Interpolation

→ **Definition of Interpolation:**

It is the process of estimating unknown values (normal or reverse) that fall betn known values.

→ **Definition of Extrapolation:**

The process of estimating unknown values that fall outside the range of known values.

→ **Equispaced & Equidistant points:**

$$X = x_0 \quad x_1 \quad x_2 \quad \cdots \quad x_{n-2} \quad x_{n-1} \quad x_n$$
$$y = f(x) = y_0 \quad y_1 \quad y_2 \quad \cdots \quad y_{n-2} \quad y_{n-1} \quad y_n$$

$x_{i+1} - x_i = h \quad \forall$ all i

(h is the length of equispace interval points) i.e. length of two successive points should be same.

$$X = x_0 \overset{x_i}{\uparrow} x_1 \quad x_2 \quad \cdot \overset{x_k}{\uparrow} \cdot \quad x_{n-2} \quad x_{n-1} \overset{x_j}{\uparrow} \quad x_n$$
$$y = y_0 \quad y_1 \quad y_2 \quad \cdots \quad y_{n-2} \quad y_{n-1} \quad y_n$$

↳ For an unknown x_i the corresponding y_i to be calculated (if x_i is located near the beginning of range) by Newton's Forward Interpolation formula if they are in equispaced for all i.

↳ For an unknown x_j the corresponding y_j to be calculated (if x_j is located near the end of range) by Newton's ~~Forward~~ Backward Interpolation formula if they are in equispaced for all j.

↳ For an unknown x_k the corresponding y_k to be calculated (if x_k is located near the middle of range) by Central Interpolation formula if they are in equispaced for all k.

→ Definition of Reverse Interpolation:

For a given range, predicting an unknown x_i ~~from~~ ~~a given~~ corresponding to a given y_i where $f(x_i) = y_i$, is called reverse interpolation

→ ~~Newt~~ Some methods:

(i) Newton's ~~Formul~~ Formulae (for equispaced point)

a) Forward Difference

b) Backward Difference

(ii) Lagrange's formula (for equispaced as well as non-equispaced points)

(iii) Cubic spline Interpolation formula

Note: Interpolation Polynomial is unique.

~~Newton~~

I.A) Newton's Forward Interpolation:

- Forward difference operator Δ (delta)

→ $\Delta f(x) = f(x+h) - f(x)$

where h is the length bet^n. two successive points

→ $\Delta f_i = f_{i+1} - f_i$

→ Example: $\Delta y_2 = y_3 - y_2$

Now,

1st Fwd. Diff : $\Delta f(x) = f(x+h) - f(x)$

2nd " " : $\Delta^2 f(x) = \Delta f(x+h) - \Delta f(x)$

i.e. $\Delta^2 f_i = \Delta f_{i+1} - \Delta f_i$

⋮

j th " " : $\Delta^j f_i = \Delta^{j-1} f_{i+1} - \Delta^{j-1} f_i$

Now, Forward difference formula:

$$f(x) = f_0 + s\Delta f_0 + \frac{s(s-1)}{2!}\Delta^2 f_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 f_0$$

$$+ \dots + \frac{s(s-1)(s-2)\dots(s-(n-1))}{n!}\Delta^n f_0$$

where $s = \dfrac{x - x_0}{h}$ [x: unknown x_0: initial]

<u>Problem</u>: If $f(x)$ is given as

x	0	1	2	3	4
f	1	7	23	55	109

Find $f(0.5)$ and $f(1.5)$ using Newton's Forward Difference Interpolation.

<u>Solution</u>:

Forward difference table:

x	f	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
0	1				
		6			
1	7		10		
		16		6	
2	23		16		0
		32		6	
3	55		22		
		54			
4	109				

[here $x_0 = 0$ & $h = 1$ (as $x_i - x_{i-1} = 1$)]

