Gauss integration assumes an approximation of the form

$$I_g = \int_{-1}^{1} f(x)\,dx \approx \sum_{i=1}^{n} w_i f(x_i) \qquad (12.28)$$

Equation (12.28) contains $2n$ unknowns to be determined. These unknowns can be determined using the condition given in the integration formula (12.28). This should give the exact value of the integral for polynomials of as high a degree as possible.

Let us find the Gaussian quadrature formula for $n = 2$. In this case, we need to find the values of w_1, w_2, x_1 and x_2. Let us assume that the integral will be exact up to cubic polynomials. This implies that the functions 1, x, x^2 and x^3 can be numerically integrated to obtain exact results.

$$w_1 + w_2 = \int_{-1}^{1} dx = 2$$

$$w_1x_1 + w_2x_2 = \int_{-1}^{1} x\,dx = 0$$

$$w_1x_1^2 + w_2x_2^2 = \int_{-1}^{1} x^2\,dx = \frac{2}{3}$$

$$w_1x_1^3 + w_2x_2^3 = \int_{-1}^{1} x^3\,dx = 0$$

Solving these simultaneous equations, we obtain

$$w_1 = w_2 = 1$$

$$x_1 = -\frac{1}{\sqrt{3}} = -0.5113502$$

$$x_2 = \frac{1}{\sqrt{3}} = 0.5773502$$

Thus, we have the Gaussian quadrature formula for $n = 2$ as

$$\int_{-1}^{1} f(x)\,dx = f\left(-1/\sqrt{3}\right) + f\left(1/\sqrt{3}\right) \qquad (12.29)$$

This formula will give correct value for integral of $f(x)$ in the range (−1, 1) for any function up to third-order. Equation (12.29) is also known as *Gauss-Legendre* formula. Two-point Gauss quadrature is illustrated in Fig. 12.7.