DS 289: Numerical Solution of Differential Equations

Assignment 2

Instructor: Konduri Aditya
Total points: 100

Due date: 24 March 2023

Please follow the below instructions in preparing the solutions:

1. Provide solutions in the same order as questions.
2. All the codes should be in C/C++/Fortran. Use Matlab or Python for plotting graphs.
3. The report should be in a PDF format with the necessary steps, plots, explanations, and discussions.
4. Compile all the solutions, including graphs, into a single PDF file.
5. Create a separate folder for each question that involves a code. Provide the code, makefile, input, and output files, which are used to obtain the solution, in the folder.
6. For submission, create a folder named *DS289_A2_firstname_lastname* (your first and last names) that includes the code folders and the report. Compress the folder in a ZIP format with the same name, and upload it into the MS Teams assignment portal.
7. Code submissions will be checked for plagiarism. Copying from external sources, including the Internet, is prohibited.
8. You may use LLMs and AI tools to generate code with proper acknowledgment, but using them to write the report is not allowed.

Questions

1. Consider a rectangular plate $R = \{(x, y) : 2 \leq x \leq 3, 4 \leq y \leq 6\}$ with the heat conduction equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0.$$

The boundary conditions are: $T(2, y) = 30$, $T(3, y) = 60$, $\partial T/\partial y(x, 4) = 0$, and $\partial T/\partial y(x, 6) = (T(x, 6) - 60)$. (a) Approximate the equation using second order central difference and obtain the algebraic equation. Use a 128×256 grid. (b) Write the structure of the coefficient matrix. (c) Solve the algebraic system using a direct solver. Standard linear solver libraries can be used. (d) Plot contours of the solution with appropriate x-y coordinates. **(15 points)**

2. Show that a two level scheme with $\theta \geq 0.5$ (as described in class) for time derivative and a second-order central difference for space derivative results in an unconditionally stable method for diffusion equation. **(10 points)**

3. (a) Use the explicit Euler and second-order central difference schemes to discretize the diffusion equation ($u_t = \alpha u_{xx}$), and obtain the modified equation. (b) Comment on the (dissipative and dispersive) nature of errors. **(10 points)**

4. (a) Derive the analytical solution for the equation in question 3 (use $\alpha = 0.5$). The initial condition is $u(x, 0) = sin(4x) + sin(x)$. The domain size is 2π with a periodic boundary condition.

   (b) Write the discretized equations for the fourth-order and sixth-order central difference schemes with the explicit Euler scheme. Perform numerical experiments with a constant r_d of $1/4$ and $1/6 (= 0.166667)$, and grid sizes $N = \{32, 64, 128, 256\}$ for the three cases with second, fourth, and sixth-order central difference schemes. Here, r_d is the stability parameter. In each simulation, evolve the solution to an end time $t_{end} = 0.4$.

(c) Use the analytical solution to compute the average of absolute error ($E(N)$) in each simulation at $t = t_{end}$.

(d) In two graphs with a logarithmic scale, plot N vs $E(N)$ for the three schemes for $r_d = 1/4$ and $1/6$, respectively. Report the order of accuracy of the three schemes for the two r_d cases.

(e) Explain the observations in the accuracy plots in both $r_d = 1/4$ and $r_d = 1/6$ case. **(30 points)**

5. (a) Perform numerical experiments with an implicit Euler scheme and second-order central difference scheme. Use the same equation and parameters provided in questions 3 and 4. Solver the linear system at each time step using the Jacobi method (tolerance $= 10^{-4}$; do not use any libraries).

   (b) In a N vs $E(N)$ graph, compare the errors with the respective explicit method for $r_d = 1/4$. Provide an explanation for your observations. **(20 points)**

6. The transient 1D heat conduction problem is modelled as

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2},$$

   where $T(x,t)$ is the temperature, and α (a constant) is the thermal diffusivity. This equation is approximated using

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \alpha \frac{(k+1)T_{i+1}^{n-k} - kT_{i+1}^{n-k-1} - 2T_i^n + T_{i-1}^n}{\Delta x^2}$$

   where $T_i^n = T(x_i, t_n)$, Δx is the grid spacing, Δt is the time step, and k is a constant ($\neq 0$). (a) Find the expression for truncation error. (b) What is the order of accuracy? (c) Is this a consistent finite difference approximation? **(15 points)**

Note: Do not hesitate to contact me if you have any questions or doubts.