For $x = 0.5$,

$$s = \frac{0.5 - 0}{1} = 0.5$$

$$\therefore f(0.5) = 1 + (0.5 \times 6) + \frac{0.5(0.5-1)}{2!} \times 10$$

$$+ \frac{0.5(0.5-1)(0.5-2)}{3!} \times 6$$

$$= 3.125$$

for, $x = 1.5$, $[x_0 = 0, h = 1]$

$$s = \frac{1.5 - 0}{1} = 1.5$$

$$\therefore f(1.5) = 1 + (1.5 \times 6) + \frac{1.5(1.5-1)}{2!} \times 10 + \frac{1.5(1.5-1)(1.5-2)}{3!} \times 6$$

$$= 13.375$$

Q. Estimate $\sin\theta$ at $\theta = 25^\circ$ using Newton-Gregory (Forward Difference) Interpolation formula with the help of following table:

θ	10	20	30	40	50
$\sin\theta$	0.1736	0.3420	0.5000	0.6428	0.7660

Ans: Forward Diff. Table:

θ	f	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
10	0.1736				
		0.1684			
			-0.0104		
20	0.3420			-0.0048	
		0.1580			0.0004
			-0.0152		
30	0.5000				
		0.1428		-0.0044	
40	0.6428		-0.0196		
		0.1232			
50	0.7660				

$$s = \frac{25 - 10}{10} \quad [x_0 = 10, h = 10]$$

$$\therefore f(25) = \sin 25 = 0.1736 + (1.5 \times 0.1684) - \frac{1.5(1.5-1)}{2!} 0.0104$$

$$- \frac{1.5(1.5-1)(1.5-2)}{3!} 0.0048 + \cancel{\frac{1.5(1.5-1)(1.5-2)(1.5-3)}{4! \times 0.000}}$$

$$+ \frac{1.5(1.5-1)(1.5-2)(1.5-3)}{4!} 0.0004$$

$$= 0.422609375$$

I. B) Newton's Backward Interpolation:

→ ~~Bs~~ Backward difference operator ∇ (del)

→ $\nabla f(x) = \cancel{\nabla} f(x) - f(x-h)$

i.e. $\nabla f_i = f_i - f_{i-1}$

$\therefore \nabla^2 f_i = \nabla f_i - \nabla f_{i-1}$

$\vdots$

$\nabla^j f_i = \nabla^{j-1} f_i - \nabla^{j-1} f_{i-1}$

Now, Backward difference formula:

$$f(x) = f_n + s\nabla f_n + \frac{s(s+1)}{2!}\nabla^2 f_n$$

$$+ \frac{s(s+1)(s+2)}{3!}\nabla^3 f_n .$$

$$+ \frac{s(s+1)(s+2)\dots(s+(n-1))}{n!}\nabla^n f_n$$

where $s = \dfrac{x - x_n}{h}$ [x_n is the last element from given data]

Q. Find $\sin 25°$, using Newton's Backward interpolat.

θ	10	20	30	40	50
$\sin\theta$	0.1736	0.3420	0.5000	0.6428	0.7660

Ans:

$$s = \frac{25-50}{10} = -2.5 \qquad [x_n = 50,\ h = 10]$$

θ	f	∇f	$\nabla^2 f$	$\nabla^3 f$	$\nabla^4 f$
10	0.1736				
		0.1684			
20	0.3420		-0.0104		
		0.1580		-0.0048	
30	0.5000		-0.0152		0.0004
		0.1428		-0.0044	
40	0.6428		-0.0196		
		0.1232			
50	0.7660				

$$\sin 25 = 0.766 - (2.5 \times 0.1232) + \frac{2.5(-2.5+1)}{2!} 0.0196$$

$$+ \frac{2.5(-2.5+1)(-2.5+2)}{3!} 0.0044$$

$$- \frac{2.5(-2.5+1)(-2.5+2)(-2.5+3)}{4!} 0.0004$$

$$= 0.422578125$$

II) Lagrange Interpolation:

→ Let, $x_0, x_1, \ldots, x_n$ denote n distinct real ~~n-distinct real~~ numbers and let $f_0, f_1, \ldots, f_n$ be arbitrary real numbers. The points $(x_0, f_0), (x_1, f_1) \ldots (x_n, f_n)$ can be imagined to be data values connected by a curve. Any function $p(x)$ satisfying the conditions

$$p(x_k) = f_k \quad \text{for } k = 0, 1, \ldots, n$$

is called an interpolation.

