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22MA101
MATRICES AND CALCULUS

Department: **MATHEMATICS**
Batch/Year:**2023 – 2024 / I**
Created by: **Department of Mathematics**
Date: **09.09.2023**

Table of Contents

Course Objectives

SI.NO	COURSE OBJECTIVES
The syllabus is designed to:	
1	Explain the concepts of matrix algebra techniques needed for practical applications.
2	Determine the curvature of the curves.
3	Illustrate the simple applications of multivariable calculus and vector calculus.
4	Elaborate the concept and application of multiple integrals.

Prerequisites

Subject Code: 22MA101
Subject Name: MATRICES AND CALCULUS
Prerequisites
To learn engineering mathematics one has to be strong in mathematics including the basic concepts of algebra, trigonometry, geometry and precalculus.
Definition of differentiation and integration from the first principle and how to use some properties and rules to find the derivatives and integration of more complicated functions.

Syllabus

UNIT I	**MATRICES**	15
Eigenvalues and Eigenvectors of a real matrix – Properties of Eigenvalues and Eigenvectors – Statement and applications of Cayley-Hamilton Theorem – Diagonalization of matrices by orthogonal transformation – Reduction of a quadratic form to canonical form by orthogonal transformation – Nature of quadratic forms. **Theory : 9** Experiments using SCILAB: 1.Introduction to SCILAB through matrices and general syntax. 2.Finding the Eigenvalues and Eigenvectors. 3.Plotting the graph of a quadratic form. **laboratory: 6**		
UNIT II	**SINGLE VARIABLE CALCULUS**	15
Curvature in Cartesian and Polar Co-ordinates – Centre and radius of curvature – Circle of curvature–Evolutes. **Theory: 9** Experiments using SCILAB: 1.Evaluating the radius of curvature. 2.Finding the coordinates of the center of curvature. 3.Tracing of Curves. **Laboratory: 6**		
UNIT III	**MULTIVARIABLE CALCULUS**	15
Partial derivatives (excluding Euler's theorem) – Total derivative – Differentiation of implicit functions – Jacobian and properties – Taylor's series for functions of two variables – Maxima and minima of functions of two variables. **Theory: 9** Experiments using SCILAB: 1.Evaluating the maxima of functions of several variables. 2.Evaluating the minima of functions of several variables. 3.Evaluation of Jacobians. **Laboratory: 6**		
UNIT IV	**MULTIPLE INTEGRALS**	15
Double integrals – Change of order of integration – Area enclosed by plane curves – Triple integrals – Volume of solids. **Theory: 9** Experiments using SCILAB 1.Evaluating area under a curve. 2.Evaluating area using double integral.. 3.Evaluation of volume by integrals. **Laboratory: 6**		
UNIT V	**VECTOR CALCULUS**	15
Gradient, divergence and curl (excluding vector identities) – Directional derivative – Irrotational and Solenoidal vector fields – Vector integration – Green's theorem in a plane and Gauss divergence theorem (Statement only) – Simple applications involving cubes and rectangular parallelopipeds. **Theory: 9** Experiments using SCILAB: 1.Evaluating gradient. 2.Evaluating directional derivative. 3.Evaluating divergent and curl. **Laboratory: 6**		

Course Outcomes

CO's	Course Outcomes	Highest Cognitive Level
After the successful completion of the course, the student will be able to:		
CO1	Use the matrix algebra methods to diagonalize the matrix.	K1, K2
CO2	Determine the evolute of the curve.	K3
CO3	Apply differential calculus ideas on the function of several variables.	K1,K3
CO4	Evaluate the area and volume by applying the concept of multiple integration.	K3
CO5	Utilize the concept of vector calculus in evaluating integrals.	K3

CO-PO/CO-PSO Mapping

CO PO	PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	PO10	PO11	PO12
CO1	3	1	1	1	-	-	-	-	-	-	-	1
CO2	3	2	-	1	-	-	-	-	-	-	-	1
CO3	3	2	1	-	-	-	-	-	-	-	-	1
CO4	3	2	-	1	-	-	-	-	-	-	-	1
CO5	3	2	1	-	-	-	-	-	-	-	-	1

CO/PSO	PSO1	PSO2	PSO3
CO1	-	-	-
CO2	-	-	-
CO3	-	-	-
CO4	-	-	-
CO5	-	-	-
CO6	-	-	-

Lecture Plan

S. No	Topics to be covered	No. of periods including Lab	Proposed Date	Actual Date	Pertain -ing CO	Taxonomy Level	Mode of Delivery
1.	Curvature in Cartesian and Polar Co-ordinates	3	29.09.23		CO2	K1	PPT, Chalk & Talk
2.	Radius of Curvature	2	04.10.23		CO2	K1	Chalk and Talk
3.	Centre of Curvature	3	06.10.23		CO2	K2	PPT, Chalk & Talk
4.	Circle of Curvature	2	10.10.23		CO2	K2	PPT, Chalk & Talk
5.	Evolute	4	12.10.23		CO2	K2	PPT, Chalk & Talk

Activity Based Learning

Free writes (Individual)

- Activity in which students write (non-stop) for 2 minute (curved material, non-curved metrical and its usage based on shape and design)

- Students are grouped into two groups. Each students should come and say at least one point and its explanation about the surface and its applications.

UNIT II

SINGLE VARIABLE CALCULUS

2. INTRODUCTION

Calculus is fundamentally different from the mathematics that you have studied previously: calculus is less static and more dynamic. It is concerned with change and motion; it deals with quantities that approach other quantities. For that reason it may be useful to have an overview of the subject before beginning its intensive study. Here we give a glimpse of some of the geometrical ideas of calculus by showing how the concept of a curvature arises when we attempt to solve a variety of problems.

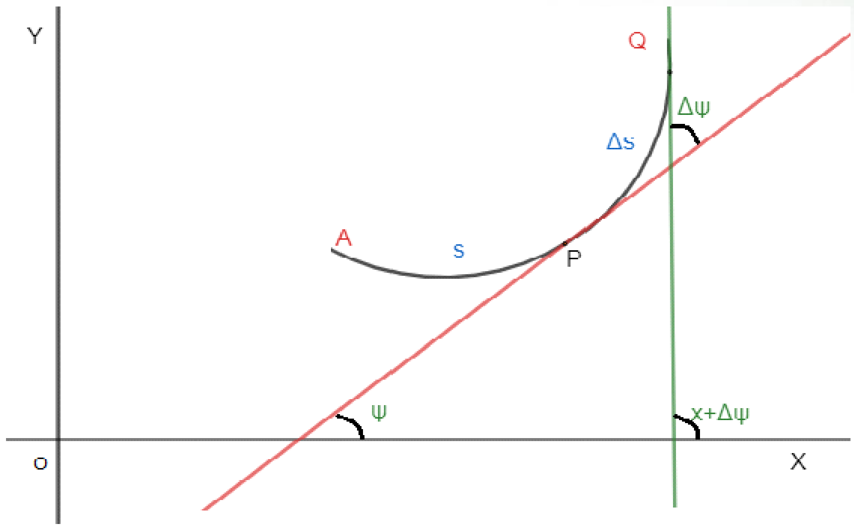
In differential geometry, it is used in Cesaro equation which tells that a plane curve is an equation that relates the curvature at a point of the curve to the arc length(s) from the start of the curve to a given point. Also, it is an equation relating to the radius of curvature to the arc length. Also, it can help to find the radius of curvature of the earth along a course at an azimuth. Besides, the radius of curvature also uses three parts equation for bending of beams. Moreover, it has a specific meaning and a sign convention in optical design. Also, spherical lenses have a centre of curvature.

2.1 Curvature in Cartesian and Polar Co-ordinates

Curvature: In Mathematics, curvature is any of several strongly related concepts in geometry. Intuitively, the curvature is the amount by which a curve deviates from being a straight line, or a surface deviates from being a plane. The curvature of a curve at a point is normally a scalar quantity, that is, it is expressed by a single real number.

Curvature of a curve: The rate of bending of a curve in any interval is called a curvature of the curve in the interval.

Note: The curvature of a circle at any point on it is the same and is equal to the reciprocal of its radius.



Note: The curvature of a straight line is zero.

Definition: Let P be any point on a given curve and Q a neighbouring point. Let arc $AP = s$ and arc $QP = \Delta s$. Let the tangents at P and Q make angle ψ and $\psi + \Delta\psi$ with the x-axis, so that the angle between the tangents at P and Q is $\Delta\psi$.

In moving from P to Q through a distance Δs, the tangent has turned through the angle $\Delta\psi$. This is called the total bending or total curvature of the arc PQ.

$$\therefore \text{ The average curvature of arc } PQ = \frac{\Delta\psi}{\Delta s}$$

The limiting value of average curvature when Q approaches P (ie., $\Delta s \to 0$ is defined as the curvature of the curve at P. It is denoted as κ (kappa).

Thus curvature $\kappa = \dfrac{d\psi}{ds}$

Note: The curvature of a circle of radius r is $\dfrac{1}{r}$.

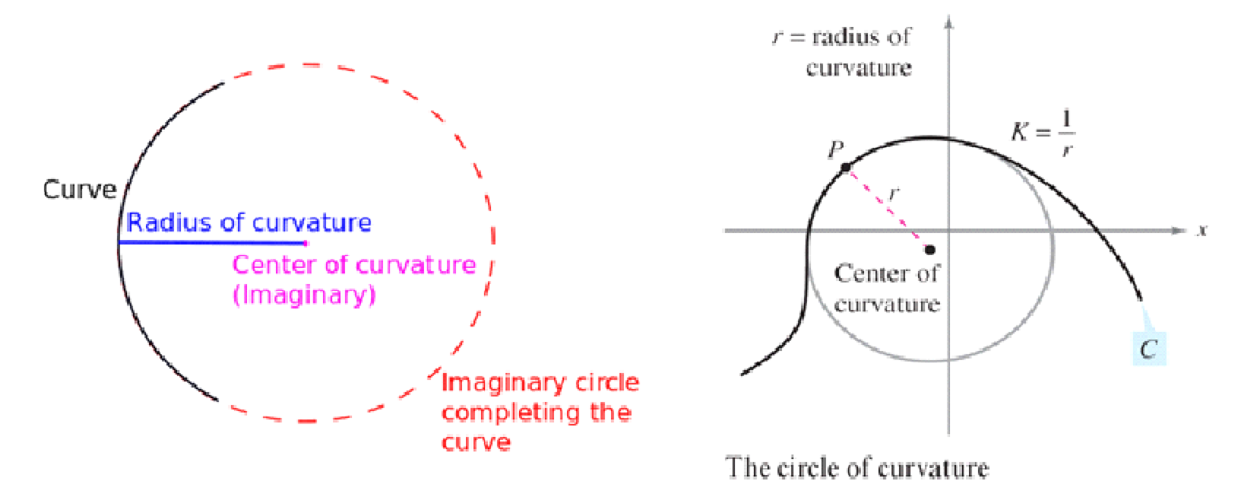
For better understanding curvature :

https://www.youtube.com/watch?v=EMo0vaphXpU: What is Curvature?

https://www.youtube.com/watch?v=ugtUGhBSeE0: Curvature intuition

2.2 Radius of curvature

The curvature at any point P of a curve is equal to the curvature of the circle which passes through P and two close points on the curve on either side of P. Such a circle exists for each point of the curve. It is called the Circle of curvature of the curve at the point. The radius of this circle is called the radius of curvature of the curve at that point.



Definition:

The reciprocal of the curvature (κ) of a curve at any point is called the radius of curvature at the point and is denoted by ρ .

Thus Radius of curvature $\rho = \dfrac{ds}{d\psi}$. i.e, $\rho = \dfrac{1}{\kappa}$.

Note: The curvature of a curve $\kappa = \dfrac{1}{\rho}$

Parts of a Circle: Try this!

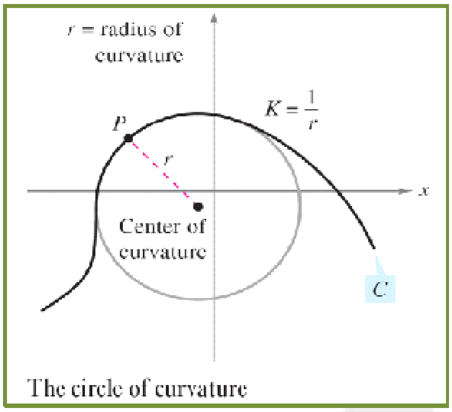
https://www.turtlediary.com/quiz/parts-of-a-circle.html

https://www.sanfoundry.com/engineering-mathematics-questions-answers-curvature-1/

Formula for the radius of curvature

1. Cartesian form:

If $y = f(x)$ be the given curve, then the radius of curvature of f is defined as



The circle of curvature

$$\rho = \frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}}{\frac{d^2y}{dx^2}}$$ i.e., $\rho = \frac{\left(1+y_1^2\right)^{3/2}}{y_2}$ where $y_1 = \frac{dy}{dx}$ and $y_2 = \frac{d^2y}{dx^2}$.

If $\frac{dy}{dx}$ becomes ∞ at the given point then the radius of curvature is defined as

$$\rho = \frac{\left(1+\left(\frac{dx}{dy}\right)^2\right)^{3/2}}{\left(\frac{d^2x}{dy^2}\right)}$$

2. Parametric form

Let $x = f(t)$ and $y = g(t)$ be the parametric equations of the given curve then

$$\rho = \frac{\left(x'^2 + y'^2\right)^{3/2}}{x'y'' - x''y'}$$, where $x'(t) = \frac{dx}{dt}$ and $y'(t) = \frac{dy}{dt}$

Note: We can convert parametric form to Cartesian form by the following way

$$y_1 = \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$ and $y_2 = \frac{d^2y}{dx^2}$.

3. Implicit form:

$$\rho=\frac{\left(f_x^{\,2}+f_y^{\,2}\right)^{3/2}}{f_{xx}f_y^{\,2}-2f_{xy}f_xf_y+f_{yy}f_x^{\,2}}\text{, where } f_x=\frac{\partial f}{\partial x}, f_y=\frac{\partial f}{\partial y}$$

4. Polar form:

Let $r=f(\theta)$ be the given curve in polar coordinates, then

$$\rho=\frac{\left(r^2+r_1^2\right)^{3/2}}{r^2+2r_1^2-rr_2}\text{, where } r_1=\frac{dr}{d\theta}\text{ and } r_2=\frac{d^2r}{d\theta^2}$$

NPTEL video lecture for radius of curvature and its applications:

https://freevideolectures.com/course/4545/nptel-mechanics-materials/52

Example 1: Find the curvature of the parabola given by $y=x-\frac{1}{4}x^2$ at $x=2$.

