Higher-order Derivatives

We can also obtain approximations to higher-order derivatives using Taylor's expansion. To illustrate this, we derive here the formula for $f''(x)$. We know that

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + R_1$$

and

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(x) + R_2$$

Adding these two expansions gives

$$f(x+h) + f(x-h) = 2f(x) + h^2 f''(x) + R_1 + R_2$$

Therefore

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} - \frac{(R_1 + R_2)}{h^2}$$

Thus, the approximation to second derivative is

$$\boxed{f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}} \tag{11.14}$$

The truncation error is

$$E_t(h) = -\frac{R_1 + R_2}{h^2}$$

$$= -\frac{1}{h^2}\frac{h^4}{4!}(f^{(4)}(\theta_1) + f^{(4)}(\theta_2))$$

$$= -\frac{h^2}{12}f^{(4)}(\theta)$$

The error is of order h^2.

Similarly, we can obtain other higher-order derivatives with the errors of order h^3 and h^4.

Example 11.4

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