# Numerical Solutions of Ordinary Differential Eq$^n$

Methods :-

① Euler's Method

(1.1) Modified Euler's Method

② Taylor's Method

③ Picard's Method

④ Heun's Method

⑤ Runge-Kutta Method (order 2 & 4)

⑥ Predictor Corrector Method (Milne-Simpson Method)

## ○ Euler's Method :

~~Euler's method~~ : The function can be expanded about the point $x = x_0$ using Taylor Theorem Eq$^n$:

$$y(x) = y(x_0) + (x-x_0)y'(x_0) + (x-x_0)^2 \frac{y''(x_0)}{2!} + (x-x_0)^3 \frac{y'''(x_0)}{3!} + \dots (x-x_0)^n \frac{y^{(n)}(x_0)}{n!} \rightarrow (1)$$

Consider the first two term of the expansion (1)

$$y(x) = y(x_0) + y'(x_0)(x-x_0)$$

given the diff. eq$^n$:

$$y'(x) = f(x,y) \text{ with } y(x_0) = y_0$$

we have

$$y'(x_0) = f(x_0, y_0)$$

and therefore

$$y(x) = y(x_0) + (x-x_0) f(x_0, y_0)$$

Then, the value of $y(x_1)$ at $x = x_1$ is given by

$$y(x_1) = y(x_0) + (x_1 - x) f(x_0, y_0)$$

Letting $h = x_1 - x_0$, we obtain

$$y_1 = y_0 + h\,f(x_0, y_0)$$

Similarly, $y(x)$ at $x = x_2$ is given by

$$y_2 = y_1 + h\,f(x_1, y_1)$$

In general, we obtain a recursive relation as

$$\boxed{y_{i+1} = y_i + h\,f(x_i, y_i)} \longrightarrow (2)$$

Q. Given the eq$^n$.

$$\frac{dy}{dx} = 3x^2 + 1 \quad \text{with} \quad y(1) = 2 \quad [x_0 = 1]$$

Estimate $y(2)$ using i) $h = 0.5$ ii) $h = 0.25$

Ans: i) $h = 0.5$

$y(1) = y_0 = 2$

$x_0 = 1$

$x_0 = 1$, $x_1 = 1.5$, $x_2 = 2$

$y_0 = 2$, $y_1 = y(1.5)$, $y_2 = y(2)$

$\therefore y(1.5) = 2 + 0.5 \times (3 \times 1^2 + 1) = 2 + (0.5 \times 4) = 4$

$\therefore y(2) = 4 + 0.5\,(3 \times (1.5)^2 + 1)$

$= 7.875$

ii) $h = 0.25$

$\therefore x_0 = 1 \quad x_1 = 1.25 \quad x_2 = 1.5 \quad x_3 = 1.75 \quad x_4 =$

