DS 298 Numerical Solution of Differential Equations
Assignment 2

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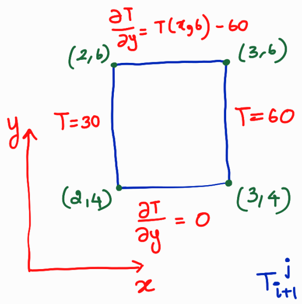
- ChatGPT was used in the code and makefile for all questions.
- GitHub Copilot was used in the code for all questions.

Question 1

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \qquad 2 \le x \le 3 \;;\; 4 \le y \le 6$$

$x \to i \;,\; y \to j$



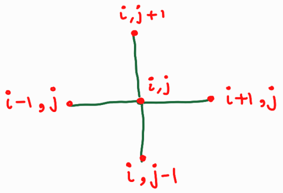
Algebraic equation:

→ For interiors:

$$\frac{T_{i+1}^{j} - 2T_i^{j} + T_{i-1}^{j}}{\Delta x^2} + \frac{T_i^{j+1} - 2T_i^{j} + T_i^{j-1}}{\Delta y^2} = 0$$

$$T_{i+1}^{j}\left[\frac{1}{\Delta x^2}\right] + T_i^{j}\left[\frac{-2}{\Delta x^2} + \frac{-2}{\Delta y^2}\right] + T_{i-1}^{j}\left[\frac{1}{\Delta x^2}\right] + T_i^{j+1}\left[\frac{1}{\Delta y^2}\right] + T_i^{j-1}\left[\frac{1}{\Delta y^2}\right] = 0$$

Stencil (5 pt.) :



Except corners:

→ Left boundary: $i = 1$

$$T_1^{j} = 30$$

→ Right boundary: $i = N_x$

$$T_{N_x}^{j} = 60$$

→ Bottom boundary: $j = 1$

$$\frac{\partial T}{\partial y} = \frac{T_i^{j+1} - T_i^{j-1}}{2\Delta y} = 0 \quad \Rightarrow \quad \boxed{T_i^{2} = T_i^{0}}$$

Imposing at $j = 1$:

$$T_{i+1}^{1}\left[\frac{1}{\Delta x^2}\right] + T_i^{1}\left[\frac{-2}{\Delta x^2} + \frac{-2}{\Delta y^2}\right] + T_{i-1}^{1}\left[\frac{1}{\Delta x^2}\right] + T_i^{2}\left[\frac{2}{\Delta y^2}\right] = 0$$

→ Top boundary: $j = N_y$

$$\frac{\partial T}{\partial y} = T(x,6) - 60 \quad \Rightarrow \quad \frac{T_i^{N_y+1} - T_i^{N_y-1}}{2\Delta y} = T_i^{N_y} - 60$$

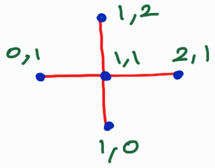
$$\Rightarrow \boxed{T_i^{N_y+1} = T_i^{N_y-1} + (2\Delta y)\left[T_i^{N_y} - 60\right]}$$

Imposing at $j = N_y$:

$$T_{i+1}^{N_y}\left[\frac{1}{\Delta x^2}\right] + T_i^{N_y}\left[\frac{-2}{\Delta x^2} + \frac{-2}{\Delta y^2}\right] + T_{i-1}^{N_y}\left[\frac{1}{\Delta x^2}\right] + T_i^{N_y-1}\left[\frac{2}{\Delta y^2}\right] + T_i^{N_y}\left[\frac{2}{\Delta y}\right] = \frac{120}{\Delta y}$$

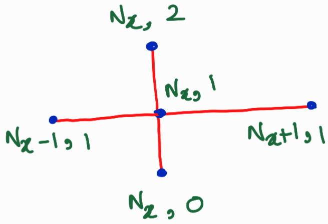
Corners:

→ **Bottom left:** Grids required to solve PDE at $i=1$, $j=1$ are:



We don't know T_0^1. So, we have to drop one of the ICs. $T_1^1 = 30$ is adopted.

