Example 11.3

Compute the approximate derivatives of $f(x) = \sin x$, at $x = 0.45$ radians, at increasing values of h from 0.01 to 0.04, with a step size of 0.005. Analyse the total error. What is the optimum step size?

$$f(x) = \sin x$$

Using two-point formula

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

Given

$$x = 0.45 \text{ radians}$$

So, $f(x) = \sin(0.45) = 0.4350$ (rounded to four digits). Exact $f'(x) = \cos x = \cos(0.45) = 0.9004$.

Table below gives the approximate derivatives of $\sin x$ at $x = 0.45$ using various values of h.

h	$f(x+h)$	$f'(x)$	*Error*
0.010	0.4439	0.8900	0.0104
0.015	0.4484	0.8933	0.0071
0.020	0.4529	0.8950	0.0054
0.025	0.4573	0.8935	0.0069
0.030	0.4618	0.8933	0.0071
0.035	0.4662	0.8914	0.0090
0.040	0.4706	0.8900	0.0104

The table shows that the total error decreases from 0.0104 (at $h = 0.01$) till $h = 0.02$ and again increases when h is increased as illustrated in Fig. 11.2.

Since we have used four significant digits, the bound for roundoff error e is 0.5×10^{-4}. For the two-point formula, the bound M_2 is given by

$$M_2 = \max |f''(\theta)|$$

$$0.41 \leq \theta \leq 0.49$$

$$= |\sin(0.49)| = 0.4706$$

Therefore, the optimum step size is

$$= \sqrt{e} \quad = \sqrt{0.5 \times 10^{-4}}$$