LAPLACE TRANSFORMATION

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Unit Step Function

A unit step function u(t – a) has a constant amplitude of unity for $t \geq a$ and zero for $t < a$. It is given by

$$u(t-a) = \begin{cases} 1 & \text{for } t \geq a \\ 0 & \text{for } t < a \end{cases} \qquad (a \geq 0)$$

In particular,

$$u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

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Laplace transform of unit-step function

We have

$$u(t-a) = \begin{cases} 0 & for\, t < a \\ 1 & for\, t > a \end{cases}$$

From the definition of Laplace transform, we get

$$L[u(t-a)] = \int_0^\infty e^{-st}\, u(t-a)\, dt$$

$$= \int_0^a e^{-st}.0\, dt + \int_a^\infty e^{-st}.1\, dt$$

$$= \left. \frac{e^{-st}}{-s} \right|_a^\infty$$

$$= 0 - \frac{e^{-sa}}{-s} = \frac{e^{-as}}{s} \qquad \therefore\ L[u(t-a)] = \frac{e^{-as}}{s}$$

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Second Shifting Theorem

Theorem:

If L [f(t)] = F(s), then

$L\,[f(t-a)\,u(t-a)] = e^{-as}\,F(s)$

Proof:

Given,

$$L\,[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)\,dt \qquad (1)$$

and we have

$$f(t-a)\,u_a(t) = \begin{cases} 0 & if\ t < a \\ f(t-a) & if\ t > a \end{cases} \qquad (2)$$

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Now, using definition

$$L\,[f(t-a)\,u(t-a)] = \int_0^\infty e^{-st} f(t-a)u(t-a)\,dt$$

$$= \int_0^a e^{-st} f(t-a)u(t-a)\,dt + \int_a^\infty e^{-st} f(t-a)u(t-a)\,dt$$

Using (2) we get

$$= \int_a^\infty e^{-st} f(t-a)\,dt \quad \left[\because u(t-a) = 0 \text{ for all } t<a\right]$$

Put $t - a = p$

$dt = dp$

when $t = a,\ p = 0$

$t = \infty,\ p = \infty$

which implies that

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which implies that

$$L\,[f(t-a)\,u(t-a)] = \int_0^\infty e^{-s(a+p)} f(p)\,dp$$

$$= e^{-as} \int_0^\infty e^{-sp} f(p)\,dp$$

Since p is dummy variable.

Hence,

$$L\,[f(t-a)\,u(t-a)] = e^{-as}\,F(s)$$

Equivalently (taking inverse of Laplace Transform)

If $L^{-1}F(s) = \{f(t)\}$ then $L^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$

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Use of Second Shifting Theorem

(i) $L[e^{-3t} u(t-2)] = L\,[e^{-3}(t-2).\, e^{-6}\, u(t-2)]$

$\left[\because e^{-3t} = e^{-3(t-2)} . e^{-6}\right]$

$= e^{-6}\, L\,[e^{-3(t-2)}\, u(t-2)]$

$= e^{-6}\, L\,[f(t-2)\, u(t-2)]$

$= e^{-6}\, e^{-2s}\, L[f(t)]$

$= e^{-2s-6}\, L(e^{-3t})$

$= e^{-2s-6} \cdot \dfrac{1}{s+3} = \dfrac{e^{-2s-6}}{s+3}$

Rough

Let $f(t) = e^{-3t}$

$\therefore\ f(t-2) = e^{-3(t-2)}$ so that

$e^{-3(t-2)}\, u(t-2) = f(t-2)\, u(t-2)$

(ii) $L[\sin 2t u(t-\pi)] = L\,[+\sin 2(t-\pi)\, u(t-\pi)]$

$= +L\,[\sin(2t-2\pi)\, u(t-\pi)]$

$= +L[f(t-\pi)\, u(t-\pi)]$

$= +e^{-\pi s}\, L[f(t)] = -e^{-\pi s}\, L(\sin 2t)$

$= +e^{-\pi s} \cdot \dfrac{2}{s^2+4} = +\dfrac{2e^{-\pi s}}{s^2+4}$

Let $f(t) = \sin 2t$ so that

$f(t-\pi) = \sin(2t-2\pi)$

$\left[\because \sin 2t = -\sin(2\pi-2t) = +\sin(2t-2\pi) = \sin\{2(t-\pi)\}\right]$