Solution: The curvature at $x=2$ is as follows.

We know that $\kappa=\frac{1}{\rho}\Rightarrow\kappa=\frac{y_2}{\left(1+y_1^{\,2}\right)^{3/2}}\left(\because\rho=\frac{\left(1+y_1^{\,2}\right)^{3/2}}{y_2}\right)$

$$y_1=\frac{dy}{dx}=1-\frac{x}{2}\text{ at } x=2,\ y_1=0$$

$$y_2=\frac{d^2y}{dx^2}=-\frac{1}{2}\text{ at } x=2,\ y_2=-\frac{1}{2}$$

$\therefore\ \kappa=\frac{-1/2}{(1+0)^{3/2}}=-\frac{1}{2}$ but curvature is a positive quantity $|\kappa|=\frac{1}{2}$.

Example 2: Find the radius of curvature of the curve $y=e^x$ at $(0,1)$.

Solution: Given $y=e^x$

$$\frac{dy}{dx}=y_1=e^x\text{ and }\frac{d^2y}{dx^2}=y_2=e^x$$

$y_1 = e^0 = 1$ and $y_2 = e^0 = 1$

$\therefore$ The radius of curvature $\rho = \dfrac{\left(1+y_1^2\right)^{3/2}}{y_2}$

$$\Rightarrow \rho = \frac{(1+1)^{3/2}}{1} = 2^{3/2} = 2\sqrt{2}.$$

Example 3: Find the radius of curvature at the point $\left(\frac{1}{4},\frac{1}{4}\right)$ on the curve $\sqrt{x}+\sqrt{y}=1$.

Solution: Given $\sqrt{x}+\sqrt{y}=1$

$$\sqrt{y} = 1-\sqrt{x} \Rightarrow y = \left(1-\sqrt{x}\right)^2$$

$$\Rightarrow y = 1 + x - 2\sqrt{x}$$

$y_1 = 1 - 2\left(\dfrac{1}{2\sqrt{x}}\right) = 1 - \dfrac{1}{\sqrt{x}}$ and $y_2 = -\left(\dfrac{-1}{2}\right)x^{\frac{-1}{2}-1} = \dfrac{1}{2}x^{\frac{-3}{2}}$

At $\left(\frac{1}{4},\frac{1}{4}\right)$

$y_1 = 1 - \dfrac{1}{\sqrt{1/4}} = 1 - \dfrac{1}{1/2} = 1 - 2 = -1$ and $y_2 = \dfrac{1}{2}\left(\dfrac{1}{4}\right)^{\frac{-3}{2}} = \dfrac{1}{2}(4)^{3/2} = \dfrac{1}{2}4\sqrt{4} = 4$

$$\therefore\ \rho = \frac{\left(1+y_1^2\right)^{3/2}}{y_2} = \frac{(1+1)^{3/2}}{4} = \frac{2^{3/2}}{4} = \frac{2\sqrt{2}}{4} = \frac{1}{\sqrt{2}}.$$

Example 4: Find the radius of curvature at the point (c,c) on the curve $xy = c^2$.

Solution: Given $xy = c^2 \Rightarrow y = \dfrac{c^2}{x}$

$y_1 = -\dfrac{c^2}{x^2}$ and $y_2 = \dfrac{2c^2}{x^3}$

At (c, c)

$$y_1 = -\frac{c^2}{c^2} = -1 \text{ and } y_2 = \frac{2c^2}{c^3} = \frac{2}{c}$$

$$\therefore\ \rho = \frac{\left(1+y_1^2\right)^{3/2}}{y_2} = \frac{(1+1)^{3/2}}{\left(2/c\right)} = \frac{2\sqrt{2}}{2}c = c\sqrt{2}.$$

Example 5: Prove that at the point $x = \frac{\pi}{2}$ of the curve $y = 4\sin x - \sin 2x$, $\rho = \frac{5\sqrt{5}}{2}$

Solution: Given $y = 4\sin x - \sin 2x$

$$y_1 = 4\cos x - 2\cos 2x \text{ and } y_2 = -4\sin x + 4\sin 2x$$

At $x = \frac{\pi}{2}$

$$y_1 = 4\cos\left(\frac{\pi}{2}\right) - 2\cos 2\left(\frac{\pi}{2}\right) = 0 - 2\cos\pi = 0 - 2(-1) = 2 \text{ and}$$

$$y_2 = -4\sin\left(\frac{\pi}{2}\right) + 4\sin 2\left(\frac{\pi}{2}\right) = -4 + 4\sin\pi = -4 + 0 = -4$$

$$\therefore\ \rho = \frac{\left(1+y_1^2\right)^{3/2}}{y_2} = \frac{\left(1+2^2\right)^{3/2}}{(-4)} = -\frac{(5)^{3/2}}{4} = -\frac{5\sqrt{5}}{4}$$

i.e., $|\rho| = \frac{5\sqrt{5}}{4}$ since ρ is positive.

Example 6: Find the radius of curvature a any point (x, y) on $y = c\log\sec\frac{x}{c}$.

Solution: Given $y = c\log\sec\frac{x}{c}$

$$y_1 = c\frac{1}{\sec\frac{x}{c}}\tan\frac{x}{c}.\sec\frac{x}{c}\frac{1}{c} = \tan\frac{x}{c} \text{ and } y_2 = \frac{1}{c}\sec^2\frac{x}{c}\ \left(\because \frac{d}{dx}(\log x) = \frac{1}{x}\right)$$

$$\left(\because \frac{d}{dx}(\sec x) = \sec x\tan x, \frac{d}{dx}(\tan x) = \sec^2 x\right)$$

$$\therefore\ \rho=\frac{\left(1+y_1^2\right)^{3/2}}{y_2}=\frac{\left(1+\tan^2\frac{x}{c}\right)^{3/2}}{\frac{1}{c}\sec^2\frac{x}{c}}=\frac{\left(\sec^2\frac{x}{c}\right)^{3/2}}{\frac{1}{c}\sec^2\frac{x}{c}}=c\frac{\sec^3\frac{x}{c}}{\sec^2\frac{x}{c}}=c\sec\frac{x}{c}.$$

Example 7: Show that the radius of curvature at any point of the catenary is $y=c\cosh\left(\frac{x}{c}\right)$. Also find ρ at $(0,c)$.

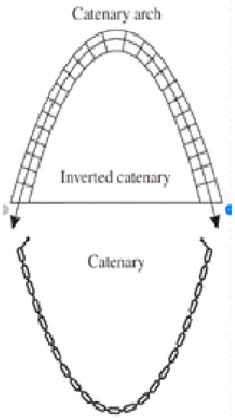
Solution: Given $y=c\cosh\left(\frac{x}{c}\right)$

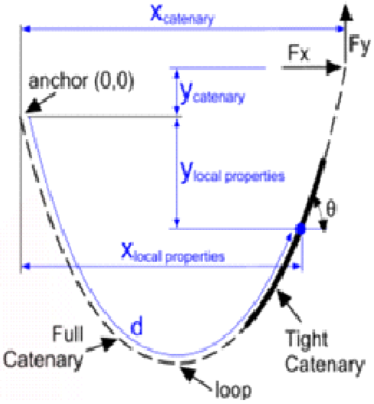
$y_1=c\sinh\left(\frac{x}{c}\right)\frac{1}{c}=\sinh\left(\frac{x}{c}\right)$ and $y_2=\cosh\left(\frac{x}{c}\right)\frac{1}{c}=\frac{1}{c}\cosh\left(\frac{x}{c}\right)$

$$\therefore\ \rho=\frac{\left(1+y_1^2\right)^{3/2}}{y_2}=\frac{\left(1+\sinh^2\left(\frac{x}{c}\right)\right)^{3/2}}{\frac{1}{c}\cosh\left(\frac{x}{c}\right)}=\frac{\left(\cosh^2\left(\frac{x}{c}\right)\right)^{3/2}}{\frac{1}{c}\cosh\left(\frac{x}{c}\right)}=c\frac{\cosh^3\left(\frac{x}{c}\right)}{\cosh\left(\frac{x}{c}\right)}=c\cosh^2\left(\frac{x}{c}\right)$$

$$\Rightarrow\qquad \rho=c\left(\frac{y^2}{c^2}\right)=\frac{y^2}{c}\qquad\left[\because\ y=c\cosh\left(\frac{x}{c}\right)\Rightarrow\cosh\left(\frac{x}{c}\right)=\frac{y}{c}\right]$$

At $(0,c)$, $\rho=c\left(\frac{c^2}{c^2}\right)=c$.





Example 8: Show that the measure of curvature of the curve $\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1$ at any point (x,y) on it is $\frac{ab}{2\left(ax+by\right)^{3/2}}$.

Solution: Given $\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1 \Rightarrow \frac{\sqrt{x}}{\sqrt{a}}+\frac{\sqrt{y}}{\sqrt{b}}=1$

$$\Rightarrow \sqrt{b}\sqrt{x}+\sqrt{a}\sqrt{y}=\sqrt{a}\sqrt{b}$$

Differentiate w.r.t. x we get

$$\sqrt{b}\,\frac{1}{2\sqrt{x}}+\sqrt{a}\,\frac{1}{2\sqrt{y}}\,y'=0$$

$$y'=-\frac{\sqrt{b}\,\frac{1}{2\sqrt{x}}}{\sqrt{a}\,\frac{1}{2\sqrt{y}}}\Rightarrow y'=-\frac{2\sqrt{b}\sqrt{y}}{2\sqrt{a}\sqrt{x}}=-\frac{\sqrt{b}}{\sqrt{a}}\frac{\sqrt{y}}{\sqrt{x}}$$

$$y'=-\frac{\sqrt{b}}{\sqrt{a}}\frac{\sqrt{y}}{\sqrt{x}}$$

Again diff. w.r.t x we get

$$y''=-\frac{\sqrt{b}}{\sqrt{a}}\left[\frac{\sqrt{x}\,\frac{1}{2\sqrt{y}}\,y'-\sqrt{y}\,\frac{1}{2\sqrt{x}}}{x}\right]=-\frac{\sqrt{b}}{\sqrt{a}}\frac{1}{x}\left[\frac{-\sqrt{b}}{2\sqrt{a}}-\frac{\sqrt{y}}{2\sqrt{x}}\right]$$

$$y''=-\frac{\sqrt{b}}{\sqrt{a}}\frac{1}{x}\left[\frac{-\sqrt{b}\sqrt{x}-\sqrt{y}\sqrt{a}}{2\sqrt{a}\sqrt{x}}\right]=\frac{\sqrt{b}}{2ax^{3/2}}\left[\sqrt{bx}+\sqrt{ay}\right]=\frac{\sqrt{b}}{2ax^{3/2}}\sqrt{ab} \qquad \left(\because \sqrt{bx}+\sqrt{ay}=\sqrt{ab}\right)$$

$$y''=\frac{b}{2\sqrt{a}x^{3/2}}$$

$$\therefore \qquad \rho=\frac{\left(1+y'^2\right)^{3/2}}{y''}=\frac{\left(1+\frac{by}{ax}\right)^{3/2}}{\left(\frac{b}{2\sqrt{a}x^{3/2}}\right)}=\frac{\left(ax+by\right)^{3/2}}{\left(ax\right)^{3/2}}\frac{2\sqrt{a}x^{3/2}}{b}=\frac{2}{ab}\left(ax+by\right)^{3/2}$$

$\therefore \quad$ Curvature $= \dfrac{1}{\rho} = \dfrac{ab}{2(ax+by)^{3/2}}$

Remark:

Name of the Curve	**Cartesian Equation**	**Parametric Equation**
Parabola	$y^2 = 4ax$ $x^2 = 4ay$	$x = at^2, y = 2at$ $x = 2at, y = at^2$
Ellipse	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$	$x = a\cos\theta, y = b\sin\theta$
Hyperbola	$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$	$x = a\sec\theta, y = b\tan\theta$
Rectangular hyperbola	$xy = c^2$	$x = ct, y = \dfrac{c}{t}$
Circle	$x^2 + y^2 = a^2$	$x = a\cos\theta, y = a\sin\theta$
Asteroid	$x^{2/3} + y^{2/3} = a^{2/3}$	$x = a\cos^3\theta, y = a\sin^3\theta$

Example 9: Find ρ at any point $P(at^2, 2at)$ on the parabola $y^2 = 4ax$.

Solution: Given $x = at^2, y = 2at$

$$x' = \frac{dt}{dx} = 2at,\ y' = \frac{dt}{dy} = 2a$$

$$x'' = 2a,\ y'' = 0$$

$$\rho = \frac{\left(x'^2 + y'^2\right)^{3/2}}{x'y'' - x''y'} = \frac{\left(4a^2t^2 + 4a^2\right)^{3/2}}{2at(0) - 2a(2a)} = \frac{\left(4a^2t^2 + 4a^2\right)^{3/2}}{-4a^2}$$

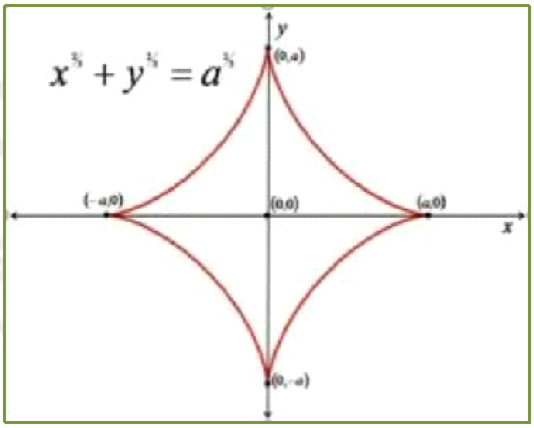
$$= \frac{\left[4a^2\left(t^2+1\right)\right]^{3/2}}{-4a^2} = \frac{8a^3\left(t^2+1\right)^{3/2}}{-4a^2} = -2a\left(t^2+1\right)^{3/2}$$

$$\therefore\ |\rho| = 2a\left(t^2+1\right)^{3/2}$$

Example 10: Find the radius of curvature at any point $x = a\cos^3\theta, y = a\sin^3\theta$ of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ also show that $\rho^3 = 27axy$.

