DS 298 Numerical Solution of Differential Equations
Assignment 3

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- ChatGPT was used in the code and makefile for all questions.
- GitHub Copilot was used in the code for all questions.

Question 1

Part (a):

The non-linear advection equation allows for strong discontinuities to be present in the solution. Since the signal is advected based on the local direction of the signal, an upwind scheme is expected to give better stability than a central scheme. Central schemes struggle to resolve strong gradients as the numerical values will be very different on either side of the node in the neighbourhood of a strong gradient. But upwind schemes preferentially collect information from the direction of the solution, so they are more stable. With the addition of artificial viscosity, the central scheme is expected to behave better than without. In this exercise, the comparison of upwind and central scheme with artificial viscosity is our aim.

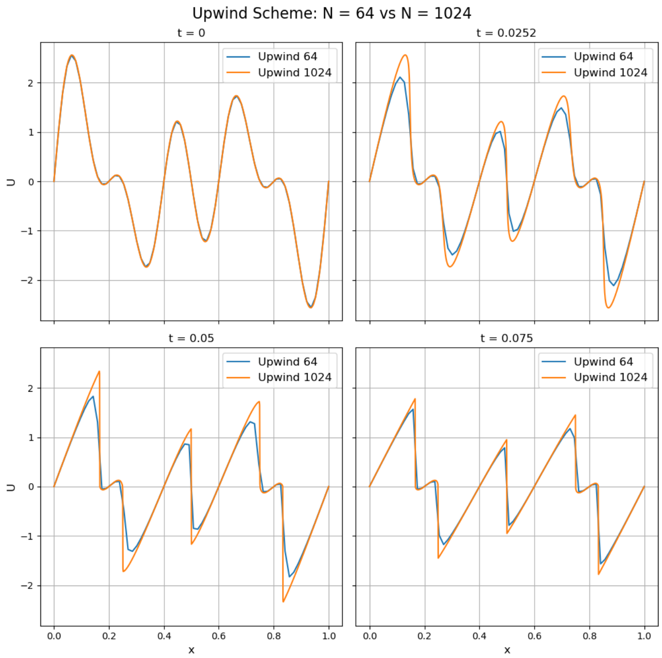

Figure 1: Part (a)

From the figure, it is clearly seen that as t approaches the end time, the solution becomes strongly discontinuous. That is, there are strong shocks present in the solution. As time evolves, strong shocks are established.

- The shock behaviour is better captured by higher resolution upwind scheme in comparison with lower resolution. This is evident from the sharpness of the peaks.

- It can also be seen that the lower resolution scheme results in solution magnitudes lower than the higher resolution case. This effect gets more and more prominent as the shock strength becomes stronger. This is due to the ability to resolve sharp gradients more accurately with higher number of points.

Part (b):

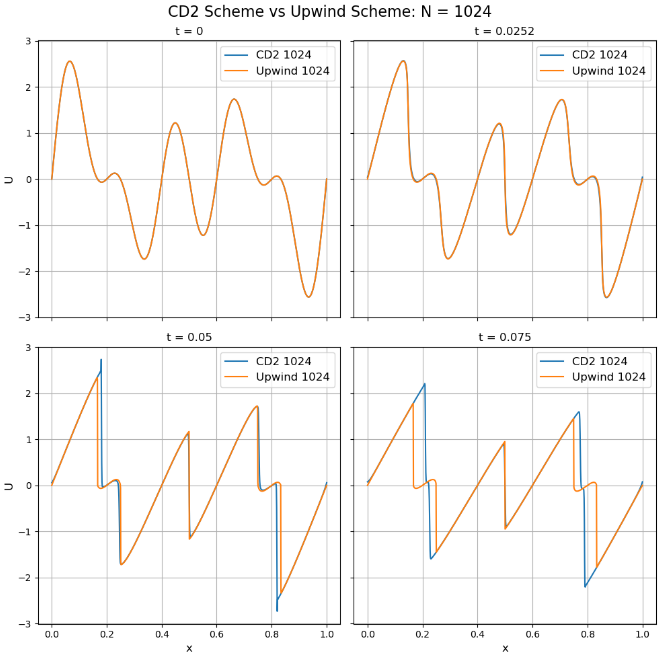

Figure 2: Part (b)

