We know that $dx = h \times ds$ and $s$ varies from 0 to 2 (when $x$ varies from $a$ to $b$). Thus,

$$I_{s11} = \int_0^2 f_0 h \, ds = 2hf_0$$

$$I_{s12} = \int_0^2 \Delta f_0 sh \, ds = 2h\Delta f_0$$

$$I_{s13} = \int_0^2 \frac{\Delta^2 f_0}{2} s(s-1)h \, ds = \frac{h}{3}\Delta^2 f_0$$

Therefore,

$$I_{s1} = h\left[ sf_0 + 2\Delta f_0 + \frac{\Delta^2 f_0}{3} \right] \tag{12.11}$$

Since $\Delta f_0 = f_1 - f_0$ and $\Delta^2 f_0 = f_2 - 2f_1 + f_0$, equation (12.11) becomes

$$\boxed{I_{s1} = \frac{h}{3}[f_0 + 4f_1 + f_2] = \frac{h}{3}[f(a) + 4f(x_1) + f(b)]} \tag{12.12}$$

This equation is called *Simpson's 1/3 rule*. Equation (12.12) can also be expressed as

$$I_{s1} = (b-a)\frac{f(a) + 4f(x_1) + f(b)}{6}$$

This shows that the area is given by *the product of total width of the segments and weighted average of heights $f(a)$, $f(x_1)$ and $f(b)$.*

## Error Analysis

Since we have used only the first three terms of Eq. (12.5), the truncation error is given by

$$E_{ts1} = \int_a^b T_3 \, dx$$

$$= \frac{f'''(\theta_s)}{6}\int_0^2 s(s-1)(s-2)h \, ds$$

$$= \frac{f'''(\theta_s)}{6}\left[\frac{s^4}{4} - s^3 + s^2\right]_0^2$$

Since the third-order error term turns out to be zero, we have to consider the next higher term for the error. Therefore,

$$= \frac{f^{(4)}(\theta_s)}{4!}\int_0^2 s(s-1)(s-2)(s-3)h \, ds$$

$$= \frac{h \times f^{(4)}(\theta_s)}{24}\left[\frac{s^5}{5} - \frac{6s^4}{4} + \frac{11s^3}{3} - \frac{6s^2}{2}\right]_0^2$$

$$= -\frac{hf^4(\theta_s)}{90}$$

Since $f^4(\theta_s) = h^4 f^{(4)}(\theta_x)$, we obtain

$$\boxed{E_{ts1} = -\frac{h^5}{90} f^{(4)}(\theta_x)} \tag{12.13}$$

where $a < \theta_x < b$. It is important to note that Simpson's 1/3 rule is exact up to degree 3, although it is based on quadratic equation.

### Example 12.3

Evaluate the following integrals using Simpson's 1/3 rule

(a) $\int_{-1}^{1} e^x \, dx$ (b) $\int_0^{\pi} \sqrt{\sin x} \, dx$

*Case (a)*

$$I = \int_{-1}^{1} e^x \, dx$$

$$I_{s1} = \frac{h}{3}[f(a) + f(b) + 4f(x_1)]$$

$$h = \frac{b-a}{2} = 1$$

$$f(x_1) = f(a+b)$$

Therefore,

$$I_{s1} = \frac{e^{-1} + 4e^0 + e^{+1}}{3} = 2.36205$$

(Note that $I_{s1}$ gives better estimate than $I_{ct}$ when $n = 2$. This is because $I_{s1}$ uses quadratic equation while $I_{ct}$ uses a linear one)

*Case (b)*

$$I = \int_0^{\pi/2} \sqrt{\sin(x)} \, dx = \pi/4$$

$$I_{s1} = \frac{\pi}{12}[f(0) + 4f(\pi/4) + f(\pi/2)]$$

$$= 0.2617993(0 + 3.3635857 + 1)$$

$$= 1.1423841$$