Numerical Integration

→ A definite integral of the form,

$$I = \int_a^b f(x)\, dx$$

which be treated as the area under the curve

$y = ax$,

→ A better alternative approach could be to use the technique that uses simple arithmetic ~~calculatio~~ operations to compute the area which can be easily implemented on computer — this approach is called "Numerical Integration" or "Numerical curbature" with the concept of continuous summation ~~of~~ to find area.

→ There is a set of method known as Newton Cotes rule; in which sampling points are equally spaced.

→ We also discuss a method Rhombus integration that is design to ~~extrapolate~~ improve the Newton Cotes formula.

○ Newton-Cote's Formula:

→ It is the most popular and widely used integration formula, it forms the ~~equal~~ basis for a number of numerical integration methods known as Newton's-Cotes method.

→ The derivation of Newton Cotes formula is based on polynomial interpolation — An n^{th} polynomial $P_n(x)$ that interpolate the value of $f(x)$ at $n+1$ evenly spaced points can be used to replace the integrant $f(x)$ of the integral $I = \int_a^b f(x)\, dx$ and the resultant formula is called $n+1$ point Newton-Cotes formula.

→ If the limits of the integration a, b are in the set of interpolating points x_i $[x_1, x_2, \dots, x_n]$ then the formula is said to closed ~~fo~~ formed.

If the limits of the integration are beyond the ~~set~~ set of interpolation points then the formula is said to open formed.

They include Trapezoidal Rule (2 point formula), Simpson's 1/3 (3 point), Simpson's 3/8 Rule (4 point formula), Weddel's Rule (6 point formula)

→ Let, $I = \int_a^b y\,dx$ where y takes the values $y_0, y_1, y_2 \dots y_n$ for $x = x_0, x_1, x_2, \dots, x_n$

Let, the interval of integration $[a, b]$ be divided into n equal sub-intervals, each of width $h = \frac{b-a}{n}$. So, that $x_0 = a$, $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, ... $x_n = x_0 + nh = b$

$$\therefore I = \int_{x_0}^{x_0+nh} f(x)\,dx$$

Since; any x is given by $x + x_0 = uh$ and $dx = h\,du$

$$I = h\int_0^n f(x_0 + uh)\,du$$

$$= h\int_0^n \left[y_0 + u\Delta y_0 + \frac{u(y-1)}{2!}\Delta^2 y_0 + \dots \right] du \quad \left[\text{Using newton's Fwd interpolation}\right]$$

$$= h\left[u.y_0 + \frac{u^2}{2!}\Delta y_0 + \frac{1}{2}\left[\frac{u^3}{3} - \frac{u^2}{2}\right]\Delta^2 y_0 + \dots\dots \right]_0^n$$

$$= nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 \dots \right]$$

This is the general formula of Newton's-Cote's.

Commission, West Bengal

(I) Trapezoidal Rule (composite):

Putting $n=1$ in formula & taking the curve through (x_0, y_0) & (x_1, y_1) as a polynomial of degree so that differences of order higher than one vanish.

we get,

$$\int_{x_0}^{x_0+h} f(x)\,dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right]$$

$$= \frac{h}{2}\left[2y_0 + (y_1 - y_0)\right]$$

$$= \frac{h}{2}\left[y_0 + y_1\right]$$

Similarly, for the next interval $(x_0 - h, x_0 + 2h)$, we get,

$$\int_{x_0+h}^{x_0+2h} f(x)\,dx = \frac{h}{2}\left[y_1 + y_2\right]$$

$\dots\dots\dots$

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx = \frac{h}{2}(y_{n-1} + y_n)$$

Adding the above integrals, we get;

$$\boxed{\int_{0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1})\right]}$$

Q. Integrate $f(x) = 4x - 3x^2$ from 0 to 1 by trapezoidal rule by taking 10 sub intervals

Ans: $h = \frac{1-0}{10} = 0.1$

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
$f(x)$	0.00	0.37	0.68	0.93	1.12	1.25	1.32	1.33	1.28

x	0.9	1.0
$f(x)$	1.17	1

$$\therefore \int_0^1 (4x-3x^2)dx = \frac{0.1}{2}\Big[0+1+2[0.37+0.68+0.93+1.112+1.25+1.32+1.33+1.28+1.17]\Big]$$

$$= 0.995$$

Again,

$$\int_0^1 (4x-3x^2)dx = \left[\frac{4x^2}{2} - \frac{3x^3}{3}\right]_0^1 = 1 \quad \text{[By calculus]}$$

