Lecture 1

n linear eqns, n unknowns

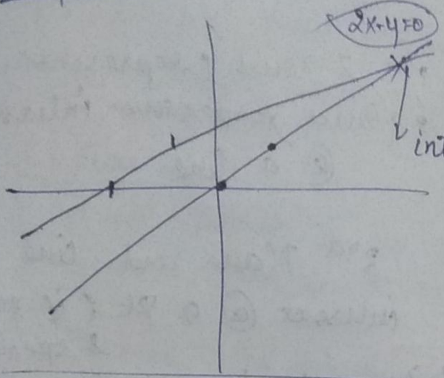
1) Row picture
2) column picture
3) matrix form.

$2x - y = 0$ ✓ (goes through origin)
$2y - x = 3$ ✗ (doesnt)

Row picture

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$\boxed{A x = b}$ Linear equations



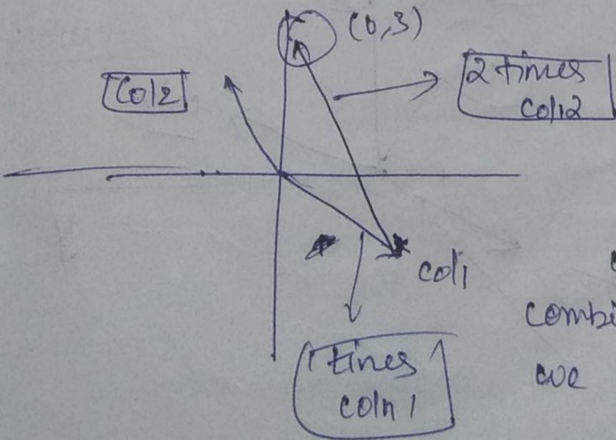

$\boxed{\begin{matrix} x=1 \\ y=2 \end{matrix}}$ solution

Column picture

$$x\begin{bmatrix} 2 \\ -1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} = [\text{Vector additions}]$$

Finding the right linear combinations of vectors that satisfy $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$ solution

As we know the soln we can put those values and visualise




if x, y (all combinations) are tried, we can fill entire 2D plane with it

## Matrix form

$2x - y = 0$

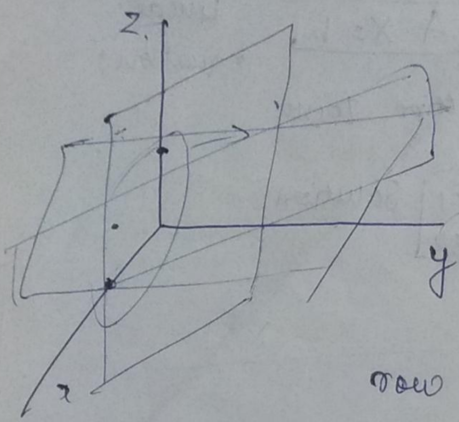
$-x + 2y - z = -1$

$-3y + 4z = 4$

$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$

each eqn is a plane.

## Row picture




1st 2 rows (representing 2 planes) ~~represent~~ intersect @ a line

3rd plane and line intersect @ a pt (if not special)

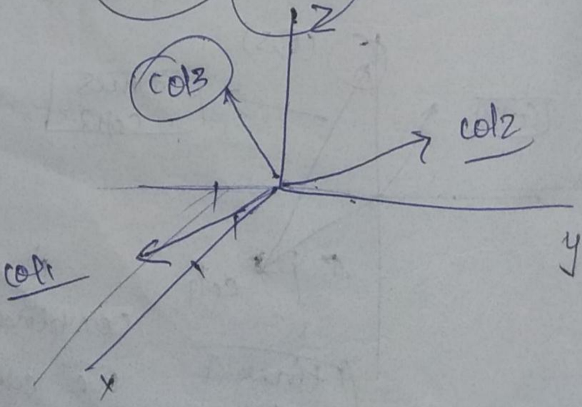
Difficult to visualise row picture

## Colum picture

$x \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + z \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} \Rightarrow$ equal

$x = 0, y = 0$

$z = 1$




Gauss elimination is used in all s/w

$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ (assume $b = A_1 + A_2$ (sum of 1st 2 columns))

$x=1, y=1, z=0$

Can we solve $Ax = b$ for every $b$?
Do linear combinations of columns fill 3d space?
non-singular (invertible) matrices.

Failure cases:

1) all vectors lie in same plane (all combinations will also lie in that plane)

2) If one colm is dependent on each other

matrix - Vector multiplication

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}$$ column perspective

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow$$ dot product

$Ax$ = a combination of columns of A.

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Lecture 2

Solving linear systems

① Elimination — Success / failure

$x + 2y + z = 2$
$3x + 8y + z = 12$
$4y + z = 2.$

$$\begin{bmatrix} \boxed{1} & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}$$ 1st pivot

~~1st pivot~~ 1st row multiplied 1st column in 2nd row

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 12 \\ 2 \end{bmatrix}$$

A b

$x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

step 1:
$$\begin{array}{r} 3 \;\; 8 \;\; 1 \\ -3[1 \;\; 2 \;\; 1] \\ \hline 0 \;\; 2 \;\; -2 \end{array} \Rightarrow A = \begin{bmatrix} 1 & 2 & \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$$

(2,1) cleaning

step 2:
(3,1) cleaning

step 3: 2nd pivot (3,2) step.

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 4 & 1 \end{bmatrix}$$ → 2x2 system

$\Rightarrow$ ~~2 -2~~

$$\begin{array}{r} [4 \;\; 1] \\ -2[2 \;\; -2] \end{array}$$

$= [0 \;\; 5]$

$$\begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 0 & \boxed{5} \end{bmatrix}$$

∴ U = upper triangular

purpose of elimination is to find U from A.