→ In general, for $(n+1)$ points we have n^{th} degree polynomial as,

$$p_n(x) = \sum_{i=0}^{n} f_i l_i(x)$$

$$\text{where } l_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

Q. Find $f(2.5)$ using Lagrange Interpolation.

x	2	3	4
f(x)	1.4142	1.7321	2

Ans: For $x = 2.5$,

$$l_0(2.5) = \frac{(2.5-3.0)(2.5-4.0)}{(2.0-3.0)(2.0-4.0)} = 0.3750$$

$$l_1(2.5) = \frac{(2.5-2.0)(2.5-4.0)}{(3.0-4.0)(3.0-2.0)} = 0.7500$$

$$l_2(2.5) = \frac{(2.5-2.0)(2.5-3.0)}{(4.0-2.0)(4.0-3.0)} = -0.125$$

$p_2(2.5) = (1.41242)(0.3750) + (1.7321)(0.7500) + (2.0)(-0.125)$

$= 0.5303 + 1.2991 - 0.250 = 1.5794$

**

Q. Find the Lagrange interpolation polynomial to fit the data.

x	0	1	2	3
e^x	0	1.7183	6.3891	19.0855

Ans: Lagrange basis polynomials are,

$$l_0(x) = \frac{(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-3)}$$

$$= \frac{x^3-6x^2+11x-6}{-3}$$

$$l_1(x) = \frac{(x-0)(x-2)(x-3)}{(1-0)(1-2)(1-3)}$$

$$= \frac{x^3-5x^2+6x}{2}$$

$$l_2(x) = \frac{(x-0)(x-1)(x-3)}{(2-0)(2-1)(2-3)}$$

$$= \frac{x^3-4x^2+3x^2}{-2}$$

$$l_3(x) = \frac{(x-0)(x-1)(x-2)}{(3-0)(3-1)(3-2)}$$

$$= \frac{x^3-3x^2+2x}{6}$$

$\therefore$ Interpolation polynomial,

$$p(x) = f_0 l_0(x) + f_1 l_1(x) + f_2 l_2(x) + f_3 l_3(x)$$

$$= 0 + \left(1.7183 \times \frac{x^3-5x^2+6x}{2}\right) + \left(6.3819 \times \frac{x^3-4x^2+3x}{-2}\right)$$

$$+ \left(19.0855 \times \frac{x^3-3x^2+2x}{6}\right)$$

$$= 0.8455x^3 - 1.0609x^2 + 1.9331x$$

Q. Find the polynomial.

x	1.0	1.1	1.2
$\cos(x)$	.5403	.4536	.3624

Hence, find $\cos(1.15)$

Ans: $p(x) = .5403 \dfrac{(x-1.1)(x-1.2)}{(1.0-1.1)(1.0-1.2)} + .4536 \dfrac{(x-1.0)(x-1.2)}{(1.1-1.0)(1.1-1.2)}$

$+ .3624 \dfrac{(x-1.0)(x-1.1)}{(1.2-1.0)(1.2-1.1)}$

$= .5403 \dfrac{x^2 - 2.3x + 1.32}{-0.1 \times -0.2} + .4536 \dfrac{x^2 - 2.2x + 1.2}{0.1 \times -0.1}$

$+ .3624 \dfrac{x^2 - 2.1x + 1.1}{0.2 \times 0.1}$

$= 27.015(x^2 - 2.3x + 1.32) - 45.36(x^2 - 2.2x + 1.2)$

$+ 18.12(x^2 - 2.1x + 1.1)$

$= -0.225x^2 - 0.3945x + 1.1598$

Now, putting $x = 1.15$,

$\cos(1.15) = -0.225(1.15)^2 - (0.3945 \times 1.15) + 1.1598$

$= 0.4085625$

Q. Find the cubic polynomial.

x	0	1	2	3
$f(x)$	1	-1	-1	0

Ans: $p(x) = \dfrac{(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-3)} - \dfrac{(x-0)(x-2)(x-3)}{(1-0)(1-2)(1-3)}$

$- \dfrac{(x-0)(x-1)(x-3)}{(2-0)(2-1)(2-3)}$

$= \dfrac{x^3 - 6x^2 + 11x - 6}{-6} - \dfrac{x^3 - 5x^2 + 6x}{2}$

$+ \dfrac{x^3 - 4x^2 + 3x}{2}$

$= x^3 - 6x^2 + 11x - 6 + 3x^3 - 15x^2 + 18x$

$- 3x^3 + 12x^2 - 9x$

$= x^3 - 9x^2 + 20x - 6$

⊛ Relation betⁿ the operator Δ, ∇, Shift (E):

