Advanced Algorithms/Analysis

BFS, Dijkstra's algorirhm and Bellman Ford algorithm
(Read Ch. 4 from reference book)
CS 721
Fal 2018

September 11, 2018

DFS and shortest path

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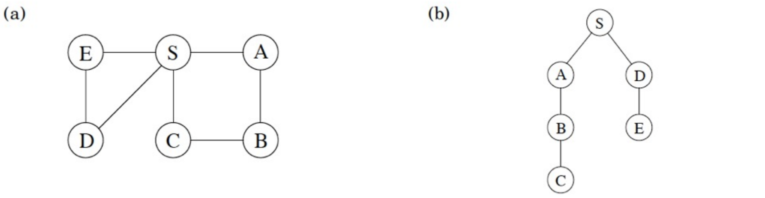
- To get a feeling for this, consider the following graph , where each edge weight is 1 and, DFS starts from node S

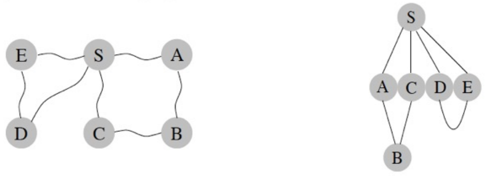
Figure 4.1 (a) A simple graph and (b) its depth-first search tree.



Breadth-first search (BFS): Basic idea

- Consider a graph where each edgee is of same unit length (weight=1)
- Consider each edge to be a string and we want to traverse the graph from a start node S
- One possibility is let the graph hang from node S and traverse the graph level by level as shown in the following figure

Figure 4.2 A physical model of a graph.



- BFS graph search tries to implement this idea

BFS Algorithm

- Here is the algorithm for BFS

Figure 4.3 Breadth-first search.

```
procedure bfs(G, s)
Input:    Graph G = (V, E), directed or undirected; vertex s ∈ V
Output:   For all vertices u reachable from s, dist(u) is set
          to the distance from s to u.

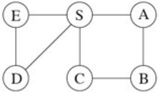
for all u ∈ V:
   dist(u) = ∞

dist(s) = 0
Q = [s] (queue containing just s)
while Q is not empty:
   u = eject(Q)
   for all edges (u, v) ∈ E:
      if dist(v) = ∞:
         inject(Q, v)
         dist(v) = dist(u) + 1
```

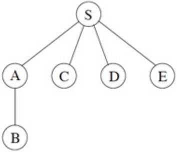
BFS Algorithm output

- Here is the result of running BFS on the same graph

Figure 4.4 The result of breadth-first search on the graph of Figure 4.1.



Order of visitation	Queue contents after processing node
	$[S]$
S	$[A\ C\ D\ E]$
A	$[C\ D\ E\ B]$
C	$[D\ E\ B]$
D	$[E\ B]$
E	$[B]$
B	$[\,]$



- Unlike DFS tree, it has the property that all paths from S are shortest possible

Running time of BFS

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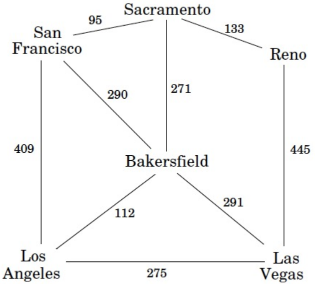
  - So total running time is $O(|V| + |E|)$

- How do we use DFS if edge weights are positive but unequal?

Figure 4.5 Edge lengths often matter.



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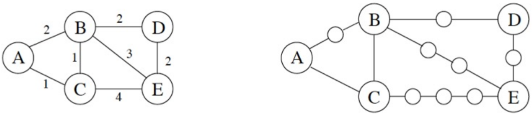

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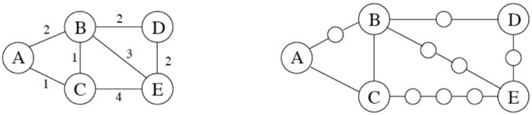

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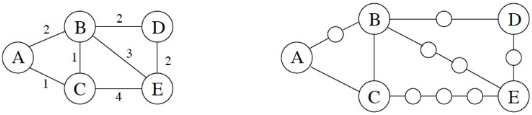

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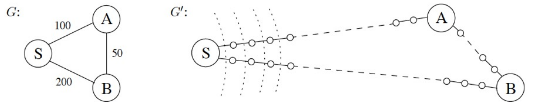
- e can perform BFS on G' and get our answer but that would be very inefficient

Alarm clock algorithm

- BFS on G' is mostly uneventful but we can set an alarm and update alarm time as we discover new nodes

Figure 4.7 BFS on G' is mostly uneventful. The dotted lines show some early "wavefronts."



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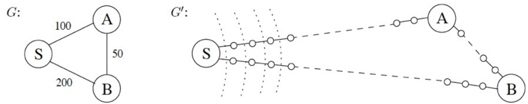

- This gives rise to alarm clock algorithm

- Set an alarm clock for node s at time 0.
- Repeat until there are no more alarms:

  Say the next alarm goes off at time T, for node u. Then:

  - The distance from s to u is T.
  - For each neighbor v of u in G:
    - If there is no alarm yet for v, set one for time $T + l(u, v)$.
    - If v's alarm is set for later than $T + l(u, v)$, then reset it to this earlier time.