→ **Bottom right:**



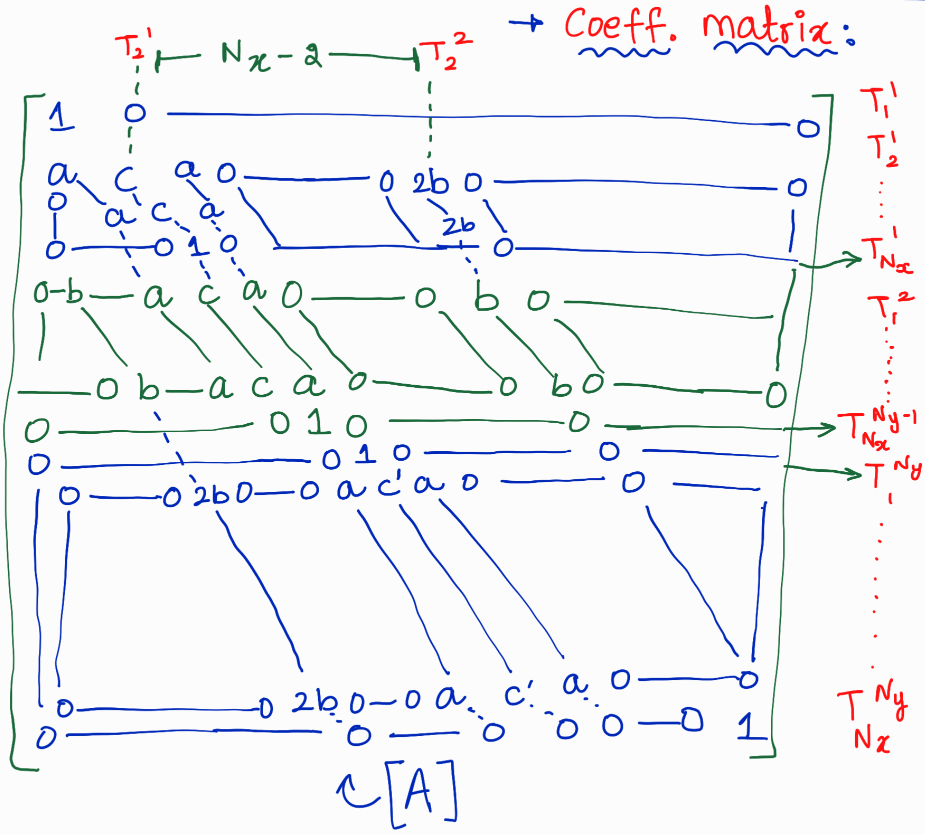
We don't know $T_{N_x+1}^0$.

$T_{N_x}^1 = 60$ is adopted.

For similar reasons, we choose the Type-1 conditions for all corner points.

$$T_1^{N_y} = 30 \quad \& \quad T_{N_x}^{N_y} = 60$$

→ Coeff. matrix:



$a = \frac{1}{\Delta x^2}$; $b = \frac{1}{\Delta y^2}$; $c = \frac{-2}{\Delta x^2} + \frac{-2}{\Delta y^2}$; $d = \frac{2}{\Delta y}$; $c' = c + d$

$$\{X\} = \begin{bmatrix} T_1^1 \\ T_2^1 \\ \vdots \\ T_{N_x}^1 \\ T_1^2 \\ \vdots \\ T_{N_x}^{N_y-1} \\ T_1^{N_y} \\ \vdots \\ T_{N_x}^{N_y} \end{bmatrix} \qquad \{b\} = \begin{bmatrix} 30 \\ 0 \\ \vdots \\ 0 \\ 60 \\ 30 \\ 0 \\ \vdots \\ 0 \\ 60 \\ 30 \\ 120/\Delta y \\ \vdots \\ 120/\Delta y \\ 60 \end{bmatrix}$$

Numerical Solution:

Using *SparseLU* solver from Eigen library, the matrix obtained from the discretization is solved. The matrix is in a special band matrix form, this is taken advantage of in this solver. The obtained solution is plotted as a contour plot below.

