Advanced Algorithms/Analysis

Graphs and DFS (Read Ch. 3 from reference book (Ch 22 from textbook))
CS 721
Fal 2018

September 4, 2018

Graph Algorithms: Why Graph?

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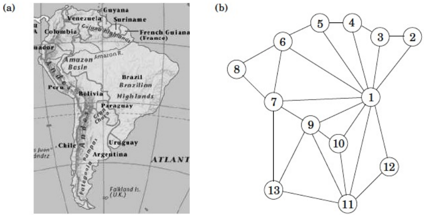

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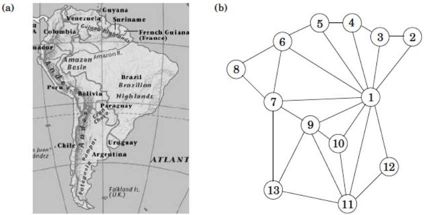

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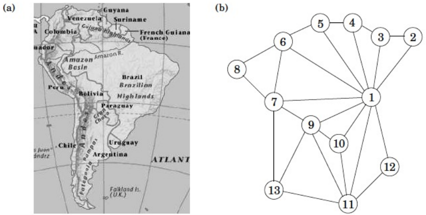

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Graph

- Formally a graph is specified by a set of vertices V and a set of edges E between the selected pair of points
  - Between two nodes x and y, if the edge is undirected we represent it as $\{x, y\}$
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    - $O(|E|)$ storage while the worst case time to check whether an edge is present in a graph or not in $O(n)$ time.

Which representation is better?

- Number of edges $|E|$ can be colse to $O(n)$ or close to $O(n^2)$.
  - Dense graph: When $|E|$ is close to $O(n^2)$.
  - Dense graph: When $|E|$ is colse to $O(n)$
- Exactly where $|E|$ lies in this range is a cruicial factor in selecting teh right graph algorithm.

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procedure explore(G, v)
Input:    G = (V, E) is a graph; v ∈ V
Output:   visited(u) is set to true for all nodes u reachable from v

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for each edge (v, u) ∈ E:
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- The previsit and postvisit procedures are optional, meant for performing operations on a vertex when it is first discovered and also when it is being left for the last time. We will soon see some creative uses for them.

Explore function

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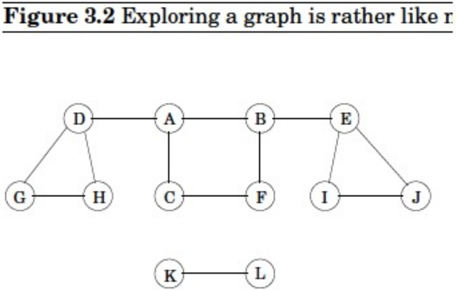
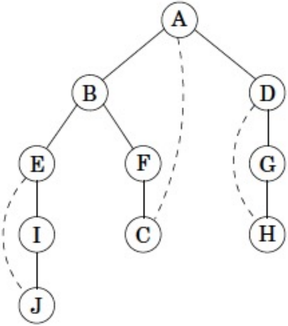
- How do we confirm explore works correctly?
  - Proof in class

Example of explore function





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  - Each vertex is explored just once
  - During exploration of a vertex there are the folowing steps
    1. Fixed amount of work to mark this vertex is visited and pre/postvisit
    2. A loop in which adjacent edges are scanned, if they lead somewhere new

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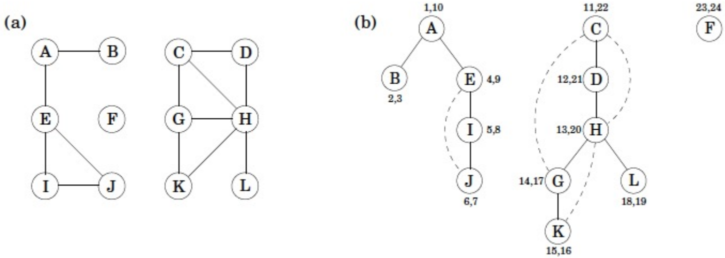
- Thus total running time of DFS is $O(|V| + |E|)$
  - This is efficient because this running time is of the same order that is required to to read the entire graph in adjacency list representation.

