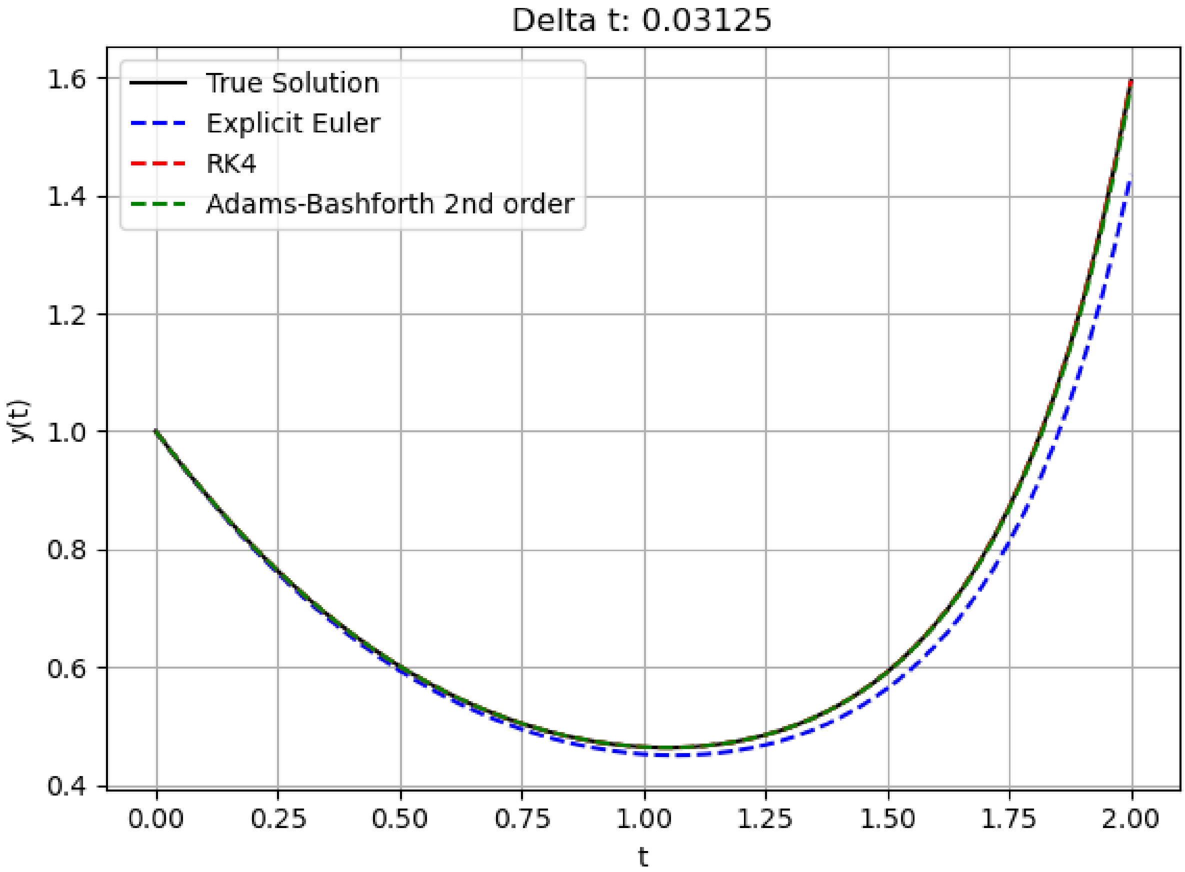
- RK4 depicts the best accuracy irrespective of the step size.
- Of all the methods, Explicit Euler is the least accurate.



Delta t: 0.125
True Solution
Explicit Euler
RK4
Adams-Bashforth 2nd order
y(t)
t



Delta t: 0.0625
True Solution
Explicit Euler
RK4
Adams-Bashforth 2nd order
y(t)
t


Delta t: 0.03125
True Solution
Explicit Euler
RK4
Adams-Bashforth 2nd order
y(t)
t
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00

Question 3

The given second order differential equation can be resolved into two first order differential equations and solved using RK4.

Given,

$$m\frac{d^2x}{dt^2} + a\frac{dx}{dt}\frac{dx}{dt} + kx = F_0 \sin(\omega t)$$

where, x is the displacement from the equilibrium position in m, t is the time in s. The other constants are $m = 2 \quad kg$, $a = 5 \quad N/m$, $k = 6 \quad N/m$, $F_0 = 2.5 \quad N$ and $\omega = 0.5 \quad rad/s$.
Given, at $t = 0 \quad s$, $v_0 = 0 \quad m/s$ and $x = 1 \quad m$.

Equation 1:

$$\frac{dx}{dt} = v$$

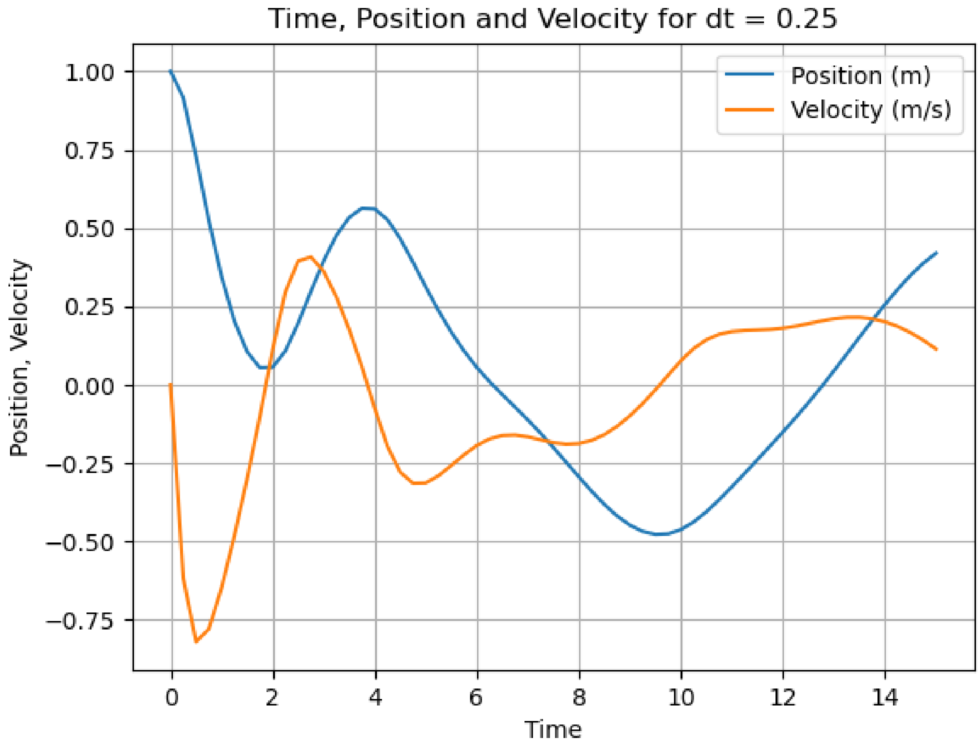
Equation 2:

$$\frac{dv}{dt} = \frac{F_0 sin(\omega t) - a|v|v - kx}{m}$$

Discussion:

- It is concluded that the grid convergence is reached at around 10^3 grid points.

Property	Minimum Value	Time (at Minimum)	Maximum Value	Time (at Maximum)
Position	-0.478014	9.58496	1	0
Velocity	-0.827922	0.551758	0.411432	2.66504



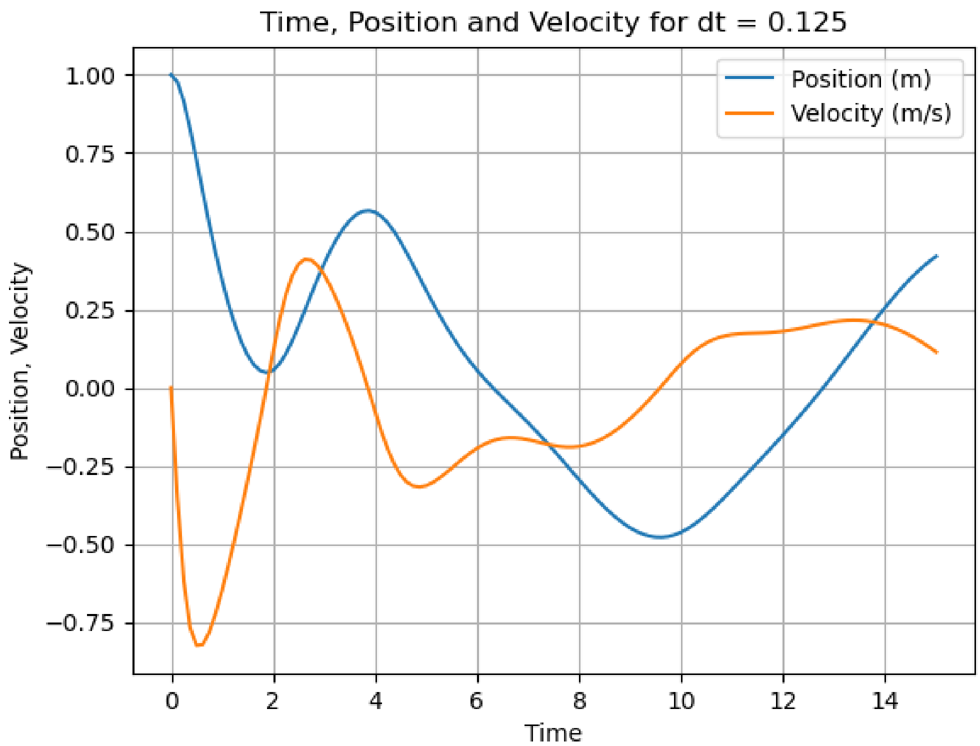
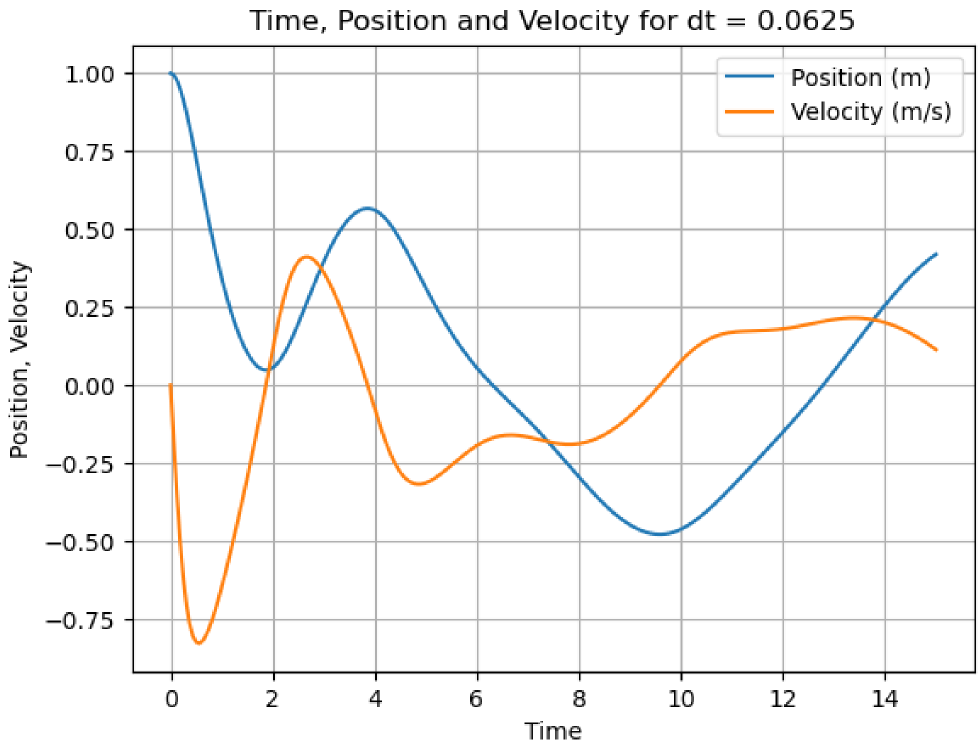
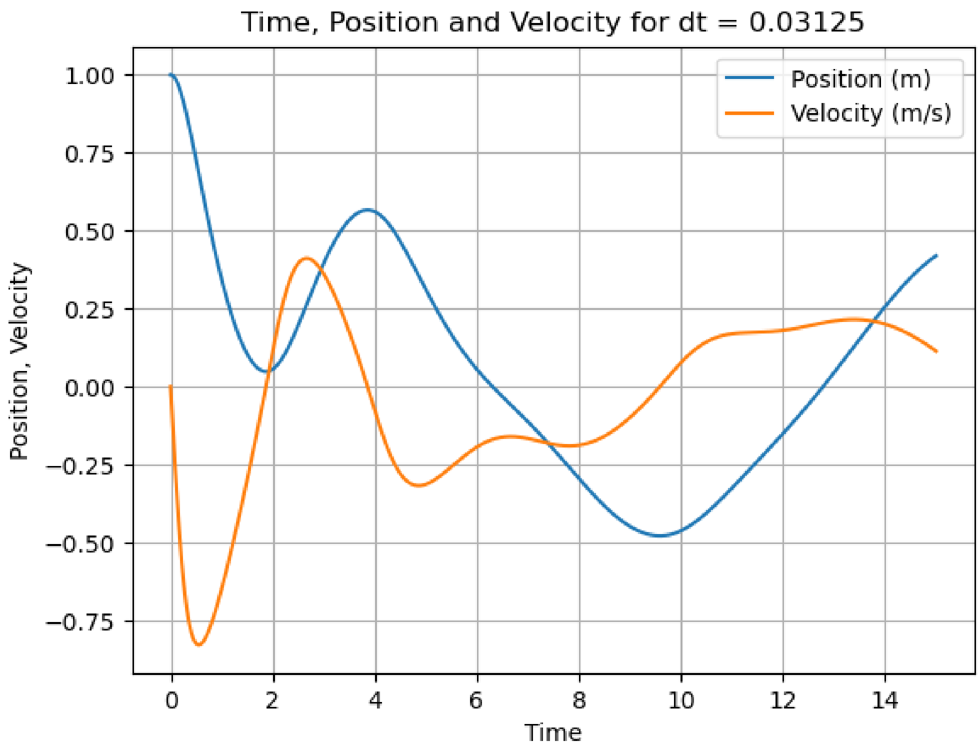