$\therefore$ Abs. error = |True Value − Appx. value|

$$= |1-0.995| = 0.005$$

Relative error $= \dfrac{\text{Abs. error}}{\text{True value.}} = \dfrac{0.005}{1} = 0.005$

Percentage error = Relative error × 100% = 0.5%

Q. Find $\int_1^2 \frac{e^x}{x} dx$ by taking $n = 4$ using trapezoidal

$$\therefore f(x) = \frac{e^x}{x}$$

$$\therefore h = \frac{2-1}{4} = 0.25$$

x	1.0	1.25	1.50	1.75	2.0
$f(x)$	2.7183	2.7923	2.9878	3.2883	3.6945

$$\therefore \int_1^2 y dx = \int_1^2 \frac{e^x}{x} dx = \frac{0.25}{2}\Big[2.7183 + 3.6945 + 2(2.7923 + 2.9878 + 3.2883)\Big]$$

$$= 3.0687$$

Note:

Error of Trapezoidal rule is of order h^2

→ **Algorithm:**

(S1) Define $f(x)$

(S2) Enter the values of upper and lower limit a, b

(S3) Enter the number of steps, N

(S4) Ns = 1

(S5). h = ~~(a-b)/N~~ (b-a)/N

(S6) sum = 0

(S7) do {

sum = sum + h/2 × ((f(a) + f(a+h)));

a = a + Ns * h;

} while (a<b);

(S8) print Sum

(S9) Stop.

(II) Simpson's 1/3 rule (composite):

Putting $n=2$ in the newton's-cotes and taking the curve through (x_0, y_0), (x_1, y_1), and (x_2, y_2) as a polynomial of degree two so, that differences of order higher than two vanish, we get.

$$\int_{x_0}^{x_0+2h} f(x)dx = 2h\left[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right]$$

$$= 2h\left[y_0 + (y_1 - y_0) + \frac{1}{6}(y_2 - 2y_1 + y_0)\right]$$

$$= \frac{h}{3}\left[y_0 + 4y_1 + y_2\right]$$

Similarly,

$$\int_{x_0+2h}^{x_0+4h} f(x)dx = \frac{h}{3}\left[y_2 + 4y_3 + y_4\right]$$

. . . . . .

$$\int_{x_0+(n-2)h}^{x_0+nh} f(x)dx = \frac{h}{3}\left[y_n + 4y_{n-1} + y_n\right]$$

Adding the above integrals, we get

$$\boxed{\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\Big[(y_0+y_n) + 4(y_1+y_3+\dots+y_{n-1}) + 2(y_2+y_4+\dots+y_{n-2})\Big]}$$

- When we use this formula, the interval (x_0, x_n) must be divided into an even number of subintervals of width h.
- Simpson's 1/3 rule (composite) is true for odd number of points.

Q. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using Simpson's 1/3 rule taking $h = \frac{1}{4} = 0.25$.

Ans:

x	0.0	0.25	0.5	0.75	1.0
$f(x)$	1.00	0.94	~~0.80~~ 0.80	0.64	0.50

$$\therefore \int_0^1 \frac{dx}{1+x^2} = \frac{h}{3}\left[1.00 + 0.5 + 4(0.94 + 0.64) + 2(0.8)\right]$$

$$= 0.785392$$

Q. Evaluate $\int_1^2 e^{-x/2}\,dx$ using Simpson's 1/3 rule with 4 intervals.

Ans: $\therefore h = \frac{2-1}{4} = 0.25$

x	1.00	1.25	1.5	1.75	2.00
$f(x)$	0.60653	0.53526	0.47235	0.41686	0.36788

$$\therefore \int_1^2 e^{-x/2}\,dx = \frac{h}{3}\Big[0.60653 + 0.36788 + 4(0.53526 + 0.41686) + 2(0.47235)\Big]$$