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Theorem:

If f(t) be a function of t and u(t – a) is unit step function

then $L\,[f(t)\,u(t-a)] = e^{-as}\,L[f(t+a)]$

Proof:

Using the definition

$$L[f(t)] = \int_0^\infty e^{-st}\, f(t)$$ [redacted] dt

We get

$$L\,[f(t)\,u(t-a)] = \int_0^\infty e^{-st} f(t)u(t-a)\,dt$$

$$= \int_0^a e^{-st} f(t).0\,dt + \int_a^\infty e^{-st} f(t).1\,dt$$

using $u(t-a) = \begin{cases} 0 & if\ t < a \\ 1 & if\ t > a \end{cases}$

$$= \int_{a}^{\infty} e^{-st} f(t)\,dt$$

Putting t – a = p, dt = dp

$$= \int_{0}^{\infty} e^{-s(a+p)} f(a+p)\,dp$$

$$= e^{-as} \int_{0}^{\infty} e^{-sp} f(a+p)\,dp$$

Since p is dummy variable,

Hence, L [f(t) u(t – a)] = e^{-as} L[f(t + a)]

Example 3.

Find the Laplace transform of

a. $\sin t\, u_\pi(t)$ b. $e^{-3t}u_2(t)$ c. $t^2u(t-3)$

Solution

a. We have

$$L[\sin t.u_\pi(t)] = -L[\sin(t-\pi)\, u_\pi(t)] \quad [\because \sin(\pi - t) = \sin t]$$

$$= -e^{-\pi s} L[\sin t] \quad \text{(Using second shifting theorem)}$$

$$= -\frac{e^{-\pi s}}{s^2+1}$$

b. We have

$$L\,[e^{-3t}\, u_2(t)] = L[e^{-3(t-2)-6}\, u_2(t)]$$

$$= e^{-6}\, L[e^{-3(t-2)}\, u_2(t)]$$

$$= e^{-6}.\, e^{-2s}\, L[e\text{-}3t]$$

$$= \frac{e^{-6-2s}}{s+3}$$

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c. We have

$L\,[t^2\,u(t-2)]$

Here $a = 2$

We know

$L[f(t)\,u(t-a)] = e^{-as}\,L[f(t+a)]$

$\left[\because L\{f(t)\,u(t-a)\} = e^{-as} L\{f(t+a)\}\right]$

Thus,

$$L[t^2\,u(t-2)] = e^{-2s}\,L[(t+2)^2]$$

$$= e^{-2s}\,L[t^2+4t+4]$$

$$= e^{-2s}\left[\frac{2}{s^3}+\frac{4}{s^2}+\frac{4}{s}\right]$$

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Example 7.

Find the inverse Laplace transform using second shifting theorem.

a. $\dfrac{e^{-s}}{(s+2)^3}$ b. $\dfrac{e^{-as}}{s^2}$ c. $\dfrac{se^{-s}}{s^2+9}$ d. $\dfrac{1+e^{-\pi s}}{s^2+1}$

Solution

a. Here a = 1, so

$$F(s) = \frac{1}{(s+2)^3}$$

$$\text{Now, } f(t) = L^{-1}\left[\frac{1}{(s+2)^3}\right] = e^{-2t} L^{-1}\left[\frac{1}{s^3}\right]$$

$$= e^{-2t}\frac{t^2}{2}$$

$$\text{Also, } f(t-a) = f(t-1) = e^{-2(t-1)}\frac{(t-1)^2}{2}$$

We have, $L^{-1}[e^{-at}F(s)] = f(t-a)\,u(t-a)$

So,

$$L^{-1}\left[e^{-s}\frac{1}{(s+2)^3}\right] = e^{-2(t-1)}\frac{(t-1)^2}{2}.u(t-1)$$

$$= \begin{cases} e^{-2(t-1)}\dfrac{(t-1)^2}{2} & if\ t>1 \\ 0 & if\ t<1 \end{cases}$$