Example of DFS

Figure 3.6 (a) A 12-node graph. (b) DFS search forest.



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- Fig 3.6 is not connected but there are three disjoint connected connected regions: $\{A, B, E, I, J\}, \{C, D, G, H, K, L\}, \{F\}$
  - these reregions are called connected components
- DFS is trivially adapted to check if a graph is connected and, more generally, to assign each node v an integer $ccnum[v]$ identifying the connected component to which it belongs.

```
procedure previsit(v)
ccnum[v] = cc
```

  - where cc needs to be initialized to zero and to be incremented each time the DFS procedure calls explore.

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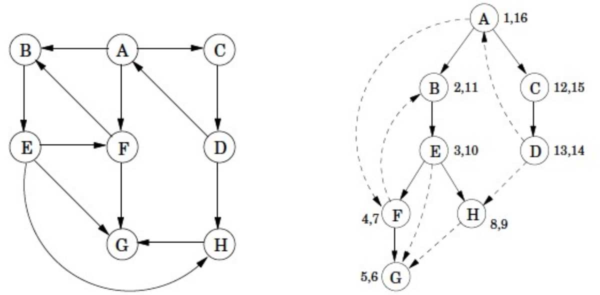
- **Property:** For any nodes u and v, the two intervals $[pre(u), post(u)]$ and $[pre(v), post(v)]$ are either disjoint or one is contained within the other.

DFS in directed graph

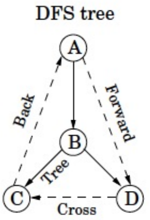
- Our depth-first search algorithm can be run verbatim on directed graphs, taking care to traverse edges only in their prescribed directions.
- Fig 3.7 shows an example

Figure 3.7 DFS on a directed graph.



Types of edges in DFS forest

- There are three types of edges
  - **Tree edges** are actually part of the DFS forest (solid lines). Edge (u, v) is a tree edge if v was discovered by exploring the edge (u, v).
  - **Forward edges** lead from a node to a nonchild descendant in the DFS tree.
  - **Back edges** lead to an ancestor in the DFS tree.
  - **Cross edges** lead to neither descendant nor ancestor; they therefore lead to a node that has already been completely explored (that is, already postvisited).



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- Because of the depth-first exploration strategy, vertex u is an ancestor of vertex v exactly in those cases where u is discovered first and v is discovered during *explore*(u). This is to say $pre(u) < pre(v) < post(v) < post(u)$
- Summary of the various possibilities for an edge (u, v) is as follows

pre/post *ordering for* (u, v)	*Edge type*
$[_u \; [_v \;]_v \;]_u$	Tree/forward
$[_v \; [_u \;]_u \;]_v$	Back
$[_v \;]_v \; [_u \;]_u$	Cross

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- **Property:** A directed graph has a cycle if and only if its depth-first search reveals a back edge.
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- Directed acyclic graphs, or DAGs for short, come up all the time. They are good for modeling relations like causalities, hierarchies, and temporal dependencies.

DAG and topological sort

- If a graph is DAG it is possible to linearize or topologically sort its vertices so that before performing a task (represented by a vertex) all other tasks (represented by other vertices) that are connected to it must be completed

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- Topological-Sort(G)
  1. Call $dfs(G)$ to compute $post[v]$ for each vertex v
  2. As each vertex is done visiting (post is updated), insert onto the front of a linked list
  3. Return linked list of vertices

Running time of topological sort

- Time for DFS is $O(|V| + E|)$ time to insert onto linked list is $O(|V|)$ (constant time for each insert)

