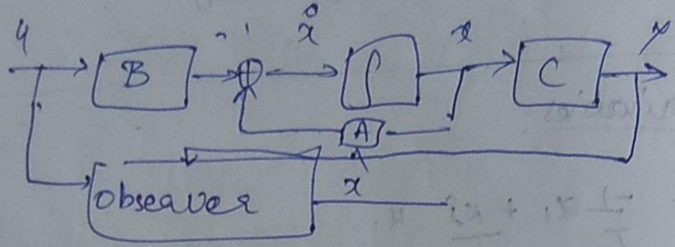
State estimation

state vector + not given always.

$u = -Kx$.

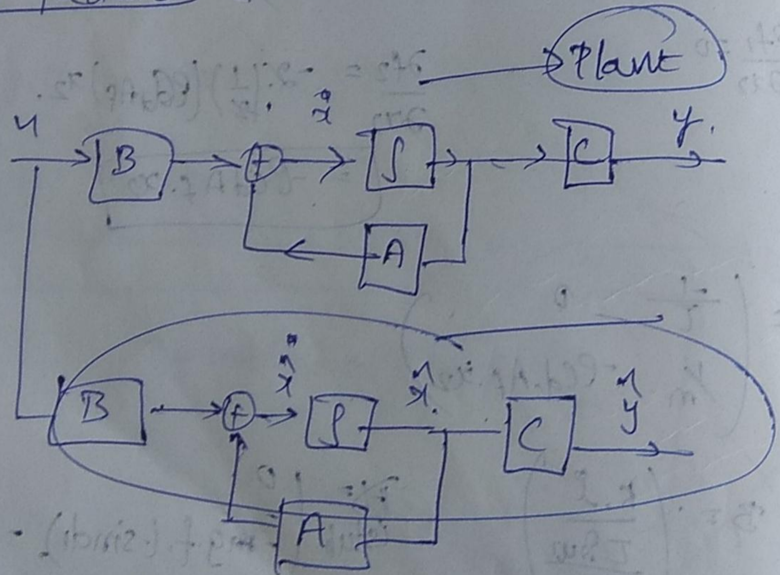


1) is fed with real input as plant.

2) can add 'y' to observer if needed

$\dot{\hat{x}} = A\tilde{x} + Bu$ $\quad \tilde{x}$ = estimated state vector

W/o feedback:



Observer design

$\tilde{x} = x - \hat{x}$

$\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}} = Ax + \cancel{Bu} - A\hat{x} - \cancel{Bu}$

$= A(x - \hat{x}) = A\tilde{x}$

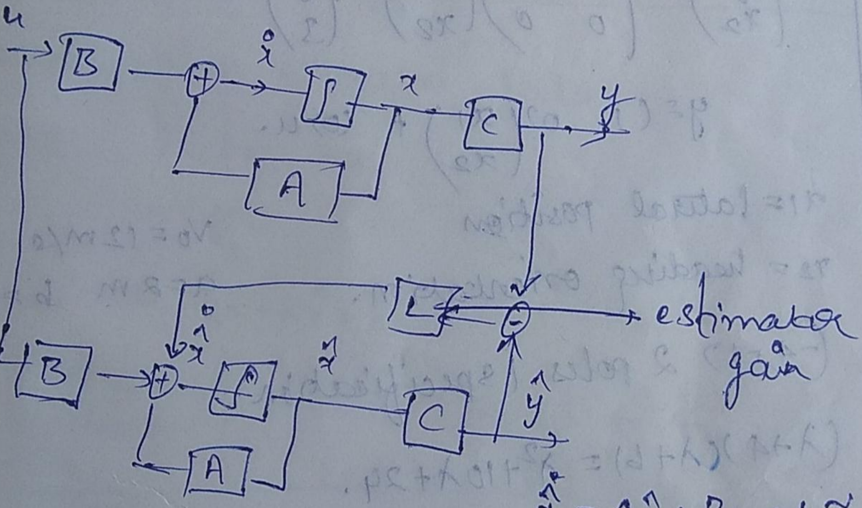
$\tilde{x}(t) = e^{At}\tilde{x}(0) = e^{At}(x(0) - \hat{x}(0))$

If A matrix has -ve eigen values [stable system]
it'll drive $\tilde{x} \to 0$

A matrix (system characterestics

initial condn of observer & plant

∴ We need system to be stable if we use an open-loop observer



$\dot{\hat{x}} = A\hat{x} + Bu + L\tilde{y}$

$\hat{y} = C\hat{x}$

$\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}}$

$= Ax + \cancel{Bu} - A\hat{x} - \cancel{Bu} - L\tilde{y}$

$= A(x - \hat{x}) - L(y - \hat{y})$

$= A(x - \hat{x}) - LC(x - \hat{x})$

$= (A - LC)(x - \hat{x})$

$= (A - LC)\tilde{x}$

solution is

$\tilde{x}(t) = e^{(A-LC)t}\tilde{x}(0)$

L can be chosen such that $\tilde{x} \to 0$
error dynamics converges to zero.
(observability)

$\rightarrow$ L chosen such that closed loop error dynamics $(A-LC)$ converges as desired rate

$$\det(sI - A + LC) = 0$$

Pole placement

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 12 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 6 \\ 3 \end{pmatrix} u.$$

$$y = (1 \quad 0) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (0) u.$$

x_1 = lateral position
x_2 = heading orientation.