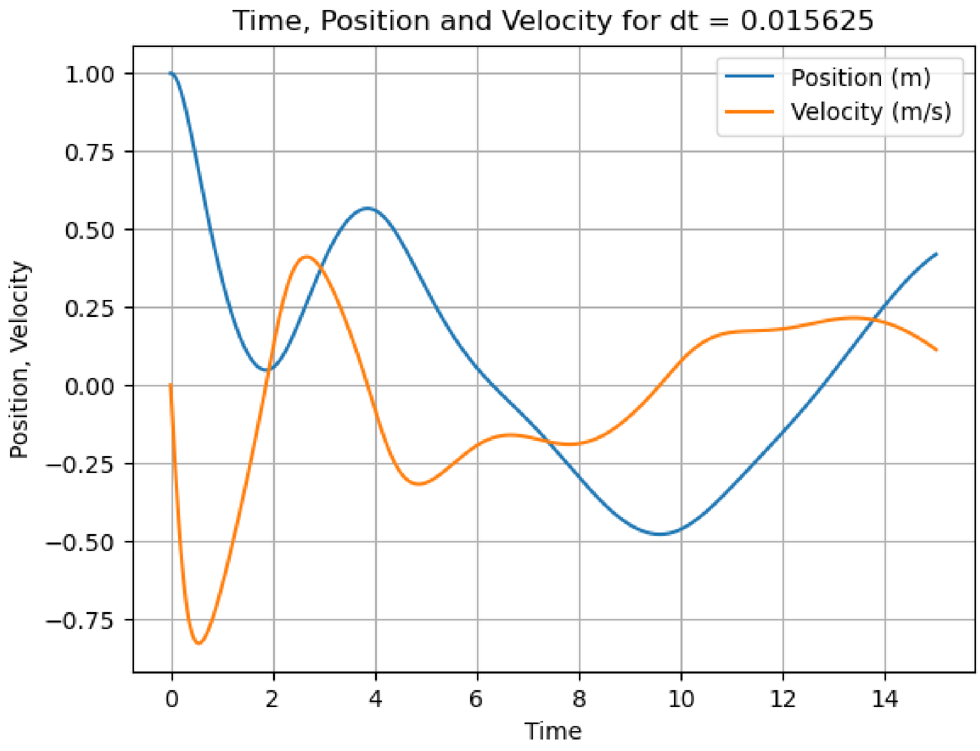
Figure 1: Enter Caption

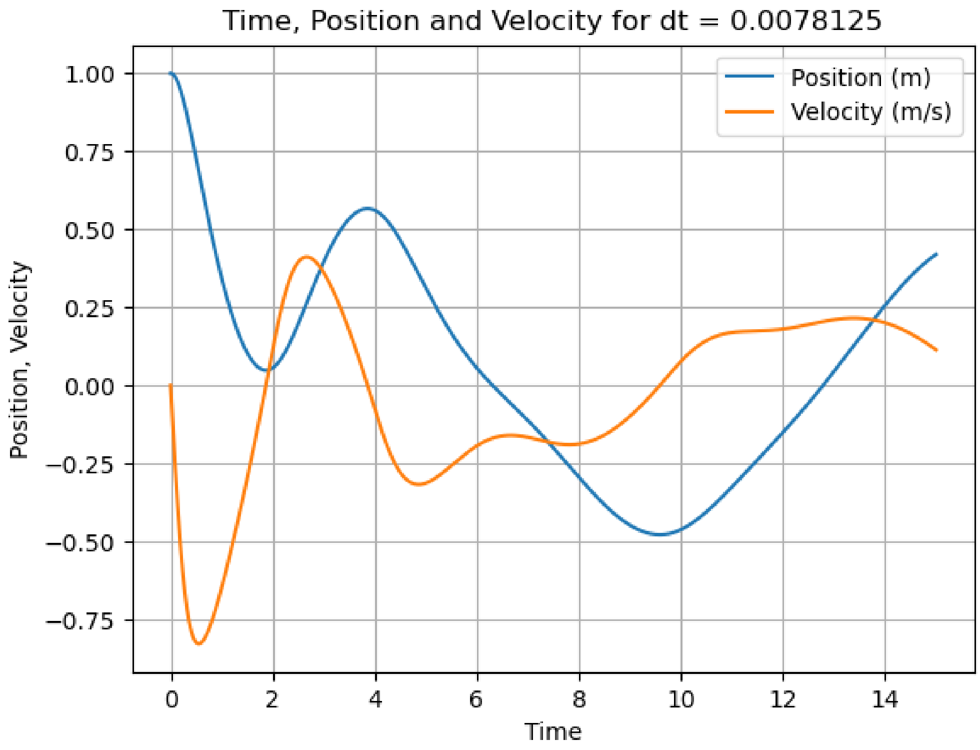


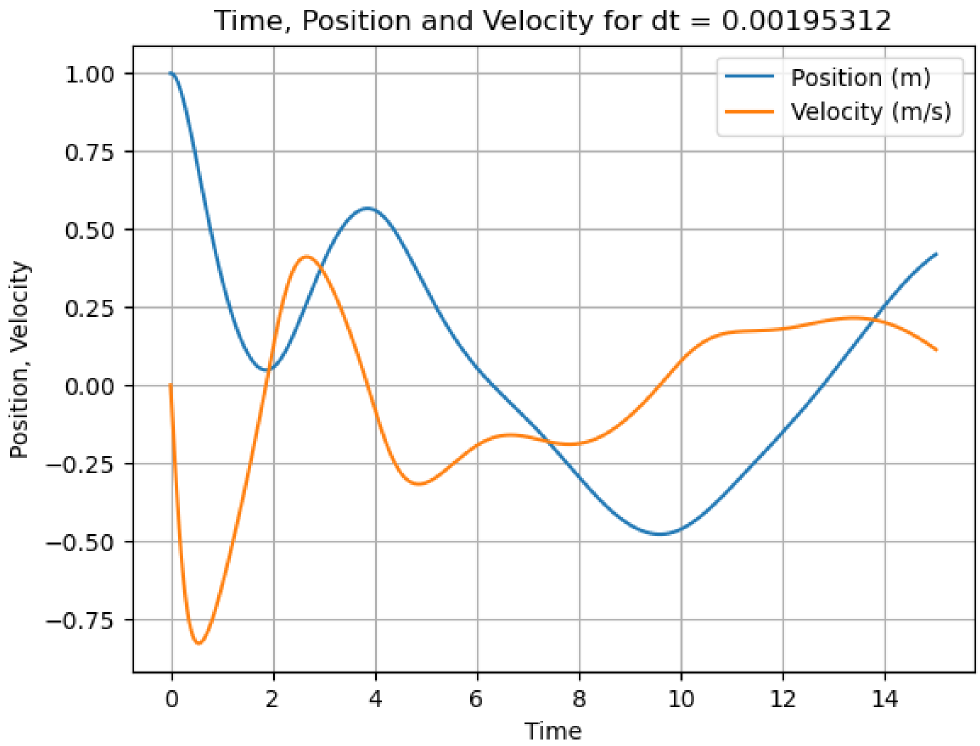
Time, Position and Velocity for dt = 0.0625
Position (m)
Velocity (m/s)
Position, Velocity
Time

