Augumented vehicle model

$\dot{x}_v = V_x \cos\theta - V_y \sin\theta$

$\dot{y}_v = V_x \sin\theta + V_y \cos\theta.$

x_v = x-direction vehicle position

y_v = y-direction vehicle position

θ = heading = $\int$ yaw rate

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = [v_y \ \dot{\psi} \ x_v \ y_v \ \theta]^T.$$

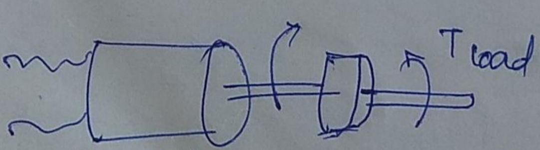
$\dot{x}_3 = \dot{x}_v = V_x \cos\theta - V_y \sin\theta$

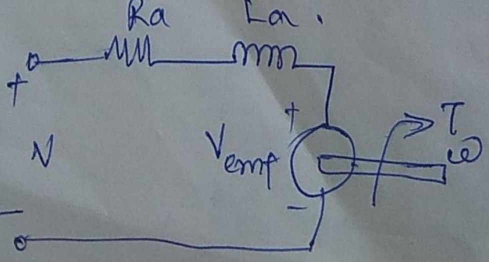
$\boxed{\dot{x}_3 = (V_x)\cos(x_5) - (x_1)\sin(x_5)}$

$\dot{x}_4 = (V_x)\sin(x_5) + (x_1)\cos(x_5).$

$\dot{x}_5 = (x_2)$

DC motor model :





$V = R_a i + L_a \frac{di}{dt} + K_e \omega$.

where $J \frac{d\omega}{dt} = T - T_{load}$.

$T = (K_t)(i)$

$T_{load} = b\omega$

$V_{emf} = K_e \omega$.

V = supply voltage

i = current

ω = rotational speed.

T = d.c. motor generating torque

T_{load} = load torque.

V_{emf} = motor induced emf.

Parameters:

R_a, L_a, K_e, K_t, J, b.

$x_1 = i$ $\quad u$ = voltage (V)

$x_2 = \omega$ $\quad y$ = rotational speed (ω)

$\dot{X} = Ax + Bu$

$Y = Cx + Du$.

$X = (2\times1)$ $\quad A = (2\times2)$ $\quad B = (2\times1)$ $\quad u = 1\times1$

$C = (1\times2)$ $\quad D = 0$

$$\dot{x}_1 = \left(\frac{di}{dt}\right) = \left(\frac{V - R_a(i) - K_e\omega}{L_a}\right)$$

$$= \left(\frac{1}{L_a}\right)(u - R_a(x_1) - K_e x_2)$$

$$\dot{x}_2 = \dot{\omega} = \left(\frac{1}{J}\right)(T - T_{load}) = \left(\frac{1}{J}\right)(K_t i - b\omega)$$

$$= \left(\frac{1}{J}\right)(K_t x_1 - b x_2)$$

$$\begin{matrix}\dot{x}_1 \\ \dot{x}_2\end{matrix} = \begin{bmatrix} -R_a/L_a & -K_e/L_a \\ K_t/J & -b/J \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1/L_a \\ 0 \end{bmatrix} u$$

Section-3:

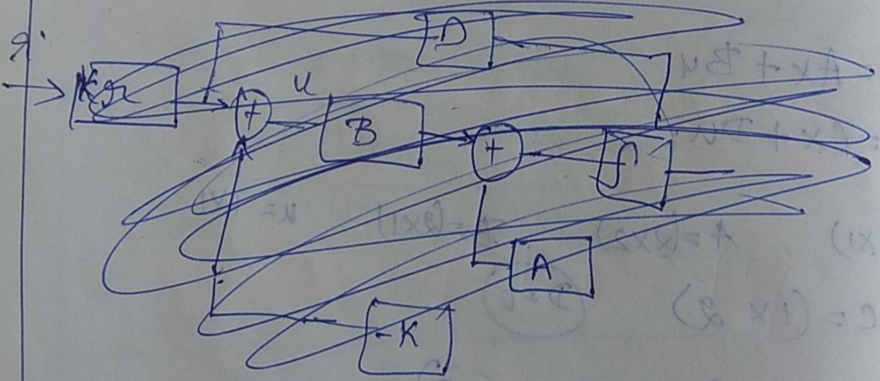
1) State feedback design
2) Linearization
3) Reachability
4) Tuning