$$= 0.4773025$$

Q. $\int_0^1 \frac{1}{1+x}\,dx$ by simpson's 1/3 rule taking 11 ordinates and hence find the value of $\log_e 2$ (correct upto five sign. figures)

$$h = \frac{1-0}{10} = 0.1$$

x	0.0	0.1	0.2	0.3	0.4	0.5
$f(x)$	1.0000	0.9091	0.8333	0.7692	0.7143	0.6667

x	0.6	0.7	0.8	0.9	0.10
$f(x)$	0.6250	0.5882	0.5558	~~0.5263~~ 0.5263	0.5000

$$\therefore \int_0^1 \frac{1}{1+x}\,dx = \frac{0.1}{3}\Big[1.000 + 0.5000 + 4(0.9091 + 0.7692 + 0.6667 + 0.5882 + 0.5263) + 2(0.8333 + 0.7143 + 0.6250 + 0.5556)\Big]$$

$$= 0.69315$$

Now, from calculas we have,

$$\int_0^1 \frac{1}{1+x}\,dx = [\log_e|1+x|]_0^1 = \log_e 2 = \ln 2$$

$$\therefore \log_e 2 = 0.69315$$

→ Algorithm:
Input: Assume function tabulated at $n+1$ (odd) number of points.

(S1) for $i = 1$ to $n+1$ do
read f_i

end for
(S2) sum ← $f_1 + f_{n+1}$
(S3) for $i = 2$ to n in steps of 2 do
sum ← sum + $4f_i$
(S4)
end for
(S5) for $i = 3$ to $n-1$ in steps of 2 do
sum ← sum + $2f_i$
(S6)
end for
(S7) Integral ← $(h \times \text{sum})/3$
(S8) write integral
(S9) stop

(IV) Simpson's 3/8 rule:
Here, we should always consider $n = 3$
∴ Points should be $(n+1) = (3+1) = 4$.
∴ $x_0 \rightarrow y_0$, $x_1 \rightarrow y_1$, $x_2 \rightarrow y_2$, $x_3 \rightarrow y_3$

∴ The formula to find the integral value using Simpson's 3/8 rule:

$$\int_{x_0}^{x_3} y\,dx = \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + y_3\right]$$

Q. Evaluate $\int_1^2 (x^3+1)\,dx$ using simpson's 3/8 rule.

Ans: ∴ $n = 3$ [for simpson's 3/8 rule]

$h = \frac{2-1}{3} = \frac{1}{3}$

x	1	$\frac{4}{3}$	$\frac{5}{3}$	2
$f(x)$	2.0000	3.3704	5.6296	9.0000

$$\therefore \int_1^2 (x^3+1)\,dx = \left[2.0000 + 9.0000 + 3\times(3.3704+5.6296)\right]\frac{3\times 1}{8\times 3}$$

$$= 4.75$$

Now, using calculus:

$$\int_1^2 (x^3+1)\,dx = \left[\frac{x^4}{4} + x\right]_1^2 = \left[\frac{2^4}{4}+2\right] - \left[\frac{1}{4}+1\right]$$

$$= \left[\frac{16}{4}+2\right] - \left[\frac{1}{4}+1\right]$$

$$= 5 - \frac{1}{4} = 4.75$$

$\therefore$ Abs error $= 0$

Q. Evaluate $\int_0^{\pi/2} \sqrt{\sin x}\,dx$ using simpson's 3/8 rule.

Ans: $h = \frac{\frac{\pi}{2} - 0}{3} = \frac{\pi}{6}$ [As, $n = 3$ for simpson's 3/8 rule]

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$f(x)$	0	0.7071	0.9306	1

$$\therefore \int_0^{\pi/2} \sqrt{\sin x}\,dx = \frac{\cancel{3}\times\pi}{8\times\cancel{6}_2}\left[0+1+3(0.7071+0.9306)\right]$$

$$= 1.1905$$

Note: It is only applicable for $n = 3$. So this is a limitation.