$\therefore y(1.25) = 2 + 0.25\,(3 + 1) = 3$

$y(1.5) = 3 + 0.25\,(3 \times 1.25^2 + 1) = 4.422$

$y(1.75) = 4.422 + 0.25\,(3 \times (1.5)^2 + 1) = 6.3595$

$y(2) = 6.3595 + 0.25\,(3 \times (1.75)^2 + 1)$

$= 8.906375$

Q. Given, $y' = -y/2y+1 \quad y(0) = 1$
estimate $y(1)$ with $h = 0.5$ & $h = 0.25$

Ans: i) $\underline{h = 0.5}$

$$y(0.5) = 1 + 0.5\left[\frac{-1}{2\times1+1}\right] = 0.833$$

$$y(1) = 0.833 + 0.5\left[\frac{-0.833}{2\times0.833+1}\right] = 0.677$$

ii) $\underline{h = 0.25}$

$$y(0.25) = 1 + 0.25\left[\frac{-1}{2\times1+1}\right] = 0.917$$

$$y(0.5) = 0.917 + 0.25\left[\frac{-0.917}{2\times0.917+1}\right] = 0.836$$

$$y(0.75) = 0.836 + 0.25\left[\frac{-0.836}{2\times0.836+1}\right] = 0.758$$

$$y(1) = 0.758 + 0.25\left[\frac{-0.758}{2\times0.758+1}\right] = 0.683$$

Q. Given, $y' = 2xy \quad y(0) = 1$
estimate $y(1)$ with $h = 0.5$ & $h = 0.25$

Ans: i) $\underline{h = 0.5}$

$$y(0.5) = 1 + 0.5(2\times0\times1) = 1$$

$$y(1) = 1 + 0.5(2\times0.5\times1) = 1.5$$

ii) $\underline{h = 0.25}$

$$y(0.25) = 1 + 0.25(2\times0\times1) = 1$$

$$y(0.5) = 1 + 0.25(2\times0.25\times1) = 1.125$$

$$y(0.75) = 1.125 + 0.25(2\times0.5\times1.125) = 1.406$$

$$y(0.5) = 1.406 + 0.25(2\times0.75\times1.406) = 1.933$$

## Heun's Method:

Also known as, "One Step Predictor-Corrector Method"

From euler's method we know,

$$y_{i+1} = y_i + m_1 h$$

→ It's an modified Euler Method.

again, if we extrapolate,

$$y_{i+1} = y_i + m_2 h$$

$$\therefore \boxed{y_{i+1} = y_i + \frac{m_1 + m_2}{2} h}$$

Prediction is done using Euler's method

Correction is done using Heun's method

$$m_1 = y'(x_i) = f(x_i, y_i)$$

$$m_2 = y'(x_{i+1}) = f(x_{i+1}, y_{i+1})$$

$$\therefore m = \frac{f(x_i, y_i) + f(x_{i+1}, y_{i+1})}{2}$$

$$\therefore \boxed{y_{i+1} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f(x_{i+1}, y^e_{i+1})\right]}$$

where $y^e_{i+1}$ has to be calculated using euler's method

i.e. $y^e_{i+1} = y_{i+1} = y_i + h \times f(x_i, y_i)$

Q. Use Heun's Method with $h = 0.25$ to solve $y' = 2xy$, $y(0) = 1$ for $y(1)$.

How do this result compare with this obtained using simple Euler's method.

Ans:- **Euler's Method:**

$$y' = 2xy \qquad y(0) = 1$$

$$y(0.25) = 1 + 0.25\,[2 \times 0 \times 1]$$
$$= 1$$

$$y(0.5) = 1 + 0.25\,[2 \times 0.25 \times 1]$$
$$= 1.125$$

$$y(0.75) = 1.125 + 0.25\,[2 \times 0.5 \times 1.125]$$
$$= 1.40625$$

$$y(1) = 1.40625 + 0.25\,[2 \times 0.75 \times 1.40625]$$
$$= 1.93359$$

**Heun's Method:**

$$m_1 = y'(x_i) = f(x_i, y_i)$$
$$m_2 = y'(x_{i+1}) = f(x_{i+1}, y_{i+1})$$

Iter 1:

$$m_1 = 2 \times 0 \times 1 = 0$$
$$y_e(0.25) = 1 + 0.25 \times 0 = 1$$
$$m_2 = 2 \times 1 \times 0.25 = 0.5$$
$$y_e(0.25) = 1 + \frac{0.25}{2}\,[0 + 0.5] = 1.0625$$

Iter 2:

$$m_1 = 2 \times 0.25 \times 1.0625 = 0.53125$$
$$y_e(0.5) = 1.0625 + (0.25 \times 0.53125)$$
$$= 1.19531$$
$$m_2 = 2 \times 0.5 \times 1.19531 = 1.19531$$
$$y_e(0.5) = 1.0625 + \frac{0.25}{2}\,[0.53125 + 1.19531]$$
$$= 1.27832$$