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b. We have

$$f(t) = L^{-1}\left[\frac{1}{s^2}\right]$$

$$= t$$

and $f(t - a) = (t - a)$

Using second shifting theorem

$$L^{-1}[e^{-as}F(s)] = f(t - a)\, u(t - a)$$

$$\therefore \quad L^{-1}\left[e^{-as}.\frac{1}{s^2}\right] = (t - a)\, u(t - a) = \begin{cases} t - a & if\ t > a \\ 0 & if\ t < a \end{cases}$$

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c. We have

$$F(s) = \frac{s}{s^2 + 9}, \quad a = 1$$

$$\therefore \quad f(t) = L^{-1}\left[\frac{s}{s^2 + 9}\right]$$

$$= \sin 3t$$

and $f(t - 1) = \cos 3(t - 1)$

Using second shifting theorem

$$L^{-1}\left[\frac{se^{-s}}{s^2 + 9}\right] = \cos 3(t - 1)\, u(t - 1)$$

d. We have

$$F(s) = \frac{1}{s^2 + 1}, \quad a = \pi$$

$$f(t) = L\text{-}1\left[\frac{1}{s^2 + 1}\right] = \sin t$$

$$f(t - \pi) = \sin(t - \pi) = -\sin t$$

By using second shifting theorem

$$L^{-1}\left[\frac{1 + e^{-\pi s}}{s^2 + 1}\right] = L^{-1}\left[\frac{1}{s^2 + 1}\right] + L^{-1}\left[\frac{e^{-\pi s}}{s^2 + 1}\right]$$

$$= \sin t - \sin t.\ u_{\pi}(t)$$

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Convolution Theorem of Laplace transform

$$If\ L[f(t)] = F(s)\ and\ L[g(t)] = G(s)$$

then,

$$L[f(t) * g(t)] = F(s)\ G(s)$$

And $$L^{-1}[F(s)\,G(s)] = \int_0^t f(T)\,g(t-T)\,dt$$

$$= f(t) * g(t) = f * g$$

Proof:

Given, from definition of Laplace transform

$$L\,[f*g] = \int_0^\infty e^{-st}(f*g)\,dt$$

and we have from definition of convolution

$$[f*g] = \int_0^t f(T)g(t-T)\,dt$$

$$\therefore\quad L[f*g] = \int_0^\infty \left[\int_0^t f(T)\,g(t-T)\,dT\right] e^{-st}\,dt$$

$$= \int_0^\infty \int_0^t e^{-st} f(T)\,g(t-T)\,dT\,dt$$

By changing the order of double integration, we get

$$= \int_0^\infty \left(\int_0^\infty e^{-st} f(T)g(t-T)\,dt\right) dT$$

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$$= \int_0^\infty \left(\int_0^\infty e^{-st} f(T) g(t-T)\, dt \right) dT$$

and put t – T = p

dt = dp

when t = T   p = 0

t = ∞   p = ∞

$$= \int_0^\infty \left(\int_0^\infty e^{-s(p+T)} f(T) g(p)\, dp \right) dT$$

$$= \int_0^\infty f(T) e^{-sT} dT \int_0^\infty e^{-sp} g(p)\, dp$$

= L[f(t)] L[g(p)]

Since p in dummy variable

= F(s) G(s)

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Example 1

Find the convolution of following functions.

a. $e^t * e^{-t}$ b. sinwt * coswt c. $u(t-3) * e^{-2t}$

Solution

a. We have

$$f(t) = e^t, \quad g(t) = e^{-t}$$

$\therefore$ The convolution of

$$et * e^{-t} = f(t) * g(t) = \int_0^t f(t-T)\,g(T)\,dT$$

$$= \int_0^t e^{t-T}\,e^{-T}\,dT$$

$$= \int_0^t e^{t-2T}\,dT$$

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$$= \int_0^t e^{t-2T}\, dT$$

