$$y_1'(x) = f_1(x, y_1, y_2), \qquad y_1(x_0) = y_{10}$$

$$y_2'(x) = f_2(x, y_1, y_2), \qquad y_2(x_0) = y_{20}$$

Assume that we want to use the Heun's method. The first stage would involve the following calculations.

$$m_1(1) = f_1(x_0, y_{10}, y_{20})$$
$$m_1(2) = f_2(x_0, y_{10}, y_{20})$$
$$m_2(1) = f_1(x_0 + h, y_{10} + hm_1(1), \quad y_{20} + hm_1(2))$$
$$m_2(2) = f_2(x_0 + h, y_{10} + hm_1(1), \quad y_{20} + hm_1(2))$$

$$m(1) = \frac{m_1(1) + m_2(1)}{2}$$

$$m(2) = \frac{m_1(2) + m_2(2)}{2}$$

$$y_1(x_1) = y_1(1) = y_1(x_0) + m(1)h = y_{10} + m(1)h$$
$$y_2(x_1) = y_2(1) = y_2(x_0) + m(2)h = y_{20} + m(2)h$$

The next stage uses $y_1(1)$ and $y_2(1)$ as initial values and, by following similar procedure, $y_1(2)$ and $y_2(2)$ are obtained.

Example 13.13

Given the equations

$$\frac{dy_1}{dx} = x + y_1 + y_2, \qquad y_1(0) = 1$$

$$\frac{dy_2}{dx} = 1 + y_1 + y_2, \qquad y_2(0) = -1$$

estimate the values of $y_1(0.1)$ and $y_2(0.1)$ using Heun's method.

Given $x_0 = 0, \quad y_{10} = 1, \quad y_{20} = -1$

$$m_1(1) = f_1(x_0, y_{10}, y_{20}) = 0 + 1 - 1 = 0$$
$$m_1(2) = f_2(x_0, y_{10}, y_{20}) = 1 + 1 - 1 = 1$$
$$m_2(1) = f_1(x_0 + h, y_{10} + hm_1(1), y_{20} + hm_1(2))$$
$$= f_1(0.1, 1 + 0.1 \times 0, -1 + 0.1 \times 1)$$
$$= f_1(0.1, 1, -0.9)$$
$$= 0.1 + 1 - 0.9 = 0.2$$
$$m_2(2) = f_2(x_0 + h, y_{10} + hm_1(1), y_{20} + hm_1(2))$$
$$= f_2(0.1, 1, -0.9)$$
$$= 1 + 1 - 0.9 = 1.1$$