As can be seen, the CD scheme with viscosity term is able to match the solution behaviour at least as long as it is smooth. But as strong discontinuities are introduced, the CD solution incurs high errors. At the end time, the CD solution deviates largely from the upwind solution. It can also be seen that the peak gradient locations incur more error than other locations in the domain for the CD solution. This is also consistent with the previously given explanation. The prevalence of shocks and the nature of the solution is also discussed in the part (a) of the question.

Question 2

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \qquad \text{CD2}: \frac{u_{i+1} - u_{i-1}}{2\Delta x} = \frac{\partial u}{\partial x}$$

$$\frac{\partial u}{\partial t} = -c\left[\frac{u_{i+1}^{n} - u_{i-1}^{n}}{2\Delta x}\right] \; ; \; \frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} = \frac{\partial u}{\partial t}$$

$$\boxed{u_j^{n+1} = u_j^{n-1} - \left[\frac{c\Delta t}{\Delta x}\right]\left\{u_{j+1}^{n} - u_{j-1}^{n}\right\}} \qquad u_j^n = e^{i(kx_j + \omega t_n)}$$

$$u_j^{n+1} = e^{i(kx_j + \omega t_{n+1})} \; ; \; u_j^{n-1} = e^{i(kx_j + \omega t_{n-1})}$$

$$u_{j+1}^{n} = e^{i(kx_{j+1} + \omega t_n)} \qquad G \equiv \frac{u_j^{n+1}}{u_j^n} = \frac{u_j^n}{u_j^{n-1}}$$

$$\frac{u_j^{n+1}}{u_j^n} = \frac{u_j^{n-1}}{u_j^n} - \gamma_c\left\{\frac{u_{j+1}^n}{u_j^n} - \frac{u_{j-1}^n}{u_j^n}\right\}$$

$$G - \frac{1}{G} = -\gamma_c\left\{e^{ik\Delta x} - e^{-ik\Delta x}\right\} = -i2\gamma_c \sin(k\Delta x)$$

$$G^2 + G\left[i2\gamma_c \sin(k\Delta x)\right] - 1 = 0$$

$$G = \frac{-2i\gamma_c \sin(k\Delta x) \pm \sqrt{-4\gamma_c^2 \sin^2(k\Delta x) + 4}}{2}$$

$$G = -i\gamma_c \sin(k\Delta x) \pm \sqrt{1 - \gamma_c^2 \sin^2(k\Delta x)}$$

$$|G| \le 1 \Rightarrow G\cdot\bar{G} \le 1 \longrightarrow \bar{G} \text{ is the conjugate of } G.$$

$$G\cdot\bar{G} = Re(G)^2 + Im(G)^2$$

$\gamma_c > 1 \Rightarrow$ No real part.

$\sqrt{1 - \gamma_c^2 \sin^2(k\Delta x)}$ can also be imaginary for some k & Δx.

For $\sin^2(k\Delta x) = 1$: $\sqrt{1 - \gamma_c^2} = i\sqrt{\underbrace{\gamma_c^2 - 1}_{\text{positive}}}$

$$G = i\left[-\gamma_c + \sqrt{\gamma_c^2 - 1}\right]$$

$$|G| = \left(-\gamma_c + \sqrt{\gamma_c^2 - 1}\right)^2 = \gamma_c^2 + \gamma_c^2 - 1 - 2\gamma_c\sqrt{\gamma_c^2 - 1}$$

$$= 2\gamma_c\left[\gamma_c - \sqrt{\gamma_c^2 - 1}\right] - 1$$

$\gamma_c > 1 \Rightarrow 2\gamma_c\left[\gamma_c - \sqrt{\gamma_c^2-1}\right] > 2$

$\Rightarrow \boxed{|G| > 1}$ So, $\gamma_c > 1$ is unstable.