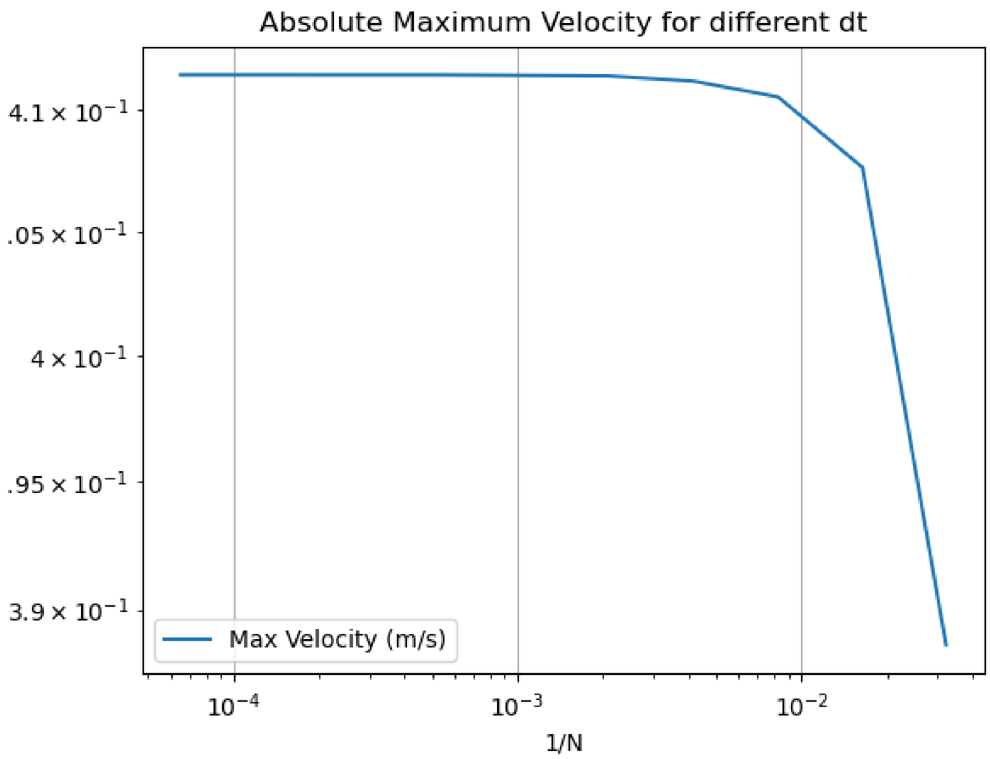


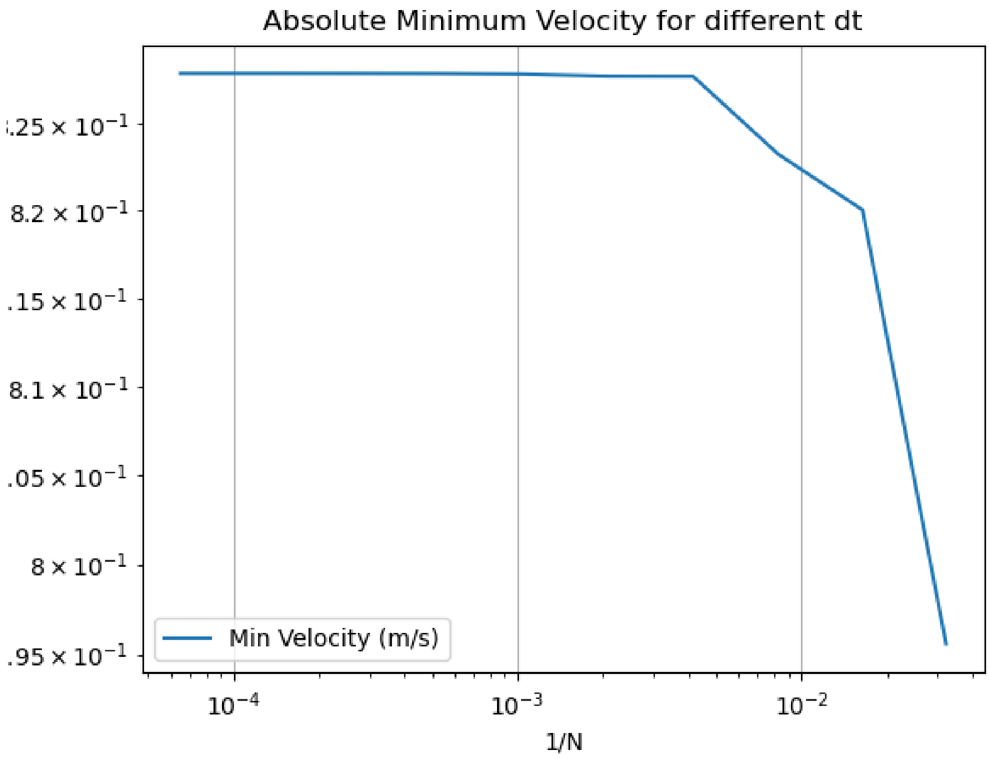
Time, Position and Velocity for dt = 0.03125
Position (m)
Velocity (m/s)
Position, Velocity
Time

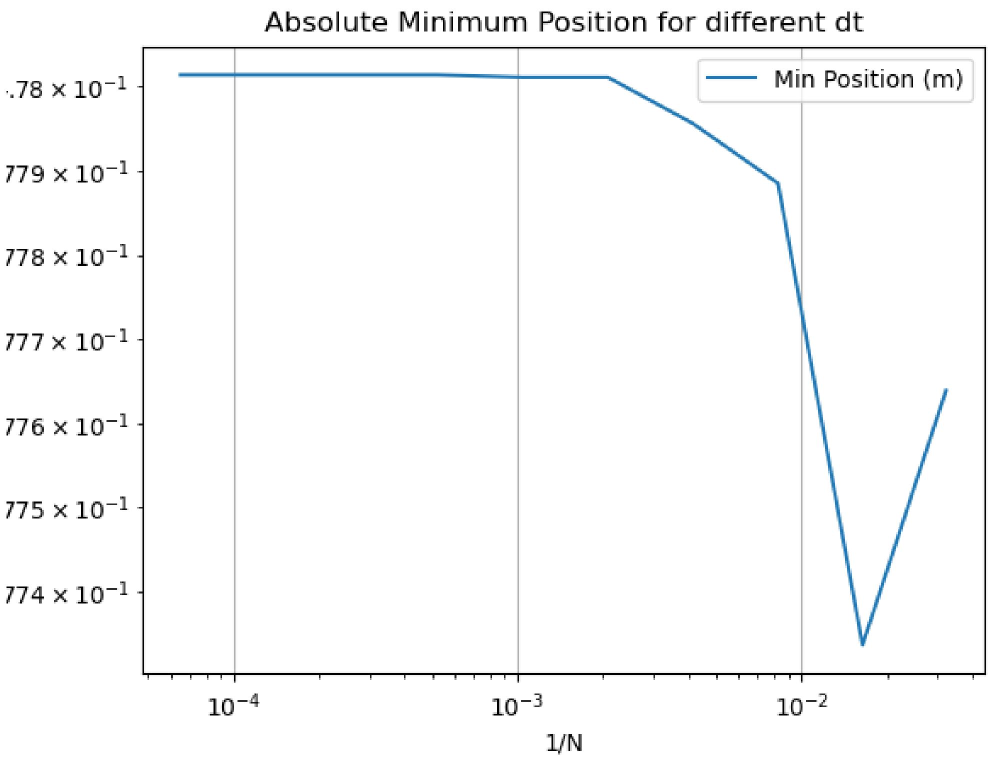

Time, Position and Velocity for dt = 0.015625
Position (m)
Velocity (m/s)
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
Position, Velocity
0
2
4
6
8
10
12
14
Time


Time, Position and Velocity for dt = 0.0078125
Position (m)
Velocity (m/s)
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
Position, Velocity
0
2
4
6
8
10
12
14
Time


Time, Position and Velocity for dt = 0.00195312
Position, Velocity
Time
Position (m)
Velocity (m/s)
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
0
2
4
6
8
10
12
14


Absolute Maximum Velocity for different dt
4.1 × 10⁻¹
.05 × 10⁻¹
4 × 10⁻¹
.95 × 10⁻¹
3.9 × 10⁻¹
Max Velocity (m/s)
10⁻⁴
10⁻³
10⁻²
1/N