$V_0 = 12\, m/s$
$a = 2\, m \quad b = 4\, m.$

$(-4, -6)$ 2 poles (specification

$$(\lambda + 4)(\lambda + 6) = \lambda^2 + 10\lambda + 24.$$

$sI - A + LC = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 12 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \cancel{l_1} & \cancel{l_2} \\ \cancel{l_3} & \cancel{l_4} \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix}$

2x2 ~~1x2~~

$$= \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 12 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix}$$

1x2

$$= \begin{pmatrix} s & -12 \\ 0 & s \end{pmatrix} + \begin{pmatrix} l_1 & 0 \\ l_2 & 0 \end{pmatrix} = \left| \begin{pmatrix} s + l_1 & -12 \\ l_2 & s \end{pmatrix} \right|$$

$(s+l_1)(s) - (-12)(l_2)$

$= s^2 + l_1 s + 12 l_2$.

$\Rightarrow \lambda^2 + 10\lambda + 24$

$l_1 = 10$ $\quad$ $l_2 = 2$

Now let $y = [0 \quad 1]$

then $(sI - A + LC) = \begin{pmatrix} s & -12 \\ 0 & s \end{pmatrix} + \begin{pmatrix} 0 & l_1 \\ 0 & l_2 \end{pmatrix}$

$= \begin{vmatrix} s & l_1 - 12 \\ 0 & s + l_2 \end{vmatrix} = s^2 + l_2 s - 0 \quad \times \Rightarrow$ $\boxed{\text{Not observable}}$

control using estimated states.

given, $\dot{x} = Ax + Bu$.

$y = Cx$.

controller $u = -K\hat{x} + k_r r$

State estimator $\Big\}$ $\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$

$$\boxed{\left[\begin{matrix} \dot{x} \\ \dot{\tilde{x}} \end{matrix}\right] = \begin{pmatrix} A - BK & BK \\ 0 & A - LC \end{pmatrix} \begin{pmatrix} x \\ \tilde{x} \end{pmatrix} + \begin{pmatrix} Bk_r \\ 0 \end{pmatrix} r}$$

Complete CL system.

$\lambda(s) = \det(sI - A + BK) \cdot \det(sI - A + LC)$

$\Rightarrow$ can be assigned arbitrary roots, if system is reachable, observable

Estimator poles = (4-5 times) state feedback design.

all in continuous form

computer - Discretise system.

$$\hat{x}(t_{k+1}) = \hat{x}(t_k) + (t_{k+1} - t_k)\left[A\hat{x}(t_k) + Bu(t_k) + L(y(t_k) - C\hat{x}(t_k))\right]$$

$(t_{k+1} - t_k)$ → Sampling time

Exercises :-

$x = (4 \times 1)$ $y = 1 \times 1$ ~~u = (1×2)~~. $u = 2 \times 1$

$u = -Kx$.
$(2 \times 4)\ (4 \times 1)$ K $(1 \times 5)(5 \times 1)$

~~L(y~~ $y = (1 \times 1)$. $y =$.

$\hat{x}$ $L(y - C\hat{x})$

$(5 \times 1) = L \times (2 \times 1)$ 2×1
5×2

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -4/m \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1/m \end{pmatrix} u.$$

$$y = (1 \quad 0)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

x_1 = position x_2 = velocity $L = \begin{pmatrix} l_{11} \\ l_{21} \end{pmatrix}$

$(s+p)^2$

Required polynomial = $\boxed{s^2 + 2ps + p^2}$

$$(sI - A + LC) = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & -c/m \end{pmatrix} + \begin{pmatrix} l_{11} \\ l_{21} \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} s & -1 \\ 0 & s+c/m \end{pmatrix} + \begin{pmatrix} l_{11} & 0 \\ l_{21} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} s+l_{11} & -1 \\ l_{21} & s+c/m \end{pmatrix} \quad s^2 + (l_{11}s) + \left(\frac{c}{m}\right)s + l_{11}\left(\frac{c}{m}\right) - l_{21}$$

$$s^2 + \left(l_{11} + \frac{c}{m}\right)s + \left(l_{11}\left(\frac{c}{m}\right) + l_{21}\right)$$

$$s^2 + 2p \cdot s + p^2$$

$$l_{11} + \frac{c}{m} = 2p$$

$$l_{11} = 2p - c/m$$

$$p^2 = l_{11}\left(\frac{c}{m}\right) + l_{21}$$

$$l_{21} = \left(2p - \frac{c}{m}\right)\left(\frac{c}{m}\right) + p^2$$

$$l_{21} = 2p\left(\frac{c}{m}\right) - \left(\frac{c}{m}\right)^2 + p^2$$

$$\cancel{= p}$$

$$= (-1)\left(p^2 + \left(\frac{c}{m}\right)^2 - 2p\left(\frac{c}{m}\right)\right)$$

$$\boxed{l_{21} = (-1)\left(p - \frac{c}{m}\right)^2}$$

DC servo motor example

$$A = \begin{pmatrix} -40 & -2 \\ 10 & -5 \end{pmatrix}$$

$$B = \begin{pmatrix} 20 \\ 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$(-50) \quad (\lambda + 50)^2 = \lambda^2 + 100\lambda + 2500$$

$$\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} -40 & -2 \\ 10 & -5 \end{pmatrix} + \begin{pmatrix} l_1 & 0 \\ l_2 & 0 \end{pmatrix}$$

$$= \begin{vmatrix} s+40+l_1 & s+2 \\ l_2-10 & s+5 \end{vmatrix} = s^2 + 40s + l_1 s + 200 + 5l_1 - (s+2)(l_2-10)$$

$$= s^2 + (4\cancel{5}+l_1)s + (200+5l_1)$$

$$- s(l_2-10) - 2(l_2-10)$$

$$\boxed{\therefore 100 = 4\cancel{5} + l_1 - l_2 + 10} \qquad 2500 = 200 + 5l_1 - 2l_2 + 20.$$

