Q 1) minimize $\frac{e^{x_1}}{e^{x_2}}$, subject to: $x_1 \geq 0 \rightarrow -x_1 \leq 0$

$e^{x_1} + e^{x_2} \leq 20$

std. Form

$$\frac{e^{x_1}}{e^{x_2}} = e^{x_1 - x_2}$$

$$g_1(\underset{\sim}{x}) = -x_1$$

$$g_2(\underset{\sim}{x}) = e^{x_1} + e^{x_2} - 20$$

$$\tilde{L}(\underset{\sim}{x}, \mu_i) = e^{x_1} \cdot \frac{1}{e^{x_2}} + \mu_1 g_1(\underset{\sim}{x}) + \mu_2 g_2(\underset{\sim}{x})$$

$\mu_1, \mu_2 > 0$

$$\nabla_{\underset{\sim}{x}, \mu_i} \left[\tilde{L}(\underset{\sim}{x}, \mu_i) \right] \Big|_{\underset{\sim}{x}^*, \mu_i^*} = 0$$

$$\frac{\partial \tilde{L}}{\partial x_1} = e^{(x_1 - x_2)} - \mu_1 + \mu_2 e^{x_1} = 0 \quad \Big| \quad \frac{\partial \hat{L}}{\partial \mu_1} = -x_1 = 0$$

$$\frac{\partial \hat{L}}{\partial x_2} = -e^{(x_1 - x_2)} + \mu_2 e^{x_2} = 0 \quad \Big| \quad \frac{\partial \hat{L}}{\partial \mu_2} = e^{x_1} + e^{x_2} - 20 = 0$$

(A) $e^{(x_1 - x_2)} - \mu_1 + \mu_2 e^{x_1} = 0$

(B) $-e^{(x_1 - x_2)} + \mu_1(0) + \mu_2 e^{x_2} = 0$

$e^{x_1} + e^{x_2} = 20 \Rightarrow e^{x_2} = 19$

$-x_1 = 0 \Rightarrow \boxed{x_1 = 0}$

$\boxed{x_2 = \ln(19)}$

(B) $\Rightarrow -e^{-\ln(19)} + \mu_2 e^{\ln(19)} = 0$

$-e^{\ln\left(\frac{1}{19}\right)} + \mu_2(19) = 0 \Rightarrow -\frac{1}{19} + 19\mu_2 = 0$

$$\boxed{\mu_2 = \frac{1}{19^2}}$$

(A) $\Rightarrow e^{(x_1 - x_2)} - \mu_1 + \mu_2 e^{x_1} = 0$

$\frac{1}{19} - \mu_1 + \frac{1}{19^2} = 0$ $\quad \boxed{\mu_1 = \frac{1}{19} + \frac{1}{19^2}}$

Q2) (a) minimize $x_1^2 + x_2^2$ subject to $-2x_1 - x_2 + 10 \le 0$

$x_2 \ge 0$

$\hookrightarrow -x_2 \le 0$

Std. form

$$g_1(\underset{\sim}{x}) = -2x_1 - x_2 + 10$$

$$g_2(\underset{\sim}{x}) = -x_2$$

$$\ell(\underset{\sim}{x}) = x_1^2 + x_2^2$$

$$\tilde{\ell}(\underset{\sim}{x}) = \ell(\underset{\sim}{x}) + \mu_1 g_1(\underset{\sim}{x}) + \mu_2 g_2(\underset{\sim}{x})$$

$$\tilde{\ell}(\underset{\sim}{x}) = x_1^2 + x_2^2 - \mu_2 x_2 + \mu_1(-2x_1 - x_2 + 10)$$

$$\nabla_{\underset{\sim}{x}, \mu_i} \tilde{\ell}(\underset{\sim}{x}, \mu_i) \Big|_{\underset{\sim}{x}^*, \mu_i^*} = 0$$

$$\frac{\partial L}{\partial x_1} = 2x_1 - 2\mu_1 = 0 \qquad \frac{\partial L}{\partial \mu_1} = -2x_1 - x_2 + 10 = 0$$

$$\frac{\partial L}{\partial x_2} = 2x_2 - \mu_2 - \mu_1 = 0 \qquad \frac{\partial L}{\partial \mu_2} = -x_2 = 0$$

$$\boxed{x_2 = 0} \Rightarrow \boxed{x_1 = 5} \Rightarrow \boxed{\mu_1 = 5} \Rightarrow \boxed{\mu_2 = -5}$$

(c) Dual given primal:

$$\tilde{\ell}(\underset{\sim}{x}, \mu_i) = x_1^2 + x_2^2 - \mu_2 x_2 + \mu_1(-2x_1 - x_2 + 10)$$

$$d(\mu_i) = \min_{\underset{\sim}{x}} \tilde{\ell}(\underset{\sim}{x}, \mu_i)$$

$$= \min_{\underset{\sim}{x}} \left\{ \underset{\sim}{x}^T \underset{\sim}{x} - \mu_2 x_2 + \mu_1(-2x_1 - x_2 + 10) \right\}$$

$$d(\mu_i) = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \mu_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \mu_1 \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

$$d^* = \max_{\mu_i} \{ d(\mu_i) \}$$