1) state feedback control design

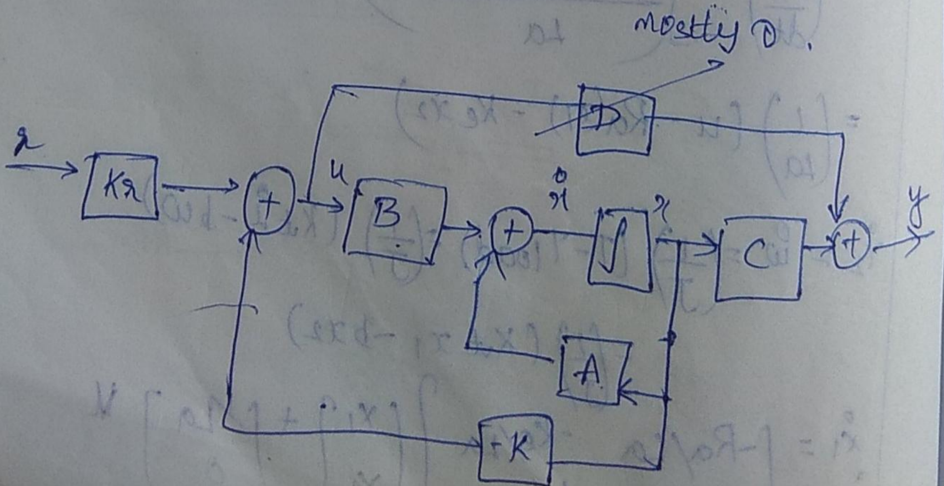
system poles = eig(A)

modifying eigen values (via) control (signal)

$u(t) = -Kx(t) + K_r r(t)$



mostly 0.



$\dot{x}(t) = Ax(t) + B(-Kx(t) + K_r r(t))$

$= (A - BK)x(t) + BK_r r(t).$

Control objective: choose K so that clo. loop dynamics satisfies stability, performance criterion

eigen value

(K_r) doesn't affect steady-state stability but affects ss-value

$y(t) \approx r(t)$ as $t \to \infty$

$\dot{x} = 0$

$\Downarrow$

in ss-condition

$$\boxed{\begin{aligned} 0 &= (A - BK)x(t) + BK_r r(t) \\ y &= Cx(t) \end{aligned}}$$

$-BK_r r(t) = (A - BK)x(t).$

$-(A - BK)^{-1} BK_r r(t) = x(t)$

$$\boxed{\therefore y(t) = -C(A - BK)^{-1} BK_r r(t)}$$

As $y(t) \approx r(t)$ as $t \to \infty$.

$K_r = -(C(A - BK)^{-1}B)^{-1}$

(Or)

$$\boxed{K_r = -1/C(A - BK)^{-1}B}$$

2-step process

1) choose K

2) Then choose K_r depending on ss-value

Integral action:

ss-error = 0 if perfect knowledge of system

idea:

Integ. term to remove ss-error

Same in state feedback.

- new state variable

$$\boxed{\dot{z}(t) = y(t) - r(t)}$$

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} Ax + Bu \\ y - r \end{bmatrix} = \begin{bmatrix} Ax + Bu \\ Cx - r \end{bmatrix}$$

$$\boxed{\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ -1 \end{bmatrix} r}$$

$$\boxed{u(t) = -Kx(t) - K_I z(t) + K_r r(t)}$$

Example:

(7.4).

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & v_0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} a v_0 / b \\ v_0 / b \end{pmatrix} u$$

$$y = (1 \quad 0) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + [0] u$$

x_1 = lateral position (Y)

x_2 = heading (θ) $\quad u$ = steering angle δ

$$\dot{x}_1 = v_0 x_2 + \left(\frac{a v_0}{b}\right) u .$$

$$\dot{x}_2 = (v_0/b)\, u .$$

$v_0 = 12$ m/s

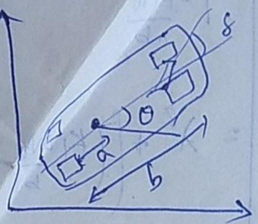
$a = 2$m

$b = 4$m

$$\dot{y} = (v_0)\psi + \left(\frac{a}{b}\right)(v_0)\,\delta .$$

$$\dot{\theta} = \left(\frac{v_0}{b}\right)(\delta)$$

State eqns.