$\gamma_c \leq 1$: $\sqrt{1 - \gamma_c^2 \sin^2(k\Delta x)}$ is real.

$$|G| = 1 - \gamma_c^2 \sin^2(k\Delta x) + \gamma_c^2 \sin^2(k\Delta x)$$

$$= 1$$

$|G| = 1$, so leapfrog method is marginally stable for $|\gamma_c| \leq 1$.

Modified Equation:

$\dot{u} + c u' = 0$

$$u_j^{n+1} = u_j^{n-1} - \left[\frac{c\Delta t}{\Delta x}\right]\left\{u_{j+1}^n - u_{j-1}^n\right\} \quad \text{— (A)}$$

$$u_j^{n+1} = u_j^n + \dot{u}_j^n \Delta t + \ddot{u}_j^n \frac{\Delta t^2}{2} + \dddot{u}_j^n \frac{\Delta t^3}{6} + \dots$$

$$u_j^{n-1} = u_j^n - \dot{u}_j^n \Delta t + \ddot{u}_j^n \frac{\Delta t^2}{2} - \dddot{u}_j^n \frac{\Delta t^3}{6} + \dots$$

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$$u_j^{n+1} - u_j^{n-1} = 2\dot{u}_j^n \Delta t + 2\dddot{u}_j^n \frac{\Delta t^3}{6} + \dots$$

$$u_{j+1}^n = u_j^n + u_j'^n \Delta x + u_j''^n \frac{\Delta x^2}{2} + u_j'''^n \frac{\Delta x^3}{6} + \dots$$

$$u_{j-1}^n = u_j^n - u_j'^n \Delta x + u_j''^n \frac{\Delta x^2}{2} - u_j'''^n \frac{\Delta x^3}{6} + \dots$$

$$- \quad - \quad + \quad - \quad + \quad -$$

$$u_{j+1}^n - u_{j-1}^n = 2u_j'^n \Delta x + 2u_j'''^n \frac{\Delta x^3}{6} + \dots$$

Substitute in (A):

$$\cancel{2}\dot{u}_j^n \cancel{\Delta t} + \cancel{2}\dddot{u}_j^n \frac{\Delta t^{\cancel{3}\,2}}{6} + \dots = -\frac{c\cancel{\Delta t}}{\cancel{\Delta x}}\left\{\cancel{2}u_j'^n \cancel{\Delta x} + \cancel{2}u_j'''^n \frac{\Delta x^{\cancel{3}\,2}}{6} + \dots\right\}$$

Ignore HOT :

$$\dot{u}_j^n + c\,u_j'^n = -\dddot{u}_j^n \frac{\Delta t^2}{6} - c\,u_j'''^n \frac{\Delta x^2}{6}$$

$$\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x} \Rightarrow \frac{\partial^2 u}{\partial t^2} = -c\frac{\partial}{\partial x}\left(-c\frac{\partial u}{\partial x}\right)$$

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \Rightarrow \frac{\partial^3 u}{\partial t^3} = c^2\frac{\partial^2}{\partial x^2}\left(-c\frac{\partial u}{\partial x}\right)$$

$$\Rightarrow \boxed{\dot{u}_j^n + c\,u_j'^n = c^3 u_j'''^n \frac{\Delta t^2}{6} - c\,u_j'''^n \frac{\Delta x^2}{6}}$$

$$= \frac{c}{6} u_j'''^n \left\{ c^2\Delta t^2 - \Delta x^2 \right\}$$

$$= \frac{c\Delta x}{6} u_j'''^n \left\{ r_c^2 - 1 \right\}$$

→ Modified equation contains 3rd derivative in leading order term. So, the nature is dispersive.

