Algorithm 14.2

14.4 SOLVING EIGENVALUE PROBLEMS

As mentioned earlier, some boundary value problems, when simplified, may result in a set of homogeneous equation of the type

$$\begin{aligned}(a_{11} - \lambda)x_1 + a_{12}x_2 + \dots + a_1 n x_n &= 0\\ a_{21}x_1 + (a_{22} - \lambda)x_2 + \dots + a_{2n}x_n &= 0\\ &\cdots\\ &\cdots\\ a_{n1}x_1 + a_{n2}x_2 + \dots + (a_{nn} - \lambda)x_n &= 0\end{aligned} \tag{14.7}$$

where λ is a scalar constant. Equation (14.7) may be expressed as

$$[\mathbf{A} - \lambda\mathbf{I}]\ [\mathbf{X}] = 0 \tag{14.8}$$

where **I** is the *identity matrix* and $[\mathbf{A} - \lambda\mathbf{I}]$ is called the *characteristic matrix* of the coefficient matrix **A**.

The homogeneous Eq. (14.8) will have a non-trivial solution if, and only if, the characteristic matrix is singular. That is, the matrix $[\mathbf{A} - \lambda\mathbf{I}]$ is not invertible. Then, we have

$$|\mathbf{A} - \lambda\mathbf{I}| \tag{14.9}$$

Expansion of the determinant will result in a polynomial of degree n in λ.

$$\lambda^n - p_1\lambda^{n-1} p_2\lambda^{n-2} - \dots - p_{n-1}\lambda - p_n = 0 \tag{14.10}$$

Equation (14.10) will have n roots $\lambda_1, \lambda_2, \dots, \lambda_n$. The equation is known as the *characteristic polynomial* (or *characteristic equation*) and the roots are known as the *eigenvalues or characteristic values* of the matrix **A**. The solution vectors $X_1, X_2, \dots, X_n$ corresponding to the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ are called the *eigenvectors*.

The roots representing the eigenvalues may be real distinct, real repeated, or complex, depending on the nature of the coefficient matrix **A**. The coefficients p_i of the characteristic polynomial are functions of the matrix elements a_{ij} and must be determined before the polynomial can be used.

Example 14.3 illustrates the procedure of evaluation of eigenvalues and eigenvectors of a simple system.