Lane-keeping system:

specification:

$$P_{des}(\lambda) = \lambda^2 + 2\xi\omega_n\lambda + \omega_n^2$$

State feedback

$$u = -Kx + K_r r = -K_1 x_1 - K_2 x_2 + K_r r$$

$u = -Kx$

$1\times1 \quad 1\times2 \quad 2\times1$

CL. dynamics

$$\dot{x} = (A - BK)x + BK_r r$$

$$BK = \begin{pmatrix} a v_0/b \\ v_0/b \end{pmatrix} (+K_1 \quad +K_2)$$

$$BK = \begin{pmatrix} +\frac{a v_0}{b} \cdot K_1 & + \frac{a v_0}{b} K_2 \\ +\frac{v_0}{b} \cdot K_1 & + \frac{v_0}{b} \cdot K_2 \end{pmatrix}$$

$$(A - BK) = \begin{pmatrix} 0 - \frac{a v_0}{b} \cdot K_1 & v_0 - \frac{a v_0}{b} \cdot K_2 \\ 0 - \left(\frac{v_0}{b}\right) K_1 & 0 - \left(\frac{v_0}{b}\right) K_2 \end{pmatrix}$$

$$\dot{x} = \begin{bmatrix} -k_1\frac{av_0}{b} & v_0 - k_2\frac{av_0}{b} \\ -k_1\frac{v_0}{b} & -k_2\frac{v_0}{b} \end{bmatrix} x + \begin{bmatrix} \frac{av_0}{b}\cdot k_r \\ \frac{v_0}{b}\cdot k_r \end{bmatrix} r$$

characteristic polynomial is

$$|\lambda - (A-BK)|$$

$$= \begin{vmatrix} \lambda + k_1\frac{av_0}{b} & v_0 - k_2\frac{av_0}{b} \\ -\frac{k_1 v_0}{b} & \lambda + k_2\frac{v_0}{b} \end{vmatrix}$$

$$= \lambda^2 + \left(\left(k_1\frac{av_0}{b} + k_2\frac{v_0}{b}\right)\lambda + \left(k_1\frac{v_0}{b}\right)\left(v_0 - k_2\frac{av_0}{b}\right)\right)$$

$$= \lambda^2 + \left(\frac{v_0}{b}\right)(k_1 a + k_2)\lambda + \left(k_1\frac{v_0^2}{b}\right) - \cancel{\frac{k_1k_2av_0^2}{b^2}} \to 0$$

$$\lambda^2 + 2\xi\omega_n\lambda + \omega_n^2$$

$$\boxed{\omega_n^2 = \frac{k_1 v_0^2}{b}}$$

$$\boxed{2\xi\omega_n = \left(\frac{v_0}{b}\right)(ak_1 + k_2)}$$

$$k_1 = \left(\frac{\omega_n^2(b)}{v_0^2}\right)$$

$$\cancel{\omega_n^2 = \frac{v_0^2}{(b^2)(4\xi^2)}(ak_1+k_2)^2}$$

$$\left(\frac{2\xi\omega_n b}{v_0}\right) = ak_1 + k_2$$

$$\boxed{k_2 = \frac{2\xi\omega_n b}{v_0} - \frac{(a)b\omega_n^2}{v_0^2}}$$

ss-reference gain

$$K_r = \frac{-1}{C(A-BK)^{-1}B} = K_1 = \frac{b\omega_n^2}{V_0^2}$$

$$u = -K_1 x_1 + K_r r - K_2 x_2$$

$$= -\frac{b\omega_n^2}{V_0^2} x_1 - \left(\frac{2\xi\omega_n b}{V_0} - \frac{ab\omega_n^2}{V_0^2}\right) x_2 + \frac{b\omega_n^2}{V_0^2} r$$

$\xi \Rightarrow$ overshoot control

<u>example 2:</u>

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad C = [1 \;\; 0]$$

$$\phi_{des}(\lambda) = \lambda^2 + 2\xi\omega_n\lambda + \omega_n^2$$

$$u = -Kx + K_r r = -K_1 x_1 - K_2 x_2 + K_r r$$

$$(A-BK) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}(K_1 \;\; K_2)$$

$$= \begin{pmatrix} -K_1 & 1-K_2 \\ 0 & 0 \end{pmatrix}$$

$$|\lambda - (A-BK)| = \begin{vmatrix} \lambda+K_1 & K_2-1 \\ 0 & \lambda \end{vmatrix} = \lambda(\lambda+K_1) - 0$$

$$= \boxed{\lambda(\lambda+K_1)}$$

can't match 2 polynomials

$\Rightarrow$ can't put eigenvalues where we want

$\Rightarrow$ can't shape dynamics as per our requirements.