Question 3

(a) Beam :

$$\frac{d^2}{dx^2}\left[b\,\frac{d^2w}{dx^2}\right] + kw = f \qquad w = b\frac{d^2w}{dx^2} = 0 \quad @\ x = 0, L$$

$$w_h^e = \sum_{1}^{N} c_j^e\,\phi_j^e(x) \qquad b, f \longrightarrow f(x)$$

$$k \longrightarrow \text{constant}$$

$$R^e = \frac{d^2}{dx^2}\left[b\,\frac{d^2 w_h^e}{dx^2}\right] + k w_h^e - f$$

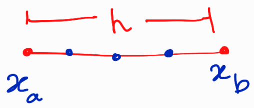
Galerkin:

$$\langle R^e, \phi_i^e \rangle = 0$$

As such, the basis functions need to be at least four times differentiable (not trivially zero).

→ Writing as weak form will reduce the constraint to three from four.

For element 'e' :



$$\int_{x_a}^{x_b} \left(\frac{d^2}{dx^2}\left[b\,\frac{d^2 w_h^e}{dx^2}\right] + k w_h^e - f \right) \phi_i^e\,dx = 0$$

$$\underbrace{\int_{x_a}^{x_b} \frac{d^2}{dx^2}\left[b\,\frac{d^2 w_h^e}{dx^2}\right] \phi_i^e\,dx}_{I} + k\int_{x_a}^{x_b} w_h^e\,\phi_i^e\,dx = \int_{x_a}^{x_b} f\,\phi_i^e\,dx \quad \text{Eq. (A)}$$

$$I = \int_{x_a}^{x_b} \frac{d^2}{dx^2}\left(b\,\frac{d^2}{dx^2} w_h^e\right) \phi_i^e\,dx \qquad u = \phi_i^e \qquad v' = \frac{d}{dx}\left\{\frac{d}{dx}\left(b w_h^{e''}\right)\right\}$$

$$\int_{x_a}^{x_b} u v'\,dx = \left[uv\right]_{x_a}^{x_b} - \int_{x_a}^{x_b} u' v\,dx$$

$$I = \left[\phi_i^e\,\frac{d}{dx}\left(b w_h^{e''}\right)\right]_{x_a}^{x_b} - \underbrace{\int_{x_a}^{x_b} \frac{d\phi_i^e}{dx}\,\frac{d}{dx}\left(b w_h^{e''}\right) dx}_{I_1}$$

$$I_1 = \int_{x_a}^{x_b} \underset{u}{\phi_i^{e\prime}} \underset{v'}{\frac{d}{dx}(b\omega_h^{e\prime\prime})}\, dx \qquad u = \phi_i^{e\prime}, \quad v' = \frac{d}{dx}(b\omega_h^{e\prime\prime})$$

$$I_1 = \left[\phi_i^{e\prime}\, b\omega_h^{e\prime\prime}\right]_{x_a}^{x_b} - \int_{x_a}^{x_b} \frac{d^2\phi_i^e}{dx^2}(b\omega_h^{e\prime\prime})\, dx$$

$$I = \left[\phi_i^e \frac{d}{dx}(b\omega_h^{e\prime\prime})\right]_{x_a}^{x_b} - \left[\phi_i^{e\prime}\, b\omega_h^{e\prime\prime}\right]_{x_a}^{x_b} + \int_{x_a}^{x_b} b\,\phi_i^{e\prime\prime}\,\omega_h^{e\prime\prime}\, dx$$

Eqn. (A):

$$I + k\int_{x_a}^{x_b} \omega_h^e \phi_i^e\, dx = \int_{x_a}^{x_b} f\,\phi_i^e\, dx$$

(From BC)

If $x_a = 0$ and $x_b = L$: $\left[\phi_i^{e\prime}\, b\, \omega_h^{e\prime\prime}\right]_{x_a}^{x_b} = 0$

The term $\left[\phi_i^e \frac{d}{dx}(b\omega_h^{e\prime\prime})\right]_{x_a}^{x_b}$ represents the flux of the quantity $(b\omega_h^{e\prime\prime})$ at both the boundary points. But we know $w = 0$ at both of these points. So, we can replace the eqns at these nodes to $w = 0$. In interior nodes & elements, the net flux will be zero. So, this term is ignored.