**

$$\Delta f(x) = f(x+h) - f(x)$$

$$\nabla f(x) = f(x) - f(x-h)$$

$$E f(x) = f(x+h)$$

(1) $\Delta \equiv E - 1$

Proof:

$$\Delta f(x) = f(x+h) - f(x)$$
$$= E f(x) - f(x)$$
$$= (E-1) f(x)$$

$\therefore \Delta \equiv E - 1$

(2) $\nabla \equiv 1 - \frac{1}{E}$

Proof:

$$\nabla f(x) = f(x) - f(x-h)$$
$$= f(x) - E^{-1}(f(x))$$
$$= (1 - E^{-1}) f(x)$$

$\therefore \nabla \equiv 1 - E^{-1}$

(3) $E(\Delta f(x)) = \Delta(E f(x))$

Proof:

$$E(\Delta f(x)) = E[f(x+h) - f(x)]$$
$$= E f(x+h) - E f(x)$$
$$= f(x+2h) - f(x+h)$$
$$= \Delta f(x+h)$$
$$= \Delta(E f(x))$$

III) Central Difference Interpolation:

$$\delta f(x) = f(x + h/2) + f(x - h/2)$$

h: const. denoting the diff. betn. successive pts. of interpolation

$$\delta \equiv \sqrt{E} + \frac{1}{\sqrt{E}}$$

Proof:

$$\delta f(x) = f(x + h/2) + f(x - h/2)$$

$$= E^{1/2} f(x) + E^{-1/2} f(x)$$

$$\therefore \delta \equiv \sqrt{E} + \frac{1}{\sqrt{E}}$$

IV) Cubic Spline Interpolation:

We consider here the construction of cubic spline which would interpolate the points $(x_0, f_0), (x_1, f_1), \ldots (x_n, f_n)$. The cubic spline $s(x)$ consists of $(n-1)$ cubics corresponding to $(n-1)$ subintervals. If we denote such cubic by $s_i(x)$, then,

$$s(x) = s_i(x) \quad ; \; i = 1, 2, \ldots, n$$

These cubics must satisfy the following conditions:

a) $s(x)$ must interpolate f at all the points $x_0, x_1, \ldots, x_n$ i.e. for each i

$$s(x_i) = f_i \quad \rightarrow (1)$$

b) The function values must be equal at all the interior knots

$$s_i(x_i) = s_{i+1}(x_i) \quad \rightarrow (2)$$

c) The first derivatives at the interior knots must be equal.

$$s_i'(x_i) = s_{i+1}'(x_i) \quad \rightarrow (3)$$

d) The second derivatives at the interior knots must be equal

$$s_i''(x_i) = s_{i+1}''(x_i) \quad \rightarrow (4)$$

e) The second derivatives at the end points are zero:

$$s''(x_0) = s''(x_n) = 0$$

Step 1:

Let us first consider second derivatives.

$\because$ $s_i(x)$ is a cubic function, then $s_i''(x)$ is a straight line.

This straight line can be represented by a first order Lagrange interpolating polynomial. Since, the line passes through the points $(x_i, s_i''(x_i))$ and $(x_{i-1}, s_i''(x_{i-1}))$, we have,

$$s_i''(x) = s_i''(x_{i-1}) \frac{x - x_i}{x_{i-1} - x_i} + s_i''(x_i) \frac{x - x_{i-1}}{x_i - x_{i-1}} \rightarrow (5)$$

The unknowns $s_i''(x_{i-1})$ and $s_i''(x_i)$ are to be determined. For the sake of simplicity, let us denote

$$s_i''(x_{i-1}) = a_{i-1} \quad \text{and} \quad s_i''(x_i) = a_i$$

$$x - x_i = u_i$$

$$x - x_{i-1} = h_i = u_{i-1} - u_i$$

Then, (5) becomes,

$$s_i''(x) = a_{i-1} \frac{u_i}{-h_i} + a_i \frac{u_{i-1}}{h_i}$$

$$= \frac{a_i u_{i-1} - a_{i-1} u_i}{h_i} \rightarrow (6)$$

Step 2:

Now we can obtain $s_i(x)$ by ~~integer~~ integrating (6) twice.