Iter 3: $m_1 = 2 \times 0.5 \times 1.27832$

$= 1.27832$

$y_e(0.75) = 1.27832 + (0.25 \times 1.27832)$

$= 1.5979$

$m_2 = 2 \times 0.75 \times 1.5979 = 2.39685$

$y_e(0.75) = 1.27832 + \frac{0.25}{2}\left[1.27832 + 2.39685\right]$

$= 1.73771625$

Iter 4: $m_1 = 2 \times 0.75 \times 1.73771625$

$= 2.606574375$

$y_e(1) = 1.73771625 + (0.25 \times m_1)$

$= 2.38936$

$m_2 = 2 \times 1 \times 2.38936 = 4.77872$

$y_e(1) = 1.73771625 + \frac{0.25}{2}\left[m_1 + m_2\right]$

$= 2.6609$

Q. Use Heuns method to solve the eq$^n$ for $y(1)$:

(i) $y' = x^2 + y^2$ $\quad y(0) = 2$ $\quad h = 0.5$

(ii) $y' = x + y + xy$ $\quad y(0) = 1$ $\quad h = 0.25$

Ans: i) Iter 1: $m_1 = 0^2 + 2^2 = 4$

$y_e(0.5) = 2 + (0.5 \times 4) = 4$

$m_2 = 0.5^2 + 4^2 = 16.25$

$y_e(0.5) = 2 + \frac{0.5}{2}\left[4 + 16.25\right] = 7.0625$

Iter 2: $m_1 = 0.5^2 + (7.0625)^2 = 50.1289$

$y_e(1) = 7.0625 + (0.5 \times 50 \times 50.1289)$

$= 32.12695$

$m_2 = 1^2 + (32.12695)^2 = 1033.141$

$y_e(1) = 7.0625 + \frac{0.5}{2}\left[50.1289 + 1033.141\right]$

$= 277.879975$

~~Iter 20~~ Iter 1:

$m_1 = (0+1+0\times1) = 1$

ii) $y_e(0.25) = 1.25$

$m_2 = (0.25 + 1.25 + (0.25\times1.25)) = 1.8125$

$y_e(0.25) = 1 + \frac{0.25}{2}[1 + 1.8125] = 1.3516$

Iter 2: $m_1 = 0.25 + 1.3516 + (0.25\times1.3516)$

$= 1.9395$

$y_e(0.5) = 1.3516 + (0.25 \times 1.9395)$

$= 1.5865$

$m_2 = 0.5 + 1.5865 + (0.5\times1.5865)$

$= 2.87975$

$y_e(0.5) = 1.3516 + \frac{0.25}{2}[1.9395 + 2.8797]$

$= 1.954$

$m_1 = 0.5 + 1.954 + (0.5\times1.954) = 3.431$

Iter 3: $y_e(0.75) = 1.954 + (0.25\times3.431) = 2.81175$

$m_2 = 0.75 + 2.81175 + (0.75\times2.81175) = 5.6706$

$y_e(0.75) = 1.954 + \frac{0.25}{2}[3.431 + 5.6706]$

$= 3.0917$

Iter 4: $m_1 = 0.75 + 3.0917 + (0.75\times3.0917)$

$= 6.1605$

$y_e(1) = 3.0917 + (0.25\times6.1605) = 4.6318$

$m_2 = 1 + 4.6318 + (1\times4.6318)$

$= 9.2636$

$y_e(1) = 3.0917 + \frac{0.25}{2}[6.1605 + 9.2638]$

$= 5.0197$

## * Note :

1. Heun's method is also called modified euler method or one-step predictor corrector method.

2. The oldest and simpliest method for numerical solution of ODE is Euler method.

3. Picard Method and Taylor's method for numerical solution of ODE are also called semi-numeric method. The major with Taylor series method is the evaluation of higher order derivative. This method is impractical from a computational point of view. Picard's method is also not convenient for computer based $sol^n$.

4. Euler method is first order Runge-Kutta method and Heun's method is the second order Runge-Kutta method. Runge-Kutta is a one-step method used for numerical $sol^n$. of initial valued problem.