Weak form:

$$\int_{x_a}^{x_b} b\,\phi_i^{e\prime\prime}\,\omega_h^{e\prime\prime}\, dx + k\int_{x_a}^{x_b} \omega_h^e \phi_i^e\, dx = \int_{x_a}^{x_b} f\,\phi_i^e\, dx$$

$i = 1$ to N

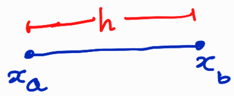
(b) Non-linear Equation:

$$-\frac{d}{dx}\left[u\frac{du}{dx}\right] + f = 0$$

$$\left.\frac{du}{dx}\right|_{x=0} = 0 \;;\; u(1) = \sqrt{2}$$

$$x \in [0,1]$$

As such, the basis function needs to be twice differentiable. But, weak form will relax it to one.



Galerkin:

$$\langle R^e, \phi_i^e \rangle = 0$$

For element 'e':

$$u_h^e = \sum_{j=1}^{N} c_j^e \phi_j^e \;;\; R^e = -\frac{d}{dx}\left(u_h^e \frac{du_h^e}{dx}\right) + f$$

$$\langle R^e, \phi_i^e \rangle = 0 \Rightarrow \int_{x_a}^{x_b} \left[-\frac{d}{dx}\left(u_h^e \frac{d}{dx} u_h^e\right) + f\right] \phi_i^e \, dx = 0$$

$$\underbrace{\int_{x_a}^{x_b} \frac{d}{dx}\overbrace{\left(u_h^e \frac{du_h^e}{dx}\right)}^{v'} \overset{u}{\phi_i^e} \, dx}_{I} = \int_{x_a}^{x_b} f \phi_i^e \, dx$$

$$I = [uv]_{x_a}^{x_b} - \int_{x_a}^{x_b} u'v \, dx = \left[\phi_i^e u_h^e \frac{du_h^e}{dx}\right]_{x_a}^{x_b} - \int_{x_a}^{x_b} \phi_i^{e\prime} u_h^e \frac{du_h^e}{dx} dx$$

$$\Rightarrow \left[\phi_i^e u_h^e \frac{du_h^e}{dx}\right]_{x_a}^{x_b} - \int_{x_a}^{x_b} \phi_i^{e\prime} u_h^e \frac{du_h^e}{dx} dx - \int_{x_a}^{x_b} f \phi_i^e dx = 0$$

If $x_a = 0$ and $x_b = 1$: $\left(\phi_i^e u_h^e \frac{du_h^e}{dx}\right)_{x = x_a} = 0$ (from BC)

At $x = x_b = 1$, the value of u itself is known through the BC. Hence, the eqn at node x_b can be replaced with the BC itself. Otherwise, knowledge of $\frac{du_h^e}{dx}$ at $x = x_b$ is required to evaluate this form.

Weak Form:

$$\int_{x_a}^{x_b} \phi_i^{e\prime} u_h^e \frac{du_h^e}{dx} dx + \int_{x_a}^{x_b} \phi_i^e f \, dx = 0$$

$i = 1$ to N

Question 4

$$A_c E \frac{d^2u}{dx^2} = P(x) \qquad 0 \le x \le 10$$

$$u(0) = u(10) = 0$$

$P(x) = 100$ N/m ; $A_c = 0.1\, m^2$; $E = 200 \times 10^9$ N/m^2

$$R^e = A_c E \frac{d^2 u_h^e}{dx^2} - P$$

$$u_h^e = \sum_{j=1}^{N} c_j^e \phi_j^e$$

Galerkin : $w_i = \phi_i^e$

$$\langle R^e, w_i \rangle = 0 \Rightarrow \int_{x_e}^{x_{e+1}} \left(A_c E \frac{d^2 u_h^e}{dx^2} - P \right) \phi_i^e \, dx = 0$$

$$\underbrace{\int_{x_e}^{x_{e+1}} A_c E \underset{v'}{\frac{d^2 u_h^e}{dx^2}} \underset{u}{\phi_i^e} \, dx}_{I} - \int_{x_e}^{x_{e+1}} P \phi_i^e \, dx = 0$$