Solution: The parametric equation of the given asteroid is $x = a\cos^3\theta, y = a\sin^3\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\tan\theta$$



i.e., $y_1 = \tan\theta$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(-\tan\theta)$$

$$= \frac{d}{d\theta}(-\tan\theta)\frac{d\theta}{dx} = \frac{-\sec^2\theta}{\left(\frac{dx}{d\theta}\right)}$$

$$y_2 = \frac{-\sec^2\theta}{-3a\cos^2\theta\sin\theta} = \frac{1}{3a\cos^4\theta\sin\theta}$$

$$\therefore\ \rho = \frac{\left(1+y_1^2\right)^{3/2}}{y_2} = \frac{\left(1+\tan^2\theta\right)^{3/2}}{\left(\frac{1}{3a\cos^4\theta\sin\theta}\right)} = \left(\sec^2\theta\right)^{3/2} 3a\cos^4\theta\sin\theta$$

$$= \left(\sec^2\theta\right)^{3/2} 3a\cos^4\theta\sin\theta = \sec^3\theta\, 3a\cos^4\theta\sin\theta = 3a\cos\theta\sin\theta$$

$$\rho = 3a\cos\theta\sin\theta \quad or \quad \rho = \frac{3a}{2}\sin 2\theta.$$

Now $\rho^3 = (3a\cos\theta\sin\theta)^3$

$$= 27a^3\sin^3\theta\cos^3\theta$$

$$= 27a^3\left(\frac{y}{a}\right)\left(\frac{x}{a}\right) \qquad \left\{\begin{array}{l} x = a\cos^3\theta \Rightarrow \cos^3\theta = \dfrac{x}{a} \\ y = a\sin^3\theta \Rightarrow \sin^3\theta = \dfrac{y}{a} \end{array}\right.$$

$\therefore \quad \rho^3 = 27axy$

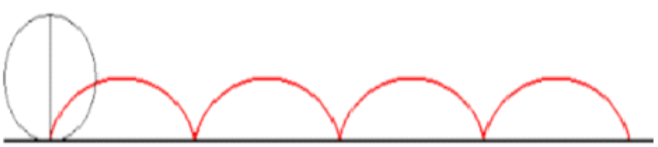
Example 11: Prove that the radius of curvature at any point of the cycloid $x = a(\theta+\sin\theta), y = a(1-\cos\theta)$ is $4a\cos\frac{\theta}{2}$

Solution: Given $x = a(\theta+\sin\theta), y = a(1-\cos\theta)$

$$\frac{dx}{d\theta} = a(1+\cos\theta), \frac{dy}{d\theta} = a\sin\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1+\cos\theta)} = \frac{\sin\theta}{1+\cos\theta}$$

$$= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$$



$$y_1 = \frac{dy}{dx} = \tan\frac{\theta}{2}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\tan\frac{\theta}{2}\right)$$

$$= \frac{d}{d\theta}\left(\tan\frac{\theta}{2}\right)\frac{d\theta}{dx} = \frac{1}{2}\sec^2\frac{\theta}{2}\frac{1}{a(1+\cos\theta)}$$

$$= \frac{1}{2}\sec^2\frac{\theta}{2}\frac{1}{a2\cos^2\frac{\theta}{2}}$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{1}{4a\cos^4\frac{\theta}{2}}$$

$$\therefore\ \rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{\left(1+\tan^2\frac{\theta}{2}\right)^{3/2}}{\left(\frac{1}{4a\cos^4\frac{\theta}{2}}\right)} = \left(\sec^2\frac{\theta}{2}\right)^{3/2} 4a\cos^4\frac{\theta}{2}$$

$$= \left(\sec^3\frac{\theta}{2}\right)4a\cos^4\frac{\theta}{2} = 4a\cos\frac{\theta}{2}$$

$$\therefore \qquad \rho = 4a\cos\frac{\theta}{2}$$

Example 12: Find the radius of curvature of the curve given by $x^3 - 2x^2y + 3xy^2 - 4y^3 + 5x^2 - 6xy + 7y^2 - 8y = 0$ at $(0,0)$.

Solution:

$f(x,y) = x^3 - 2x^2y + 3xy^2 - 4y^3 + 5x^2 - 6xy + 7y^2 - 8y$	At $(0,0)$
$f_x(x,y) = 3x^2 - 4xy + 3y^2 + 10x - 6y$	$f_x = 0$
$f_y(x,y) = -2x^2 + 6xy - 12y^2 - 6x + 14y - 8$	$f_y = -8$
$f_{xx}(x,y) = 6x - 4y + 10$	$f_{xx} = 10$
$f_{yy}(x,y) = 6x - 24y + 14$	$f_{yy} = 14$

$f_{xy}(x,y) = -4y + 6y - 6$	$f_{xy} = -6$

Radius of curvature $\rho = \dfrac{\left(f_x^2 + f_y^2\right)^{3/2}}{f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2}$

$$= \frac{\left(0+8^2\right)^{3/2}}{(10)(-8)^2 - 2(-6)(0)(-8) + (14)(0)}$$

$$= \frac{8^3}{10\left(8^2\right)} = \frac{8}{10}$$

$\therefore \qquad \rho = \dfrac{4}{5}$

Example 13: Find the radius of curvature at the point $\left(\dfrac{3a}{2}, \dfrac{3a}{2}\right)$ on the curve $x^3 + y^3 = 3axy$

Solution:

$f(x,y) = x^3 + y^3 - 3axy$	At $\left(\dfrac{3a}{2}, \dfrac{3a}{2}\right)$
$f_x(x,y) = 3x^2 - 3ay$	$f_x = 3\left(\dfrac{3a}{2}\right)^2 - 3a\left(\dfrac{3a}{2}\right) = \dfrac{27a^2}{4} - \dfrac{9a^2}{2} = \dfrac{9a^2}{4}$
$f_y(x,y) = 3y^2 - 3ax$	$f_y = 3\left(\dfrac{3a}{2}\right)^2 - 3a\left(\dfrac{3a}{2}\right) = \dfrac{27a^2}{4} - \dfrac{9a^2}{2} = \dfrac{9a^2}{4}$
$f_{xx}(x,y) = 6x$	$f_{xx} = 6\left(\dfrac{3a}{2}\right) = 9a$

$f_{yy}(x,y)=6y$	$f_{yy}=6\left(\frac{3a}{2}\right)=9a$
$f_{xy}(x,y)=-3a$	$f_{xy}=-3a$

Radius of curvature $\rho=\dfrac{\left(f_x^{\,2}+f_y^{\,2}\right)^{3/2}}{f_{xx}f_y^{\,2}-2f_{xy}f_xf_y+f_{yy}f_x^{\,2}}$

$$=\frac{\left(\left(\frac{9a^2}{4}\right)^2+\left(\frac{9a^2}{4}\right)^2\right)^{3/2}}{(9a)\left(\frac{9a^2}{4}\right)^2-2(-3a)\left(\frac{9a^2}{4}\right)\left(\frac{9a^2}{4}\right)+(9a)\left(\frac{9a^2}{4}\right)^2}$$

$$=\frac{\left(2\left(\frac{9a^2}{4}\right)^2\right)^{3/2}}{\left(\frac{9a^2}{4}\right)^2(9a+6a+9a)}=\frac{2^{3/2}\left(\frac{9a^2}{4}\right)^3}{\left(\frac{9a^2}{4}\right)^2(24a)}$$

$$=\frac{2\sqrt{2}\left(\frac{9a^2}{4}\right)}{24a}=\frac{\sqrt{2}\left(\frac{3a^2}{2}\right)}{8a}$$

$\therefore$ $\qquad \rho=\dfrac{3\sqrt{2}a}{16}$

Example 14: Find the radius of curvature at the pole for the curve $r=a\sin n\theta$

Solution: Given $r=a\sin n\theta$

$$r_1=\frac{dr}{d\theta}=na\cos n\theta,\ r_2=\frac{d^2r}{d\theta^2}=-n^2a\sin n\theta$$

At the pole, i.e., At $\theta=0$

$$r_1=\frac{dr}{d\theta}=na,\ r_2=\frac{d^2r}{d\theta^2}=0 \text{ and } r=0 \qquad [\because \cos 0=1,\ \sin 0=0]$$

Radius of curvature $\rho = \dfrac{\left(r^2 + r_1^2\right)^{3/2}}{r^2 + 2r_1^2 - rr_2}$

$$= \frac{\left(0 + n^2 a^2\right)^{3/2}}{0 + 2n^2 a^2 - 0} = \frac{\left[(na)^2\right]^{3/2}}{2(na)} = \frac{(na)^3}{2(na)}$$

$$\therefore \qquad \rho = \frac{na}{2}$$

Example 15: Find the radius of curvature at any point on the curve $r = e^{\theta}$

Solution: Given $r = e^{\theta}$

$$r_1 = \frac{dr}{d\theta} = e^{\theta}, \; r_2 = \frac{d^2 r}{d\theta^2} = e^{\theta}$$

Radius of curvature $\rho = \dfrac{\left(r^2 + r_1^2\right)^{3/2}}{r^2 + 2r_1^2 - rr_2}$

$$= \frac{\left(\left(e^{\theta}\right)^2 + \left(e^{\theta}\right)^2\right)^{3/2}}{\left(e^{\theta}\right)^2 + 2\left(e^{\theta}\right)^2 - \left(e^{\theta}\right)\left(e^{\theta}\right)}$$

$$= \frac{\left(2\left(e^{\theta}\right)^2\right)^{3/2}}{2\left(e^{\theta}\right)^2} = \frac{2^{3/2}\left(e^{\theta}\right)^3}{2\left(e^{\theta}\right)^2}$$

$$= \frac{2\sqrt{2}\left(e^{\theta}\right)^3}{2\left(e^{\theta}\right)^2} = \sqrt{2}e^{\theta}$$

$$\therefore \qquad \rho = \sqrt{2}r$$

Example:16 Find the radius of curvature at $(a,0)$ on the curve $xy^2 = a^3 - x^3$.

Solution: Given $xy^2 = a^3 - x^3$

Differentiating with respect to x,

$$x\left(2y\frac{dy}{dx}\right) + (1)y^2 = 0 - 3x^2$$

$$\frac{dy}{dx} = -\frac{(3x^2 + y^2)}{2xy} \quad \rightarrow (1)$$

$$\left(\frac{dy}{dx}\right)_{(a,0)} = \infty$$

We use the formula $\rho = \dfrac{\left[1 + \left(\dfrac{dx}{dy}\right)^2\right]^{3/2}}{\dfrac{d^2x}{dy^2}}$

$$\frac{dx}{dy} = -\frac{2xy}{3x^2 + y^2} \quad \rightarrow (2)$$

$$\left(\frac{dx}{dy}\right)_{(a,0)} = 0$$

Differentiating (2) with respect to y,

$$\frac{d^2x}{dy^2} = -2\left[\frac{(3x^2 + y^2)\left\{x(1) + y\dfrac{dx}{dy}\right\} - xy\left\{6x\dfrac{dx}{dy} + 2y\right\}}{(3x^2 + y^2)^2}\right]$$

$$\left(\frac{d^2x}{dy^2}\right)_{(a,0)} = -2\left[\frac{(3a^2 + 0)\{a + 0\} - 0}{(3a + 0)^2}\right] = \frac{-2}{3a}$$

$$= \frac{-3a}{2}$$

$$\rho = \frac{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}^{3/2}}{\frac{d^2x}{dy^2}} = \frac{(1+0)^{3/2}}{\left(\frac{-2}{3a}\right)}$$

$$|\rho| = \frac{3a}{2}.$$

Example:17 For the curve $y=\frac{ax}{a+x}$ if ρ is the radius of curvature at any point (x, y). Show that $\left(\frac{2\rho}{a}\right)^{2/3} = \left(\frac{y}{x}\right)^2 + \left(\frac{x}{y}\right)^2$.

Solution: Given $y=\frac{ax}{a+x}$

Differentiating w.r.t. x,

$$\frac{dy}{dx} = \frac{(a+x)a-ax(1)}{(a+x)^2} = \frac{a^2}{(a+x)^2}$$

$$y_1 = \frac{1}{x^2}\frac{a^2x^2}{(a+x)^2} = \frac{y^2}{x^2}$$

Differentiating (1) with respect to x,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[a^2(a+x)^{-2}\right] = -2a^2(a+x)^{-3} = \frac{-2a^2}{(a+x)^3}$$

$$y_2 = \frac{1}{ax^3}\frac{-2a^3x^3}{(a+x)^3} = \frac{-2y^3}{ax^3}$$

The radius of curvature is $\rho = \frac{\left[1+(y_1)^2\right]^{3/2}}{y_2}$

$$= \frac{\left(1+\left(\frac{y^2}{x^2}\right)^2\right)^{3/2}}{\frac{-2y^3}{ax^3}} = \frac{-ax^3}{2y^3}\left(1+\frac{y^4}{x^4}\right)^{3/2}$$

$$\left(\frac{2\rho}{a}\right)^{2/3} = \frac{x^2}{y^2}\left(1+\frac{y^4}{x^4}\right) = \frac{x^4+y^4}{x^2y^2}$$

$$\left(\frac{2\rho}{a}\right)^{2/3}=\left(\frac{x}{y}\right)^2+\left(\frac{y}{x}\right)^2.$$

Example:18 Find the points on the parabola $y^2=4x$ at which radius of curvature is $4\sqrt{2}$.

Solution: Given $y^2=4x \rightarrow (1)$

Differentiating w.r.t. x,

$$2yy_1=4 \Rightarrow y_1=\frac{2}{y} \rightarrow (2)$$

$$y_2=\frac{d}{dx}\left(2y^{-1}\right)=2\left\{-y^{-2}\frac{dy}{dx}\right\}=\frac{-2}{y^2}\frac{2}{y}=\frac{-4}{y^3}$$

Let $P(a,b)$ be the point on the parabola $y^2=4x$.

$$\left(y_1\right)_{(a,b)}=\frac{2}{b} \ \& \ \left(y_2\right)_{(a,b)}=\frac{-4}{y^3}$$

$$\rho=\frac{\left(1+y_1^2\right)^{3/2}}{y_2}=4\sqrt{2}$$

$$\frac{\left(1+\frac{4}{b^2}\right)^{3/2}}{\frac{-4}{b^3}}=4\sqrt{2} \Rightarrow \frac{-b^3}{4}\left(1+\frac{4}{b^2}\right)^{3/2}=4\sqrt{2}$$

$$\frac{-b^3}{4}\left(\frac{b^2+4}{b^2}\right)^{3/2}=4\sqrt{2} \Rightarrow \left(b^2+4\right)^{3/2}=-16\sqrt{2}$$

$$b^2+4=\left(-16\sqrt{2}\right)^{2/3} \Rightarrow b^2+4=8$$

$$b^2=4 \Rightarrow b=\pm 2.$$

But $P(a,b)$ be the point on the parabola $y^2=4x \Rightarrow b^2=4a$

When $b=\pm 2 \Rightarrow a=1$

The points are $(1,2)$ and $(1,-2)$.