$$= e^t \int_0^t e^{-2T}\, dT$$

$$= e^t \left. \frac{e^{-2T}}{-2} \right|_0^t$$

$$= \frac{e^t}{-2} [e^{-2t} - 1]$$

$$= \frac{-e^{-t} + e^t}{2}$$

$$= \frac{e^t - e^{-t}}{2}$$

$$= \sinh t$$

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b. We have

$$f(t) * g(t) = \sin wt * \cos wt$$

$\therefore$ The convolution of f * g is

$$f(t) * g(t) = \int_0^t f(t-T)\,g(T)\,dT$$

$$= \int_0^t \sin w(t-T)\cos wT\,dT$$

$$= \int_0^t (\sin wt\cos wT - \sin wT\cos wt)\cos wT\,dT$$

$$= \sin wt\int_0^t \cos^2 wT\,dT - \cos wt\int_0^t \sin wT\cos wT\,dT$$

$$= \sin wt\int_0^t \left(\frac{1+\cos 2wT}{2}\right)dT - \cos wt\int_0^t \left(\frac{1+\sin 2wT}{2}\right)dT$$

$$= \frac{\sin wt}{2}\left[T + \frac{\sin 2wT}{2w}\right]_0^t - \frac{\cos wt}{2}\left[-\frac{\sin 2wT}{2w}\right]_0^t$$

$$= \frac{\sin wt}{2}\left(t + \frac{\sin 2wt}{2w}\right) + \frac{\cos wt}{4w}(\cos 2wt - 1)$$

$$= \frac{t}{2}\sin wt + \frac{1}{4w}\left(\sin wt \sin 2wt + \cos wt \cos 2wt - \frac{\cos 2t}{4w}\right)$$

$$= \frac{t}{2}\sin wt - \frac{\cos wt}{4w} + \frac{1}{4w}\cos(2wt - wt)$$

$$= \frac{t}{2}\sin wt - \frac{\cos wt}{4w} + \frac{\cos wt}{4w}$$

$$= \frac{t}{2}\sin wt$$

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c. We have

$$f(t) * g(t) = u(t-3) * e^{-2t}$$

and $f(t) = 1$ for $t \geq 3$

$\therefore$ The convolution of $f(t) * g(t)$ is

$$= \int_0^t u(t-3).e^{-2(t-T)}dT$$

$$= \int_3^t e^{-2}(t-T)dT$$

$$= e^{-2t}\int_3^t e^{2T}\,dT$$

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$$= e^{-2t}\int_{3}^{t} e^{2T}\,dT$$

$$= e^{-2t}\left.\frac{e^{2T}}{2}\right|_{3}^{t}$$

$$= \frac{e^{-2t}}{2}\left(e^{-2t} - e^{6}\right) \text{ for } t > 3$$

$$= \frac{1}{2}\left[1 - e^{-2(t-3)}\right] \text{ for } t > 3$$

$$= \frac{1}{2}\left[1 - e^{-2(t-3)}\right] u(t-3)$$

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Example 2.

Find the inverse Laplace transform of following functions.

a. $\dfrac{1}{s(s^2+4)}$ b. $\dfrac{s}{(s^2+a^2)^2}$ c. $\dfrac{1}{(s^2+a^2)^2}$ d. $\dfrac{w}{s^2(s^2+w^2)}$

Solution

a. Let $F(s) = \dfrac{1}{s}$, $G(s) = \dfrac{1}{s^2+4}$

So, $f(t) = L^{-1}\left(\dfrac{1}{s}\right) = 1 \quad g(t) = L^{-1}\left(\dfrac{1}{s^2+4}\right) = \dfrac{1}{2}\sin 2t$

Therefore

$$f(T) = 1,\ g(t-T) = \frac{1}{2}\sin 2(t-T)$$

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Now,

$$L^{-1}[F(s)\,G(s)] = \int_0^t f(T)\,g(t-T)\,dT$$

$$= \int_0^t \frac{1}{2}\sin 2(t-T)\,dT$$

$$= \frac{1}{2}\left[\frac{-\cos 2(t-T)}{-2}\right]_0^t$$

$$= \frac{1}{4}(1 - \cos 2t)$$

$$\therefore\ L^{-1}\left[\frac{1}{s(s^2+4)}\right] = \frac{1}{4}(1 - \cos 2t)$$

b. Let $F(s) = \frac{s}{s^2 + a^2}$; $G(s) = \frac{1}{s^2 + a^2}$

$f(t) = \cos at$; $g(t) = \frac{1}{a}\sin at$

and $f(T) = \cos aT$, $g(t - T) = \frac{1}{a}\sin a(t - T)$

Here,

$$L^{-1}[F(s)\,G(s)] = \int_0^t f(T)\,g(t-T)\,dT$$

$$= \int_0^t \cos aT \frac{1}{a}\sin a(t-T)\,dT$$

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$$= \frac{1}{2a} \int_0^t [\sin(aT + at - aT) - \sin(aT - at + aT)] dT$$