The pivots can't be 0

Determinant of matrix = ~~multiply~~ $\prod$ Diag (U)

Failurecases:

① If $A(1,1) == 0$ (exchange rows)

② If there's a non-zero below a 0 pivot then U can always exchange rows

If there's 0 and nothing to exchange, its a permanent failure.

Back-substitution:

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 4 & 1 & 2 \end{bmatrix}$$

Augumented matrix

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 5 & -10 \end{bmatrix}$$

$\boxed{C}$

$$\begin{aligned} x + 2y + z &= 2 \\ 2y - 2z &= 6 \\ 5z &= -10 \end{aligned}$$

$\boxed{x = 2 \quad y = 1 \quad z = -2}$

$\boxed{Ux = C}$

Back substitute from last row to get values

Matrices:

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}$$

matrix vector multiplication (column picture)

# there's a corresponding row picture also

$$[1\ 2\ 7] \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}_{3\times3}$$
$1\times3$

1x row 1
2x row2
7x row3

Elimination in matrix steps

(1) step1

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}$$ subtract 3x Row1 from row 2

$E_{2,1}$

becoz they're unchanged

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$$

$[-3\ 1\ 0]$

(2) step2:

$$\begin{bmatrix} 1 & 2 & -2 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$$ subtract 2xRow2 from row 3

becoz rows 1,2 remain unchanged

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

$E_{32}$

Summary:

$$E_{32}(E_{21}A) = U$$

How can I compress or find (E) matrix that does this at once (tranforming A to U)

Permutation matrix (exchanges rows/columns)

$$\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{P} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

exchange columns

$$\begin{bmatrix} & \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

row operations always on left
column operations on right

Inverses: a matrix which undoes

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

(here it undoes elimination)

Adding (3×row1 to row2)

$E^{-1}$ $E = I$

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Lectures: Multiplication with inverse matrices:

$AB = C$. · 1st way

$c_{ij}$ row number, coln number

$A_{3,n}$ $B_{n,4}$ $C_{34} = (\text{row 3 of } A) \cdot (\text{col 4 of } B)$.

$$[a_{31} \dots] \begin{bmatrix} b_{14} \\ \vdots \\ \vdots \end{bmatrix}$$

$b_{14}$

$\Rightarrow a_{31} \cdot b_{14} + a_{32} \cdot b_{24} + \dots$

$= \boxed{\sum_{k=1}^{n} a_{3k} b_{k4}}$

$A = m \times n$
$B = n \times p$ $\Rightarrow$ $\boxed{C = m \times p}$

2nd way

$A \times \text{col}1(B) = \text{col}1(C)$
$A \times \text{col}2(B) = \text{col}2(C)$

→ series of matrix-vector multiplications (column picture) columns of C are combinations of columns

3rd way (row perspective)

each row of A multiplying B matrix

each row of C is a linear combination of rows of B.

4th way (column by row)

~~(m×1) × (1×p)~~

$$\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}_{3\times1} \begin{bmatrix} 1 & 6 \end{bmatrix}_{1\times2} = \begin{bmatrix} 2 & 12 \\ 1 & 6 \\ 3 & 18 \end{bmatrix}_{3\times2}$$

AB = sum of (columns of A)(rows of B)

$$\begin{bmatrix} 2 & 7 \\ 3 & 8 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix} + \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \begin{bmatrix} 0 & 0 \end{bmatrix}$$

special case where all rows (lie on line [1 6]; all columns (column space) lie on line $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

Block multiplication

$$\left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right] \left[\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right] = \left[\begin{array}{c|c} & \\ \hline & \end{array}\right]$$

$\boxed{A_1B_1 + A_2B_3}$

Inverses (square matrix)

most imp. question for a matrix (is it invertible or not?)

(if it exists how do we find it?)

singular (non-invertible)

$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ why is this not invertible?

Reasoning: I must have a matrix (combination of columns of A) that I can to create $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

But since both ~~vectors~~ cols are on same line, its not possible.

If you can find a non-zero vector that solves $AX=0$ then it is not invertible.

$X = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ gives $AX=0$

if $A^{-1}$ exists then $AX=0$ ~~for~~

on multiplying by $A^{-1}$

$A^{-1}AX = A^{-1}0$

$\Rightarrow \boxed{X=0}$ (But $X = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$)

$$\overset{A}{\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}} \overset{A^{-1}}{\begin{bmatrix} a & c \\ b & d \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A (col j of $A^{-1}$) = col j of I

so for each column you get system of 'n' equations (2 times here)

Gaussian elimination

That's where Jordan comes in

Gauss - Jordan method (solves 2 eqns @ once)

$\begin{bmatrix}1 & 3\\2 & 7\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$ $\Rightarrow$ Augment matrices

$\begin{bmatrix}1 & 3\\2 & 7\end{bmatrix}\begin{bmatrix}c\\d\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$

$\left[\begin{array}{cc|cc}1 & 3 & 1 & 0\\2 & 7 & 0 & 1\end{array}\right]$

$\Downarrow$ elimination

$\left[\begin{array}{cc|cc}1 & 3 & 1 & 0\\0 & 1 & -2 & 1\end{array}\right]$

upper $\Delta$lar form (Gauss ask to quit

But Jordan says keep going

Elimination upwards (subtract multiple of equation 2 from equation 1)

row1 − 3 (row2)

$\left[\begin{array}{cc|cc}1 & 0 & 7 & -3\\0 & 1 & -2 & 1\end{array}\right]$

$I$ $\quad A^{-1}$

$[A\ I]$

(E's lot of)

$\Downarrow$

$[I\ A^{-1}]$

$E[A\ I]=[I\ ?]$

$EA=I$ means $\boxed{A^{-1}=E}$