Example:19 Find ρ for the curve $x=a(\cos t+t\sin t)$, $y=a(\sin t-t\cos t)$.

Solution: Given $x=a(\cos t+t\sin t)$

$$\frac{dx}{dt}=a\left(-\sin t+\{t\cos t+\sin t\}\right)=at\cos t$$

$$y=a(\sin t-t\cos t)$$

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$$y_1 = \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{at\sin t}{at\cos t} = \tan t$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$

$$= \sec^2 t\,\frac{1}{at\cos t} = \frac{\sec^3 t}{at}$$

The radius of curvature $\rho = \frac{\left(1+y_1^2\right)^{3/2}}{y_2}$

$$= \frac{\left(1+\tan^2 t\right)^{3/2}}{\left(\frac{\sec^3 t}{at}\right)} = \frac{at\sec^3 t}{\sec^3 t} = at.$$

Exercise:

1. Find the radius of curvature at $x=\frac{\pi}{2}$ on the curve $y=4\sin x$.

2. Find the radius of curvature at $x=1$ on $y=\frac{\log x}{x}$.

3. Find the radius of curvature at any point (x,y) on the curve $y=\frac{1}{2}a\left(e^{x/a}+e^{-x/a}\right)$.

4. Find the radius of curvature at any point $P(a\cos\theta, b\sin\theta)$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

5. Show that the radius of curvature at any point of the curve $x=ae^{\theta}(\sin\theta-\cos\theta)$, $y=ae^{\theta}(\sin\theta+\cos\theta)$ is twice the perpendicular distance of the tangent at the point from the origin.

6. Find the curvature of the curve $f(x,y)=x^3+3xy^2+5x^2+7y^2-6x=0$ at $(0,0)$.

7. Find the curvature and radius of curvature of the curve $2x^2+2y^2+2x-5y+1=0$ at any point.

8. Find ρ at $(1,1)$ on $x^3+y^3=2$.

Answers:

1. $\rho=\frac{-1}{4}$ 2. $\rho=\frac{2\sqrt{2}}{3}$ 3. $\rho=a\cosh^2\frac{x}{a}$ 4. $\rho=\frac{\left(a^2\sin^2\theta+b^2\cos^2\theta\right)^{3/2}}{ab}$

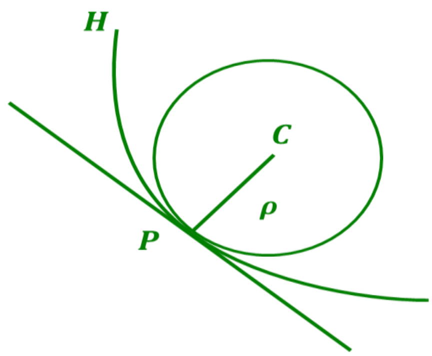
6. $\kappa=\frac{7}{3}$ 7. $\kappa=\frac{4}{\sqrt{(4x+2)^2+(4y-5)^2}}$ 8. $\frac{1}{\sqrt{2}}$

To Know More:

https://mathalino.com/content/curvature-and-radius-curvature

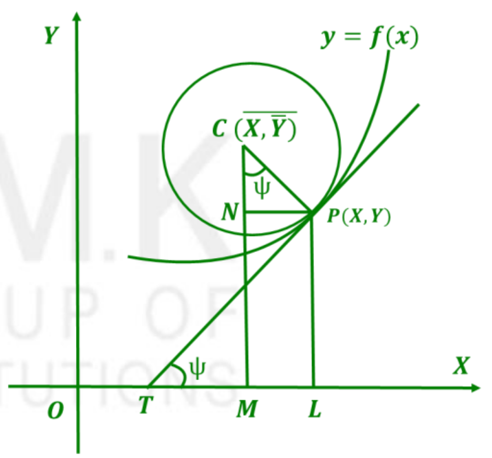
2.3 Centre of Curvature and Circle of Curvature

Definition: Let P any point on the simple curve $y = f(x)$.Let PC be a normal at P such that $PC = \rho$ the radius of curvature at P. Draw the circle with centre C and radius $= \rho$ C is called the centre of curvature and the circle is called the circle of curvature.



Coordinates of the centre of curvature

Let $y = f(x)$ be the given curve. Let $C\ \bar{x}, \bar{y}$ be the centre of curvature to the point P. Let PT be the tangent at P, which makes an angle ψ with the X-axis. Draw PL and CM perpendicular to the X-axis from P and C respectively. Let PN be drawn perpendicular to CM. From $\triangle PCN, \angle PCN = \psi$



Since P is the point (x, y) from the figure,

we have $OL = x, LP = y$

$$\text{Now,}\quad \bar{x} = OM = OL - LM = x - PC\sin\psi$$

$$\bar{x} = x - PC\sin\psi$$

$$\bar{y} = MC = MN + NC = LP + NC = y + PC\cos\psi$$

$$\bar{y} = y + PC\cos\psi$$

$$\text{We know that, } \tan\psi = y_1$$

$$\therefore \sin\psi = \frac{y_1}{\sqrt{1 + y_1^2}}$$

$$\cos\psi = \frac{1}{\sqrt{1 + y_1^2}}$$

Substituting the value of $\sin\psi$ and $\cos\psi$ in $\bar{x}$ and $\bar{y}$ respectively.

We get

$$\bar{x} = x - \rho\frac{y_1}{\sqrt{1+y_1^2}} = x - \frac{\left[1+y_1^2\right]^{3/2}}{y_2}\frac{y_1}{\sqrt{1+y_1^2}}$$

$$\bar{x} = x - \frac{y_1\left[1+y_1^2\right]}{y_2} \text{..............(1)}$$

$$\bar{y} = y + \rho\frac{1}{\sqrt{1+y_1^2}} = y + \frac{\left[1+y_1^2\right]^{3/2}}{y_2}\frac{1}{\sqrt{1+y_1^2}}$$

$$\bar{y} = y + \frac{\left[1+y_1^2\right]}{y_2} \text{....................(2)}$$

(1) and (2) gives the x and y coordinates of the centre of curvature.

The equation of the circle of curvature is $(x-\bar{x})^2+(y-\bar{y})^2=\rho^2$

Example 1: Find the Circle of Curvature of the curve $y = mx + \frac{x^2}{a}$ at the origin.

Solution: Given that $y = mx + \frac{x^2}{a}$

Differentiating w.r.to x we get

$$y_1 = m + \frac{2x}{a}; \quad y_2 = \frac{2}{a}$$

$$y_{1\,(0,0)} = m$$

$$\text{Now, } \rho = \frac{\left[1+y_1^2\right]^{3/2}}{y_2} = \frac{\left[1+m^2\right]^{3/2}}{\frac{2}{a}} = \frac{a\left[1+m^2\right]^{3/2}}{2}$$

Let $(\bar{x}, \bar{y})$ be the coordinates of the centre of curvature.

$$\bar{x} = x - \frac{y_1\left[1+y_1^2\right]}{y_2} = 0 - \frac{m\left[1+m^2\right]}{\frac{2}{a}} = -\frac{am\left[1+m^2\right]}{2}$$

$$\bar{y}=y+\frac{[1+y_1^2]}{y_2}=0+\frac{[1+m^2]}{\frac{2}{a}}=\frac{a[1+m^2]}{2}$$

The equation of the circle of curvature is $(x-\bar{x})^2+(y-\bar{y})^2=\rho^2$

$$\left[x+\frac{am[1+m^2]}{2}\right]^2+\left[y-\frac{a[1+m^2]}{2}\right]^2=\rho^2=\frac{a^2[1+m^2]^3}{4}$$

Example 2: Find the centre of curvature of $y=x^2$ at the origin.

Solution: Given that, $y=x^2$

Differentiating w.r.to x we get

$y_1=2x;\quad y_2=2$

$(y_1)_{(0,0)}=0;\ (y_2)_{(0,0)}=2$

Now, $\rho=\frac{(1+y_1^2)^{3/2}}{y_2}=\frac{1}{2}$

Let $(\bar{x},\bar{y})$ be the coordinates of the centre of curvature.

$$\bar{x}=x-\frac{y_1(1+y_1^2)}{y_2}=0-0=0$$

$$\bar{y}=y+\frac{1+y_1^2}{y_2}=0+\frac{1}{2}=\frac{1}{2}$$

$\therefore$ The centre of curvature is $\left(0,\frac{1}{2}\right)$

Example 3: Find the centre of curvature of $y=x^3-6x^2+3x+1$ at the (1,-1).

Solution: Given that, $y=x^3-6x^2+3x+1$

Differentiating w.r.to x we get

$$y_1 = 3x^2 - 12x + 3; \quad y_2 = 6x - 12$$

$$y_{1\,(1,-1)} = 3 - 12 + 3 = -6; \; y_{2\,(1,-1)} = 6 - 12 = -6$$

Let $(\bar{x}, \bar{y})$ be the coordinates of the centre of curvature.

$$\bar{x} = x - \frac{y_1\left(1 + y_1^2\right)}{y_2} = 1 - \frac{-6\left(1+36\right)}{-6} = 1 - 37 = -36$$

$$\bar{y} = y + \frac{1 + y_1^2}{y_2} = -1 + \frac{1+36}{-6} = -1 - \frac{37}{6} = \frac{-43}{6}$$

$\therefore$ The centre of curvature is $\left(-36, \frac{-43}{6}\right)$

Example 4: Find the Circle of Curvature at (3,4) on $xy = 12$.

Solution: Given that $xy = 12$

Differentiating w.r.to x we get

$$y_1 = -12x^{-2}; \quad y_2 = 24x^{-3}$$

$$y_{1\,(3,4)} = \frac{-12}{9} = \frac{-4}{3}; \; y_{2\,(3,4)} = \frac{24}{27} = \frac{8}{9}$$

Let $(\bar{x}, \bar{y})$ be the coordinates of the centre of curvature.

$$\bar{x} = x - \frac{y_1\left(1 + y_1^2\right)}{y_2} = 3 + \frac{\frac{4}{3}\left(1 + \frac{16}{9}\right)}{\frac{8}{9}} = 3 + \frac{25}{6} = \frac{43}{6}$$

$$\bar{y} = y + \frac{1 + y_1^2}{y_2} = 4 + \frac{\left(1 + \frac{16}{9}\right)}{\frac{8}{9}} = 4 + \frac{25}{8} = \frac{57}{8}$$

$$\text{Now, } \rho = \frac{\left(1 + y_1^2\right)^{3/2}}{y_2} = \frac{\left(1 + \frac{16}{9}\right)^{3/2}}{\frac{8}{9}} = \frac{125}{24}$$

The equation of the circle of curvature is $\left(x - \bar{x}\right)^2 + \left(y - \bar{y}\right)^2 = \rho^2$

$$\left(x - \frac{43}{6}\right)^2 + \left(y - \frac{57}{8}\right)^2 = \rho^2 = \left(\frac{125}{24}\right)^2$$

Example 5: Find the equation of Circle of Curvature at (3,6) on $y^2 = 12x$.

Solution: Given that $y^2 = 12x$

Differentiating w.r.to x we get

$$2yy_1 = 12 \Rightarrow y_1 = \frac{6}{y} = 6y^{-1} \Rightarrow y_1\,_{(3,6)} = 1$$

$$y_2 = \frac{-6}{y^2} y_1 \Rightarrow y_2\,_{(3,6)} = \frac{-6}{36}.1 = \frac{-1}{6}$$

$$\text{Now, } \rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+1)^{3/2}}{\frac{-1}{6}} = -6(2\sqrt{2}) = 12\sqrt{2} \quad (\because \rho \text{ is not negative})$$

Let $(\bar{x}, \bar{y})$ be the coordinates of the centre of curvature.

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2} = 3 - \frac{1(2)(6)}{-1} = 3 + 12 = 15$$

$$\bar{y} = y + \frac{1+y_1^2}{y_2} = 6 - 2(6) = -6$$

The equation of the circle of curvature is $(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$

$$(x-15)^2 + (y+6)^2 = \rho^2 = (2\sqrt{2})^2 = 288$$

Example 6: Show that the circle of curvature $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at $\left(\frac{a}{4}, \frac{a}{4}\right)$ is $\left(x - \frac{3a}{4}\right)^2 + \left(y - \frac{3a}{4}\right)^2 = \frac{a^2}{2}$.

Solution: Given that $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$

Differentiating w.r.to x we get

$$\frac{1}{2}x^{\frac{1}{2}-1} + \frac{1}{2}y^{\frac{1}{2}-1} y_1 = 0 \Rightarrow \frac{1}{\sqrt{y}} y_1 = \frac{-1}{\sqrt{x}} \Rightarrow y_1 = \frac{-\sqrt{y}}{\sqrt{x}} \Rightarrow y_1\,_{\left(\frac{a}{4},\frac{a}{4}\right)} = -1$$

$$y_2 = \left[\frac{\frac{\sqrt{x}}{2\sqrt{y}}y_1 - \frac{\sqrt{y}}{2\sqrt{x}}}{x}\right] \Rightarrow \; y_2\left(\frac{a}{4},\frac{a}{4}\right) = \frac{\frac{-1}{2}-\frac{1}{2}}{\frac{a}{4}} = \frac{4}{a}$$

$$\text{Now, } \rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{a(2)^{3/2}}{4} = \frac{a}{\sqrt{2}}$$

Let $(\bar{x},\bar{y})$ be the coordinates of the centre of curvature.

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2} = \frac{a}{4} + \frac{2a}{4} = \frac{3a}{4}$$

$$\bar{y} = y + \frac{1+y_1^2}{y_2} = \frac{a}{4} + \frac{2a}{4} = \frac{3a}{4}$$

The equation of the circle of curvature is $(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$

$$\left(x-\frac{3a}{4}\right)^2 + \left(y-\frac{3a}{4}\right)^2 = \rho^2 = \frac{a^2}{2}$$

Example 7: Find the equation of the Circle of Curvature at (c,c) on $xy = c^2$.

Solution: Given that $xy = c^2 \Rightarrow y = \dfrac{c^2}{x} = c^2 x^{-1}$

Differentiating w.r.to x we get

$$y_1 = -c^2 x^{-2}; \quad y_2 = 2c^2 x^{-3}$$

$$\left(y_1\right)_{(c,c)} = \frac{-c^2}{c^2} = -1; \left(y_2\right)_{(c,c)} = \frac{2c^2}{c^3} = \frac{2}{c}$$

Now, $\rho = \dfrac{\left(1+y_1^2\right)^{3/2}}{y_2} = \dfrac{c\left(2\right)^{3/2}}{2} = c\sqrt{2}$

Let $(\bar{x}, \bar{y})$ be the coordinates of the centre of curvature.

$$\bar{x} = x - \frac{y_1\left(1+y_1^2\right)}{y_2} = c + \frac{2c}{2} = 2c$$

$$\bar{y} = y + \frac{1+y_1^2}{y_2} = c + \frac{2c}{2} = 2c$$

The equation of the circle of curvature is $\left(x-\bar{x}\right)^2 + \left(y-\bar{y}\right)^2 = \rho^2$

$$\left(x-2c\right)^2 + \left(y-2c\right)^2 = \rho^2 = 2c^2$$

Example 8: Find the centre and circle of Curvature at the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ of the Folium $x^3 + y^3 = 3axy$.