$$5\cancel{5} = l_1 - l_2 \qquad \cancel{2300}\ (2280 = 5l_1 - 2l_2)$$

$$2280 = 5l_1 - 2l_2$$

$$250 = 5l_1 - 5l_2$$

$$(-) \quad (-) \quad (+)$$

$$l_1 =$$

$$2030 = 3l_2$$

$$l_2 = \frac{2030}{3} = 676.66$$

<u>observer.</u>

for every $T>0$, we can find $x(T)$ by $y(t)$, $u(t)$ on interval of $[0, T]$

$$y(t) = Ce^{At}x(0).$$

$$\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} x^* = 0 \longrightarrow \text{if } W_0 \text{ has full rank only soln is } x^* = 0$$

$$\Downarrow W_0$$

$$W_0 = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 12 \end{vmatrix} = 12 \neq 0 \quad \text{observable}$$

if $y = [0 \;\; 1]$

$$W_0 = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0 \Rightarrow \text{unobservable}$$

Sensors to use in system design

<u>Exercise</u>: $A = \begin{pmatrix} 0 & 1 \\ 0 & -c/m \end{pmatrix}$ $C = [1 \;\; 0]$

$$W_0 = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{pmatrix} 1 & 0 \\ \underbrace{(1, 0)}_{1\times 2} \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & -c/m \end{pmatrix}}_{2\times 2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$$

$$W_0 \begin{pmatrix} 0 & 1 \\ 0 & -c/m \end{pmatrix} \begin{pmatrix} x^* \end{pmatrix} = 0$$

<u>Driveline model</u>

$$A = \begin{pmatrix} d_s/J_f i^2 & d_s/J_f i & -c_s/J_f i \\ d_s/J_c i & -d_s/J_c & c_s/J_c \\ 1/i & -1 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} \frac{1000}{(0.625)(57)^2} & \frac{1000}{(0.625)(57)} & \frac{-75000}{(57)(0.625)} \\ \frac{1000}{(6250)(57)} & \frac{-1000}{6250} & \frac{75000}{6250} \\ 1/57 & -1 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 0.4924 & 28.07 & -2105.26 \\ 0.00296 & -0.16 & 12 \\ 0.0175 & -1 & 0 \end{pmatrix}$$

$C = [\,0 \;\; 1 \;\; 0\,]$

$A^2 =$

Kalman filter

discrete system (mainly)

$\dot{x} = Ax + Bu + v \rightarrow$ disturbance

$y = Cx + w \rightarrow$ noise $\sim N(0, \sigma^2)$

Gaussian.

Active suspension:

$A = (4 \times 5)$ $\quad B = 5 \times 1$ $\quad H = 5 \times 1$.

$C = (1 \times 5)$

C
CA
CA^2
CA^3
CA^4

$1 \times 5 \; (1 \times 5) \; (\cancel{4} \times 5)$

1×5

State estimation: $E(v(s) v^T(t)) = R_v \, \delta(t-s)$.

$E(w(s) w^T(t)) = R_w \, \delta(t-s)$

where δ is dirac fn (unit impulse)

state estimator:- $\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$

estimation error $\tilde{x} = x - \hat{x}$

$\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}} = Ax + \cancel{Bu} + v - A\hat{x} - \cancel{Bu} - L(Cx + w - C\hat{x})$

$\cancel{(A - LC)x}$

$\dot{\tilde{x}} = Ax + v - A\hat{x} - L(Cx + w - C\hat{x})$

$\Rightarrow (A - LC)(x - \hat{x}) - Lw + v$

If $(A - LC)$ is stable - then $\tilde{x}$ = stationary stochastic process.

$P_{\tilde{x}} = E(\tilde{x}(t)\tilde{x}^T(t)) \Rightarrow$ covariance matrix

$$0 = (A - LC)P_{\tilde{x}} + P_{\tilde{x}}(A - LC)^T + R_v + LR_wL^T$$

Minimize $P_{\tilde{x}}$ with L.

If system is non-LTI, $P_{\tilde{x}}$ is also non-LTI

$\boxed{L = P_{\tilde{x}} C^T R_w^{-1}}$ where $P_{\tilde{x}} = P_{\tilde{x}}^T \geq 0$

is soln to Riccati eqn

$$0 = AP_{\tilde{x}} + P_{\tilde{x}}A^T + R_v - P_{\tilde{x}}C^T R_w^{-1} CP_{\tilde{x}}$$

(Kalman - Bucy filter). $\Rightarrow$ LQR

LQR	Kalman
$A \leftrightarrow A^T$	$B \leftrightarrow C^T$
$S \leftrightarrow P$	$K \leftrightarrow L^T$
$Q_x \leftrightarrow R_v$	$Q_u \leftrightarrow R_w$

$\Rightarrow$ LQG

Kalman-Bucy
- always stable
- $\rightarrow$ optimal linear filter
- $\rightarrow R_v, R_w$ - design parameters

$\downarrow$

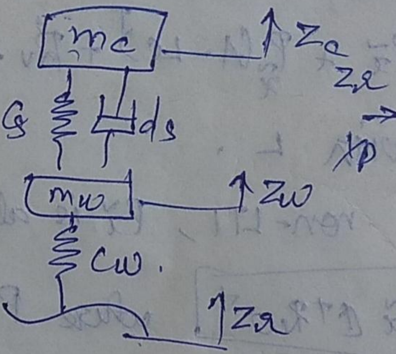
difficult spec shalas

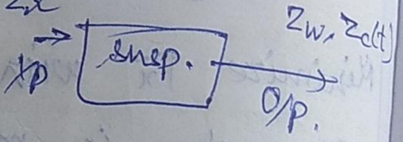
Suspension:

- connect veh. body to wheels
- transmit traction force to road.