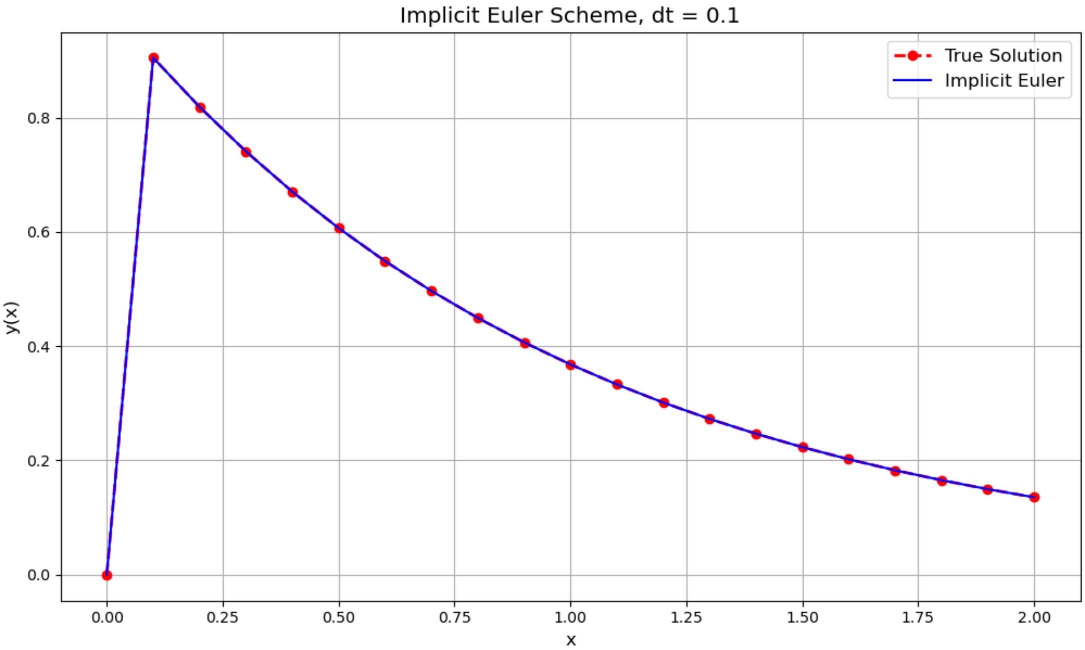

Absolute Minimum Velocity for different dt
.25 × 10^-1
8.2 × 10^-1
.15 × 10^-1
8.1 × 10^-1
.05 × 10^-1
8 × 10^-1
.95 × 10^-1
Min Velocity (m/s)
10^-4
10^-3
10^-2
1/N


Absolute Minimum Position for different dt
.78 × 10^-1
779 × 10^-1
778 × 10^-1
777 × 10^-1
776 × 10^-1
775 × 10^-1
774 × 10^-1
Min Position (m)
10^-4
10^-3
10^-2
1/N

Question 4

$$\frac{dy}{dx} = -2,00,000y + 2,00,000e^{-x} - e^{-x}$$

with $y(0) = 0$, from $x = 0$ to $x = 2$ with step size of 0.1.



- Implicit is better than explicit as step size is not as restrictive.
- For 10000 times greater step size than required for explicit scheme, we are able to obtain a good approximate solution using implicit scheme.

Explicit Euler: $\dfrac{dy}{dx} = f(x, y) \quad a \le x \le b$

$y(a) = y_0$; $y_b = ?$

(N-1) sub-intervals
with N-gridpoints.

Uniform $\Rightarrow$ $\Delta x = h = \dfrac{b-a}{N-1}$

$y(x)$ is the true solution

$$y_{i+1} = y_i + h f(x_i, y_i)$$

Stability: $|y_{i+1}| \le |y_i|$

$$\frac{dy}{dx} = -2\times10^5 \left[y - e^{-x} \right] - e^{-x}$$

$$y_{i+1} = y_i + h \left\{ -2\times10^5 \left[y_i - e^{-x_i} \right] - e^{-x_i} \right\}$$

$h > 0$ & $e^{-x_i} > 0 \quad \forall \; x_i \neq \infty$

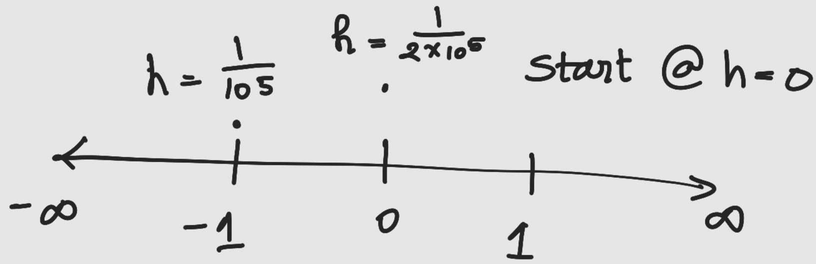
So, $|y_{i+1}| < |y_i - 2\times10^5 h \, y_i|$

$$|y_{i+1}| < |y_i| \, |1 - 2\times10^5 h|$$

$$\frac{|y_{i+1}|}{|y_i|} < 1 \implies |1 - 2\times10^5 h| < 1$$

$$-1 < 1 - 2\times10^5 h < 1$$

Start @ h=0



$h < \dfrac{1}{10^5}$

$$y' + ay = (a-1)\, e^{-x} \qquad a = 2\times 10^5$$

$$IF = e^{\int a\,dx} = e^{ax}$$

$$y e^{ax} = \int e^{(a-1)x} \cdot (a-1)\, dx + c$$

$$y e^{ax} = e^{(a-1)x} + c$$

$$y = e^{-x} + c\, e^{-ax}$$

$$y(0) = 0 \Rightarrow 0 = 1 + c \Rightarrow c = -1$$

$$y = e^{-x} - e^{-ax}$$

↳ Exact Solution

Question 5

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} + S = 0 \quad \text{over the range } 0 \leq r \leq 1,$$

$$\text{with the boundary conditions:} \quad T(r=1) = 1 \quad \text{and} \quad \left.\frac{dT}{dr}\right|_{r=0} = 0.$$

The above equation is solved using second order central difference.