Thus,

$$s_i(x) = \frac{a_i u_{i-2}^2 - a_{i-1} u_i^3}{6h_i} + C_1 x + C_2 \rightarrow (7)$$

where C_1 & C_2 are integral constants. [observe, $du_i/dx = 1$ and therefore, differentiation and integration with respect to u_i will be equivalent, respect to x & with respect to u_i will be equivalent.

The linear part $C_1 x + C_2$ can be expressed,

$$b_1 (x - x_{i-1}) + b_2 (x - x_i)$$

with suitable choice of b_1 & b_2.

$$\therefore \quad C_1 x + C_2 = b_1 (x - x_{i-1}) + b_2 (x - x_i)$$

$$= b_1 u_{i-1} + b_2 u_i$$

Then, (7) becomes,

$$s_i(x) = \frac{a_i u_{i-1}^3 - a_{i-1} u_i^3}{6h_i} + b_1 u_{i-1} + b_2 u_i$$

Step 3:

Now we must determine the coefficients b_1 and b_2. We know that, by condition 1,

$$s(x_i) = f_i \quad \text{and} \quad s(x_{i-1}) = f_{i-1}$$

At $x = x_i$,

$$u_i = 0, \quad u_{i-1} = h_i$$

$$f_i = \frac{a_i h_i^2}{6} + b_1 h_i$$

Similarly, at $x = x_{i-1}$

$$u_{i-1} = 0, \quad u_i = -h_i$$

and therefore,

$$f_{i-1} = \frac{a_{i-1} h_i^2}{6} - b_2 h_i$$

Thus, we get

$$b_1 = \frac{f_i}{h_i} - \frac{a_i h_i}{6} \rightarrow (8.a)$$

$$b_2 = \frac{f_{i-1}}{h_i} + \frac{a_{i-1} h_i}{6} \rightarrow (8.b)$$

Substituting for b_1 and b_2 in (8) and after rearrangement of terms, we get

$$s_i(x) = \frac{a_{i-1}}{6h_i}(h_i^2 u_i - u_i^3) + \frac{a_i}{6h_i}(u_{i-1}^3 - h_i^2 u_{i-1})$$
$$+ \frac{1}{h_i}(f_i u_{i-1} - f_{i-1} u_i) \rightarrow (9)$$

Note, (9) has only 2 unknowns a_{i-1} & a_i

Step 4:

The final step is to evaluate these constants. This can be done by invoking the condition

$$s_i'(x_i) = s_{i+1}'(x_i)$$

Differentiating (9),

$$s_i'(x) = \frac{a_{i-1}}{6h_i}(h_i^2 - 3u_i^2) + \frac{a_i}{6h_i}(3u_{i-1}^2 - h_i^2)$$
$$+ \frac{1}{h_i}(f_i - f_{i-1})$$

Setting $x = x_i$,

$$s_i'(x_i) = \frac{a_{i-1}h_i}{6} + \frac{a_i h_i}{3} + \frac{f_i - f_{i-1}}{h_i}$$

Similarly.

$$s_{i+1}'(x_i) = -\frac{a_i h_{i+1}}{3} - \frac{a_{i+1} h_{i+1}}{6} + \frac{f_{i+1} - f_i}{h_{i+1}}$$

Since, $s_i'(x_i) = s_{i+1}'(x_i)$

We have,

$$h_i a_{i-1} + 2(h_i + h_{i+1})a_i + h_{i+1}a_{i+1} = 6\left[\frac{f_{i+1} - f_i}{h_{i+1}} - \frac{f_i - f_{i-1}}{h_i}\right] \rightarrow (10)$$

(10), when written for all interior knots ($i = 1, \dots, n-1$) we get $n-1$ simultaneous eq^n containing $n+1$ unknowns ($a_0, a_1, \dots, a_n$). Now applying the condition that the second derivatives at the end points are zero, we get,

$$a_0 = a_n = 0$$

→ The concepts of spline originated from a mechanical drafting tool (spline) used by designers for drawing smooth curves.