Solution: Given that, $x^3 + y^3 = 3axy$

Differentiating with respect to x, we get

$$3x^2 + 3y^2 y_1 = 3a(y + xy_1)$$

$$(y^2 - ax)y_1 = ay - x^2 \text{.............}(1)$$

$$y_1 = \frac{ay - x^2}{y^2 - ax} \Rightarrow (y_1)_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = -1$$

Differentiating (1) we get,

$$(2yy_1 - a)y_1 + (y^2 - ax)y_2 = ay_1 - 2x$$

$$\therefore (y_2)_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = \frac{-32}{3a}$$

Hence, $\rho_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = \dfrac{(1 + y_1^2)^{3/2}}{y_2} = \dfrac{[1+(-1)^2]^{3/2}}{-32/3a} = \dfrac{3a}{8\sqrt{2}}$

Let $(\bar{x}, \bar{y})$ be the coordinates of the centre of curvature.

$$\bar{x} = x - \frac{y_1(1 + y_1^2)}{y_2} = \frac{3a}{2} - \frac{-1(2)}{\frac{-32}{3a}} = \frac{3a}{2} - \frac{3a}{16} = \frac{21a}{16}$$

$$\bar{y} = y + \frac{1 + y_1^2}{y_2} = \frac{3a}{2} + \frac{(2)}{\frac{-32}{3a}} = \frac{3a}{2} - \frac{3a}{16} = \frac{21a}{16}$$

The equation of the circle of curvature is $(x - \bar{x})^2 + (y - \bar{y})^2 = \rho^2$

$$\left(x - \frac{21a}{16}\right)^2 + \left(y - \frac{21a}{16}\right)^2 = \rho^2 = \frac{9a^2}{128}$$

Example 9: Find the radius of curvature and Centre of Curvature at any point (x,y) on the curve $y = c \log \sec\left(\frac{x}{c}\right)$.

Solution: Given that $y = c \log \sec\left(\frac{x}{c}\right)$

Differentiating w.r.to x we get

$$y_1 = \left(\frac{c}{\sec\frac{x}{c}}\right)\left(\sec\frac{x}{c}\right)\left(\tan\frac{x}{c}\right)\left(\frac{1}{c}\right) \Rightarrow y_1 = \left(\tan\frac{x}{c}\right)$$

$$y_2 = \left(\frac{1}{c}\right)\left(\sec^2\frac{x}{c}\right)$$

$$\text{Now, } \rho = \frac{1+y_1^2 \;{}^{3/2}}{y_2} = \frac{\left(1+\tan^2\frac{x}{c}\right)^{3/2}}{\left(\frac{1}{c}\right)\left(\sec^2\frac{x}{c}\right)} = \frac{c\ \sec^3\frac{x}{c}}{\left(\sec^2\frac{x}{c}\right)} = c\ \sec\frac{x}{c}$$

Let $(\bar{x}, \bar{y})$ be the coordinates of the centre of curvature.

$$\bar{x} = x - \frac{y_1\ 1+y_1^2}{y_2} = x - \frac{\tan\frac{x}{c}\sec^2\frac{x}{c}}{\sec^2\frac{x}{c}} = x - c\tan\frac{x}{c}$$

$$\bar{y} = y + \frac{1+y_1^2}{y_2} = y + \frac{c\sec^2\frac{x}{c}}{\sec^2\frac{x}{c}} = y + c$$

Example 10: Find the Centre of Curvature at $\theta = \left(\frac{\pi}{2}\right)$ on the curve $x = 2\cos t + \cos 2t, y = 2\sin t + \sin 2t.$

Solution: Given that

$$x = 2\cos t + \cos 2t$$
$$y = 2\sin t + \sin 2t$$

Differentiating w.r.to t we get

$$\dot{x} = -2\sin t - 2\sin 2t$$
$$\dot{y} = 2\cos t + 2\cos 2t$$

$$y_1 = \frac{\dot{y}}{\dot{x}} = -\left(\frac{2\cos t + 2\cos 2t}{2\sin t + 2\sin 2t}\right) = -\left(\frac{\cos t + \cos 2t}{\sin t + \sin 2t}\right) = -\left(\frac{2\cos\frac{3t}{2}\cos\frac{t}{2}}{2\sin\frac{3t}{2}\sin\frac{t}{2}}\right) = -\cot\frac{3t}{2}$$

$$y_2 = \frac{d}{dt}\left(y_1\right) = \frac{d}{dt}\left(-\cot\frac{3t}{2}\right)\frac{dt}{dx} = \frac{3}{2}\operatorname{cosec}^2\frac{3t}{2}\left(\frac{1}{-2\sin t+\sin 2t}\right) = \frac{-3}{8\sin^3\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)}.$$

Here $x_{\left(\theta=\frac{\pi}{2}\right)} = -1;\ y_{\left(\theta=\frac{\pi}{2}\right)} = 2$

$$y_{1\left(\theta=\frac{\pi}{2}\right)} = 1;\ y_{2\left(\theta=\frac{\pi}{2}\right)} = \frac{-3}{2}$$

Let $(\bar{x}, \bar{y})$ be the coordinates of the centre of curvature.

$$\bar{x} = x - \frac{y_1\left(1+y_1^2\right)}{y_2} = -1 - \frac{1+1}{\frac{-3}{2}} = -1 + \frac{4}{3} = \frac{1}{3}$$

$$\bar{y} = y + \frac{1+y_1^2}{y_2} = 2 + \frac{1+1}{\frac{-3}{2}} = 2 - \frac{4}{3} = \frac{2}{3}$$

$\therefore$ Required centre of curvature is $\left(\frac{1}{3}, \frac{2}{3}\right)$

Example 11: Find the Centre of Curvature and equation of circle of the curvature at the point P on the curve at $y = e^x$ where the curve crosses the y-axis.

Solution: Given that $y = e^x$

Differentiating w.r.to x we get

$y_1 = e^x$

$y_2 = e^x$

Consider the point $(x, y) = (0,1)$ [$\because$ The graph is continuus curve that passes through $(0,1)$]

$$\left(y_1\right)_{(0,1)} = 1 = \left(y_2\right)_{(0,1)}$$

$$\text{Now, } \rho = \frac{\left(1+y_1^2\right)^{3/2}}{y_2} = \frac{\left(1+1\right)^{3/2}}{1} = \left(2\right)^{3/2} = 2\sqrt{2}$$

Let $(\bar{x}, \bar{y})$ be the coordinates of the centre of curvature.

The equation of the circle of curvature is $(x-\bar{x})^2+(y-\bar{y})^2=\rho^2$

$(x+2)^2+(y-3)^2=\rho^2=8$

Exercise:

1. Find the radius of curvature and centre of curvature of $x^4+y^4=2$ at the (1, 1)
2. Show that the two parabolas $y=1+x-x^2$ *and* $x=1+y-y^2$ have the same circle of curvature at the point (1,1)
3. Find the circle of curvature $\sqrt{x}+\sqrt{y}=\sqrt{1}$ at $\left(\frac{1}{4},\frac{1}{4}\right)$
4. Find the coordinates of the Centre of curvature at $(at^2,2at)$ on the parabola $y^2=4ax$
5. Find the coordinates of the centre of curvature of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at $a\cos\theta, b\sin\theta$

Answers:

1. $\rho=\frac{\sqrt{2}}{3}$; Centre: $\left(\frac{4}{3},\frac{4}{3}\right)$

3. $\left(x-\frac{3}{4}\right)^2+\left(y-\frac{3}{4}\right)^2=\frac{1}{2}$

4. $\bar{x}=a(2+3t)t^2$

   $\bar{y}=-4\sqrt{2}at^{3/2}$

5. $\bar{x}=\frac{a^2-b^2}{a}\cos^3\theta$

   $\bar{y}=b\sin\theta-\frac{a^2}{b}\sin^3\theta-b\sin\theta\cos^2\theta$

To Know More:

Spherical mirrors- centre of curvature, pole , focus, focal length , principle axis:

https://www.youtube.com/watch?v=5Pkbjdb6DXk

Define Pole centre of curvature Radius of curvature Principal Axis Principal:

https://www.youtube.com/watch?v=fxZCymKj3SY

2.4 Evolute

Definition: The locus of the centre of curvature of a curve is called the evolute of the curve.

Definition: If the evolute itself be regarded as the original curve, a curve of which it is the evolute is called an involute.

Method of finding Evolute of a curve $y = f(x)$:

Step 1: Write the parametric form of the given equation $x = f(\theta), y = g(\theta)$.

Step 2: Find the centre of curvature $\left(\overline{X}, \overline{Y}\right)$ at θ.

Step 3: Eliminate the parameter θ and find the equation in terms of $\overline{X}$ and $\overline{Y}$ only.

Step 4: Replace $\overline{X}$ by x and $\overline{Y}$ by y in the equation obtained in step 3, and that equation is the equation of evolute of the given curve.

Parametric form of Standard Curves:

Curve	**Parametric Form**
Parabola: $y^2 = 4ax$ (Symmetric about x axis)	$x = at^2, y = 2at$
Circle: $x^2 + y^2 = a^2$	$x = a\cos\theta, y = a\sin\theta$
Ellipse: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$	$x = a\cos\theta, y = b\sin\theta$
Hyperbola: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$	$x = a\sec\theta, y = b\tan\theta$
Rectangular Hyperbola: $xy = c^2$	$x = ct, y = \dfrac{c}{t}$
Parabola: $x^2 = 4ay$ (Symmetric about y axis)	$x = 2at, y = at^2$
Ostrick: $x^{2/3} + y^{2/3} = a^{2/3}$	$x = a\cos^3\theta, y = a\sin^3\theta$

Example 1:

Find the equation of the evolute of the parabola $y^2 = 4ax$

Solution: The parametric equations of the Parabola are

$$x = at^2, y = 2at.$$

Let $\left(\overline{X}, \overline{Y}\right)$ be the centre of the curvature where

$$\overline{X} = x - \frac{y_1\left(1+y_1^2\right)}{y_2}; \; y_1 = \frac{dy}{dx}$$

$$\overline{Y} = y + \frac{1+y_1^2}{y_2} \quad ; \; y_2 = \frac{d^2y}{dx^2}$$

Now

$$y_1 = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t}$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \frac{d}{dt}\left(\frac{1}{t}\right)\left(\frac{1}{\left(\frac{dx}{dt}\right)}\right) = -\frac{1}{t^2}\left(\frac{1}{2at}\right) = -\frac{1}{2at^3}$$

$$\overline{X} = x - \frac{y_1\left(1+y_1^2\right)}{y_2}$$

$$\overline{X} = at^2 - \left(\frac{\frac{1}{t}\left(1+\frac{1}{t^2}\right)}{-\frac{1}{2at^3}} \right)$$

$$= at^2 + 2at^2\left(1+\frac{1}{t^2}\right)$$

$$\overline{X} = 3at^2 + 2a \qquad \text{--------} (1)$$

$$\overline{Y} = y + \frac{1+y_1^2}{y_2}$$

$$\overline{Y} = 2at + \frac{1+\frac{1}{t^2}}{-\frac{1}{2at^3}}$$

$$= 2at - 2at^3\left[1+\frac{1}{t^2}\right]$$

$$= 2at - 2at^3 - 2at$$

$$= -2at^3 \qquad \text{-------} (2)$$

Now we have to eliminate the parameter 't' between (1) and (2).

Equation (1)

$$\Rightarrow 3at^2 = \overline{X} - 2a$$

$$t^2 = \frac{\overline{X} - 2a}{3a}$$

$$\left(t^2\right)^3 = \left(\frac{\overline{X} - 2a}{3a}\right)^3$$

$$t^6 = \frac{\left(\overline{X} - 2a\right)^3}{27a^3} \qquad \text{-----} (3)$$

Equation (2)

$$\Rightarrow t^3 = \frac{\overline{Y}}{-2a}$$

$$\left(t^3\right)^2 = \frac{\overline{Y}^2}{4a^2}$$

$$t^6 = \frac{\overline{Y}^2}{4a^2} \qquad ------(4)$$

From (3) and (4),

$$\frac{\overline{Y}^2}{4a^2} = \frac{\left(\overline{X}-2a\right)^3}{27a^3}$$

$$\frac{\overline{Y}^2}{4} = \frac{\left(\overline{X}-2a\right)^3}{27a}$$

$$27a\overline{Y}^2 = 4\left(\overline{X}-2a\right)^3$$

Therefore, Locus of $\left(\overline{X},\overline{Y}\right)$ is,

$$4\left(x-2a\right)^3 = 27ay^2$$

which is the evolute of the parabola.