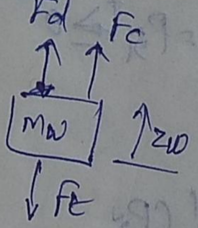
Wheel follows uneven road

1/4 model:

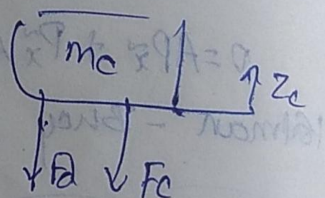




wheel



chassis:



conservation:

wheel:

$$m_w a_w = F_d + F_c - F_t \quad (1)$$

$$= d_s(\dot{Z}_w - Z_r$$

$$m_c a_c = -F_d - F_c \quad (2)$$

Constitution:

$$V_w = \dot{Z}_w \qquad a_w = \dot{V}_w \quad (3)$$

$$V_c = \dot{Z}_c \qquad a_c = \dot{V}_c$$

$$F_d = d_s(V_c - V_w)$$

$$F_t = C_w(Z_w - Z_r) \text{ (Hookes law)}$$

$$F_c = C_s(Z_c - Z_w)$$

Form state-space model

select variables

Z_w, Z_c, V_w, V_c

time derivatives.

$\dot{z}_w = v_w$

$\dot{v}_w = a_w = \frac{(F_d + F_c + F_t)}{m_w} = \frac{(d_s(v_c - v_w) + c_w(z_w - z_r) + c_s(z_c - z_w))}{m_w}$

$\dot{z}_c = v_c$

$\dot{v}_c = a_c = \left(\frac{-F_d - F_c}{m_c}\right) = \left(\frac{d_s(v_c - v_w) - c_s(z_c - z_w)}{m_c}\right)$

$x_1 = z_w$
$x_2 = v_w$
$x_3 = z_c$
$x_4 = v_c$
$u = z_r$
$y = (x_1 \quad x_3)$

$$\boxed{\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \left(\frac{d_s(x_4 - x_2) + c_w(x_1 - u) - c_s(x_3 - x_1)}{m_w}\right) \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \frac{d_s(x_4 - x_2) - c_s(x_3 - x_1)}{m_c} \end{aligned}}$$

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{c_w - c_s}{m_w} & \frac{-d_s}{m_w} & \frac{+c_s}{m_w} & \frac{d_s}{m_w} \\ 0 & 0 & 0 & 1 \\ \frac{+c_s}{m_c} & \frac{+d_s}{m_c} & \frac{-c_s}{m_c} & \frac{-d_s}{m_c} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}}_{x} + \underbrace{\begin{pmatrix} 0 \\ +\frac{c_w}{m_w} \\ 0 \\ 0 \end{pmatrix}}_{B} \underbrace{u}_{u}$$

$$y = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}}_{C} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}}_{x} + \underbrace{[0]}_{D}(u)$$

K,S Body movement specification Passive system

Active suspension} ⇒ Then z_r is disturbance
an actuator to control it
movement.

New output:

$(z_w - z_c)$ $\quad$ $\cancel{a_c = \frac{1}{m_c}\left(d_s(x_4 - x_2) + c_w(x_1 - u) - c_s(x_3 - x_1)\right)}$

$(x_1 - x_3)$ $\quad x_4$

$x_3 - x_1$

$$y = \begin{pmatrix} 1 & 0 & -1 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{1} \\ \frac{c_w - c_s}{m_c} & \frac{-d_s}{m_c} & \frac{c_s}{m_c} & \frac{d_s}{m_c} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \dot{x}_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{-c_w}{m_c} \end{pmatrix} u$$

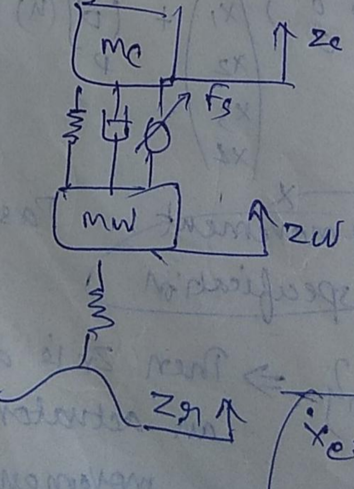
C $\qquad$ D

Now v can measure just $(\dot{x}_3 - x_1)$ and acceleration of chassis a_c

$$a_c = \left(\frac{1}{m_c}\right)\left(-d_s(x_4 - x_2) + c_s(x_3 + x_1)\right)$$

$$y\text{-new} = \begin{pmatrix} -1 & 0 & 1 & 0 \\ \frac{+c_s}{m_c} & \frac{+d_s}{m_c} & \frac{-c_s}{m_c} & \frac{-d_s}{m_c} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} [u]$$

Augmented ss-model:



$$\dot{F}_s = -\frac{1}{\tau} F_s + \frac{1}{\tau} u$$

$$\dot{x}_e = \begin{bmatrix} \dot{x} \\ \dot{F}_s \end{bmatrix}$$

$$\dot{x}_e = A x_e + B u + H d$$

$$Y = C x_e + D u$$

$\dot{x}_1 = \dot{z}_w = V_w = x_2$
$\dot{x}_3 = \dot{z}_c = V_c = x_4.$

$$\dot{x}_2 = a_w = \left(\frac{\Sigma F_{wheles}}{m_w} \right)$$

$$m_w (a_w) = F_d + F_c + F_s - F_t.$$

$$= d_s (v_c - v_w) + c_s (z_c - z_w) - c_w (z_w - z_r) + x_5.$$

$$a_w = \frac{d_s (x_4 - x_2) + c_s (x_3 - x_1) - c_w (x_1) - c_w (u)}{m_w} + \left(\frac{x_5}{m_w} \right)$$

$$a_c = -F_d - F_c - F_s = \left(\frac{-1}{m_c} \right)$$

Long. vehicle dynamics linearized model.