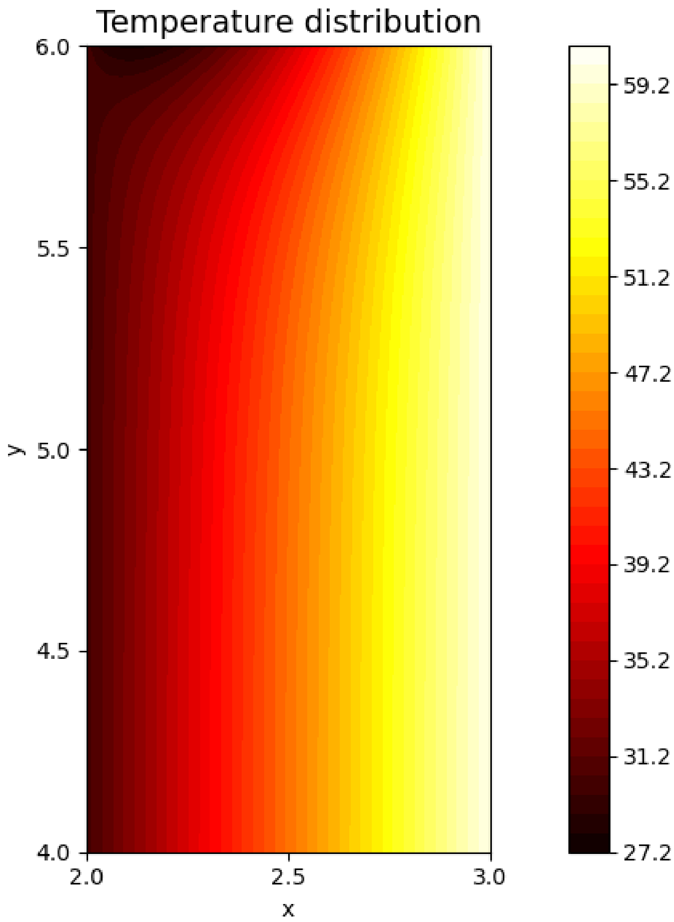


Figure 1: Temperature distribution for the given boundary conditions

This plot represents the steady state solution of temperature distribution on a flat, rectangular plate of given dimensions and boundary conditions. This result can be considered as the equilibrium state of temperature for the given boundary conditions. Points near the bottom face show no gradient in y, adhering to the BC. Large gradients in y can be seen near the top left. As the BC dictates, the gradients in y near the top wall depend on the local temperature itself. The lower the temperature, the stronger the gradient. So, as can be expected from the BC, the top left region depicts the highest thermal gradients in y direction.

Question 2

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \alpha \left[\theta \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2} + (1-\theta) \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} \right]$$

$$u_j^{n+1} - u_j^n = r_d \left[\theta \left(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1} \right) + (1-\theta)\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n \right) \right]$$

Say, $u_j^n = e^{i(kx_j - \omega t_n)} \Rightarrow \begin{matrix} u_{j+1}^n = u_j^n e^{ik\Delta x} \\ u_j^{n+1} = u_j^n e^{-i\omega \Delta t} \end{matrix}$

$$u_j^{n+1} - u_j^n = r_d \theta u_j^{n+1} \left(e^{ik\Delta x} + e^{-ik\Delta x} - 2 \right) + r_d (1-\theta) u_j^n \left(e^{ik\Delta x} + e^{-ik\Delta x} - 2 \right)$$

$$e^{ik\Delta x} + e^{-ik\Delta x} = 2\cos(k\Delta x)$$

$$u_j^{n+1} - u_j^n = u_j^{n+1} r_d \theta (2\cos(k\Delta x) - 2) + u_j^n r_d (1-\theta)(2\cos(k\Delta x) - 2)$$

$$u_j^{n+1}\left(1 - r_d\theta \left[2\cos(k\Delta x) - 2\right]\right) = u_j^n \left(r_d (1-\theta)\left[2\cos(k\Delta x) - 2\right] + 1 \right)$$

$$\frac{u_j^{n+1}}{u_j^n} = \frac{1 + r_d(1-\theta)\left[2\cos(k\Delta x) - 2\right]}{1 - r_d\theta\left[2\cos(k\Delta x) - 2\right]}$$

$$= \frac{1 - 4r_d(1-\theta)\sin^2(k\Delta x/2)}{1 + 4r_d\theta\sin^2(k\Delta x/2)}$$

$G = \frac{u_j^{n+1}}{u_j^n}$; for stability : $|G| \le 1$

$$G = \underbrace{1}_{\text{term 1}} - \underbrace{\frac{4r_d \sin^2(k\Delta x/2)}{1 + 4r_d\theta\sin^2(k\Delta x/2)}}_{\text{term 2}}$$

term 2 has to be greater than zero & lesser than 2.

$$\cancel{2}\,1 \geq \frac{\cancel{4}\,2\, r_d \sin^2(k\Delta x/2)}{1 + 4 r_d \theta \sin^2(k\Delta x/2)} \geq 0$$

This value will always be greater than or equal to zero. It is sufficient to check the left condition.

$$1 + 4 r_d \theta \sin^2(k\Delta x/2) \geq 2 r_d \sin^2\left(\frac{k\Delta x}{2}\right)$$

$$2 r_d \sin^2\left(\frac{k\Delta x}{2}\right) - 4 r_d \theta \sin^2\left(\frac{k\Delta x}{2}\right) \leq 1$$

$$2 r_d \sin^2\left(\frac{k\Delta x}{2}\right)(1 - 2\theta) \leq 1$$

If $(1-2\theta)$ is negative, this inequality will be satisfied for any value of r_d, k & Δx.

$$1 - 2\theta \leq 0 \Rightarrow \boxed{\theta \geq \frac{1}{2}}$$

Question 3 (a) and (b)

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \Rightarrow u_i^{n+1} = r_d\, u_{i+1}^n + (1-2r_d)\, u_i^n + r_d\, u_{i-1}^n$$

$$u_i^{n+1} = u_i^n + \Delta t\, \dot{u}_i^n + \frac{\Delta t^2}{2} \ddot{u}_i^n + \frac{\Delta t^3}{3!} \dddot{u}_i + \frac{\Delta t^4}{4!} \ddddot{u}_i + \dots$$

$$u_i^n = u_i^n$$

$$u_{i+1}^n = u_i^n + \Delta x\, u_i'^{\,n} + \frac{\Delta x^2}{2} (u_i'')^n + \frac{\Delta x^3}{3!} (u_i''')^n + \frac{\Delta x^4}{4!} (u_i'''')^n + \dots$$