Example 2: Find the equation of the evolute of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Solution: The parametric equation of the ellipse are

$$x = a\cos\theta, \quad y = a\sin\theta$$

Let $\left(\overline{X},\overline{Y}\right)$ be the centre of curvature where

$$\overline{X} = x - \frac{y_1\left(1+y_1^2\right)}{y_2}; \; y_1 = \frac{dy}{dx}$$

$$\overline{Y} = y + \frac{1+y_1^2}{y_2} \qquad ; \; y_2 = \frac{d^2y}{dx^2}$$

$$y_1 = \frac{dy}{dx}$$

$$= \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$= \frac{b\cos\theta}{-a\sin\theta}$$

$$= -\frac{b}{a}\cot\theta$$

$$y_2 = \frac{d^2y}{dx^2}$$

$$= \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\frac{d\theta}{dx}$$

$$= \frac{d}{d\theta}\left(-\frac{b}{a}\cot\theta\right)\left(\frac{1}{\frac{dx}{d\theta}}\right)$$

$$= \frac{b}{a}\cos ec^2\theta \, \frac{1}{-a\sin\theta}$$

$$= -\frac{b}{a^2}\cos ec^3\theta$$

$$\overline{X} = a\cos\theta - \frac{-\frac{b}{a}\cot\theta\left[1+\frac{b^2}{a^2}\cot^2\theta\right]}{-\frac{b}{a^2}\cos ec^3\theta}$$

$$= a\cos\theta - a\cot\theta\sin^3\theta\left[1+\frac{b^2}{a^2}\cot^2\theta\right]$$

$$= a\cos\theta - a\sin^2\theta\cos\theta\left[1+\frac{b^2}{a^2}\frac{\cos^2\theta}{\sin^2\theta}\right]$$

$$= a\cos\theta - a\sin^2\theta\cos\theta - \frac{b^2}{a}\cos^3\theta$$

$$= a\cos\theta\left(1-\sin^2\theta\right) - \frac{b^2}{a}\cos^3\theta$$

$$= a\cos^3\theta - \frac{b^2}{a}\cos^3\theta$$

$$= \left(\frac{a^2-b^2}{a}\right)\cos^3\theta \qquad ------(1)$$

$$\overline{Y} = b\sin\theta + \frac{1+\frac{b^2}{a^2}\cot^2\theta}{-\frac{b}{a^2}\cos ec^3\theta}$$

$$= b\sin\theta - \frac{a^2}{b}\sin^3\theta\left[1+\frac{b^2}{a^2}\cot^2\theta\right]$$

$$= b\sin\theta - \frac{a^2}{b}\sin^3\theta - b\sin^3\theta\cot^2\theta$$

$$= b\sin\theta - \frac{a^2}{b}\sin^3\theta - b\cos^2\theta\sin\theta$$

$$= b\sin\theta\left(1-\cos^2\theta\right) - \frac{a^2}{b}\sin^3\theta$$

$$= b\sin\theta\sin^2\theta - \frac{a^2}{b}\sin^3\theta$$

$$= b\sin^3\theta - \frac{a^2}{b}\sin^3\theta$$

$$\overline{Y} = \left(\frac{b^2-a^2}{b}\right)\sin^3\theta \qquad ------(2)$$

To find the equation of the evolute we have to eliminate the parameter θ between (1) and (2).

The equation (1)

$$\Rightarrow a\overline{X} = \left(a^2-b^2\right)\cos^3\theta$$

$$\left(a\overline{X}\right)^{2/3} = \left(a^2-b^2\right)^{2/3}\cos^2\theta \qquad ------(3)$$

The equation (2)

$$\Rightarrow b\overline{Y} = \left(b^2 - a^2\right)\sin^3\theta$$

$$\left(b\overline{Y}\right)^{2/3} = \left(b^2 - a^2\right)^{2/3}\sin^2\theta$$

$$\left(b\overline{Y}\right)^{2/3} = \left(a^2 - b^2\right)^{2/3}\sin^2\theta \qquad ------(4)$$

$(3)+(4)$ gives

$$\left(a\overline{X}\right)^{2/3} + \left(b\overline{Y}\right)^{2/3} = \left(a^2 - b^2\right)^{2/3}\left[\sin^2\theta + \cos^2\theta\right]$$

$$\left(a\overline{X}\right)^{2/3} + \left(b\overline{Y}\right)^{2/3} = \left(a^2 - b^2\right)^{2/3}$$

Locus of $\left(\overline{X},\overline{Y}\right)$ is

$$(ax)^{2/3} + (by)^{2/3} = \left(a^2 - b^2\right)^{2/3}$$

which is the evolute.

Example 3:

Show that the evolute of the cycloid

$x = a(\theta - \sin\theta), y = a(1-\cos\theta)$ is another cycloid $x = a(\theta + \sin\theta), y = a(1-\cos\theta)$.

Solution: Given

$$x = a(\theta - \sin\theta)$$

$$y = a(1-\cos\theta)$$

Differentiating w.r.t. 'θ', we get

$$\frac{dx}{d\theta} = a(1-\cos\theta)$$

$$\frac{dy}{d\theta} = a\sin\theta$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$= \frac{a\sin\theta}{a(1-\cos\theta)}$$

$$=\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$$

$$y_1=\frac{dy}{dx}=\cot\frac{\theta}{2}$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$=\frac{d}{dx}\left(\cot\frac{\theta}{2}\right)$$

$$=\frac{d}{d\theta}\left(\cot\frac{\theta}{2}\right)\frac{d\theta}{dx}$$

$$=-\cos ec^2\left(\frac{\theta}{2}\right)\left(\frac{1}{2}\right)\frac{1}{\frac{dx}{d\theta}}$$

$$=-\cos ec^2\left(\frac{\theta}{2}\right)\left(\frac{1}{2}\right)\frac{1}{a(1-\cos\theta)}$$

$$\frac{d^2y}{dx^2}=-\frac{1}{2\sin^2\frac{\theta}{2}}\cdot\frac{1}{a\left(2\sin^2\frac{\theta}{2}\right)}$$

$$y_2=-\frac{1}{4a\sin^4\frac{\theta}{2}}$$

Let $\left(\overline{X},\overline{Y}\right)$ be the centre of curvature where

$$\overline{X}=x-\frac{y_1\left(1+y_1^2\right)}{y_2};\ y_1=\frac{dy}{dx}$$

$$\overline{Y}=y+\frac{1+y_1^2}{y_2}\qquad;\ y_2=\frac{d^2y}{dx^2}$$

$$\overline{X} = a(\theta - \sin\theta) - \frac{\cot\frac{\theta}{2}\left[1+\cot^2\frac{\theta}{2}\right]}{\left[-\frac{1}{4a\sin^4\frac{\theta}{2}}\right]}$$

$$= a(\theta - \sin\theta) + 4a\sin^4\frac{\theta}{2}\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\cos ec^2\frac{\theta}{2}$$

$$= a(\theta - \sin\theta) + 4a\sin^4\frac{\theta}{2}\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}.\frac{1}{\sin^2\frac{\theta}{2}}$$

$$= a\theta - a\sin\theta + 4a\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= a\theta - a\sin\theta + 2a\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)$$

$$= a\theta + a\sin\theta$$

$$= a(\theta + \sin\theta) \qquad --------(1)$$

$$\overline{Y} = a(1-\cos\theta) + \left[\frac{1+\cot^2\frac{\theta}{2}}{\frac{-1}{4a\sin^4\frac{\theta}{2}}}\right]$$

$$= a - a\cos\theta - 4a\sin^4\frac{\theta}{2}.\cos ec^2\frac{\theta}{2}$$

$$= a - a\cos\theta - 4a\sin^4\frac{\theta}{2}.\frac{1}{\sin^2\frac{\theta}{2}}$$

$$= a - a\cos\theta - 4a\sin^2\frac{\theta}{2}$$

$$= a - a\cos\theta - 2a\left(2\sin^2\frac{\theta}{2}\right)$$

$$= a - a\cos\theta - 2a(1-\cos\theta)$$

$$= a - a\cos\theta - 2a + 2a\cos\theta$$

$$= -a + a\cos\theta$$

$$\overline{Y} = -a(1-\cos\theta) \qquad ------(2)$$

The locus of centre of curvature is

$x = a(\theta + \sin\theta)$ and $y = -a(1-\cos\theta)$

which is the required evolute.

Example 4:

Show that the evolute of the tractrix $x = a\left(\cos t + \log \tan\left(\frac{t}{2}\right)\right)$, $y = a \sin t$ is the catenary $y = a\cosh\left(\frac{x}{a}\right)$.

Solution: Given $x = a\left(\cos t + \log \tan\left(\frac{t}{2}\right)\right)$

$$\frac{dx}{dt} = a\left(-\sin t + \frac{1}{\tan\left(\frac{t}{2}\right)}\sec^2\left(\frac{t}{2}\right)\times\frac{1}{2}\right)$$

$$= a\left(-\sin t + \frac{\cos\left(\frac{t}{2}\right)}{2\sin\left(\frac{t}{2}\right)\cos^2\left(\frac{t}{2}\right)}\right) = a\left(-\sin t + \frac{1}{2\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)}\right)$$

$$= a\left(-\sin t + \frac{1}{\sin t}\right) = a\left(\frac{1-\sin^2 t}{\sin t}\right)$$

$$= \frac{a\cos^2 t}{\sin t}$$

$$\frac{dy}{dt} = a\cos t$$

$$y_1 = \frac{dy}{dx} = \frac{\left(dy/dt\right)}{\left(dx/dt\right)} = \frac{a\cos t}{\left(\frac{a\cos^2 t}{\sin t}\right)} = \frac{a\sin t}{\cos t} = \tan t$$

$$y_2 = \frac{d^2 y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \sec^2 t\,\frac{\sin t}{a\cos^2 t} = \frac{\sin t}{a\cos^4 t}$$

Let $(\bar{x}, \bar{y})$ be the centre of curvature.

$$\bar{x} = x - \frac{y_1\left(1+y_1^2\right)}{y_2}$$

$$= a\left(\cos t + \log\tan\left(\frac{t}{2}\right)\right) - \frac{\tan t\left(1+\tan^2 t\right)}{\left(\frac{\sin t}{a\cos^4 t}\right)}$$

$$= a\left(\cos t + \log\tan\left(\frac{t}{2}\right)\right) - \frac{\sin t\left(\sec^2 t\right)a\cos^4 t}{\cos t\left(\sin t\right)}$$

$$\bar{y} = y + \frac{\left(1+y_1^2\right)}{y_2}$$

$$= a\sin t + \frac{1+\tan^2 t}{\left(\frac{\sin t}{a\cos^4 t}\right)} = a\sin t + \frac{a\cos^4 t\sec^2 t}{\sin t}$$

$$= a\sin t + \frac{a\cos^2 t}{\sin t} = a\left(\frac{\sin^2 t + \cos^2 t}{\sin t}\right) = \frac{a}{\sin t} \rightarrow (2)$$

$$(1) \Rightarrow \frac{\bar{x}}{a} = \log\tan\left(\frac{t}{2}\right)$$

$$e^{\left(\frac{\bar{x}}{a}\right)} = \tan\left(\frac{t}{2}\right)$$

$$(2) \Rightarrow \bar{y} = \frac{a}{\sin t} = \frac{a}{\left(\frac{2\tan\left(\frac{t}{2}\right)}{1+\tan^2\left(\frac{t}{2}\right)}\right)} = \frac{a\left[1+\tan^2\left(\frac{t}{2}\right)\right]}{2\tan\left(\frac{t}{2}\right)}$$

$$= \frac{a}{2}\left[\frac{1}{\tan\left(\frac{t}{2}\right)} + \tan\left(\frac{t}{2}\right)\right] = \frac{a}{2}\left[e^{-\left(\frac{\bar{x}}{a}\right)} + e^{\left(\frac{\bar{x}}{a}\right)}\right]$$

$$\bar{y} = a\cosh\left(\frac{\bar{x}}{a}\right).$$

The locus of $\left(\bar{x}, \bar{y}\right)$ is $y = a\cosh\left(\frac{x}{a}\right)$, which is the required catenary.

Example 5:

Find the equation of evolute of the curve $x^{2/3} + y^{2/3} = a^{2/3}$.

Solution:

The parametric coordinates of the curve is $x = a\cos^3\theta$ & $y = a\sin^3\theta$

$$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta\ , \qquad \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

$$y_1 = \frac{dy}{dx} = \frac{\left({dy}/{d\theta}\right)}{\left({dx}/{d\theta}\right)} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\tan\theta$$

$$= -\sec^2\theta\left(\frac{1}{-3a\cos^2\theta\sin\theta}\right)$$

$$y_2 = \frac{\sec^4\theta}{3a\sin\theta}$$

$$\bar{x} = x - \frac{y_1\left(1+y_1^2\right)}{y_2}$$

$$= a\cos^3\theta - \frac{(-\tan\theta)\left(1+\tan^2\theta\right)}{\frac{\sec^4\theta}{3a\sin\theta}} = a\cos^3\theta - \frac{(-\tan\theta)\left(\sec^2\theta\right)}{\frac{\sec^4\theta}{3a\sin\theta}}$$

$$= a\cos^3\theta + \frac{\left(\frac{\sin\theta}{\cos\theta}\right)\left(\frac{1}{\cos^2\theta}\right)}{\frac{\sec^4\theta}{3a\sin\theta}} = a\cos^3\theta + \frac{3a\sin\theta\sin\theta}{\cos\theta\cos^2\theta\left(\frac{1}{\cos^4\theta}\right)}$$

$$\bar{x} = a\cos^3\theta + 3a\sin^2\theta\cos\theta \rightarrow (1)$$

$$\bar{y} = y + \frac{\left(1+y_1^2\right)}{y_2}$$

$$= a\sin^3\theta - \frac{\left(1+\tan^2\theta\right)}{\frac{\sec^4\theta}{3a\sin\theta}} = a\sin^3\theta + \frac{\sec^2\theta}{\frac{\sec^4\theta}{3a\sin\theta}}$$

$$= a\sin^3\theta + \frac{\left(\frac{1}{\cos^2\theta}\right)}{\frac{\sec^4\theta}{3a\sin\theta}} = a\cos^3\theta + \frac{3a\sin\theta\cos^4\theta}{\cos^2\theta}$$

$$\bar{y} = a\sin^3\theta + 3a\cos^2\theta\sin\theta \rightarrow (2)$$

$(1)+(2) \Rightarrow$

$$\bar{x}+\bar{y} = a\cos^3\theta + 3a\sin^2\theta\cos\theta + a\sin^3\theta + 3a\cos^2\theta\sin\theta$$

$$= a\left(\cos^3\theta + 3\sin^2\theta\cos\theta + 3a\cos^2\theta\sin\theta + \sin^3\theta\right)$$

$$\bar{x}+\bar{y} = a(\cos\theta+\sin\theta)^3 \quad \rightarrow (3)$$

$$\bar{x}-\bar{y} = a\cos^3\theta + 3a\sin^2\theta\cos\theta - a\sin^3\theta - 3a\cos^2\theta\sin\theta$$

$$= a\left(\cos^3\theta + 3\sin^2\theta\cos\theta - 3\cos^2\theta\sin\theta - \sin^3\theta\right)$$

$$\bar{x}-\bar{y} = a(\cos\theta-\sin\theta)^3 \rightarrow (4)$$

From (3) and (4)

$$(\bar{x}+\bar{y})^{2/3} + (\bar{x}-\bar{y})^{2/3} = \left[a(\cos\theta+\sin\theta)^3\right]^{2/3} + \left[a(\cos\theta-\sin\theta)^3\right]^{2/3}$$

$$= a^{2/3}(\cos\theta+\sin\theta)^2 + a^{2/3}(\cos\theta-\sin\theta)^2$$

$$= a^{2/3}\left[\left(\cos^2\theta+\sin^2\theta+2\cos\theta\sin\theta\right)+\left(\cos^2\theta+\sin^2\theta-2\cos\theta\sin\theta\right)\right]$$

$$= a^{2/3}\left[2\left(\cos^2\theta+\sin^2\theta\right)\right] = a^{2/3}\left[2(1)\right]$$

$$(\bar{x}+\bar{y})^{2/3} + (\bar{x}-\bar{y})^{2/3} = 2a^{2/3}$$

The locus of $(\bar{x},\bar{y})$ is $(x+y)^{2/3}+(x-y)^{2/3} = 2a^{2/3}$ is the evolute of the astroid.

