$$p_n(s) = f_0 + \Delta f_0 s + \frac{\Delta^2 f_0}{2!} s(s-1) + \frac{\Delta^3 f_0}{3!} s(s-1)(s-2) + \ldots$$

$$= T_0 + T_1 + T_2 + \ldots + T_n \qquad (12.5)$$

where $s = (x - x_0)/h$

and $h = x_{i+1} - x_i$

12.3 TRAPEZOIDAL RULE

The trapezoidal rule is the first and the simplest of the Newton-Cotes formulae. Since it is a two-point formula, it uses the first order interpolation polynomial $p_1(x)$ for approximating the function $f(x)$ and assumes $x_0 = a$ and $x_1 = b$. This is illustrated in Fig. 12.2. According to Eq. (12.5), $p_1(x)$ consists of the first two terms T_0 and T_1. Therefore, the integral for trapezoidal rule is given by

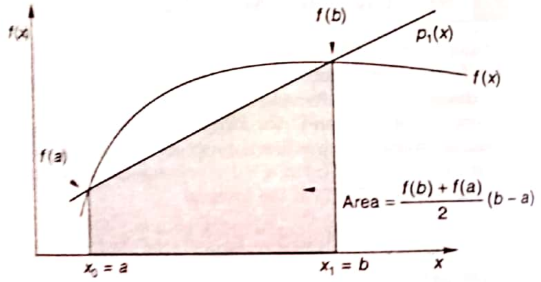

Fig. 12.2 Representation of trapezoidal rule

$$I_t = \int_a^b (T_0 + T_1)\,dx$$

$$= \int_a^b T_0\,dx + \int_a^b T_1\,dx = I_{t1} + I_{t2}$$

Since T_i are expressed in terms of s, we need to use the following transformation:

$dx = h \times ds$

$x_0 = a, \quad x_1 = b \quad$ and $\quad h = b - a$

At $x = a, \quad s = (a - x_0)/h = 0$

At $x = b, \quad s = (b - x_0)/h = 1$

Then,

$$I_{t1} = \int_a^b T_0\,dx = \int_0^1 h f_0\,dx = h f_0$$

$$I_{t2} = \int_a^b T_1\,dx = \int_0^1 \Delta f_0\, s h\,ds = h\frac{\Delta f_0}{2}$$

Therefore,

$$I_t = h\left[f_0 + \frac{\Delta f_0}{2}\right] = h\left[\frac{f_0 + f_1}{2}\right]$$

Since $f_0 = f(a)$ and $f_1 = f(b)$, we have

$$\boxed{I_t = h\,\frac{f(a) + f(b)}{2} = (b - a)\,\frac{f(a) + f(b)}{2}} \qquad (12.6)$$

Note that the area is the *product of width of the segment* $(b - a)$ *and average height of the points* $f(a)$ *and* $f(b)$.

Error Analysis

Since only the first two terms of eq. (12.5) are used for I_t, the term T_2 becomes the remainder and, therefore, the truncation error in trapezoidal rule is given by

$$E_{tt} = \int_a^b T_2\,dx = \frac{f''(\theta_s)}{2}\int_0^1 s(s-1)h\cdot ds$$

$$= \frac{f''(\theta_s)h}{2}\left[\frac{s^3}{3} - \frac{s^2}{2}\right]_0^1 = -\frac{f''(\theta_s)}{12}h$$

Since $dx/ds = h$,

$$f''(\theta_s) = h^2 f''(\theta_x),$$

we obtain

$$\boxed{E_{tt} = -\frac{h^3}{12} f''(\theta_x)} \qquad (12.7)$$

where $a < \theta_x < b$

Example 12.1

Evaluate the integral

$$I = \int_a^b (x^3 + 1)\,dx$$