→ ~~Algorithm~~:

(V) Weddle's Rule :

Substituting $n=6$ in the New's Cot's formula, we have

$$\int_{x_0}^{x_0+6h} f(x)\,dx = h\Big[6y_0 + 18\Delta y_0 + 27\Delta^2 y_0 + 24\Delta^3 y_0 + \frac{123}{10}\Delta^4 y_0 + \frac{33}{10}\Delta^5 y_0 + \frac{41}{140}\Delta^6 y_0\Big]$$

Then if we write in terms of y_i,

$$\int_{x_0}^{x_0+6h} f(x)\,dx = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\right]$$

Similarly, for x_6 to x_{12} we have

$$\int_{x_6}^{x_0+12h} f(x)\,dx = \frac{3h}{10}\left[y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}\right]$$

$\vdots$

For x_{n-6} to x_n we have

$$\int_{x_{n-6}}^{x_n} f(x)\,dx = \frac{3h}{10}\left[y_{n-6} + 5y_{n-5} + y_{n-4} + 6y_{n-3} + y_{n-2} + 5y_{n-1} + y_n\right]$$

$$\therefore \int_{x_0}^{x_n} f(x)\,dx = \frac{3h}{10}\Big[(y_0 + y_2 + y_4 + y_8 + y_{10} + \cdots) + 5(y_1 + y_5 + y_7 + y_{11} + \cdots) + 6(y_3 + y_9 + \cdots) + 2(y_6 + y_{12} + \cdots)\Big]$$

(*) $n = 6k$ always, k = constant (any)

→ Algorithm :

(S1) Define $f(x)$

(S2) Enter the values of upper and lower limit of b, a

(S3) Enter the number of steps, N

(S4) Ns = 6

(S5) $h = \{(b-a)/N\}/N_s$

(S6) sum = 0

(S7) do {

sum = sum + $3h/10$ (($f(a) + 5(f(a+h)) + f(a+2h) + 6(f(a+3h))$
$+ f(a+4h) + 5(f(a+5h)) + f(a+6h)$);

a = a + Ns * h;

} while (a<b);

(S8) Print sum

(S9) stop

Q. $\int_4^{5.2} \log_2 x\,dx$ using weddles rule. taking $n=6$.

$\therefore\ h = \frac{5.2-4}{6} = 0.2$

x	4.0	4.2	4.4	4.6	4.8
$f(x)$	0.6020	0.6232	0.6435	~~0.6812~~ 0.6627	~~0.6990~~ 0.6812

x	5.0	5.2
$f(x)$	0.6990	0.7160

$$\therefore\ I = \int_4^{5.2} \log_2 x\,dx = \frac{3\times 0.2}{10}\left[0.6020 + 5\times 0.6232 + 0.6435 + 6\times 0.6627 + 0.6812 + 5\times 0.6990 + 0.7160\right]$$

$= 0.793794$

Q. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ where $h = 1/12$ using weddles rule.

x	0	1/12	1/6	1/4	1/3	5/12
$f(x)$	0	0.9931	0.9730	0.9412	0.9000	0.8521

x	$10/12$	$7/12$	$2/3$	$3/4$	$5/6$	$11/12$
$f(x)$	0.8000	0.7461	0.6923	0.6400	0.5902	0.5434

x	1
$f(x)$	0.5000

$$\therefore \int_0^1 \frac{dx}{1+x^2} = \frac{3\times 1}{10\times 12}\Big[0 + 5\times 0.9331 + 0.9730 + 6\times 0.9412 + 0.9 + 5\times 0.852 + 0.8 + 5\times 0.7461 + 0.6923 + 6\times 0.6400 + 0.5902 + 5\times 0.5434 + 0.5\Big]$$

$$= 0.75878 = 0.7$$

Note: subinterval of 6k.

1. Weddels rule computes using subinterval of 6k.
2. Error in weddles rule is given by:

Romberg Integration:

→ Why we need Romberg Integration?

① By increasing the number of subintervals (i.e. by decreasing h) this decreases the magnitude of error terms. Here, the order of the method is fixed.

② By using higher-order methods — this eliminates the lower-order error terms. Here, the ~~the~~ order of the method is varied and, therefore this method is known as variable-order approach.

