DS 289: Numerical Solution of Differential Equations

Assignment 1

Instructor: Konduri Aditya — Due date: 18 Feb 2024
Total points: 100

Please follow the below instructions in preparing the solutions:

1. Provide solutions in the same order as questions.
2. All the codes should be in C/C++/Fortran. Use Matlab or Python for plotting graphs.
3. The report should be in a PDF format with the necessary steps, plots, explanations and discussions.
4. Compile all the solutions, including graphs, into a single PDF file.
5. Create a separate folder for each question that involves a code. Provide the code, makefile, input and output files, which are used to obtain the solution, in the folder.
6. For submission, create a single ZIP file that includes the code folders and the report. Name the ZIP file as *DS289_A1_firstname_lastname.zip* (your first and last names) and upload into MS Teams assignment portal.
7. Code submissions will be checked for plagiarism. Copying from external sources, including the internet, is prohibited.
8. You may use LLMs and AI tools to generate code with proper acknowledgment, but using them to write the report is not allowed.

Questions

1. (a) For a function $u(x)$, derive a second order accurate finite difference approximation to compute d^2u/dx^2 at a point i on a non-uniform grid. The scheme should use a one-sided stencil to the left i. **(10 points)**
   (b) Derive a second order accurate finite difference approximation to compute the cross derivative $\partial^2 u/\partial x \partial y$ of a function $u(x, y)$ on a uniform grid. Obtain the leading order terms in the truncation error.

2. Consider the initial value problem over the interval $t = 0$ to 20 where $y(0) = 1$. **(30 points)**

$$\frac{dy}{dt} = yt^2 - 1.1y \tag{1}$$

   Solve this ODE:

   (a) Analytically and plot the solution.
   (b) Obtain the numerical solution for the step sizes = 0.5, 0.25, 0.125, 0.0625 and 0.03125 using:
      i. Euler's explicit method.
      ii. Second-order Adams-Bashforth.
      iii. Fourth-order RK method.
   (c) Compute the maximum and average errors for each step size using the analytical solution and obtain the error graph demonstrating the order of accuracy for each method. Note: use separate graphs for maximum and average errors to compare all the three methods.
   (d) Discuss the results.

3. The motion of a forced damped spring-mass system (Fig. 1) is described by the following ordinary differential equation: **(20 points)**

$$m\frac{d^2x}{dt^2} + a\left|\frac{dx}{dt}\right|\frac{dx}{dt} + kx = F_o sin(\omega t) \tag{2}$$

where x = displacement from equilibrium position (m), t = time (s), m = 2kg, $a = 5N/(m/s)^2$, and $k = 6N/m$. The initial velocity is zero, and the initial displacement is $x = 1$ m. (a) Solve this equation using RK4 method over the time period $0 \leq t \leq 15s$. (b) Starting with time step $\Delta t = 0.5s$, progressively reduce the time step size by half to show the convergence/grid independence of results. Plot the distance-time and displacement-time curves, and discuss your observations. (c) Find the maximum and minimum values of velocity and displacement, and the corresponding times.

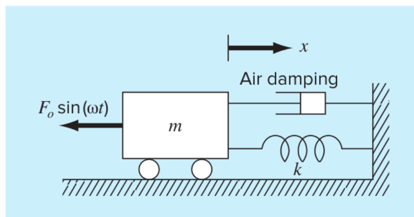

Figure 1

4. Given **(20 points)**

$$\frac{dy}{dx} = -2,00,000y + 2,00,000e^{-x} - e^{-x} \tag{3}$$

(a) Estimate the step-size required to maintain stability using the explicit Euler method. (b) If $y(0) = 0$, use the implicit Euler to obtain a solution from $x = 0$ to 2 using a step size of 0.1. Plot the solution.

5. Solve the ODE using finite difference methods that describe the temperature distribution in a circular rod with internal heat source S **(20 points)**

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} + S = 0 \tag{4}$$

over the range $0 \leq r \leq 1$, with the boundary conditions $T(r = 1) = 1$ and dT/dr at $(r = 0) = 0$.

(a) Solve this equation using the second order central difference scheme and find temperature distribution along the radial direction for S = 100, 500, 1000, 1500. Use a grid size of 1024 points.

(b) Plot the temperature distribution for each case and discuss the results.

(c) Compare these plots for different S and find maximum S such that the peak temperature in the domain does not exceed 100.