$$\dot{x}_1 = \frac{-1}{\tau} x_1 + \frac{k \dot{i}}{\tau r_w} (u) - \qquad \boxed{\frac{\partial f_1}{\partial u} = \frac{K . \dot{i}}{\tau r_w}} \qquad \boxed{\frac{\partial f_1}{\partial d_1} = 0}$$

$$\dot{x}_2 = \frac{1}{m} (x_1 - mgf \cos d_1 - \frac{1}{2} \rho C_d A_f x_2^2 - mg \sin d_1)$$

$y = x_2.$

$$\frac{\partial f_2}{\partial u} = 0 \qquad \frac{\partial f_2}{\partial d_1} = + gf \sin d_1 - g \cos d_1$$

$$\frac{\partial f_1}{\partial x_1} = \frac{-1}{\tau}. \qquad \frac{\partial f_1}{\partial x_2} = 0.$$

$$\frac{\partial f_2}{\partial x_1} = \left(\frac{1}{m} \right) \qquad \frac{\partial f_2}{\partial x_2} = \left(\frac{1}{m} \right) \left(\frac{-1}{2} \rho C_d A_f \right) (2 x_2)$$

$$= \left(- \frac{\rho C_d A_f}{m} \right) x_2$$

$$\dot{A} = \begin{bmatrix} \frac{-1}{\tau} & 0 \\ \left(\frac{1}{m}\right) & \left(\frac{-\rho \cdot c_d \cdot A_f}{m}\right) x_2 \end{bmatrix} \quad B = \begin{bmatrix} \frac{k_i^{\circ}}{\tau \cdot r_w} \\ 0 \end{bmatrix} \quad H' = \begin{bmatrix} 0 \\ g f \sin d_1 \\ - g \cos d_1 \end{bmatrix}$$

$$C = [0 \quad 1]$$

$x_{1e} \quad x_{2e} \quad u_e \quad d_e$

$(3216, 15, 0.2010, 0)$

$m = 20000 \quad k = 2000 \quad i = 4 \quad \tau = 0.8$

$g = 9.8 \quad \rho = 1.2 \quad c_d = 0.5 \quad A_f = 4 \quad r_w = 0.5$

$f = 0.015$

$$A = \begin{bmatrix} -1.25 & 0 \\ 0.00005 & \frac{-1.2 \times 0.5 \times 4}{2000} \times 15 \end{bmatrix} = \begin{bmatrix} -1.25 & 0 \\ 0.005 & -0.0072 \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{20000 \times 4}{0.8 \times 0.5} \\ 0 \end{bmatrix} \quad H = \begin{bmatrix} 0 \\ -9.81 \end{bmatrix} \quad C = [0 \quad 1]$$

Reachability

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -c/m \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1/m \end{bmatrix}$$

$$W_R = [B \quad AB \ \ldots] \qquad AB = \underset{2\times 2}{\begin{bmatrix} 0 & 1 \\ 0 & -c/m \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} 0 \\ 1/m \end{bmatrix}}$$

$$W_r = \begin{bmatrix} 0 & 1/m \\ 1/m & -c/m^2 \end{bmatrix} \qquad = \begin{bmatrix} 1/m \\ -c/m^2 \end{bmatrix}$$

Case A   Case B

Tuning

given :- $T_s = 2.0s$ $T_s = 4s$

$M_p = 0\%$ $M_p = 10\%$

$T_s = 4/\xi\omega_n$ $M_p = e^{-\pi\xi/\sqrt{1-\xi^2}}$

Case A :

$\xi = \dfrac{4}{\xi\omega_n}$

$\xi\omega_n = 2$

$\xi = 1$

$\omega_n = 2$

$M_p = \dfrac{1}{e^{\pi\xi/\sqrt{1-\xi^2}}} = 0$

$e^{\pi\xi/\sqrt{1-\xi^2}} = \infty$.

Case B :

$M_p = 10\% = e^{-\pi\xi/\sqrt{1-\xi^2}}$

$\xi\omega_n = 1$ from $T_s = 4s$.

$\cancel{\Rightarrow 0.1 =}$ $e^{\pi\xi/\sqrt{1-\xi^2}} = 10$

$\log e^{\pi\xi/\sqrt{1-\xi^2}} = \log(10)$

$= 1.00$

$(\pi\xi/\sqrt{1-\xi^2}) = 1$

$a = \dfrac{1.7712}{2}$ or $\dfrac{0.228}{2}$

$\pi^2\xi^2(1-\xi^2) = 1.$

$a = 0.8856$ $\xi = 0.941$

$\xi^2(1-\xi^2) = 1/\pi^2$

$a = 0.114$

$\xi^2 - \xi^4 = 0.1013$

$a = \xi^2$ $\cancel{a^2}$ $-a^2 + a = 0.1013$

$a^2 - a + 0.1013 = 0$

$a = \dfrac{1 \pm \sqrt{1 - 4(1)(0.1013)}}{2(1)}$

Given: $\omega_n = 1.69$ $\xi = 0.59$

$s^2 + 2\xi\omega_n s + \omega_n^2$

$\Rightarrow s^2 + 2(0.59)(1.69) + (1.69)^2$

$= s^2 + 1.9942s + 2.8561$

$\Rightarrow -0.9971 + 1.3645i$

$-0.9971 - 1.3645i$

Third pole (suppose min 4 times >> 2 poles)

$\Rightarrow -4 \pm x$ value

Driveline model:

Given: $\dot{x} = Ax + Bu + Hd$

$\dot{y} = Cx$

$x = 3\times1$ states - 3 poles.