The variable-order method can be implemented using Richardson extrapolation technique. As we know this techique involves combining two estimates of a given order to obtain a third estimate of higher order. The method that incorporates this process (i.e. Richardson's extrapolation) to the trapezoidal rule is called Romberg Integration

→ **Formulae:**

① $h = b - a$

$$R(0,0) = \frac{h}{2}\left[f(a) + f(b)\right]$$

② $\cancel{R_{i,0}}$ $R(i,0) = \dfrac{R(i-1,0)}{2} + h_i \sum_{k=1}^{2^{i-1}} f(x_{2k-1})$ for $i = 1, 2, \ldots$

$$h_i = (b-a)/2^i$$

$$x_k = a + kh_i$$

③ $$R_{ij} = \frac{4^j R_{i,j-1} - R_{i-1,j-1}}{4^j - 1}$$

→ **A short Illustration:**

$$\begin{tikzcd}
R(0,0) \arrow[d] \arrow[dr] & & & \\
R(1,0) \arrow[r] \arrow[d] \arrow[dr] & R(1,1) \arrow[dr] & & \\
R(2,0) \arrow[r] \arrow[d] \arrow[dr] & R(2,1) \arrow[r] \arrow[dr] & R(2,2) \arrow[dr] & \\
R(3,0) \arrow[r] & R(3,1) \arrow[r] & R(3,2) \arrow[r] & R(3,3)
\end{tikzcd}$$

Q. Evaluate

$$\int_0^{\pi/2} \frac{\cos x}{\sqrt{1+\sin x}}\, dx$$ using Romberg integration.

Ans: ~~R(0,0) =~~ $h = \dfrac{\frac{\pi}{2} - 0}{2} = \dfrac{\pi}{4}$

$$R(0,0) = \frac{\pi}{8}\left[\frac{1}{\sqrt{1+0}} + \frac{0}{\sqrt{1+1}}\right]$$

$$= \frac{\pi}{8} = 22.5$$

$$\therefore \; h_1 = \left(\frac{\pi}{2} - 0\right)/2^1 = \frac{\pi}{4}$$

$$h_2 = \left(\frac{\pi}{2} - 0\right)/2^2 = \frac{\pi}{8}$$

and

$$x_1 = 0 + 1\frac{\pi}{4} = \frac{\pi}{4}$$

$$x_3 = 0 + 3\frac{\pi}{8} = \frac{3\pi}{8}$$

$$\therefore \; R(1,0) = \frac{R(0,0)}{2} + h_1 f(x_1)$$

$$= \frac{22.5}{2} + \frac{\pi}{4} \times 0.5412$$

$$= 35.604$$

$$\therefore \; R(2,0) = \frac{R(1,0)}{2} + h_2 \left[f(x_1) + f(x_3)\right]$$

$$= \frac{35.604}{2} + \frac{\pi}{8}\left[0.5412 + 0.2759\right]$$

$$= \cancel{\frac{35.604}{2} + 080 \cdot \frac{\pi}{8}} = 36.18675$$

Now,

$$R(1,1) = \frac{4R(1,0) - R(0,0)}{3}$$

$$= \frac{(4 \times 35.604) - (22.5)}{3} = 39.972$$

$$R(2,1) = \frac{4R(2,0) - R(1,0)}{3}$$

$$= \frac{(4 \times 36.18675) - 35.604}{3} =$$

$$= 36.381$$

$$R(2,2) = \frac{16R(2,1) - R(1,1)}{15}$$

$$= \frac{(16 \times 36.381) - 39.972}{15} = 36.1416$$

- Gaussian Quadrature:

→ Also known as Gaussian method / Gauss-Legendre formula.

→ Gaussian quadrature formula for $n = 2$:

$$w_1 = w_2 = 1$$

$$x_1 = -\frac{1}{\sqrt{3}}$$

$$x_2 = +\frac{1}{\sqrt{3}}$$

gauss integration assumes an approximation of the form

$$I_g = \int_{-1}^{1} f(x)\,dx = \sum_{i=1}^{n} w_i f(x_i)$$

$$\therefore I_g = \int_{-1}^{1} f(x)\,dx = w_1 f(x_1) + w_2 f(x_2)$$

$$= 1 \times f\left(\frac{-1}{\sqrt{3}}\right) + 1 \times f\left(\frac{1}{\sqrt{3}}\right)$$

$$\boxed{\Rightarrow I_g = \int_{-1}^{1} f(x)\,dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)}$$