Example 6:

Find the equation of evolute of the curve $xy = c^2$.

Solution:

The parametric coordinates of the curve is $x = ct$ & $y = \frac{c}{t}$

$$\frac{dx}{dt} = c, \qquad \frac{dy}{dt} = \frac{-c}{t^2}$$

$$y_1 = \frac{dy}{dx} = \frac{\left(dy/dt\right)}{\left(dx/dt\right)} = \frac{\left(-c/t^2\right)}{c} = \frac{-1}{t^2}$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$

$$= \left(\frac{2}{t^3}\right)\frac{1}{c} = \frac{2}{ct^3}$$

$$\bar{x} = x - \frac{y_1\left(1+y_1^2\right)}{y_2}$$

$$= ct - \frac{\left(\frac{-1}{t^2}\right)\left(1+\frac{1}{t^4}\right)}{\frac{2}{ct^3}} = ct + \frac{ct^3}{2t^2}\left(\frac{t^4+1}{t^4}\right)$$

$$= ct + \frac{c\left(t^4+1\right)}{2t^3} = \frac{2ct^4 + c\left(t^4+1\right)}{2t^3}$$

$$\bar{x} = \frac{c\left(3t^4+1\right)}{2t^3} \rightarrow (1)$$

$$\bar{y} = y + \frac{\left(1+y_1^2\right)}{y_2}$$

$$= \frac{c}{t} + \frac{\left(1 + \frac{1}{t^4}\right)}{\frac{2}{ct^3}} = \frac{c}{t} + \frac{ct^3}{2}\left(\frac{t^4+1}{t^4}\right)$$

$$= \frac{c}{t} + \frac{c\left(t^4+1\right)}{2t} = \frac{2c + c\left(t^4+1\right)}{2t}$$

$$\bar{y} = \frac{c\left(t^4+3\right)}{2t} \rightarrow (2)$$

$(1)+(2) \Rightarrow$

$$\bar{x} + \bar{y} = \frac{c\left(3t^4+1\right)}{2t^3} + \frac{c\left(t^4+3\right)}{2t}$$

$$= \frac{c\left(3t^4+1+t^6+3t^2\right)}{2t^3} = \frac{c\left(t^6+3t^4+3t^2+1\right)}{2t^3}$$

$$= \frac{c\left(t^2+1\right)^3}{2t^3} = \frac{c}{2}\left(\frac{t^2+1}{t}\right)^3 = \frac{c}{2}\left(t + \frac{1}{t}\right)^3$$

$$\bar{x} + \bar{y} = \frac{c}{2}\left(t + \frac{1}{t}\right)^3 \rightarrow (3)$$

$$\bar{x} - \bar{y} = \frac{c\left(3t^4+1\right)}{2t^3} - \frac{c\left(t^4+3\right)}{2t}$$

$$= \frac{c\left(3t^4+1-t^6-3t^2\right)}{2t^3} = \frac{-c\left(t^6-3t^4+3t^2-1\right)}{2t^3}$$

$$= \frac{-c\left(t^2-1\right)^3}{2t^3} = \frac{-c}{2}\left(\frac{t^2-1}{t}\right)^3 = \frac{-c}{2}\left(t - \frac{1}{t}\right)^3$$

$$\bar{x} + \bar{y} = \frac{-c}{2}\left(t - \frac{1}{t}\right)^3 \rightarrow (4)$$

From (3) and (4)

$$\left(\bar{x}+\bar{y}\right)^{2/3} - \left(\bar{x}-\bar{y}\right)^{2/3} = \left[\frac{c}{2}\left(t+\frac{1}{t}\right)^3\right]^{2/3} - \left[\frac{-c}{2}\left(t-\frac{1}{t}\right)^3\right]^{2/3}$$

$$= \left(\frac{c}{2}\right)^{2/3}\left(t+\frac{1}{t}\right)^2 - \left(\frac{-c}{2}\right)^{2/3}\left(t-\frac{1}{t}\right)^2$$

$$= \left(\frac{c}{2}\right)^{2/3}\left(t+\frac{1}{t}\right)^2 - \left(\frac{c}{2}\right)^{2/3}\left(t-\frac{1}{t}\right)^2 \qquad \left(\square \left(\frac{-c}{2}\right)^{2/3} = \left(\frac{c}{2}\right)^{2/3}\right)$$

$$=\left(\frac{c}{2}\right)^{2/3}\left[\left(t^2+\frac{1}{t^2}+2\right)-\left(t^2+\frac{1}{t^2}-2\right)\right]$$

$$=\left(\frac{c}{2}\right)^{2/3}(4)=\left(\frac{c}{2}\right)^{2/3}(2)^2$$

$$=\left(\frac{c}{2}\right)^{2/3}(2)^2=\frac{c^{2/3}}{2^{2/3}}2^2=c^{2/3}2^{2-(2/3)}=c^{2/3}2^{(4/3)}$$

$$\left(\overline{x}+\overline{y}\right)^{2/3}-\left(\overline{x}-\overline{y}\right)^{2/3}=\left(4c\right)^{2/3}$$

The locus of $\left(\overline{x},\overline{y}\right)$ is $\left(x+y\right)^{2/3}-\left(x-y\right)^{2/3}=\left(4c\right)^{2/3}$ is the evolute of the rectangular hyperbola.

Exercise:

1. Find the evolute of the parabola $x^2 = 4ay$.

2. Find the evolute of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

3. Find the equation of the evolute of the curve $x^{2/3} + y^{2/3} = a^{2/3}$.

4. Prove that the evolute of the rectangular hyperbola $xy = c^2$ is

$$(x+y)^{2/3} - (x-y)^{2/3} = (4c)^{2/3}$$

Answers:

1. $27ax^2 = 4(y-2a)^3$ 2. $(ax)^{2/3} - (by)^{2/3} = \left(a^2+b^2\right)^{2/3}$ 3. $(x+y)^{2/3} + (x-y)^{2/3} = 2a^{2/3}$

Practice Quiz

Multiple Choice Questions

1.	The curvature of a function $f(x)$ is zero. Which of the following functions could be $f(x)$? (a) $ax+b$ (b) ax^2+bx+c (c) $\sin x$ (d) $\cos x$
2.	The curvature of the function $f(x)=x^2+2x+1$ at $x=0$ is? (a) $\frac{3}{2}$ (b) 2 (c) $\left\lvert \frac{2}{5^{3/2}} \right\rvert$ (d) 0
3.	Find the curvature of the function $f(x)=3x^3+4680x^2+1789x+181$ at $x=-520$ (a) 1 (b) 0 (c) ∞ (d) -520
4.	Given $x=k_1e^{a_1t}$, $y=k_2e^{a_2t}$ it is observed that the curvature function obtained is zero. What is the relation between a_1 and a_2? (a) $a_1 \neq a_2$ (b) $a_1 = a_2$ (c) $a_1=(a_2)^2$ (d) $a_2=(a_1)^2$
5.	Let $c(f(x))$ denote the curvature function of given curve $f(x)$. The value of $c(c(f(x)))$ is observe to be zero. Then which of the following functions could be $f(x)$? (a) $f(x)=x^2+y^2-23400$ (b) $f(x)=x^3+x+1$ (c) $f(x)=x^{19930}+x+90903$ (d) No such function exist
6.	The radius of curvature for y is given by the formula (a) $\rho=\dfrac{\frac{d^2y}{dx^2}}{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}}$ (b) $\rho=\dfrac{\frac{d^2y}{dx^2}}{\left(1+\frac{dy}{dx}\right)^{3/2}}$ (c) $\rho=\dfrac{\left(1+\frac{dy}{dx}\right)^{3/2}}{\frac{d^2y}{dx^2}}$ (d) $\rho=\dfrac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}}{\frac{d^2y}{dx^2}}$

Practice Quiz

7.	The radius of curvature of $y=x^3$ at $(1,1)$ is (a) 4.27 (b) 5.27 (c) 5 (d) 4
8.	The radius of curvature of $y=e^x$ at $(0,1)$ is (a) 2^2 (b) $2^{2/3}$ (c) $2^{3/2}$ (d) 2
9.	A telegraph wire hangs in the form of a curve $y=a\log\left(\sec\frac{x}{a}\right)$ where a is constant. Then the radius of curvature at any point is (a) $a\cos\left(\frac{x}{a}\right)$ (b) $\frac{1}{a}\sec\left(\frac{x}{a}\right)$ (c) $\frac{1}{a}\cos\left(\frac{x}{a}\right)$ (d) $a\sec\left(\frac{x}{a}\right)$
10.	The radius of curvature of the curve $\sqrt{x}+\sqrt{y}=1$ at the point $\left(\frac{1}{4},\frac{1}{4}\right)$ is (a) 2 (b) $\frac{1}{2}$ (c) $\sqrt{2}$ (d) $\frac{1}{\sqrt{2}}$
11.	Find the centre of curvature of $y=x^2$ at the origin. a) $\left(0,\frac{1}{2}\right)$ b) $\left(0,y+\frac{1}{2}\right)$ c) $\left(0,y+\frac{1}{2}\right)$
12.	Find the centre of curvature of the curve $y=3x^3+2x^2-3$ *at* $(0,-3)$. a) $\left(0,\frac{-11}{4}\right)$ b) $\left(0,\frac{1}{2}\right)$ c) $\left(\frac{-11}{4},\frac{1}{2}\right)$
13.	Involute is also known as (a) Evolute (b) Evolvent (c) Envelope (d) Tangent
14.	What is the evolute of parabola called? a) Cycloid b) Spiral c) Congruent parabola d) Semicubical parabola

Practice Quiz

15.	The name of the evolute of an ellipse is a) Centroid b) Astroid c) Asteroid d) Cycloid
16.	Definition of evolute of a curve is a) locus of centre of the given curve b) locus of centre of tangential curve c) locus of circumferential point on the curve d) locus of tangent to the curve
17.	Number of possible evolutes for a curve is a) Two b) Equal to radius c) One d) Infinity

Answers:

1. a	**2. c**	**3. b**	**4. b**	**5. a**	**6. d**
7. b	**8. c**	**9. c**	**10. d**	**11. b**	**12. a**
13. b	**14. d**	**15. b**	**16. a**	**17. c**	

Assignments

Assignment - 1

Q. No	Questions	K level	CO
1.	Find the radius of curvature at $(a,0)$ of the curve $xy^2 = a^3 - x^3$.	K1	CO2
2.	Show that the radius of curvature of the curve $x^2 y = a\left(x^2 + y^2\right)$ at $(-2a,-2a)$ is $-2a$.	K1	CO2
3.	Show that at any point on the cycloid $x = a(\theta - \sin\theta),\ y = a(1-\cos\theta)$, the radius of curvature is twice the normal at that point.	K1	CO2
4.	Find the radius of curvature at any point θ on the curve $x = 3a\cos\theta - a\cos 3\theta,\ y = 3a\sin\theta - a\sin 3\theta$.	K2	CO2
5.	Find the radius of curvature at any point on $x = e^t \cos t,\ y = e^t \sin t$.	K2	CO2
6.	Find the curvature for $r = ae^{\theta\cot\alpha}$.	K2	CO2
7.	For the cardioid $r = a(1+\cos\theta)$ prove that $\frac{\rho^2}{r}$ is constant.	K2	CO2
8.	Find the radius of curvature of the curve $r^n = a^n \cos n\theta$ at any point (r,θ). Hence prove that the radius of curvature of the curve $r^2 = a^2\cos 2\theta$ is $\frac{a^2}{3r}$.	K2	CO2

Assignments

Assignment - 2

Q. No	Questions	K level	CO
1.	Find the evolute of the tractrix $x = a\left(\cos t + \log\tan\frac{t}{2}\right), y = a\sin t.$	K1	CO2
2.	Find the evolute of the curve $x^2 + y^2 + 2x + 4y + 1 = 0.$	K1	CO2
3.	Show that the evolute of the curve $x = a(\cos\theta + \theta\sin\theta), y = a(\sin\theta - \theta\cos\theta)$ is a circle.	K1	CO2
4.	Show that the evolute of the cycloid $x = a(t + \sin t), y = a(1 - \cos t)$ is given by $x = a(t - \sin t), y - 2a = a(1 + \cos t).$	K2	CO2

Answers:

1. $y = a\cosh\frac{x}{a}$

2. $x^2 + y^2 = 5$

Part A

Q. No.	Questions	K Level	CO
1.	Find the curvature of the curve given by $s = c\tan\psi$ at $\psi = 0$. **Solution:** $s = c\tan\psi$ $\frac{ds}{d\psi} = c\sec^2\psi$ $\frac{d\psi}{ds} = \frac{1}{c\sec^2\psi} = \frac{\cos^2\psi}{c}$ $\left(\frac{d\psi}{ds}\right)_{\psi=0} = \frac{1}{c}.$ Curvature at $\psi = 0$ is $\frac{1}{c}.$	K1	CO2
2.	What is the curvature at any point on the curve $x^2 + y^2 - 4x - 6y + 11 = 0$. **Solution:** We know that the curvature of a circle of radius 'r' at any point is $\frac{1}{r}$ The given circle is $x^2 + y^2 - 4x - 6y + 10 = 0$ $(x-2)^2 + (y-3)^2 - 4 - 9 + 10 = 0$ $(i.e)\ (x-2)^2 + (y-3)^2 = 2$ The radius of the given circle r = $\sqrt{2}$ $\therefore$ Curvature $= \frac{1}{\sqrt{2}}$.	K1	CO2
3.	Find the radius of curvature of the curve $y = c\cosh\left(\frac{x}{c}\right)$ $at\ (0,c)$. **Solution:** $y = c\cosh\left(\frac{x}{c}\right)$ $y' = c\sin h\left(\frac{x}{c}\right)\left\{\frac{1}{c}\right\} = \sinh\left(\frac{x}{c}\right)$	K1	CO2