Integral by parts : $I = A_c E \left\{ \left[\phi_i^e \frac{du_h^e}{dx} \right]_{x_e}^{x_{e+1}} - \int_{x_e}^{x_{e+1}} \frac{du_h^e}{dx} \frac{d\phi_i^e}{dx} \, dx \right\}$

→ We have BC as $u(0) = u(L) = 0$. For the interior nodes, fluxes will cancel out by the neighbouring elements. And the boundary node equations can be replaced with the BC itself. So, $\left[\phi_i^e \frac{du_h^e}{dx} \right]_{x_e}^{x_{e+1}}$ can be ignored.

Weak form :

$$-A_c E \int_{x_e}^{x_{e+1}} \phi_i^{e\,\prime} u_h^{e\,\prime} \, dx - \int_{x_e}^{x_{e+1}} P \phi_i^e \, dx \quad \text{— (A)}$$

→ Choosing basis functions as first degree Lagrange polynomial.

$$\phi_1^e = \frac{x - x_{e+1}}{x_e - x_{e+1}} \quad ; \quad \phi_2^e = \frac{x - x_e}{x_{e+1} - x_e}$$

$$\phi_1^{e\,\prime} = \frac{1}{x_e - x_{e+1}} = \frac{-1}{h} \quad ; \quad \phi_2^{e\,\prime} = 1/h$$

$$(A) \Rightarrow \quad A_c E \int_{x_e}^{x_{e+1}} \phi_i^{e\,\prime} u_h^{e\,\prime}\, dx + \int_{x_e}^{x_{e+1}} P \phi_i^e \, dx = 0$$

$$A_c E \sum_{j=1}^{2} c_j^e \int_{x_e}^{x_{e+1}} \phi_i^{e\,\prime} \phi_j^{e\,\prime} dx + \int_{x_e}^{x_{e+1}} P \phi_i^e \, dx = 0$$

$$K_{ij}^e = \int_{x_e}^{x_{e+1}} \phi_i^{e\,\prime} \phi_j^{e\,\prime} dx \Rightarrow \sum_{j=1}^{2} K_{ij}^e c_j^e + \int_{x_e}^{x_{e+1}} \frac{P}{A_c E} \phi_i^e dx = 0$$

$$K_{ij}^e = \begin{cases} 1/h & \text{if } i = j \\ -1/h & \text{if } i \neq j \end{cases} \qquad i, j \in \{1, 2\}$$

