Sunday, 7 December 2025 4:59 PM

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}$$

5.1: $A^TA = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$

Now, $\det(A^TA - \lambda I) = 0$

$\Rightarrow (17-\lambda)^2 - 64 = 0$

$\Rightarrow 17 - \lambda = \pm 8$ → $\lambda_1 = 25$, $\lambda_2 = 9$ } Eigenvalues.

Eigenvectors →

① $\lambda_1 = 25$: $\begin{bmatrix} -8 & 8 \\ 8 & -8 \end{bmatrix} \longrightarrow -8x + 8y = 0 \Rightarrow -x = y$ : $v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

② $\lambda_2 = 9$: $\begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix} \longrightarrow 8x + 8y = 0 \Rightarrow x = y$ : $v_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

Singular values →

$\sigma_1 = \sqrt{\lambda_1} = 5$

$\sigma_2 = \sqrt{\lambda_2} = 3$

$$\Sigma_{3\times 2} = \begin{bmatrix} 5 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}$$

Now, $V = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Rightarrow V^T = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

**U:** $u_i = \frac{1}{\sigma_i} A v_i$

$$u_1 = \frac{1}{5}\begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$= \frac{1}{5\sqrt{2}}\begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$u_2 = \frac{1}{3}\begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$= \frac{1}{3\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

$$\therefore \boxed{U = \begin{bmatrix} 1/\sqrt{2} & 1/3\sqrt{2} & u_1 \times u_2 \\ 1/\sqrt{2} & -1/3\sqrt{2} & \\ 0 & 4/3\sqrt{2} & \end{bmatrix}}$$

**5.2:** $A_1 = \sigma_1 u_1 v_1^T$

$$= 5\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= 5\begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \\ 0 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 2.5 & 2.5 \\ 2.5 & 2.5 \\ 0 & 0 \end{bmatrix}}$$

**5.3:** $\|A\|_F = \sqrt{\sum \sigma_i^2} = \sqrt{25+9} = \sqrt{34} \approx \boxed{5.83}$

Energy by Rank-1 approx. $= \sigma_1^2 = \boxed{25}$

Total energy $= 34$

$\therefore$ % transmitted $= \frac{25}{34} \times 100 = \boxed{73.5\%}$