Discussion:

- All the plots show the same trend with only scaling due to the source term.
- The peak temperatures always occur at $r = 0$.
- The peak values are always $0.25 \times S$.
- From this, it can be easily concluded that $S = 400$ for a peak temperature of 100.

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} = -S \qquad (1)\ \left.\frac{dT}{dr}\right|_{r=0} = 0$$

$$\left. T \right|_{r=1} = 1$$

$$\frac{d^2T}{dr^2} = \frac{T_{i+1} - 2T_i + T_{i-1}}{h^2} \qquad 0 \le r \le 1$$

$$(1) \Rightarrow T_2 = T_0$$

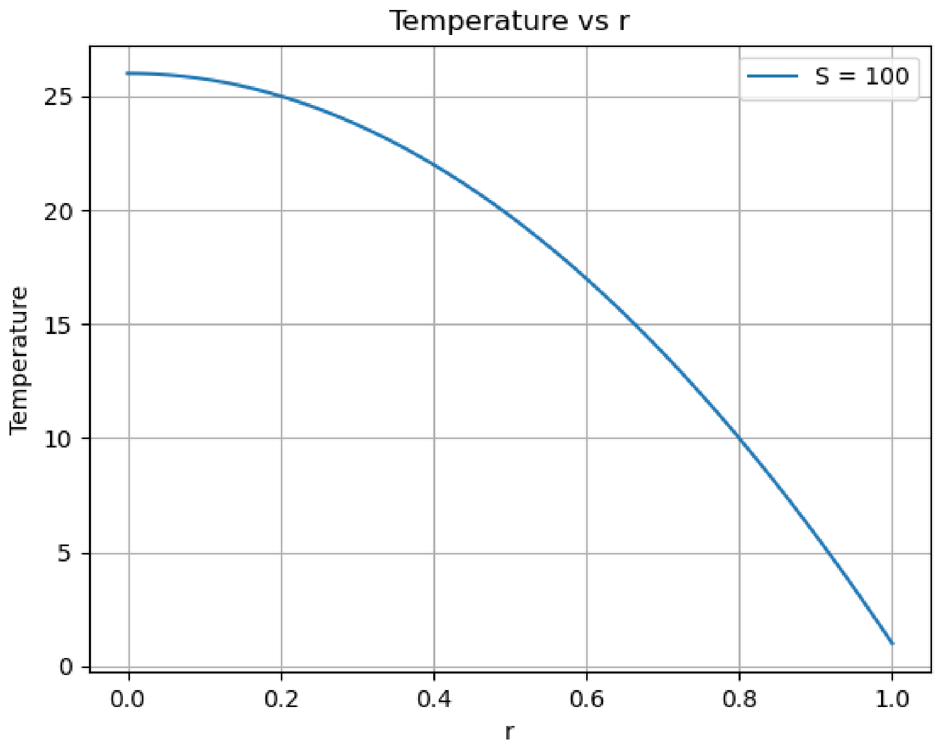
$$\frac{dT}{dr} = \frac{T_{i+1} - T_{i-1}}{2h}$$

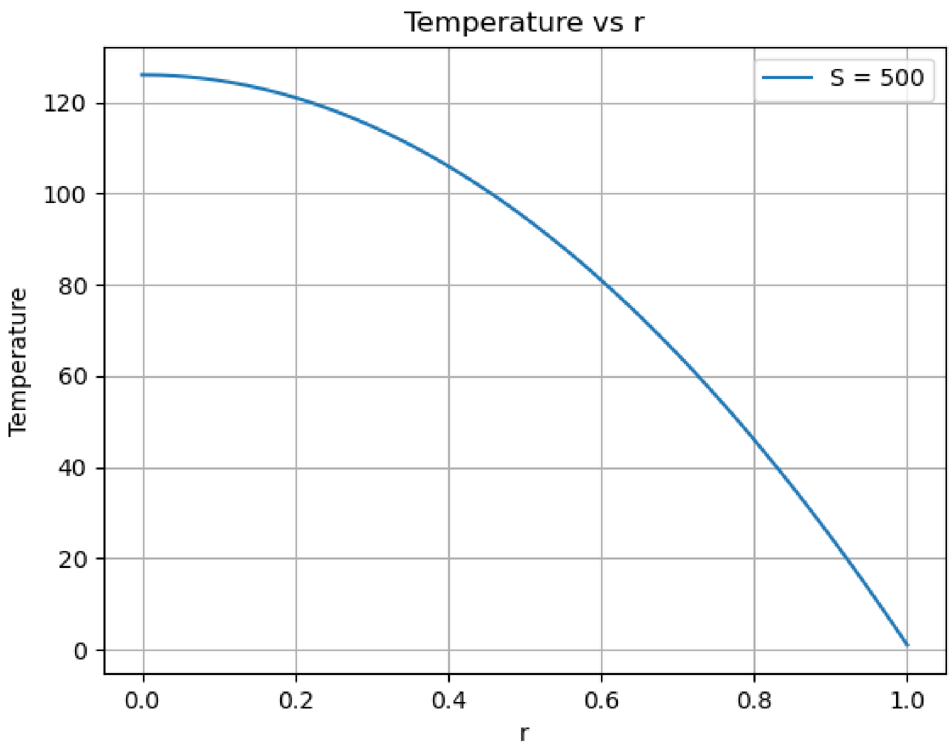
$$T_{i+1}\overset{A}{\left[\frac{1}{h^2} + \frac{1}{2hr_i}\right]} + T_{i-1}\overset{C}{\left[\frac{1}{h^2} - \frac{1}{2hr_i}\right]} + T_i\underset{B}{\left[\frac{-2}{h^2}\right]} = -S$$