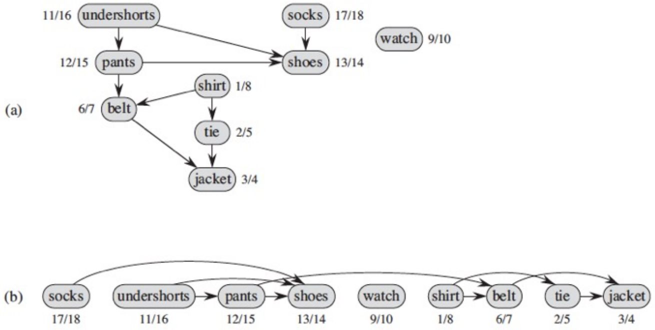

Figure 22.7 (a) Professor Bumstead topologically sorts his clothing when getting dressed. Each directed edge (u, v) means that garment u must be put on before garment v. The discovery and finishing times from a depth-first search are shown next to each vertex. **(b)** The same graph shown topologically sorted, with its vertices arranged from left to right in order of decreasing finishing time. All directed edges go from left to right.

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  - If an edge is (u, v) is a tree/forward edge then $post[v] < post[u]$.
  - If an edge (u, v) is a cross edge (u, v) can not be an edge of G. So order of their post values does not matter.

Source and sink in a DAG

- Since DAG can be linearized by decreasing *post* number, the vertex with smallest post number comes last in linearization
  - It must be a "sink"– no outgoing edges
- Similarly, the node with highest *post* is a "source", a node with no incoming edge.

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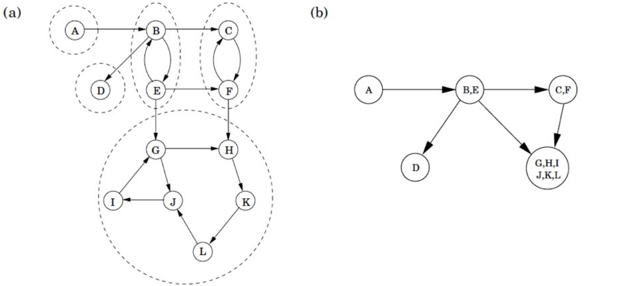
- This relation partitions V into disjoint sets that we call "strongly connected components"
- In the following graph there are five strongly connected components

Figure 3.9 (a) A directed graph and its strongly connected components. (b) The meta-graph.



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  - At the top level we have a DAG (which is rather simple structure and can be linearize)
  - If we want further details, we can look inside one of this nodes of this DAG and examine ful-fledged strongly connected component within

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  - **Property 1:** If the *explore* subroutine is started at a node u, then it will terminate precisely when all nodes reachable from u has been visited.
    - Therefore if we call *explore* on a node that lies somewhere in a sink strongly connected component, then we will retrieve exactly the component

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- We can find strongly connected components of a directed graph in linear time making use of depth first search
- The algorithm is based on some properties that we have seen before
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    - (B) How we we continue once the first component is discovered?

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- Property 3 can be restated as ,- *The strongly connected components can be linearize by arranging them in decreasing order of their highest post numbers.*

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  - This can be done by considering the reverse graph G^R, which is same as G except direction of each edge is reversed
  - G^R has exactly the same strongly connected components as G
  - So if we do a depth first search on G^R, the node with highest *post* number comes from a source strongly connected component of G^R which is a sink strongly connected component of G (Problem A solved!)

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  - This solution is also provided by property 3
  - Once we have found the first strongly connected component and deleted it from graph, the node with the highest *post* number among those remaining will belong to sink strongly connected component of whatever is remains of G
  - Therefore we can keep using the *post* numbers from our initial depth first search on G^R to successfully output the second strongly connected component, the third strongly connected component and so on.

Efficient algorithm for strongly connected components

- The resulting algorithm is as follows:
- $\underline{StronglyConnectedComponent(G)}$
    1. Run depth first search on G^R
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- This algorithm runs in linear time. Only the constant in linear time is about twice that of straight depth first search.