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  - **Delete-min:** Return the element with smallest key and remove it from the queue
  - **Make-queue:** Build a priority queue with given elements and their key values

Dijkstra's algorithm

- Dijkstra's shortest path algorithm

Figure 4.8 Dijkstra's shortest-path algorithm.

```
procedure dijkstra(G, l, s)
Input:    Graph G = (V, E), directed or undirected;
          positive edge lengths {l_e : e ∈ E}; vertex s ∈ V
Output:   For all vertices u reachable from s, dist(u) is set
          to the distance from s to u.

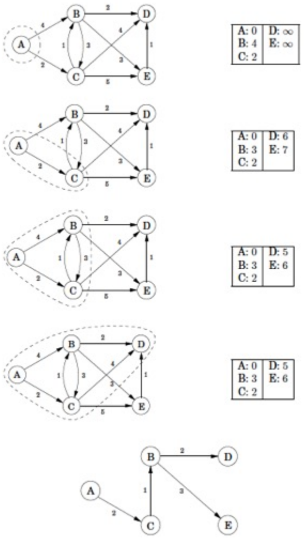
for all u ∈ V:
   dist(u) = ∞
   prev(u) = nil
dist(s) = 0

H = makequeue(V)  (using dist-values as keys)
while H is not empty:
   u = deletemin(H)
   for all edges (u, v) ∈ E:
      if dist(v) > dist(u) + l(u, v):
         dist(v) = dist(u) + l(u, v)
         prev(v) = u
         decreasekey(H, v)
```

Dijkstra's algorithm result

- A complete run of Dijkstra's shortest path algorithm

Figure 4.9 A complete run of Dijkstra's algorithm, with node A as the starting point. Also shown are the associated `dist` values and the final shortest-path tree.



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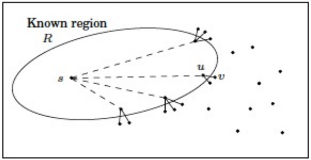
- An alternative way of computing shortest path is as follows:
  - Expand outward from the starting point s, steadily growing the region of the graph to which distances and shortest paths are known.
    - This growth should be orderly, first incorporating the closest nodes and then moving on to those further away.
    - More precisely, when the known region is some subset of vertices R that includes s, the next addition to it should be the node outside R that is closest to s. Let us call this node v; now the question is: how do we identify it?

Figure 4.10 Single-edge extensions of known shortest paths.



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    - The answer is, the shortest of all extended paths: it is the node outside R for which the smallest value of $dist(s, u) + l(u, v)$ is attained, as u ranges over R.
    - In other words, try all single-edge extensions of the currently known shortest paths, find the shortest such extended path, and proclaim its endpoint to be the next node of R.

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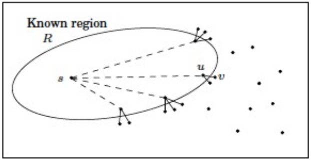