$u = -K\Delta x + k_r r$.

Dominant poles $s^2 + 2\xi\omega_n s + \omega_n^2$

$\omega_n = 6$ $\xi = \frac{1}{\sqrt{2}}$

$\Rightarrow s^2 + 2(6)(1/\sqrt{2})s + 36$

$= s^2 + 8.4852s + 36$

Roots = $-4.2426 + 4.2427i$

$-4.2426 - 4.2427i$

$s = -2\xi\omega_n$

$\boxed{s = -8.4852}$

$(s^2 + 8.4852s + 36)(s + 8.4852)$

$s^3 + 8.4852s^2 + 36s + 8.4852s^2 + (8.4852)^2 s$

$+ 36 \times 8.4852$

$$\overset{P_1}{} \quad \overset{P_2}{} \quad \overset{P_3}{}$$

$s^3 + 16.97056s^2 + 108s + 305.4672 = 0$

desired polynomial

(a terms) $|sI - A|$

$$= \begin{vmatrix} s + \frac{d_s}{J_f i^2} & -\frac{d_s}{J_f i} & +\frac{C_s}{(J_f i)} \\ -\frac{d_s}{J_c i} & s + \frac{d_s}{J_c} & \frac{-C_s}{J_c} \\ -\frac{1}{i} & +1 & s \end{vmatrix}$$

$$= \left(s + \frac{d_s}{J_f i^2}\right)\left(s^2 + \frac{d_s}{J_c} s + \frac{C_s}{J_c}\right) + \left(\frac{d_s}{J_f i}\right)\left(s \cdot \left(\frac{-d_s}{J_c i}\right) - \frac{C_s}{J_c i}\right)$$

$$+ \left(\frac{C_s}{J_f i}\right)\left[\frac{-d_s}{J_c i} + \left(\frac{1}{i}\right)\left(s + \frac{d_s}{J_c}\right)\right]$$

$$\cancel{= \left(s + \frac{1000}{(57^2)(0.625)}\right)\left(s^2 + \frac{1000}{6250} s + 12\right) + \frac{1000}{(0.625)(57)}\left(s\left(\frac{-1000}{(6250)57}\right) - \left(\frac{12}{57}\right)\right)}$$

$$= (s + 0.4924)(s^2 + 0.16s + 12) +$$

$$(28.07)(s(-0.002807) - 0.2105)$$

$$+ (2105.26)\left[(-0.002807) + \left(\frac{1}{57}\right)(s + 0.16)\right]$$

$$= s^3 + 0.16s^2 + 12s + 0.4924s^2 + \cancel{0.0787843}s + 5.9088$$

$$- \cancel{0.078799}s - \cancel{5.908} - 5.909 + 36.9\ldots$$

$$\begin{array}{cccc} & a_1 & a_2 & a_3 \\ s^3 + & 0.6524s^2 + & 48.998s & + 0 \\ s^3 + & 16.9705s^2 + & 108s & + 305.46 \\ & P_1 & P_2 & P_3 \end{array}$$

$(P_1 - a_1) = [16.31 \quad 59.0856 \quad 305.47]$

W_R = Reachability matrix

$= [B \;\; AB \;\; A^2B]$

$$\tilde{W}_R = \begin{bmatrix} 1 & a_1 & a_2 \cdots a_{n-1} \\ & 1 & a_1 \cdots a_{n-2} \\ & & 1 \quad a_1 \end{bmatrix}^{-1} \qquad a_1 \;\; a_2 \;\; a_3$$

$$\tilde{W}_R = \begin{bmatrix} 1 & a_1 & a_2 \\ 0 & 1 & a_1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0.6525 & 48.9344 \\ 0 & 1 & 0.6525 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

$$\tilde{W}_R = \begin{pmatrix} 1 & -0.6525 & -48.5086 \\ 0 & 1 & -0.6525 \\ 0 & 0 & 1 \end{pmatrix}$$

$K = [P_1 - a_1 \quad P_2 - a_2 \quad P_3 - a_3]\, \tilde{W}_R \, W_R^{-1}$

$= [16.31 \quad 59.0656 \quad 305.47]\,[\tilde{W}_R]\,[W_R]^{-1}$

$\quad 1\times3 \qquad 3\times3 \qquad 3\times3$

$= \underline{1\times3}$

$$k_r = \frac{-1}{C(A-BK)^{-1}B}$$

$\underset{3\times3}{C}$ $\underset{3\times3}{(A-BK)}$ $\underset{(3\times1)}{(B)}$

(3×3) (3×1)

Poo6

~~A=~~ 3×1

SA3:

Cruise control system:

$$\dot{x}_1 = \frac{-1}{\tau}x_1 + \frac{k_i}{\tau . r_w}(u)$$

$$\dot{x}_2 = \left(\frac{1}{m}\right)\left(x_1 - mgf\cos(d_1) - 0.5\rho C_d A_f x_2^2 - mg\sin(d_1)\right)$$

$x_{2e} = 20$ m/s (given)