$$u_{i-1}^n = u_i^n - \Delta x\, u_i'^{\,n} + \frac{\Delta x^2}{2} (u_i'')^n - \frac{\Delta x^3}{3!} (u_i''')^n + \frac{\Delta x^4}{4!} (u_i'''')^n - \dots$$

$$\left[\Delta t\, \dot{u}_i^n + \frac{\Delta t^2}{2} \ddot{u}_i^n + \dots\right] = r_d \left[\Delta x\, u_i'^{\,n} + \frac{\Delta x^2}{2} u_i''^{\,n} + \frac{\Delta x^3}{3!} u_i'''^{\,n} + \frac{\Delta x^4}{4!} u_i''''^{\,n} + \dots\right]$$
$$+ r_d \left[-\Delta x\, u_i'^{\,n} + \frac{\Delta x^2}{2} u_i''^{\,n} - \frac{\Delta x^3}{3!} u_i'''^{\,n} + \frac{\Delta x^4}{4!} u_i''''^{\,n} \dots\right]$$

$$\cancel{\Delta t} \left[\dot{u}_i^n + \frac{\Delta t}{2} \ddot{u}_i^n + \dots\right] = \frac{\alpha \cancel{\Delta t}}{\cancel{\Delta x^2}} \left[\cancel{\Delta x^2}\, u_i''^{\,n} + \frac{2\Delta x^{\cancel{4}\,2}}{4!} u_i''''^{\,n} + \dots\right]$$

Leading order terms

$$\boxed{\dot{u}_i^n - \alpha\, u_i''^{\,n} = \frac{\alpha \Delta x^2}{12} u_i''''^{\,n} - \frac{\Delta t}{2} \ddot{u}_i^n + \dots}$$

$$\frac{\partial}{\partial t}\left[\frac{\partial u}{\partial t} + \alpha \frac{\partial^2 u}{\partial x^2}\right] = \frac{\partial^2 u}{\partial t^2} + \frac{\partial}{\partial t}\left(\alpha \frac{\partial^2 u}{\partial x^2}\right)$$

$$= \frac{\partial^2 u}{\partial t^2} - \alpha \frac{\partial^2}{\partial x^2}\left[\alpha \frac{\partial^2 u}{\partial x^2}\right]$$

$$= \frac{\partial^2 u}{\partial t^2} - \alpha^2 \frac{\partial^4 u}{\partial x^4} = 0$$

$$\dot{u}_i^n - \alpha\, u_i''^{\,n} = \frac{\alpha \Delta x^2}{12} u_i''''^{\,n} - \frac{\Delta t}{2} \ddot{u}_i^n + \dots$$

$$= \alpha \frac{\Delta x^2}{12} u_i''''^{\,n} - \alpha^2 \frac{\Delta t}{2} u_i''''^{\,n}$$

Leading order terms are of even order derivative.
So, the error is dissipative in nature.

Question 4

(a) Analytical Solution:

$u_t = \alpha u_{xx}$ $\quad \alpha = 0.5$ $\quad u(x,0) = \sin(x) + \sin(4x)$

$u(0,t) = u(L,t)$ $\quad I = [0, 2\pi]$

Separation of variables: $u(x,t) = X(x)\,T(t)$

$\Rightarrow XT' = \alpha T X'' \Rightarrow \boxed{\frac{X''}{X} = \frac{1}{\alpha}\frac{T'}{T} = -k}$

$X'' + kX = 0$

$T' + \alpha k T = 0$

Solve for X using the BC: $u(0,t) = u(L,t)$

It has to hold for all t, so @ $t=0$: $u(0,0) = u(L,0)$

From BC: $u(0,0) = u(L,0) = 0 \Rightarrow X(0) = X(L) = 0$
as $T(t)$ is not 0 everywhere

So, $X'' + kX = 0$; $0 \le x \le L$; $X(0) = X(L) = 0$

for $k > 0$: $k = \lambda^2 \rightarrow X = A\cos(\lambda x) + B\sin(\lambda x)$

$X(0) = 0 \Rightarrow A\cos(0) = 0 \Rightarrow A = 0$

$L = 2\pi$

$X(L) = 0 \Rightarrow B\sin(2\pi\lambda) = 0 \Rightarrow \sin(2\pi\lambda) = 0$
for non-trivial sol.