$$= \frac{1}{2a} \int_0^t [\sin at - \sin(2aT - at)] dT$$

$$= \frac{1}{2a} \left[T \sin at + \frac{\cos(2aT - at)}{2a} \right]_0^t$$

$$= \frac{1}{2a} \left[\text{tsinat} + \frac{\text{cosat}}{2a} - \frac{\text{cosat}}{2a} \right]$$

$$= \frac{t}{2a} \text{sinat}$$

$$\therefore L^{-1}\left[\frac{s}{(s^2 + a^2)^2} \right] = \frac{t}{2a} \text{sinat}$$

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c. Let $F(s) = \frac{1}{s^2 + a^2}$; $G(s) = \frac{1}{s^2 + a^2}$

$$f(t) = \frac{\sin at}{a}; \quad g(t) = \frac{\sin at}{a}$$

$$\therefore \quad f(T) = \frac{\sin aT}{a}; \; g(t - T) = \frac{\sin a(t - T)}{a}$$

Therefore,

$$L^{-1}[F(s)\,G(s)] = \int_0^t f(T)\,g(t - T)\,dT$$

$$= \frac{1}{2a^2}\int_0^t \frac{\sin aT}{a}.\frac{\sin a(t-T)}{a}\,dt$$

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$$= \frac{1}{2a^2} \int_0^t [\cos(2aT - at) - \cos at]\, du$$

$$= \frac{1}{2a^2} \left[\frac{\sin(2aT - at)}{2a} - T\cos at \right]_0^t$$

$$= \frac{1}{2a^2} \left[\frac{\sin at}{2a} - t\cos t + \frac{\sin at}{2a} \right]$$

$$= \frac{1}{2a^2} \left[\frac{\sin at}{a} - t\cos at \right]$$

$$= \frac{1}{2a^3} (\sin at - at\cos at)$$

$$\therefore \left[\frac{1}{(s^2 + a^2)^2} \right] = \frac{1}{2a^3} (\sin at - at\cos at)$$

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d. Let $F(s) = \frac{1}{s^2}$; $G(s) = \frac{w}{s^2 + w^2}$

$f(t) = t$; $g(t) = \sin wt$

and $f(t - T) = t - T$, $g(T) = \sin wt$

Therefore,

$$L^{-1}[F(s)\,G(s)] = \int_0^t (t-T)\sin wT\,dT$$

$$= \left[(t-T)\frac{\cos wT}{-w} + \left(\frac{\sin wT}{-w^2}\right)\right]_0^t$$

$$= -\frac{\sin wt}{w^2} - \left(-\frac{t}{w} - 0\right)$$

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$$L^{-1}[F(s)\,G(s)] = \int_0^t (t-T)\sin wT\,dT$$

$$= \left[(t-T)\frac{\cos wT}{-w} + \left(\frac{\sin wT}{-w^2}\right)\right]_0^t$$

$$= -\frac{\text{sinwt}}{w^2} - \left(-\frac{t}{w} - 0\right)$$

$$= \frac{t}{w} - \frac{\text{sinwt}}{w^2}$$

$$= \frac{\text{wt} - \text{sinwt}}{w^2}$$

$$\therefore\ L\text{-}1\left[\frac{w}{s^2(s^2+w^2)}\right] = \frac{\text{wt} - \text{sinwt}}{w^2}$$

APPLICATION OF LAPLACE TRANSFORM TO THE DIFFERENTIAL EQUATION

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4. Laplace Transform of the Derivative of a Function

Let f(t) be continuous for all $t \geq 0$ and be of exponential order. If f'(t) is piecewise continuous on every finite interval in the range $t \geq 0$, then the Laplace transform of the derivative f'(t) exists when $s > k$, and

$$L[f'(t)] = s\,L[f(t)] - f(0)$$

Proof: Let f'(t) be continuous for all $t \geq 0$. Then

$$L[f'(t)] = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= [e^{-st} f(t)]\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt \quad \text{[Integrating by parts]}$$

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$$= [0 - f(0)] + s\int_0^{\infty} e^{-st} f(t)\,dt$$

$$= sL[f(t)] - f(0)$$

The above theorem may be extended to piecewise continuous function f(t).