$x_{1e} = 0.5\rho C_d A_f (x_{2e}^2) + mgf$

$u_e = \frac{r_w}{k_i}(x_{1e})$

$d_{1e} = 0$

$\Delta u = -k\Delta x + k_r \Delta r$

$P(s) = s^2 + 2\xi\omega_n + \omega_n^2$

$\Rightarrow p(s) = s^2 + 0.8485s + 0.36$

roots $= -0.4243 \pm 0.4243i$

$\omega_n = 0.6$ $\xi = \frac{1}{\sqrt{2}} \Rightarrow 45^\circ \Rightarrow \text{Re} = \text{Im}$

$$|sI - A| = \begin{vmatrix} s+1.25 & 0 \\ -0.0001 & s+0.0024 \end{vmatrix}$$

$$= s^2 + 1.25s + 0.0024s + (1.25)(0.0024)$$

$$\boxed{= s^2 + \underset{a_1}{1.2524}s + \underset{a_2}{0.003}} \quad a(s)$$

$$\boxed{s^2 + \underset{p_1}{0.8485}s + \underset{p_2}{0.36}} \quad p(s).$$

$$\Rightarrow k = [p_1 - a_1 \quad p_2 - a_2] \tilde{W}_x W_x^{-1}$$

$$= 0.000 \quad +0.358$$

$$k_r = \frac{1}{C(A-Bk)^{-1}B} = 0.36$$

Actual solution.

$$(1\times3)(3\times3)(3\times1)$$

$$(1\times3)$$

$$\Delta\dot{x} = (A - Bk)\Delta x$$

$$\Rightarrow |sI - (A - Bk)|$$

$$\therefore (A - Bk) = \cancel{(-1.25}$$

$$Bk = \underset{2\times1}{\begin{pmatrix} 20000 \\ 0 \end{pmatrix}} (k_{11} \quad k_{12}) = \begin{pmatrix} 20000k_{11} & 20000k_{12} \\ 0 & 0 \end{pmatrix}$$

$$(A - Bk) = \begin{pmatrix} -1.25 - 20000k_{11} & -20000k_{12} \\ -0.001 & -0.0024 \end{pmatrix}$$

$$= \begin{vmatrix} s + 1.25 + 20000k_{11} & 20000k_{12} \\ \cancel{s + 0.0} \\ 0.001 & (s + 0.0024) \end{vmatrix}$$

$$= \cancel{s^2 + 1.25s +}$$

$$s^2 + (1.25 + 20000k_{11} + 0.0024)s + 48k_{11} + 0.003 - 20k_{12} =$$

$$s^2 + 0.8485s + 0.36$$

$$1.25 + 20000k_{11} + 0.0024 = 0.8485$$

$$48k_{11} + 0.003 - 20k_{12} = 0.36.$$

$$k_{11} = -0.000020195 \qquad k_{12} = 0.3580$$

$$\cancel{20k_{12} = 48(\;) + 0.0036 - 0.36}$$

Exercise:

Now assume a disturbance and

At $t = 1s$, $r = 20 \rightarrow 25\, m/s$

At $t = 20s$, slope occurs (disturbance).

At 20s, the vehicle speed drops and assumes a speed less than reference speed since disturbance rejection is not included

Soln: integral action.

$\dot{x}_I = \Delta y - \Delta r$ $\Delta y = \Delta x_2$

$\dot{x}_{Aug} = A_{aug} x_{aug} + B_{aug} \Delta u + F_{aug} \Delta r$

$x_{Aug} = \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ x_I \end{bmatrix}$ $A_{aug} = \begin{bmatrix} -1.25 & 0 & a_{13} \\ 0.0001 & -0.0024 & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$B_{aug} = \begin{bmatrix} 20000 \\ 0 \\ b_{31} \end{bmatrix}$, $F_{aug} = \begin{bmatrix} 0 \\ 0 \\ f_{31} \end{bmatrix}$

$A_{aug} = \begin{bmatrix} -1.25 & 0 & 0 \\ 0.0001 & -0.0024 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $B_{aug} = \begin{bmatrix} 20000 \\ 0 \\ 0 \end{bmatrix}$ $F_{aug} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$

Now statefeedback becomes

$\Delta u = -K_{aug} \Delta x_{aug}$

$K_{aug} = [K_{11aug} \;\; K_{12aug} \;\; K_{13aug}]$

Ackermann formula

Third poles $-3\xi\omega_n$

$(s^2 + 0.8485s + 0.36)(s + 1.2728)$

$s^3 + (0.8485 + 1.2728)s^2$

$+ \left(0.36 + (1.2728)(0.8485)\right)s + (0.36)(1.2728)$

$\Rightarrow$ $s^3 + 2.1213s^2 + 1.4399s + 0.458208$

p_1 p_2 p_3

$$|sI - A| = \begin{vmatrix} s + 1.25 & 0 & 0 \\ -0.0001 & s + 0.0024 & 0 \\ 0 & -1 & s \end{vmatrix}$$

$= (s + 1.25)(s^2 + 0.0024s)$

$= s^3 + 1.25s^2 + 0.0024s^2 + (1.25)(0.0024s)$

$\boxed{= s^3 + 1.2524s^2 + 0.003s + 0}$

a_1 a_2 a_3

$K = [(p_1 - a_1)\ (p_2 - a_2)\ (p_3 - a_3)]\ \tilde{W}_a W_a^{-1}$

$K = [4.5566 \times 10^{-5},\ 1.4349,\ 0.459]$

is the answer

Vehicle steering

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 12 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \end{bmatrix} u + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$V_0 = 12\ m/s$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + w$$

$a = 2m$

$b = 4m.$

x_1 = lateral position

x_2 = heading

$$R_v = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad R_w = \rho$$

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

$$L = P_{\tilde{x}} C^T R_w^{-1}$$

$$P_{\tilde{x}} = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix} \quad [P, E, L] = \text{care}(A', C', [[1\ 0];[0\ 1]], ...)$$

Continuous time Ricatti eqn.