Part A

	$y'' = \frac{1}{c}\cosh(\frac{x}{c})$ At $(0,c)$, $y' = 0$, $y'' = \frac{1}{c}$ $\rho = \frac{\left[1+(y')^2\right]^{3/2}}{y''} = \frac{[1+0]^{3/2}}{1/c} = c$ $\rho = c$		
4.	Find the radius of curvature at x =1 on $y = \frac{\log x}{x}$. **Solution:** $y = \frac{\log x}{x}$ $y' = \frac{x\left(1/x\right) - \log x\{1\}}{x^2} = \frac{1-\log x}{x^2}$ $y'' = \frac{x^2\left(-1/x\right) - (1-\log x)\ (2x)}{x^4} = \frac{-x-2x+2x\log x}{x^4}$ $= \frac{-3x+2x\log x}{x^4}$ $\qquad y'' = \frac{2\log x - 3}{x^3}$ $\rho = \frac{\left[1+(1)^2\right]^{3/2}}{-3} = \frac{2\sqrt{2}}{-3}$ $\qquad \rho = \frac{2\sqrt{2}}{3}$.	K1	CO2
5.	Find the radius of curvature for $y = e^x$ at the point where it cuts the y-axis. **Solution:** The equation of y-axis is x=0 Put x =0 in $y = e^x \Rightarrow y = 1$. The curve $y = e^x$ cuts the y-axis at (0,1) $y = e^x$ $y' = e^x$ $y'' = e^x$ At (0,1) ρ $\rho_{(0,1)} = \frac{(1+1)^{3/2}}{1} = 2\sqrt{2}.$	K1	CO2

Part A

6.	Find the radius of curvature of the curve for $y^2 = 4ax$ at $y = 2a$. **Solution:** Given $y^2 = 4ax$ Diff. w. r. t. 'x', $2yy' = 4a \Rightarrow y' = \frac{4a}{2y} = \frac{2a}{y}$ $y'' = 2a\left(\frac{-1}{y^2}\right)y' = \frac{-2ay'}{y^2}$ At $y = 2a$, $(2a)^2 = 4ax \Rightarrow x = a$ At $(a, 2a)$ $y'_{(a,2a)} = 1$, $y''_{(a,2a)} = -\frac{1}{2a}$ $\therefore \rho = \frac{[1+1]^{3/2}}{\left(-1/2a\right)} = 2\sqrt{2} \times (-2a) = -4a\sqrt{2}$ Hence $\rho = 4a\sqrt{2}$.	K1	CO2
7.	Find ρ for the curve $x = at^2$, $y = 2at$ at any point 't'. **Solution:** Given $x = at^2$, $y = 2at$ $\frac{dx}{dt} = 2at$, $\frac{dy}{dt} = 2a.$ $\frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}.$ $\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right).\frac{dt}{dx} = \frac{d}{dt}\left(\frac{1}{t}\right)\frac{1}{2at}$ $= \left(-\frac{1}{t^2}\right)\frac{1}{2at} = -\frac{1}{2at^3}$ $\rho = \frac{\left[1+\left(\frac{1}{t}\right)^2\right]^{3/2}}{-1/2at^3} = \frac{(t^2+1)^{3/2}}{t^3} \times (-2at^3)$ $= -2a(1+t^2)^{3/2}$ $\therefore \rho = 2a(1+t^2)^{3/2}$	K1	CO2

Part A

8.	Find 'ρ' for the curve $x=a\cos\theta, y=a\sin\theta$, at '$\theta$'. **Solution:** Given $x=a\cos\theta$, $y=a\sin\theta$ $\frac{dx}{d\theta}=-a\sin\theta \quad , \quad \frac{dy}{d\theta}=a\cos\theta$ $\frac{dy}{dx}=\frac{a\cos\theta}{-a\sin\theta}=-\cot\theta$ $\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)$ $=\frac{d}{d\theta}\left(\frac{dy}{dx}\right)\frac{d\theta}{dx}=\frac{d}{d\theta}(-\cot\theta)\frac{1}{-a\sin\theta}$ $=\operatorname{cosec}^2\theta\left(\frac{1}{-a\sin\theta}\right)=\frac{-1}{a\sin^3\theta}$ $\therefore \rho=\frac{\left[1+(-\cot\theta)^2\right]^{\frac{3}{2}}}{\left(\frac{-1}{a\sin^3\theta}\right)}=(\operatorname{cosec}\theta)^3\times(-a\sin^3\theta)=-a$ $\therefore \rho=a$	K1	CO2
9.	Define Evolute Definition: The locus of the centre of curvature of a given curve is called evolute.	K2	CO2
10.	Given the co-ordinates of the centre of curvature of the curve as $\bar{x}=2a+3at^2$, $\bar{y}=-2at^3$, determine the evolutes of the curve. **Solution :** Let $\bar{x}=2a+3at^2$ $3at^2=\bar{x}-2a \quad \Rightarrow t^2=\frac{\bar{x}-2a}{3a}$ $(t^2)^3=\frac{(\bar{x}-2a)^3}{27a^3} \quad ------>(1)$ $\bar{y}=-2at^3 \quad \Rightarrow t^3=\frac{-\bar{y}}{2a}$ $(t^3)^2=\frac{+\bar{y}^2}{4a^2} \quad ------>(2)$ From (1) and (2) $\frac{(\bar{x}-2a)^3}{27a^3}=\frac{\bar{y}^2}{4a^2}$ $27a\bar{y}^2=4(\bar{x}-2a)^3$ Evolute of the Curve is $27ay^2=4(x-2a)^3$	K2	CO2

Part B

Q. No.	Questions	K Level	CO
1.	If ρ is the radius of curvature at any point (x, y) on the curve $y = \frac{ax}{a+x}$, Prove that $\left(\frac{2\rho}{a}\right)^{2/3} = \left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2$.	K2	CO2
2.	Find the radius of curvature at $(a,0)$ on the curve $xy^2 = a^3 - x^3$ **Ans:** The radius of curvature $\rho = \frac{3a}{2}$	K2	CO2
3.	Show that the measure of curvature of the curve $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$ at any point (x,y) on it is $\frac{ab}{2(ax+by)^{3/2}}$	K2	CO2
4.	Show that the radius of curvature at the point 'θ' on the curve $x = 3a\cos\theta - a\cos 3\theta, y = 3a\sin\theta - a\sin 3\theta$ is $3a\sin\theta$.	K2	CO2
5.	Find the centre of curvature at $t = \pi/2$ on the curve $x = 2\cos t + \cos 2t, y = 2\sin t + \sin 2t.$ **Ans:** $(\bar{x}, \bar{y}) = (1/3, 2/3).$	K1	CO2
6.	Find the equation of the circle of curvature of the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at $\left(\frac{a}{4}, \frac{a}{4}\right)$ **Ans:** The equation of circle of curvature is $\left(x - \frac{3a}{4}\right)^2 + \left(y - \frac{3a}{4}\right)^2 = \frac{a^2}{2}$	K1	CO2

Part B

7.	Find the equation of the circle of curvature of the curve $xy = 12$ Ans: The equation of circle of curvature is $$\left(x-\frac{43}{6}\right)^2+\left(y-\frac{57}{8}\right)^2=\left(\frac{125}{24}\right)^2$$	K2	CO2
8.	Find the evolute of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ **Ans:** The evolute of the hyperbola is $(ax)^{2/3}-(by)^{2/3}=\left(a^2+b^2\right)^{2/3}$	K2	CO2
9.	Show that the evolute of the cycloid $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$ is another cycloid ,given by $x=a(\theta-\sin\theta),\quad y-2a=a(1+\cos\theta)$.	K1	CO2
10.	Find the evolute of the curve $x^{2/3}+y^{2/3}=a^{2/3}$ **Ans:** The evolute of the curve is $(x+y)^{2/3}+(x-y)^{2/3}=2a^{2/3}$	K1	CO2
11.	Find the evolute of the parabola $x^2=4ay$ **Ans:** The evolute of the parabola $x^2=4ay$ is $27ax^2=4(y-2a)^3$	K2	CO2
12.	Show that the evolute of the tractrix $x=a\left(\cos\theta+\log\tan\frac{\theta}{2}\right)$, $y=a\sin\theta$ is the caternary $y=a\cosh\left(\frac{x}{a}\right)$	K2	CO2

- Online Course: NPTEL
  Course Name: :Curves and Surfaces
  Course Instructor: By Prof. Sudipta Dutta, Department of Mathematics and Statistics, IIT Kanpur
  Duration :4 weeks
  https://nptel.ac.in/courses/111/104/111104095/

- Online Course: SWAYAM
  Course Name: Differential Calculus
  Course Instructor: Dr. Bijumon Ramalayathil, Mahatma Gandhi College, Iritty, Keezhur PO, Kannur Dt – 670 703, Kerala
  Duration: 14 weeks (See Week 12)
  https://onlinecourses.swayam2.ac.in/cec20_ma22/preview

Real Time Applications

Practical method for estimating road curvatures using on board GPS and IMU equipment

Abstract:

This paper describes an experimental method to determine with high accuracy the curvature of a road segment, the turning radius of a car, and the discomfort level perceived by the passengers in the vehicle cabin when passing through a curve. For these experiments we used professional equipment provided with two GPS active antennas with 13 dB gain featuring non-contact 100 Hz speed and distance measurement, and a ten degree Inertial Measurement Unit (IMU) with dynamic orientation outputs. The same experimental measurements also used the low cost GPS equipment available on smartphones, domestic vehicle GPS devices, as well as an Arduino GPS shield in order to compare the results generated by professional equipment. The purpose of these experiments was also to establish if certain road curve sections were correctly executed in order to ensure the safety and comfort of passengers. Another use of the proposed method relates to the road accident reconstruction field, providing experts and forensics with an accurate method of measuring the roadway curvature at accident scenes or traffic events. The research and equipment described in this paper have been acquired and developed under a PhD study and a European funded project won and elaborated by the authors.

Real Time Applications

2 The impact of the road geometry in curves on the passengers' safety

Figure1 presents the ideal curvature of a road curve section by unifying two alignments, composed by two clothoid joints and a circular arc.

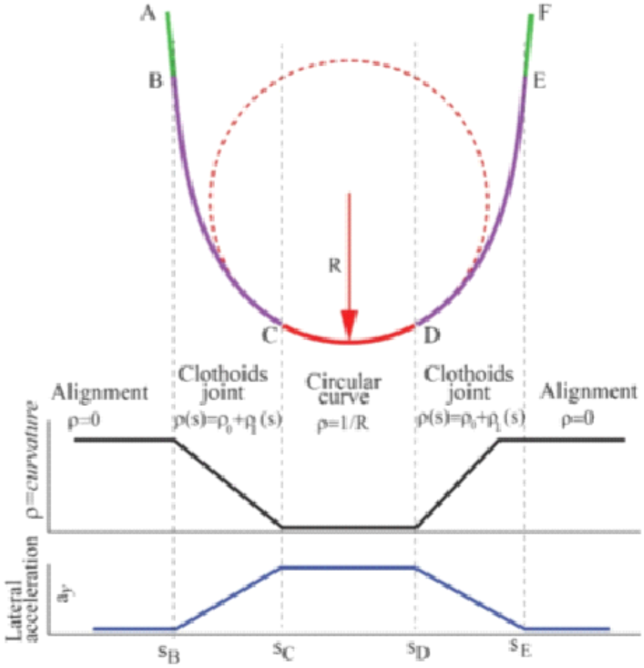


Figure 1. Transition to a very tight curve using two clothoid arcs before the alignment.

To Know More:

- **IOP Conference Series:** Materials Science and Engineering: Practical method forestimating road curvatures using on board GPS and IMU equipment
  https://iopscience.iop.org/article/10.1088/1757-899X/147/1/012114/pdf

Contents Beyond Syllabus

- **Research Article:** Radius of curvature analysis of oscillating follower cylindrical cam's expansion pitch curve

  https://journals.sagepub.com/doi/pdf/10.1177/1687814015596471

- **IOP Conference Series:** Materials Science and Engineering: Practical method forestimating road curvatures using onboard GPS and IMU equipment
  https://iopscience.iop.org/article/10.1088/1757-899X/147/1/012114/pdf

- **Research Article:** Influence of Curvature Radius on Jacking Pipe Construction on Mechanical Behavior
  http://www.ejge.com/2017/Ppr2017.0062ma.pdf

- **Research Article:** A study on the optimum curvature for the curved monitor
  https://www.tandfonline.com/doi/pdf/10.1080/15980316.2015.1111847?needAccess=true

Additional Resources

- https://www.math24.net/curvature-radius/
- https://www.khanacademy.org/math/multivariable-calculus/multivariable-derivatives/differentiating-vector-valued-functions/a/curvature
- https://www.math24.net/envelope-family-curves/
- https://www.math24.net/evolute-involute/
- Envelope and Evolutes, Evolute, Evolute of Curve Differential Calculus: https://www.youtube.com/watch?v=Yh1TQcS_byE

Mini Project

- **Calculate:** The local radius of curvature as well as the cumulative arc length and the curvature vector along a 1D curve in 2D or 3D space.

  Reference:

  https://in.mathworks.com/matlabcentral/fileexchange/69452-curvature-of-a-1d-curve-in-a-2d-or-3d-space

- **2D Line Curvature and Normals:** Accurate Curvature and Normals of a line/contour consisting of 2D points.

  Reference:

  https://in.mathworks.com/matlabcentral/fileexchange/32696-2d-line-curvature-and-normals

- **Radius of Curvature:** Use the concept of Radius of curvature to design a simple roller coaster track. Calculate the radius of curvature for a circular segment of the track.

- **Radius of Curvature:** Explore the properties of curved mirrors, such as concave and convex mirrors and calculate their focal points using the radius of curvature.

Prescribed Text Books & Reference Books

MATRICES AND CALCULUS	**22MA101**
S.No.	**TEXT BOOKS**
1	Erwin Kreyszig, "AdvancedEngineering Mathematics", John Wiley and Sons, 10^{th} Edition, New Delhi, 2016.
2	B.S. Grewal, "Higher Engineering Mathematics", Khanna Publishers, New Delhi, 43rd Edition, 2014.
REFERENCES :	
1	M. K. Venkataraman, "Engineering Mathematics, Volume I", 4^{th}Edition, The National Publication Company, Chennai, 2003.
2	SivaramakrishnaDass, C. Vijayakumari, "Engineering Mathematics", Pearson Education India, 4^{th}Edition 2019.
3	H. K. Dass, and Er. RajnishVerma, "Higher Engineering Mathematics", S. Chand Private Limited, 3^{rd} Edition 2014.
4	B.V. Ramana, "Higher Engineering Mathematics", Tata McGraw Hill Publishing Company,6^{th} Edition, New Delhi, 2008.
5	S.S. Sastry, "Engineering Mathematics", Vol.I&II, PHI Learning PrivateLimited,4^{th} Edition, New Delhi, 2014.
6	James Stewart, "Calculus: Early Transcendentals", Cengage Learning, 7^{th} Edition, New Delhi, 2015.

Thank you

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