Applying (1) to the second derivative, we get

$$L[f''(t)] = s\,L[f'(t)] - f'(0) = s\,[s\,L[f(t)] - f(0)] - f'(0)$$

or, $\quad L[f''(t)] = s^2\,L[f(t)] - s\,f(0) - f'(0)$

Similarly, $L[f'''(t)] = s^2\,L[f'(t)' - s\,f'(0) - f''(0)$

or, $\quad L[f'''(t)] = s^3\,L[f(t)] - s^2\,f(0) - s\,f'(0) - f''(0)$

In general, $L[f^n(t) = s^n\,L[f(t)] - s^{n-1}\,f(0) - s^{n-2}\,f'(0) \ldots.. - f^{(n-1)}(0)$

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Example 2

Using the Laplace transform of the derivatives find the Laplace transform of following functions.

a. e^{-at} b. sinat c. $\sin^2 t$ d. L[sinat + at cosat]

Solution

a. We have

$$f(t) = e^{-at} \Rightarrow f'(t) = -ae^{-at},\ f(0) = 1$$

we have, $L[f'(t)] = sL[f(t)] - f(0)$

$$\text{so, } L[-ae^{-at}] = sL[e^{-at}] - 1$$

$$\Rightarrow sL[e^{-at}] = -aL[e^{-at}] + 1$$

$$\Rightarrow (s + a)\, L[e^{-at}] = 1$$

$$\therefore L[e^{-at}] = \frac{1}{s + a}$$

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b. We have, $f(t) = \sin at$

$f'(t) = a\cos at, \quad f''(t) = -a^2 \sin at$

where, $f(0) = 0, \quad f'(0) = a$

We know, $L[f''(t)] = s^2 L[f(t)] - sf(0) - f'(0)$

Thus, $L[-a^2 \sin at] = s^2 L[\sin at] - 0 - a$

$$\Rightarrow a = (s^2 + a^2) L[\sin at]$$

$$\therefore L[\sin at] = \frac{a}{s^2 + a^2}$$

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c. We have, $f(t) = \sin^2 t$, $f'(t) = 2\sin t \cos t = \sin 2t$

Now, $L[f'(t)] = sL[f(t)] - f(0)$

Thus, $L[\sin 2t] = sL[\sin^2 t] - 0$

$$\Rightarrow \frac{2}{s^2+4} = sL\,[\sin^2 t]$$

$$\therefore L[\sin 2t] = \frac{2}{s(s^2+4)}$$

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d. We have

$$f(t) = \sin at + at\cos at$$

For (sinat)

Let $f(t) = \sin at,\ f(0) = 0$

$$f'(t) = a\cos at,\ f''(t) = -a^2\sin at,\ f'(0) = a$$

We know,

$$L[f''(t)] = s^2L[f(t)] - sf(0) - f'(0)$$

$$\Rightarrow \quad L[-a^2\sin at] = s^2L[\sin at] - a$$

$$\Rightarrow \quad L[\sin at] = \frac{a}{s^2 + a^2}$$

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For (tcosat)

Let $f(t) = t\cos at$, $f(0) = 0$

$f'(t) = \cos at - ta\sin at$, $f'(0) = 1$

$f''(t) = -a\sin at - a\sin at - ta^2\cos at$

$= -2a\sin at - a^2t\cos at$

We know, $L[f''f(t)' = s^2L[f(t)] - sf(0) - f'(0)$

$\Rightarrow L[-2a\sin at - a^2t\cos at] = s^2 L[t\cos at] - 1$

or, $-2a\, L[\sin at] - a^2L[t\cos at] = s^2L[t\cos at] - 1$

or, $\dfrac{-2a^2}{s^2 + a^2} = (s^2 + a^2)\, L[t\cos at] - 1$

or, $\dfrac{s^2 + a^2 - 2a^2}{(s^2 + a^2)^2} = L[t\cos at]$

$$\therefore\ L[t\cos at] = \frac{s^2 - a^2}{(s^2 + a^2)^2}$$

$$\therefore\ L\,[\sin at + at\cos at] = \frac{a}{s^2 + a^2} + \frac{a(s^2 - a^2)}{(s^2 + a^2)^2}$$

$$= \frac{a(s^2 + a^2) + as^2 - a^3}{(s^2 + a^2)^2}$$

$$= \frac{2as^2}{(s^2 + a^2)^2}$$

Application of Laplace Transform to Differential Equation

Laplace transform can be used to solve ordinary and partial differential equations.