$\sin(2\pi\lambda) = 0 \Rightarrow 2\pi\lambda = n\pi$ $\quad n = 1, 2, 3, \ldots$

$\boxed{\lambda = \frac{n}{2}}$ $\quad \boxed{k = \frac{n^2}{4}}$

$X_n = B\sin\left(\frac{nx}{2}\right)$

for $k < 0$: $k = -\lambda^2 \Rightarrow X = A\cosh(\lambda x) + B\sinh(\lambda x)$

$X(0) = 0 \Rightarrow A = 0$

$X(L) = 0 \Rightarrow B\sinh(2\pi\lambda) = 0 \Rightarrow B = 0$ as $\lambda \neq 0$

$\therefore X(x) = 0$ (The trivial case) for $k=0$: $X''=0 \Rightarrow X = A+Bx$

$X(0)=0$ & $X(L)=0 \Rightarrow A=B=0$

$X(x) = B\sin\left(\frac{nx}{2}\right) \qquad 0 \le x \le 2\pi$

For $T(t)$: $T' + k\alpha T = 0$

$$\frac{T'}{T} = -k\alpha \Rightarrow T(t) = e^{-k\alpha t}$$

$$k = \frac{n^2}{4} \Rightarrow T(t) = e^{-n^2\alpha t/4}$$

Complete solution: $u_n(x,t) = X_n(x)\,T_n(t)$

$$u_n(x,t) = b_n \sin\left(\frac{nx}{2}\right) e^{-n^2\alpha t/4}$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{nx}{2}\right) e^{-n^2\alpha t/4}$$

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{nx}{2}\right)$$ This has to equal the given IC

$$\Rightarrow \sum_{n=1}^{\infty} b_n \sin\left(\frac{nx}{2}\right) = \sin(x) + \sin(4x)$$

By equating component-wise:

$b_1 = 0$; $b_2 = 1$; $b_3 = b_4 \ldots = b_7 = 0$; $b_8 = 1$; $b_9 = \ldots b_\infty = 0$

Therefore, $\boxed{u(x,t) = \sin(x)\,e^{-\alpha t} + \sin(4x)\,e^{-16\alpha t}}$

(b) CD Schemes:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \qquad \gamma_d = \frac{\alpha \Delta t}{\Delta x^2}$$

4th order:

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_i = \frac{1}{12 \Delta x^2}\left(-u_{i-2} + 16 u_{i-1} - 30 u_i + 16 u_{i+1} - u_{i+2}\right) + \frac{\Delta x^4}{90} \frac{\partial^6 u}{\partial x^6}$$

$$u_i^{n+1} = u_i^n + \frac{\gamma_d}{12}\left(-u_{i-2}^n + 16 u_{i-1}^n - 30 u_i^n + 16 u_{i+1}^n - u_{i+2}^n\right)$$

$$u_i^{n+1} = u_{i-2}^n\left(\frac{-\gamma_d}{12}\right) + u_{i-1}^n\left(\frac{16\gamma_d}{12}\right) + \left(1 - \frac{30\gamma_d}{12}\right)u_i^n + u_{i+1}^n\left(\frac{16\gamma_d}{12}\right) + u_{i+2}^n\left(\frac{-\gamma_d}{12}\right)$$

6th order:

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_i = \frac{-u_{i-3} + 9u_{i-2} - 45 u_{i-1} + 45 u_{i+1} - 9u_{i+2} + u_{i+3}}{60 \Delta x^2}$$

$$u_i^{n+1} = u_{i-3}^n\left[\frac{-\gamma_d}{60}\right] + u_{i-2}^n\left[\frac{9\gamma_d}{60}\right] + u_{i-1}^n\left[\frac{-45\gamma_d}{60}\right] + u_{i+1}^n\left[\frac{45\gamma_d}{60}\right] + u_{i+2}^n\left[\frac{-9\gamma_d}{60}\right] + u_{i+3}^n\left[\frac{\gamma_d}{60}\right]$$

2nd order:

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}$$

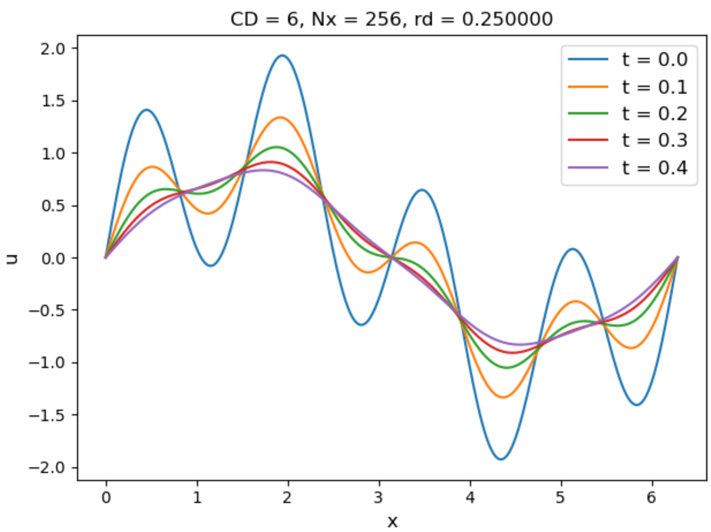
$$u_i^{n+1} = u_{i+1}^n(\gamma_d) + (1 - 2\gamma_d)u_i^n + (\gamma_d)u_{i-1}^n$$

Numerical Solution (c), (d), (e):

Using the schemes described in the earlier part for CD2, CD4 and CD6 and explicit Euler for time-stepping, numerical solutions are obtained for various grid sizes. Periodic boundary condition is used. A sample numerical solution for CD6 with a grid size of 256 is presented below. The plot does not change appreciably for other schemes, grid sizes or values of stability parameter. But the mean error shows the variation. The mean errors obtained at the final time level is tabulated below. The inferences are written based on the plots and tables provided below.