For $\rho = 1$.

$$P_x = \begin{bmatrix} 5 & 1 \\ 1 & 0.416 \end{bmatrix} \quad L = \begin{pmatrix} 5.0 \\ 1.0 \end{pmatrix}$$

eg: $x(0) = (5, 0.1)$ $\hat{x}(0) = (0, 0)$;

Exercise

$$\dot{x}(t) = ax(t) + bu(t) + v(t)$$

$$y(t) = cx(t) + w(t).$$

$$L = P C^T R_w^{-1}$$

$a = 1$, $c = 2$, $R_v = 0.1$ $R_w = 0.01$

$$AP + PA^T + R_v - PC^T R_w^{-1} CP = 0$$

$$pa + pa + r_v - pCR_w^{-1}(c)p = 0$$

$$2pa + r_v - p^2 c^2 r_w^{-1} = 0.$$

$$p^2 c^2 r_w^{-1} \cancel{+} 2a\,p \cancel{+} r_v = 0$$

$\times$ by $\frac{r_w c}{c^2}$

$$p^2 \cancel{+} \frac{2a r_w p}{c^2} \cancel{+} \frac{r_v r_w}{c^2} = 0.$$

$$p = \frac{\left(+\frac{2a r_w}{c^2}\right) \pm \sqrt{\frac{4a^2 r_w^2}{c^4} + \frac{(4)(1)(r_w r_w)}{c^2}}}{2(1)}$$

$$p = \frac{a r_w}{c^2} \pm \sqrt{\frac{a^2 r_w^2}{c^4} + \frac{r_v r_w}{c^2}}$$

if $p > 0$

$$\cancel{P}\; L = \frac{pc}{R_w}\left(\frac{a r_w}{c^2} \pm \sqrt{\frac{a^2 r_w^2}{c^4} + \frac{r_w r_v}{c^2}}\right)$$

$$= pc\left(\frac{a}{c^2} \pm\right.$$

Actual Riccati solves in MATLAB:

$$A^T P + PA - PBR_w^{-1}B^T P + Q = 0.$$

given system form.

$$AP + PA^T + HR_VH^T - PC^TR_W^{-1}CP = 0.$$

(A^T instead of A) + (B instead of C^T) + ($Q = HRH^T$)

MSD system with noise

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}u + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{2\times 1} V$$

H. process disturbance

$$y = (0 \ \ 1)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{w}_{1\times 1}$$

→ noise

R_1

$$Q = \cancel{H^T} HRH^T = (2\times 1) R_V \times 1\times 2 = (2\times 2) R_V.$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}(0 \ \ 1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & R_V \end{pmatrix} \Rightarrow Q.$$

SA4:

Driveline estimation

Poleplacement design:

$\omega_n = 5 \qquad \xi = 1$

Third pole is @ $-3\xi\omega_n$.

$a = s^3 + 0.6525s^2 + 48.9344s + 0.00$

$\therefore k = 15.217,\ 245.89\ \ 4312$

$K_r = 1113.$

$(A-BK)$ eigenvalues -15000
$-5.000 + 0.005j$
$-5.000 - 0.005j$

$\therefore$ Estimator poles placed @ $4\times(-15) \approx -60$
all 3 poles placed @ $s = -60$ or $(s+60)^3$

$s^3 + 180s^2 + 10800s + 216000$

$|sI - A + LC|$

$LC = \begin{bmatrix} l_{11} \\ l_{21} \\ l_{31} \end{bmatrix} [1\ 0\ 0]$

$$= \begin{vmatrix} s+0.5+l_{11} & -28.1 & 2105.3 \\ l_{21} & s+0.2 & -12.0 \\ l_{31} & +1.0 & s \end{vmatrix}$$

$(LC) = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & 0 & 0 \\ l_{31} & 0 & 0 \end{pmatrix}$

$= (s+0.5+l_{11})(s^2+0.2s+12)$
$+ 28.1(l_{21}s - l_{31}12)$
$+ 2105.3(l_{21} - l_{31}s - 0.2l_{31})$

$= s^3 + 0.5s^2 + l_{11}s^2 + 0.2s^2 + 0.1s + (l_{11})(0.2)s$
$+ 12s + 6 + 12l_{11} + (28.1)(l_{21})s - 337.2(l_{31})$
$+ 2105.3l_{21} - 2105.3(l_{31})s - 421.06(l_{31})$

$$s^3 + s^2(0.7 + l_{11}) + s(0.1 + 0.2\,l_{11} + 12 + 28.1\,l_{21} - 2105.3\,l_{31})$$
$$+ (6 + 12\,l_{11} - 337.2\,l_{31} + 2105.3\,l_{21} - 421.06\,l_{31})$$

$0.7 + l_{11} = 180$

$l_{11} = 179.3$

$10800 = 0.1 + 35.86 + 12 + 28.1\,l_{21} - 2105.3\,l_{31}$

$\cancel{1080}$

$10800 - 47.96 = 10752.04$

(1) $= 28.1\,l_{21} - 2105.3\,l_{31}$

216000
$-2151.6 - 6$

$\cancel{= -337.2\,l_{31} + 2105.3\,l_{21}}$

213842.4

$= -758.26\,l_{31} + 2105.3\,l_{21}$ (2)

(1) $10752.04 = -2105.3\,l_{31} + 28.1\,l_{21}$

(2) $\times 3 \Rightarrow 592343.448 = -2105.3\,l_{31} + 5831.681\,l_{21}$

$(-) \quad (+) \quad (-)$

$-581591.408 = -5803.58\,l_{21}$

$l_{21} = 100.21$

From (1) $l_{31} = -3.769$