Steps to solve the problems are

(i) First we take Laplace transform on both sides of the given differential equations

(ii) Use the formulae of transform of derivatives with given conditions.

(iii) Finally, we find the inverse Laplace transform $L^{-1}(y)$ to get the solution

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Example 1

Solve $y'' + 4y' + 4y = e^{-t}$, $y(0) = y'(0) = 0$

Solution

Given,

$$y'' + 4y' + 4y = e^{-t}$$

Taking Laplace transform on both sides, we get

$$L[y''] + 4L[y'] + 4L[y] = L[e^{-t}]$$

$$\Rightarrow \quad s^2L[y] - sy(0) - y'(0) - 4\{sL[y] - y(0)\} + 4L[y] = \frac{1}{s+1}$$

We have

$$y(0) + y'(0) = 0$$

$$\Rightarrow \quad s^2L[y] + 4sL[y] + 4L[y] = \frac{1}{s+1}$$

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$$\Rightarrow \quad s^2L[y] + 4sL[y] + 4L[y] = \frac{1}{s+1}$$

$$\Rightarrow \quad L[y] = \frac{1}{(s+1)(s^2+4s+4)}$$

$$= \frac{1}{(s+1)(s+2)^2}$$

$$\Rightarrow \quad y = L^{-1}\left[\frac{1}{(s+1)(s+2)^2}\right]$$

So, using partial fraction, we get

$$\frac{1}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$$

$$\Rightarrow \quad 1 = A(s+2)^2 + B(s+1)(s+2) + C(s+1)$$

Put $s = -1$; $\quad 1 = A$

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$$\Rightarrow \quad 1 = A(s+2)^2 + B(s+1)(s+2) + C(s+1)$$

Put $s = -1$; $\quad 1 = A$

Put $s = -2$; $\quad 1 = -C \quad \therefore C = -1$

Put $s = 0$; $\quad 1 = A4 + 2B + C$

$$\Rightarrow 1 = 4 + 2B - 1 \quad \Rightarrow -2 = 2B$$

$$\Rightarrow B = -1$$

$$\therefore \quad y = L^{-1}\left[\frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}\right]$$

$$= L^{-1}\left[\frac{1}{s+1} - \frac{1}{s+2} - \frac{1}{(s+2)^2}\right]$$

$$= e^{-t} - e^{-2t} - e^{-2t}t$$

$$= e^{-t} - (1+t)\,e^{-2t}$$

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Example 2.

Solve the initial value problem

$$y'' + 2y' + 2y = 5\sin t, \quad y(0) = y'(0) = 0$$

Solution

Given,

$$y'' + 2y' + 2y = 5\sin t$$

Taking Laplace transform on both sides, we get

$$L[y''] + 2L[y'] + 2L[y] = 5L[\sin t]$$

$$\Rightarrow \quad s^2L[y] - sy(0) - y'(0) + 2\{sL[y] - y(0)\} + 2L[y] = \frac{5}{s^2 + 1}$$

$$\Rightarrow \quad L[y]\,[s^2 + 2s + 2] = \frac{5}{s^2 + 1}$$

$$\therefore\ L[y] = \frac{5}{(s^2 + 1)\,(s^2 + 2s + 2)}$$

Resolving into partial fraction

$$\frac{5}{(s^2+1)(s^2+2s+2)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2s+2}$$

$$\Rightarrow \qquad 5 = \frac{(As+B)(s^2+2s+2)+(Cs+D)(s^2+1)}{(s^2+1)(s^2+2s+2)}$$

$$\Rightarrow \qquad 5 = (As+B)(s^2+2s+2)+(Cs+D)(s^2+1)$$

Solving we get

$$A = -2,\ B = 1,\ C = 2\ \&\ D = 3$$

$$\therefore \quad y = L^{-1}\left[\frac{5}{(s^2+1)(s^2+2s+2)}\right]$$

$$= L^{-1}\left[\frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2s+2}\right]$$