- CD-2 scheme actually displays fourth order accuracy while CD-4 and 6 schemes display only second order accuracy.
- Since the time integration scheme (Explicit Euler) is second order in spatial equivalence, only second order slopes are expected.
- But for CD-2, $r_d = 1/6$ cancels the second order terms and delivers fourth order accuracy for CD-2. This does not occur in CD-4 or 6, so they stay second order accurate.
- Furthermore, if $r_d = 1/4$, the second order term for CD-2 does not completely cancel out. But there is still a subtraction effect. Hence we see CD-2 with lesser absolute error than CD-4 and 6.
- For both r_d values, CD-4 and CD-6 always display same magnitude of error and order of accuracy as their leading order term is $O(\triangle t)$ and the front coefficient depends on r_d the same way for both the schemes. Hence, they almost overlap.

Figure 2: Numerical solution



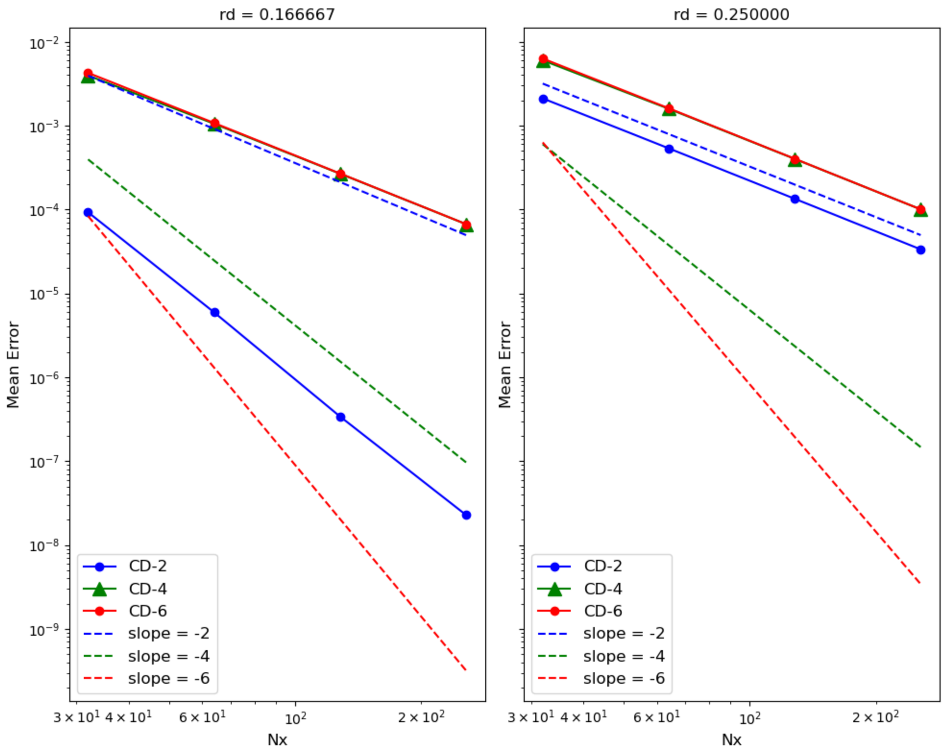


Figure 3: E(N) vs N

Table 1: Mean Error Values for Different CD Schemes with $rd = \frac{1}{4}$

Grid Size	**CD = 2**	**CD = 4**	**CD = 6**
32	2.1153×10^{-3}	6.0187×10^{-3}	6.3129×10^{-3}
64	5.3692×10^{-4}	1.5897×10^{-3}	1.6112×10^{-3}
128	1.3443×10^{-4}	4.0169×10^{-4}	4.0307×10^{-4}
256	3.3509×10^{-5}	1.0044×10^{-4}	1.0053×10^{-4}

Table 2: Mean Error Values for Different CD Schemes with $rd = \frac{1}{6}$

Grid Size	**CD = 2**	**CD = 4**	**CD = 6**
32	9.2813×10^{-5}	3.9592×10^{-3}	4.2659×10^{-3}
64	5.9522×10^{-6}	1.0555×10^{-3}	1.0771×10^{-3}
128	3.4012×10^{-7}	2.6745×10^{-4}	2.6883×10^{-4}
256	2.2793×10^{-8}	6.6944×10^{-5}	6.7030×10^{-5}

Modified equation: $r_d = \frac{\alpha \Delta t}{\Delta x^2}$

$$\dot{u}_i^n - \alpha u_i''^n = \frac{\alpha \Delta x^2}{12} u_i''''^n - \alpha^2 \frac{\Delta t}{2} u_i''''^n + O(\Delta x^4, \Delta t^2)$$

$$= \underbrace{\left[\frac{\alpha \Delta x^2}{12} - \frac{\alpha r_d \Delta x^2}{2}\right]}_{\text{LoT}} u_i''''^n + O(\Delta x^4, \Delta t^2)$$

at $r_d = \frac{1}{6}$, the LoT is zero. So,

for $r_d = \frac{1}{6}$, the order of accuracy is $O(\Delta x^4)$ as $\Delta t \sim \Delta x^2$ for stability.