5. Methods that used information from more than one previous point to compute the next point are called multistep method. It improves the efficiency of estimation by using the information at several previous points. Popular multistep methods are :

   a) Milne - Simpson Method
   b) Adam - Bashforth - Moultan Method.

## Algorithm of Euler's Method:

Step 1: Start

Step 2: Define function

Step 3: Get the values $x_0$, $y_0$, $h$ and $x_n$

Step 4: $n = \dfrac{(x_n - x_0)}{h} + 1$

Step 5: for $i = 1$ to $n$:

Step 6: $y_0 = y_0 + hf(x_0, y_0)$

$x = x + h$

Step 7: Printy $y_0$ and $x_0$

Step 8: Check if $x < x_n$

if true:

assign $x_0 = x$ and $y_0 = y$

Go to step 9

Step 9: End of loop $i$

Step 10: Stop.

## Runge-Kutta Method:

Rung-Kutta Method

Order 2 | Order 4

- Order 2 RK Method formula:

$$y_{i+1} = y_i + \frac{m_1 + 2m_2}{3} h$$

where

$$m_1 = f(x_i, y_i)$$

$$m_2 = f\left(x_i + \frac{3}{4}h, y_i + \frac{3}{2} m_1 h\right)$$

- Order 4 RK Method formula:

$$y_{i+1} = y_i + \frac{m_1 + 2m_2 + 2m_3 + m_4}{6} h$$

where, $m_1 = f(x_i, y_i)$

$m_2 = f\left(x_i + \frac{h}{2}, y_i + \frac{m_1 h}{2}\right)$

$m_3 = f\left(x_i + \frac{h}{2}, y_i + \frac{m_2 h}{2}\right)$

$m_4 = f\left(x_i + h, y_i + m_3 h\right)$

Q. Solve using RK Method (Order 4):

$y'(x) = x^2 + y^2 \qquad y(0) = 0$

estimate $y(0.4)$ where $h = 0.1$

Ans: $x_0 = 0 \qquad y_0 = 0$

Iter 1:

$m_1 = 0 \qquad m_2 = 0.0025$

$m_3 = 0.0025 \qquad m_4 = 0.01$

$y(0.1) = 0.00033$

Iter 2: $x_1 = 0.1 \qquad y_1 = 0.00033$

$m_1 = 0.01 \qquad m_2 = 0.0225$

$m_3 = 0.0225 \qquad m_4 = 0.04$

$y(0.2) = 0.00266$

Iter 3: $x_2 = 0.2 \qquad y_2 = 0.00266$

$m_1 = 0.04 \qquad m_2 = 0.0625$

$m_3 = 0.0625 \qquad m_4 = 0.0901$

$y(0.3) = 0.008995$

Iter 4: $x_3 = 0.3 \qquad y_3 = 0.008995$

$m_1 = 0.0901 \qquad m_2 = 0.1227$

$m_3 = 0.1227 \qquad m_4 = 0.1604$

$y(0.4) = 0.02135$

Q. Use classical Runge Kutta method to estimate $y(0.5)$ of the following eqn. with $h=0.25$

[Order 2]

i) $y' = x+y$ $\quad y(0)=1$

ii) $y' = x/y$ $\quad y(0)=1$

Ans: i)

