

$T(h)$ is the trapezoidal approximation with step size $= (b - a)/n = h$.

Let us define

$$T(h, 0) = T(h)$$

to indicate that $T(h)$ is the trapezoidal rule with no Richardson's extrapolation being applied (zero level extrapolation). Thus, Eq. (12.20) can be written as

$$I = T(h, 0) + a_2 h^2 + a_4 h^4 + a_6 h^6 + \dots \quad (12.21)$$

Let us have another estimate with step size $= (b - a)/2n = h/2$ (at zero level extrapolation) as

$$I = T(h/2, 0) + \frac{a_2}{4} h^2 + \frac{a_4}{16} h^4 + \frac{a_6}{32} h^6 + \dots \quad (12.22)$$

By multiplying Eq. (12.22) by 4 and then subtracting Eq. (12.21) from the resultant equation, we obtain (after rearranging terms),

$$\begin{aligned} I &= \frac{4T(h/2, 0) - T(h, 0)}{4 - 1} + b_4 h^4 + b_6 h^6 + \dots \\ &= T(h/2, 1) + b_4 h^4 + b_6 h^6 + \dots \end{aligned} \quad (12.23)$$

where

$$T(h/2, 1) = \frac{4T(h/2, 0) - T(h, 0)}{3}$$

is the *corrected* trapezoidal formula using Richardson's extrapolation technique "once" (level 1). Note that its truncation error is of the order h^4 , instead of h^2 which is the order in the "uncorrected" trapezoidal formula.

Now, we can apply Richardson's extrapolation technique once more to Eq. (12.23) to eliminate the error term containing h^4 . The result would be

$$\begin{aligned} I &= \frac{16T(h/4, 1) - T(h/2, 1)}{16 - 1} + C_6 h^6 + \dots \\ &= T(h/4, 2) + C_6 h^6 + \dots \end{aligned} \quad (12.24)$$

where

$$T(h/4, 2) = \frac{16T(h/4, 1) - T(h/2, 1)}{16 - 1}$$

is the estimate, refined again by applying Richardson's extrapolation a second time (level 2). Similarly, we can obtain an estimate with third-level correction as

$$T(h/8, 3) = \frac{64T(h/8, 2) - T(h/4, 2)}{64 - 1}$$