This is why we observe -4 slope for $r_d = 1/6$ case.

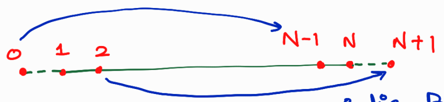
→ No such cancellation for $r_d = \frac{1}{2}$
So the order of accuracy is $O(\Delta x^2)$ as expected.

→ Even for other r_d values, there is still a subtraction effect in the coefficient of leading order term. This is the reason for lower absolute error than higher order schemes where this subtraction effect may not exist.

Question 5

$$u_t = \alpha u_{xx} \Rightarrow \frac{u_i^{n+1} - u_i^n}{\Delta t} = \alpha \left[\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2} \right]$$

$$r_d = \frac{\alpha \Delta t}{\Delta x^2} \; ; \quad u_{i+1}^{n+1}\left[-r_d\right] + u_i^{n+1}\left[1+2r_d\right] + u_{i-1}\left[-r_d\right]$$



At the boundaries: Using periodic BC

$$\begin{aligned} u_1^n &= -r_d u_2^{n+1} + (1+2r_d) u_1^{n+1} - r_d u_0^{n+1} \\ &= -r_d u_2^{n+1} + (1+2r_d) u_1^{n+1} - r_d u_{N-1}^{n+1} \end{aligned}$$

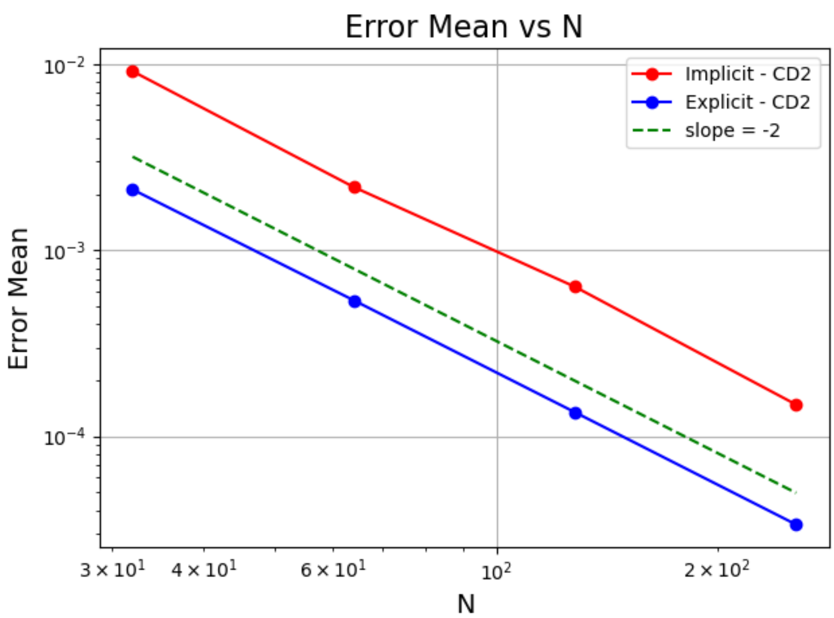
$$\begin{aligned} u_N^n &= -r_d u_{N+1}^{n+1} + (1+2r_d) u_N^{n+1} - r_d u_{N-1}^{n+1} \\ &= -r_d u_2^{n+1} + (1+2r_d) u_N^{n+1} - r_d u_{N-1}^{n+1} \end{aligned}$$

$$A = \begin{bmatrix} 1+2r_d & r_d & 0 & \cdots & 0 & -r_d & 0 \\ -r_d & 1+2r_d & -r_d & 0 & \cdots & \cdots & 0 \\ & \ddots & \ddots & \ddots & & & \\ & & & & & & -r_d \\ 0 & -r_d & 0 & \cdots & & -r_d & 1+2r_d \end{bmatrix}$$

$$x = \begin{Bmatrix} u_1^{n+1} \\ \vdots \\ \vdots \\ u_N^{n+1} \end{Bmatrix} \; ; \quad b = \begin{Bmatrix} u_1^n \\ \vdots \\ \vdots \\ u_N^n \end{Bmatrix} \qquad \boxed{Ax = b}$$

- The implicit and explicit schemes both have the same order of accuracy, as expected, 2.
- Contrary to expectations, the absolute errors of implicit is higher than that of explicit.
- This is explained by the fact that the leading order terms of the modified equation can be grouped for the explicit scheme as both time and spatial derivatives are written at the n^{th} time level.
- But in the implicit scheme, the time derivative is at the n^{th} time level whereas the spatial derivative is at the $(n+1)^{th}$ time level. This leaves no hope for subtractive cancelling.
- Hence, the explicit method displays better accuracy in terms of mean absolute error than the implicit method. This will not be the case for other CD schemes as there is no subtractive cancelling possible in their explicit implementations.