Iter 1: $m_1 = 1$ $\quad m_2 = 1.5625$

$y(0.25) = 1.34375$

Iter 2: $m_1 = 1.59375$ $\quad m_2 = 2.37891$

$y(0.5) = 1.87305$

ii) Iter 1: $m_1 = 0$ $\quad m_3 = 0.1231$

$m_2 = 0.125$ $\quad m_4 = 0.24254$

$y(0.25) = 1.03078$

Iter 2: $m_1 = 0.24253$ $\quad m_3 = 0.34885$

$m_2 = 0.35341$ $\quad m_4 = 0.459892$

$y(0.5) = 1.118569$

## * Algorithm of RK Method (Order 4):

step 1: Start

step 2: Define function

step 3: Read values of initial condition $(x_0, y_0)$

step 4: Calculate step size $(h) = \frac{x_n - x_0}{n}$

step 5: $i=0$

step 6: Loop begins

$K1 = h \times f(x_0, y_0)$

$K2 = h \times f(x_0 + \frac{h}{2}, y_0 + \frac{K1 \times h}{2})$

$K2 = h \times f(x_0 + \frac{h}{2}, y_0 + \frac{K2 \times h}{2})$

$K3 = h \times f(x_0 + \frac{h}{2}, y_0 + \frac{K2 \times h}{2})$

$K4 = h \times f(x_0 + \frac{h}{2}, y_0 + K3 \times h)$

$K = \frac{K_1 + 2K_2 + 2K_3 + K_4}{6}$

$y_N = y_0 + K$

$x_0 = x_0 + h$

$y_0 = y_n$

Step 7: Display $y_n$

Step 8: Stop

## ⊛ Algorithm for Taylor Series:

Step 1: Start

Step 2: Read $x_0, x_n, y_0, h$

Step 3: Repeat.

compute $f(x_i, y_i), f''(x_i, y_i), f'''(x_i, y_i), \ldots$

compute ~~y(x~~ $y(x_i + h) =$

$$y(x_i) + hf(x_i y_i) + \frac{h^2 f(x_i y_i)}{2!} + \frac{h^3 f(x_i, y_i)}{3!} + \ldots$$

$x_i = x_i + h$

until $x_i = x_n$

Step 4: Display $y(x_i + h)$

Step 5: Stop.

## ○ Polygon Method:

→ Its also an modified euler method.

→ It is to use the slope of the function at the estimated mid points of $(x, y)$ and $(x_{i+1}, y_{i+1})$ to approximate $y_{i+1}$. Thus,

$$y_{i+1} = y_i + f\left(\frac{x_i + x_{i+1}}{2}, \frac{y_i + y_{i+1}}{2}\right)h$$

$$= y_i + f(x_i + h/2, y_i + \Delta y/2)h$$

$\Delta y$ is the estimated incremental value of $y$ from $y_i$ & can be obtained using Euler's formula as

$\Delta y = h f(x_i, y_i$

$$y_{i+1} = y_i + hf\left(x_i + h/2,\ y_i + h/2\, f(x_i, y_i)\right)$$

$$\Rightarrow y_{i+1} = y_i + hf\left(x_i + h/2,\ y_i + m_1 h/2\right)$$

$$\boxed{\begin{aligned} &\Rightarrow y_{i+1} = y_i + m_2 h \\ &\text{where} \quad m_1 = f(x_i, y_i) \\ &\qquad\quad\ \ m_2 = f\left(x_i + \frac{h}{2},\ y_i + \frac{m_1 h}{2}\right) \end{aligned}}$$

Q. Solve using Polygon method.

$$y'(x) = \frac{2y}{x} \quad \text{with } y(1) = 2$$

Estimate $y(1.5)$ with $h = 0.25$

Ans: Iter 1: $x_0 = 1 \quad y_0 = 2$

$$m_1 = \frac{2\times 2}{1} = 4$$

$$m_2 = f\left(1 + \frac{0.25}{2},\ 2 + \frac{4\times 0.25}{2}\right)$$

$$= f(1.125, 2.5) = 4.4444$$

$$\therefore y(1.25) = 2 + (4.4444 \times 0.25) = 3.1111$$

Iter 2: $x_0 = 1.25 \quad y_0 = 3.1111$

$$m_1 = \frac{2\times 3.1111}{1.25} = 4.97776$$

$$m_2 = f\left(1.25 + \frac{0.25}{2},\ 3.1111 + \frac{4.97776\times 0.25}{2}\right)$$

$$= 5.4302$$

$$y(1.5) = 3.1111 + (5.4302 \times 0.25)$$

$$= 4.46865$$