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  - All other extensions will have been assessed previously and do not need to be recomputed.
  - In the following pseudocode, $dist(v)$ is the length of the currently shortest single-edge-extended path leading to v; it is ∞ for nodes not adjacent to R.

```
Initialize dist(s) to 0, other dist(·) values to ∞
R = { } (the ''known region'')
while R ≠ V:
    Pick the node v ∉ R with smallest dist(·)
    Add v to R
    for all edges (v, z) ∈ E:
        if dist(z) > dist(v) + l(v, z):
            dist(z) = dist(v) + l(v, z)
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- For priority queue implementation all three operations (insert, deletemin,decrease-key) take $O(\log(|V|))$ time.
- Thus total running time of Dijkstra's algorithm is $O\left((|V| + |E|)\log(|V|)\right)$.

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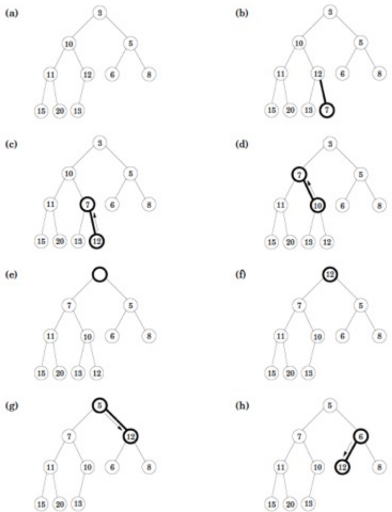
  - To **deletemin**, return the root value, remove this element from the heap, then take the last node in the tree (in the rightmost position in the bottom row) and place it at the root and let it sift down: if it is bigger than either child, swap it with the smaller child and repeat. Again this takes O(log n) time.

Binary heap

- The regularity of a complete binary tree makes it easy to represent using an array.
  - If there are n nodes, this ordering specifies their positions $1, 2, ..., n$ within the array.
  - Moving up and down the tree is easily simulated on the array, using the fact that node number j has parent $\lfloor j/2 \rfloor$ and children $2j$ and $2j + 1$.

Binary heap

Figure 4.11 (a) A binary heap with 10 elements. Only the key values are shown. (b)–(d) The intermediate "bubble-up" steps in inserting an element with key 7. (e)–(g) The "sift-down" steps in a delete-min operation.



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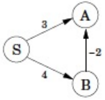
- This no longer holds when edge lengths can be negative.



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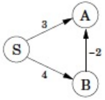

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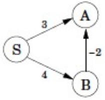

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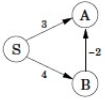

- What needs to be changed in order to accommodate this new complication?
  - To answer this, observe that dist values it maintains are always either overestimates or exactly correct.
  - They start off at ∞, and the only way they ever change is by updating along an edge:

<u>procedure update</u>$((u, v) \in E)$
$\texttt{dist}(v) = \min\{\texttt{dist}(v), \texttt{dist}(u) + l(u, v)\}$

Shortest path in presence of negative edge contd.

- The update operation says that the distance to v cannot possibly be more than the distance to u, plus $l(u, v)$. It has the following properties.

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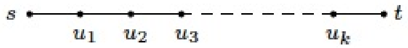
- This operation is extremely useful: it is harmless, and if used carefully, will correctly set distances.
- In fact, Dijkstras algorithm can be thought of simply as a sequence of updates.
- We know this particular sequence doesnt work with negative edges, but is there some other sequence that does?

Shortest path in presence of negative edge contd.

- To get a sense of the properties this sequence must possess, lets pick a node t and look at the shortest path to it from s.



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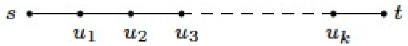

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- However, if we dont know all the shortest paths beforehand, how can we be sure to update the right edges in the right order?

Shortest path in presence of negative edge: Bellman Ford algorithm

- Here is an easy solution:
  - simply update all the edges, $|V| - 1$ times! The resulting $O(|V||E|)$ procedure is called the Bellman-Ford algorithm.

Figure 4.13 The Bellman-Ford algorithm for single-source shortest paths in general graphs.

```
procedure shortest-paths(G, l, s)
Input:    Directed graph G = (V, E);
          edge lengths {l_e : e ∈ E} with no negative cycles;
          vertex s ∈ V
Output:   For all vertices u reachable from s, dist(u) is set
          to the distance from s to u.

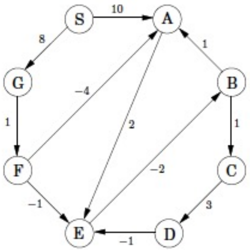
for all u ∈ V:
   dist(u) = ∞
   prev(u) = nil

dist(s) = 0
repeat |V| − 1 times:
   for all e ∈ E:
      update(e)
```

```
procedure update((u, v) ∈ E)
dist(v) = min{dist(v), dist(u) + l(u, v)}
```

Bellman Ford algorithm

Figure 4.14 The Bellman-Ford algorithm illustrated on a sample graph.



	Iteration							
Node	0	1	2	3	4	5	6	7
S	0	0	0	0	0	0	0	0
A	∞	10	10	5	5	5	5	5
B	∞	∞	∞	10	6	5	5	5
C	∞	∞	∞	∞	11	7	6	6
D	∞	∞	∞	∞	∞	14	10	9
E	∞	∞	12	8	7	7	7	7
F	∞	∞	9	9	9	9	9	9
G	∞	8	8	8	8	8	8	8

- A note about implementation:
  - for many graphs, the maximum number of edges in any shortest path is substantially less than $|V| - 1$, with the result that fewer rounds of updates are needed.
  - Therefore, it makes sense to add an extra check to the shortest-path algorithm, to make it terminate immediately after any round in which no update occurred.

Negative cycle

- The shortest-path problem is ill-posed in graphs with negative cycles.
  - As might be expected, Bellman Ford algorithmworks only in the absence of such cycles.
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  - But where did this assumption appear in the derivation of the algorithm?
    - Well, it slipped in when we asserted the existence of a shortest path from s to t.
- Fortunately, it is easy to automatically detect negative cycles and issue a warning.
  - Instead of stopping after $|V| - 1$ iterations, perform one extra round. There is a negative cycle if and only if some dist value is reduced during this final round.

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    - Notice that our scheme doesnt require edges to be positive.
    - In particular, we can find longest paths in a DAG by the same algorithm: just negate all edge lengths.

Single source shortest paths in DAGs

Figure 4.15 A single-source shortest-path algorithm for directed acyclic graphs.

```
procedure dag-shortest-paths(G, l, s)
Input:    Dag G = (V, E);
          edge lengths {l_e : e ∈ E}; vertex s ∈ V
Output:   For all vertices u reachable from s, dist(u) is set
          to the distance from s to u.

for all u ∈ V:
   dist(u) = ∞
   prev(u) = nil

dist(s) = 0
Linearize G
for each u ∈ V, in linearized order:
   for all edges (u, v) ∈ E:
      update(u, v)
```