Q. Compute $\int_{-1}^{1} e^x\,dx$ using two-point Gauss-Legendre formula.

Ans: $I = \int_{-1}^{1} \exp(x)\,dx = f(x_1) + f(x_2)$

where x_1 & x_2 are gaussian quadrature points and are given by,

$x_1 = -\frac{1}{\sqrt{3}} = -0.5773502$ | $x_2 = \frac{1}{\sqrt{3}} = 0.5773502$

Therefore,

$$I = \exp(-0.5773502) + \exp(0.5773502)$$

$$= 0.5613839 + 1.7813122 = 2.3426961$$

→ Now for any interval a to b.

Assumption of formula $I_g = \int_a^b f(x)dx = c\int_{-1}^{1} g(z)dz$

Now, ~~x = b~~

Assume the following transformation betn a & the new variable

$$x = Az + B$$

This must satisfy the following conditions:

At:

$x = a$, $z = -1$ and $x = b$, $z = 1$

i.e.

$$B - A = a$$
$$A + B = b$$

Then,

$$A = \frac{b-a}{2} \quad \& \quad B = \frac{a+b}{2}$$

$$\therefore \boxed{x = \frac{b-a}{2} z + \frac{a+b}{2}}$$

$$dx = \frac{b-a}{2} dz$$

∴ This implies that, $c = \frac{b-a}{2}$

∴ Then the integral becomes $\boxed{\frac{b-a}{2}\int_{-1}^{1} g(z)dz}$

The Gaussian for the integration is

$$\boxed{\frac{b-a}{2}\int_{-1}^{1} g(z)\,dz = \frac{b-a}{2}\sum_{i=1}^{n} w_i\, g(z_i)}$$

8. Compute the integral

$$I = \int_{-2}^{2} e^{-x/2}\, dx$$

using Gaussian two-point formula.

Ans: $I_g = \frac{b-a}{2}\left[w_1 g(z_1) + w_2 g(z_2)\right]$ $\quad [\because n=2]$

$$x = \frac{b-a}{2} z + \frac{b+a}{2} = \frac{2+2}{2} z + \frac{0}{2} = 2z$$

$$\therefore g(z) = e^{-2z/2} = e^{-z}$$

$\because n = 2$, ~~the formula~~:

$w_1 = w_2 = 1$

$z_1 = -\frac{1}{\sqrt{3}} \qquad z_2 = \frac{1}{\sqrt{3}}$

$\frac{b-a}{2} = \frac{2+2}{2} = 2$

$$\therefore I_g = \cancel{2[\exp} \; 2\left[\exp(-1/\sqrt{3}) + \exp(1/\sqrt{3})\right]$$

$$= 4.6853922$$

→ Now for $n > 2$, we just need to know w_i's & z_i's & then $\sum_{i=1}^{n} w_i f(z_i)$ as given earlier [w_i & z_i will be given]

→ Note:

The difference bet^n analytical answer of an integration (by calculas) & numerical answer is due to round off error. Round off error can be minimized by increasing the precision.

→ Algorithm:

(S1) Define function $f(x)$

(S2) Obtain integration ~~lib~~ limit a, b

(S3) Decide number of inter ~~pto~~ polating point n

(S4) Read the gaussian parameter (w_i, z_i)

(S5) compute x using $x = \frac{b-a}{2} z + \frac{b+a}{2}$

(S6) Compute I_g, $I_g = \frac{b-a}{2} \sum_{1}^{n} w_i f(z_i)$

(S7) Write Result

(S8) Stop.

Numerical Integration

- Newton cotes
  - Trapezoidal
  - Simpson 1/3
  - Simpson 3/8
  - weddle's rule
  - Boole's rule
- Richardson Extrapolation
- Gaussian Legendre

* Boole's Rule: $(n=4)$ $\int_a^b y\,dx = \frac{2h}{45}\left[7f_0 + 32f_1 + 12f_2 + 32f_3 + 7f_4\right]$

Error: $-h.h^7 \cdot 945 f^6 \theta_x$ [where $0 < \theta_x < 1$]