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$$= L^{-1}\left[\frac{-2s+1}{s^2+1} + \frac{2s+3}{s^2+2s+2}\right]$$

$$= L^{-1}\left[\frac{-2s}{s^2+1}\right] + L^{-1}\left[\frac{1}{s^2+1}\right] + L^{-1}\left[\frac{1}{(s^2+1)+1}\right]$$

$$= -2\cos t + \sin t + 2L^{-1}\left[\frac{(s+1)}{(s+1)^2+1}\right] + L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$= -2\cos t + \sin t + 2e^{-t}\cos t + e^{-t}\cos t$$

$$= 2\cos t\,(e^{-t} - 1) + (e^{-t} + 1)\sin t$$

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Example 3.

Solve the initial value problem

$$y'' - 2y' + 10y = 0,\ y(0) = 3,\ \ y'(0) = 3$$

Solution

Given,

$$y'' - 2y' + 10y = 0$$

Taking Laplace transform on both sides

$$L[y''] - 2L[y'] + 10L[y] = 0$$

$$\Rightarrow \quad s^2L[y] - sy(0) - y'(0) + 2\{sL[y] - y(0)\} + 10L[y] = 0$$

$$\Rightarrow \quad s^2L[y] - 3s - 3 - 2\{sL[y] - 3\} + 10L[y] = 0$$

$$\Rightarrow \quad L[y]\ (s^2 - 2s + 10) = 3s - 3$$

$$\Rightarrow \quad L[y] = \frac{3(s-1)}{s^2 - 2s + 10} = \frac{3(s-1)}{(s-1)^2 + 3^2}$$

$$\therefore \quad y = L^{-1}\left[\frac{3(s-1)}{(s-1)^2 + 3^2}\right]$$

$$y = 3e^t \cos 3t$$

Example 4

Solve the simultaneous differential equations $y' + 2x = \sin 2t$ and $x' - 2y = \cos 2t$ given that $x(0) = 1$, $y(0) = 0$

Solution

Given,

$$y' + 2x = \sin 2t \qquad \text{and} \qquad x' - 2y = \cos 2t$$

Taking Laplace transform on both sides, we get

$$L[y'] + 2L[x] = L[\sin 2t]$$

and $\quad L[x'] - 2L[y] = L[\cos 2t]$

$$\Rightarrow \qquad sL[y] - y(0) + 2L[x] = \frac{2}{s^2 + 4}$$

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$$sL[x] - x(0) - 2L[y] = \frac{s}{s^2 + 2}$$

Using initial conditions, we get

$$sL[y] + 2L[x] = \frac{2}{s^2 + 4} \qquad (1)$$

and $$sL[x] - 1 - 2L[y] = \frac{s}{s^2 + 4}$$

or, $$sL[x] - 2L[y] = \frac{s}{s^2 + 4} + 1$$

$$= \frac{s^2 + s + 4}{s^2 + 4} \qquad (2)$$

Solving (1) and (2), we get

$$L[x] = \frac{1}{s^2+4} + \frac{s}{s^2+4}$$

and $$L[y] = -\frac{2}{s^2+4}$$

Taking inverse Laplace transform, we get

$$x = L^{-1}\left[\frac{1}{s^2+4}\right] + L^{-1}\left[\frac{s}{s^2+4}\right]$$

$$= \frac{1}{2}\sin 2t + \cos 2t$$

and $$y = L^{-1}\left[-\frac{2}{s^2+4}\right]$$

$$= -\sin 2t$$

Unit Step Function:

The Unit step function $u_a(t)$ is defined as,

$$u_a(t) = \begin{cases} 0 & if\ t < a \\ 1 & if\ \ t > a \end{cases} \qquad (a \geq 0)$$

$u_a(t)$ is also denoted by $u(t-a)$

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ANY QUESTIONS...?

THANK YOU VERY MUCH