$$[K^e]\{c^e\} + \{P^e\} = \{0\}$$

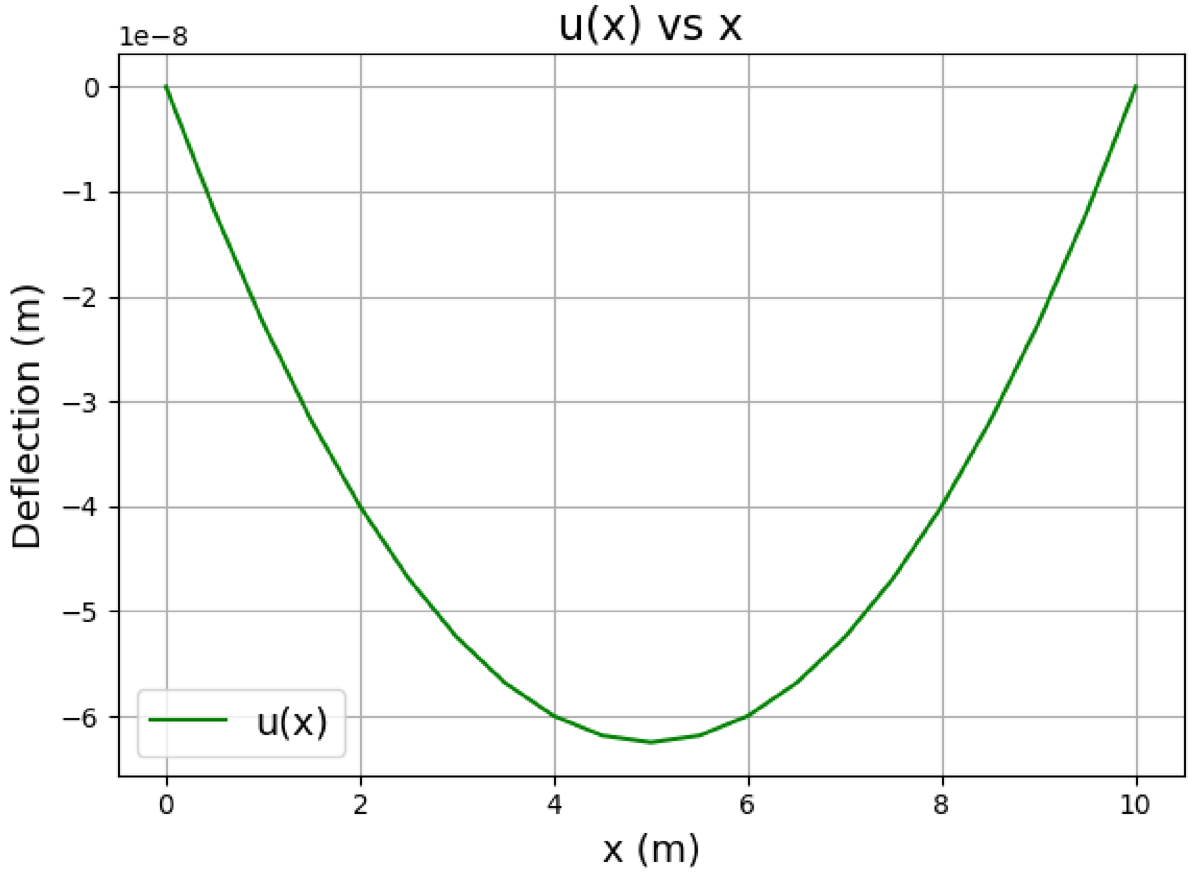
$$\{P^e\} = \frac{1}{AE} \int_{x_e}^{x_{e+1}} P \phi_i^e \, dx \Rightarrow P_1^e = \frac{P}{AE} \int_{x_e}^{x_{e+1}} \phi_1^e \, dx = \frac{Ph}{2 A_c E}$$

$$\Rightarrow \{P^e\} = \frac{Ph}{2A_c E} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Global matrix: N_e elements (with changed BC eqns at 0 & L)

$$\frac{1}{h} \begin{bmatrix} h & 0 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & \cdots & \cdots & 0 \\ 0 & -1 & 2 & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & & -1 & 2 & -1 \\ 0 & 0 & \cdots & 0 & 0 & h \end{bmatrix} \begin{Bmatrix} c_1 \\ \vdots \\ \vdots \\ \vdots \\ c_{N_e+1} \end{Bmatrix} + \frac{Ph}{2A_c E} \begin{Bmatrix} 0 \\ 2 \\ 2 \\ \vdots \\ \vdots \\ 2 \\ 2 \\ 0 \end{Bmatrix} = \{0\}$$

$$(N_e+1) \times (N_e+1) \qquad (N_e+1) \times 1 \qquad (N_e+1) \times 1$$



As expected intuitively, the beam follows a parabolic profile for uniformly distributed loading of 100 N/m. The endpoints have zero deflection, as expected. The mid-point of the beam takes the maximum amount of deflection. This is also expected from physical intuition. It is important to note that the order of deflection is $10^{-8}\,m$. This results from the stiffness of the elements.