Figure 4: Implicit vs Explicit Euler for $r_d = 1/4$



Question 6

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \alpha \frac{(k+1)\, T_{i+1}^{n-k} - k\, T_{i+1}^{n-k-1} - 2T_i^n + T_{i-1}^n}{\Delta x^2}$$

(a)

Add & subtract αT_{i+1}^n to the numerator:

$$\underbrace{\frac{T_i^{n+1} - T_i^n}{\Delta t}}_{O(\Delta t)} = \alpha \underbrace{\left[\frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta x^2}\right]}_{O(\Delta x^2)} + \alpha \underbrace{\left[\frac{(k+1) T_{i+1}^{n-k} - k T_{i+1}^{n-k-1} - T_{i+1}^n}{\Delta x^2}\right]}_{O(?)}$$

$$T_{i+1}^{n-k-1} = T_{i+1}^{n-k} - \Delta t\, \dot{T}_{i+1}^{n-k} + \frac{\Delta t^2}{2!} \ddot{T}_{i+1}^{n-k} - \dots$$

add & sub. T_{i+1}^{n-k-1}

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \alpha \left[\frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta x^2}\right] + \frac{\alpha (k+1)}{\Delta x^2}\left[T_{i+1}^{n-k} - T_{i+1}^{n-k-1}\right]$$

Multiply & divide by Δt

$$- \frac{\alpha}{\Delta x^2}\left[T_{i+1}^n - T_{i+1}^{n-(k+1)}\right]$$

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \alpha \left[\frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta x^2}\right] + \frac{\alpha (k+1)}{\Delta x^2} \Delta t \left(\frac{T_{i+1}^{n-k} - T_{i+1}^{n-k-1}}{\Delta t}\right)$$

$$- \frac{\alpha \Delta t}{\Delta x^2}(k+1)\left[\frac{T_{i+1}^n - T_{i+1}^{n-(k+1)}}{(k+1)\,\Delta t}\right]$$

$$T_{i+1}^{n-(k+1)} = T_{i+1}^n - (k+1)\Delta t\, \dot{T}_{i+1}^n + (k+1)^2 \frac{\Delta t^2}{2!} \ddot{T}_{i+1}^n - \dots$$

$$\dot{T}_{i+1}^n = \frac{T_{i+1}^n - T_{i+1}^{n-(k+1)}}{(k+1)\,\Delta t} + O[(k+1)\Delta t]$$

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} + O(\Delta t) = \alpha \left[\frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta x^2} + O(\Delta x^2)\right]$$

$$+ \frac{\alpha \Delta t}{\Delta x^2}(k+1)\left[\frac{T_{i+1}^{n-k} - T_{i+1}^{n-k-1}}{\Delta t} + O(\Delta t)\right]$$

$$- \frac{\alpha \Delta t}{\Delta x^2}(k+1)\left[\frac{T_{i+1}^n - T_{i+1}^{n-(k+1)}}{(k+1)\,\Delta t} + O[(k+1)\Delta t]\right]$$

Truncation Error : $O\left(\Delta t,\ \Delta x^2,\ (k+1)\dfrac{\Delta t^2}{\Delta x^2},\ -(k+1)^2\dfrac{\Delta t^2}{\Delta x^2}\right)$

$$\Rightarrow \ TE : O\left(\Delta t,\ \Delta x^2,\ k(k+1)\frac{\Delta t^2}{\Delta x^2}\right)$$

$$(k+1)\frac{\Delta t^2}{\Delta x^2} - (k^2+2k+1)\frac{\Delta t^2}{\Delta x^2} = -k(k+1)\frac{\Delta t^2}{\Delta x^2}$$

(b) For diffusion problems, $r_d = \dfrac{\alpha \Delta t}{\Delta x^2}$.

$\Rightarrow \Delta t \sim \Delta x^2$ for stability.

In this case, the order of accuracy is $O(\Delta x^2)$.

(c) For consistency, $TE \to 0$ as $\Delta t, \Delta x \to 0$ independently. In this case:

$$TE = O\left(\Delta t,\ \Delta x^2,\ \underline{k(k+1)\frac{\Delta t^2}{\Delta x^2}}\right)$$

Because of the third term, the scheme can only be consistent if Δt reduces faster than Δx. So, the scheme is conditionally consistent.