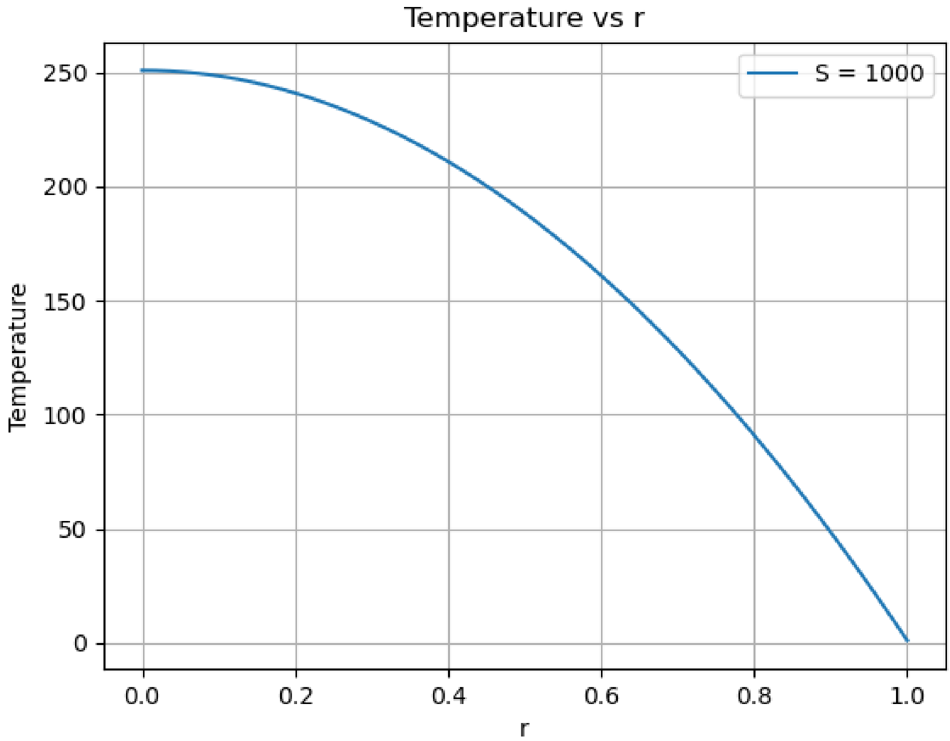
$$\begin{array}{c} T_1 \\ T_2 \\ \vdots \\ T_i \\ \vdots \\ T_{N-1} \\ T_N \end{array}
\begin{bmatrix} B & -B & \cdots & & & & 0 \\ C & B & A & 0 & \cdots & & \\ & \ddots & \ddots & \ddots & & & \\ 0 & \cdots & C & B & A & \cdots & 0 \\ & & & \ddots & \ddots & \ddots & \\ \cdots & & & & C & B & A \\ & & & & & & 1 \end{bmatrix}
\begin{bmatrix} T_1 \\ \vdots \\ T_{i-1} \\ T_i \\ T_{i+1} \\ \vdots \\ T_N \end{bmatrix} = \begin{bmatrix} \\ -S \\ \\ \end{bmatrix}$$

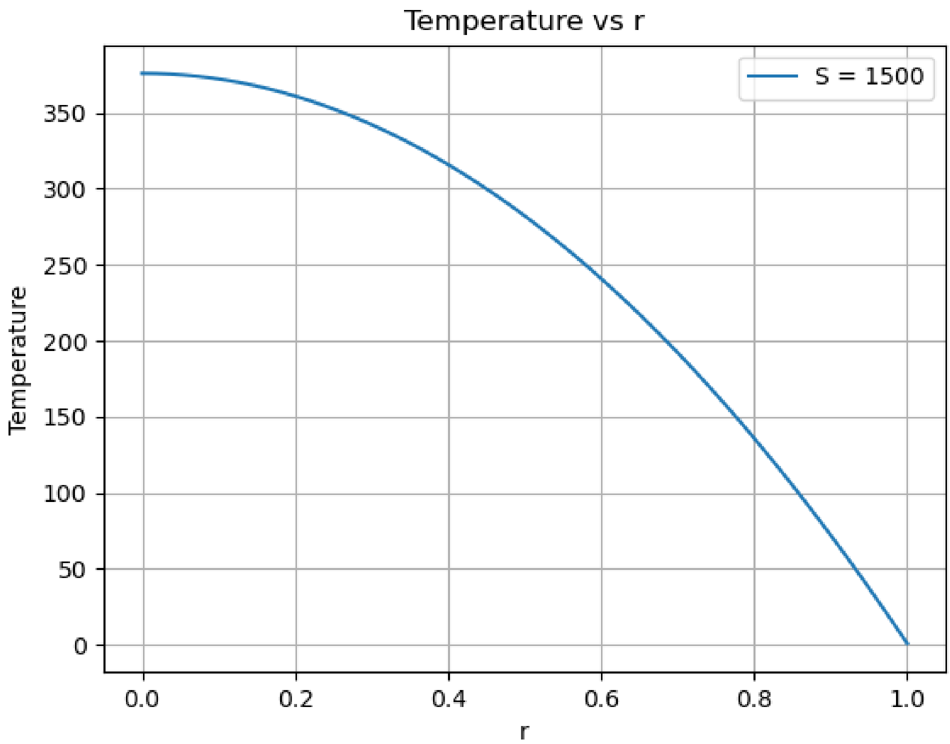
$$T_2\overset{A}{\left[\frac{1}{h^2} + \frac{1}{2hr_i}\right]} + T_0\overset{C}{\left[\frac{1}{h^2} - \frac{1}{2hr_i}\right]} + T_1\left[\frac{-2}{h^2}\right] = -S \qquad T_0 = T_2$$

$$T_2\left[\frac{2}{h^2}\right] + T_1\left[\frac{-2}{h^2}\right] = -S$$


Temperature vs r
S = 100
Temperature
r
0
5
10
15
20
25
0.0
0.2
0.4
0.6
0.8
1.0


Temperature vs r
S = 500
Temperature
r
0
20
40
60
80
100
120
0.0
0.2
0.4
0.6
0.8
1.0


Temperature vs r
S = 1000
Temperature
250
200
150
100
50
0
0.0
0.2
0.4
0.6
0.8
1.0
r


Temperature vs r
S = 1500
Temperature
350
300
250
200
150
100
50
0
0.0
0.2
0.4
0.6
0.8
1.0
r