→ This curve ressambles cubic curves and hence it names 'Cubic Spline', has been given to the 'piece wise cubic interpolating polynomial'

→ Cubic Spline are popular because of this ability to interpolate data with smooth curve

Q. Given the points below, fit quadratics spline, to the data given below:

x	1	2	3
f(x)	1	1	2

Predict $f(2.5)$

Ans: $h_1 = x_1 - x_0 = 2 - 1 = 1$ $\quad h_2 = x_2 - x_1 = 3 - 2 = 1$

$f_0 = 1 \quad f_1 = 1 \quad f_2 = 2$

Now, we have for $i=1$,

$$h_1 a_0 + 2(h_1+h_2)a_1 + h_2 a_2 = 6\left[\frac{f_2-f_1}{h_2} - \frac{f_1-f_0}{h_1}\right]$$

we know that, $a_0 = a_2 = 0$

$$\therefore\ 2(h_1+h_2)a_1 = 6\left[\frac{f_2-f_1}{h_2} - \frac{f_1-f_0}{h_1}\right]$$

$$\Rightarrow 2(1+1)a_1 = 6\left[\frac{1}{1} - \frac{0}{1}\right]$$

$$\Rightarrow a_1 = 1.5$$

Since, $n=3$, there are two splines, namely.

$s_1(x)$ $\quad x_0 \le x \le x_1$

$s_2(x)$ $\quad x_1 \le x \le x_2$

The target point $x = 2.5$ is in the domain of $s_2(x)$ & therefore we need to use only $s_2(x)$ for estimation.

$$\therefore\ s_2(x) = \frac{a_1}{6h_2}(h_2^2 u_2 - u_2^3) + \frac{1}{h_2}(f_2 u_1 - f_1 u_2)$$

& $u_2 = x - x_2$ $\quad u_1 = x - x_1$

$$\therefore\ s_2(2.5) = \frac{1.5}{6\times1}\left((1)^2(-0.5) - (-0.5)^3\right) + \frac{1}{1}\left(2\times(0.5) + 1\times(0.5)\right)$$

$$= 1.40625$$

Q. Given the data points,

x	1.0	3.0	4.0
$f(x)$	1.5	4.5	9.0

Estimate $f(1.5)$ using cubic's spline.

Ans: $h_1 = 2.0$ $\quad h_2 = 1.0$

$f_0 = 1.5$ $\quad f_1 = 4.5$ $\quad f_2 = 9.0$

Now we have for $i=1$,

$$h_1 a_0 + 2(h_1+h_2)a_1 + h_2 a_2 = 6\left[\frac{f_2-f_1}{h_2} - \frac{f_1-f_0}{h_1}\right]$$

we know that, $a_0 = a_2 = 0$

$$\therefore\ 2(h_1+h_2)a_1 = 6\left[\frac{f_2-f_1}{h_2} - \frac{f_1-f_0}{h_1}\right]$$

$\Rightarrow 2(2.0+1.0)a_1 = 6\left[\frac{4.5}{1} - \frac{3}{2.0}\right]$

$\Rightarrow a_1 = 3$

$\because n = 3$, we have two cubic spline.

$s_1(x) \qquad x_0 \le x \le x_1$

$s_2(x) \qquad x_1 \le x \le x_2$

$\therefore$ The target point $x = 1.5$ is in the domain of $s_1(x)$ & therefore we need to use only $s_1(x)$ for estimation.

$$s_1(x) = \frac{a_1(u_0^3 - h_1^2 u_0)}{6h_1} + \frac{1}{h_1}(f_1 u_0 - f_0 u_1)$$

$u_1 = x - x_1 \qquad u_0 = x - x_0$

$$\therefore s_1(1.5) = \frac{3}{6\times 2}\left((0.5)^3 - (2^2 \times (0.5))\right) + \frac{1}{2}\left((4.5\times 0.5) + (1.5\times 1.5)\right)$$

$= 1.78125$

→ Algorithm:

1) Provide input data
2) Compute step lengths and form function differences
3) Obtain the co-efficients of the traditional matrix
4) Compute the right-hand side (D array) of the system
5) Compute the elements a_i using Gauss elimination method.
6) Evaluate the co-efficients of natural cubic's spline.
7) Evaluate the spline function at the point of interest.
8) Print result
9) Stop