$\Rightarrow$ Not controllable

always one eigenvalue $= 0$, unstable

model

$\dot{x} = f(x, u, d)$

$y = h(x, u, d)$

↓

Lineariation

$\dot{x} = Ax + Bu$

$y = Cx + Du$

Analysis

↓

Reachability

Design

↓

Tuning

control design

Lineariation

$\dot{x} = f(x, u)$

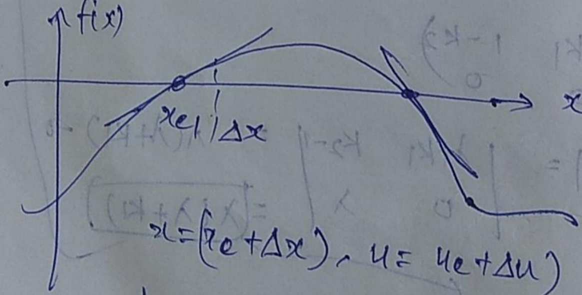
$y = h(x, u)$.

Locally approximate around eqbm pt (x_e, u_e)

$\dot{x} = f(x_e, u_e) = 0$

ss-solution to

(pts where $f(x) = 0$)



$x = (x_e + \Delta x)$, $u = u_e + \Delta u$)

$\frac{d}{dt}(x_e + \Delta x) = f(x_e + \Delta x, u_e + \Delta u)$

Dynamics close to (x_e, u_e) is described by above eqn

$\frac{d}{dt}(x_e + \Delta x) = f(x_e, u_e) + \frac{\partial f}{\partial x}\Big|_{x_e, u_e} \Delta x + \frac{\partial f}{\partial u}\Big|_{x_e, u_e} \Delta u$

(1)

$f(x_e, u_e) = \dot{x}_e = 0$.

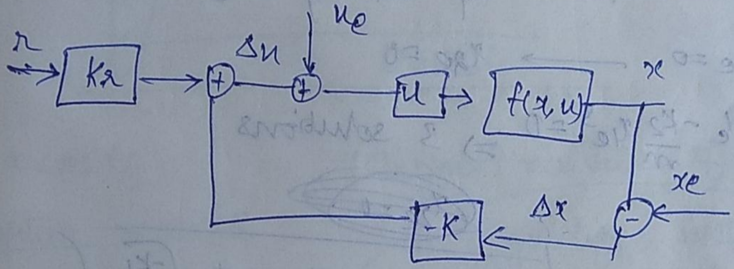
① ⇒ $$\frac{d}{dt}\Delta x = \underbrace{\frac{\partial f}{\partial x}\Big|_{(x_e,u_e)}}_{A}\Delta x + \underbrace{\frac{\partial f}{\partial u}\Big|_{(x_e,u_e)}}_{B}\Delta u$$

$$\Delta\dot{x} = A\Delta x + B\Delta u$$
$$\Delta y = C\Delta x + D\Delta u.$$

where $\Delta x = x - x_e$ = state vector.

$\Delta u = u - u_e$ = i/p control signal

$\Delta y = y - y_e$ = o/p signal.



$$A = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & & & \\ \vdots & & & \\ \frac{\partial f_n}{\partial x_1} & \cdots & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix} \qquad B = \begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_n} \\ \vdots & & \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_n} \end{pmatrix}$$

$$C = \begin{pmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial h_n}{\partial x_1} & \cdots & \frac{\partial h_n}{\partial x_n} \end{pmatrix} \qquad D = \begin{pmatrix} \frac{\partial h_1}{\partial u_1} & \cdots & \frac{\partial h_1}{\partial u_n} \\ \vdots & & \\ \frac{\partial h_n}{\partial u_1} & \cdots & \frac{\partial h_n}{\partial x_n} \end{pmatrix}$$

example leafspring.

$$\boxed{m\ddot{x} = -k_1 x - k_2 x^3}$$

x_1 = horizontal pos. of chassis

$x_2 = \dot{x}_1$ = " velocity

$\dot{x}_1 = x_2$

$\dot{x}_2 = \ddot{x} = \frac{-k_1 x - k_2 x^3}{m} = \left(\frac{1}{m}\right)\left(-k_1 x_1 - k_2 x_1^3\right)$

$\Rightarrow f_1(x_1, x_2)$

$\Rightarrow f_2(x_1, x_2)$

eqn. point found by solving d.e.

$x_{2e} = 0 \longrightarrow x_{2e} = 0$

$-\frac{k_1}{m} x_{1e} - \frac{k_2}{m} x_{1e}^3 = 0 \Rightarrow$ 3 solutions

$x_{2e} = 0$

$x_{1e} = 0, \quad x_{1e} = \pm \sqrt{\frac{-k_1}{k_2}}$

Consider operating pt $(x_{1e}, x_{2e}) = 0$.

$f_1 = x_2$

$f_2 = -\frac{k_1}{m} x_1 - \frac{k_2}{m} x_1^3$

$\left.\frac{\partial f_1}{\partial x_1}\right|_{(0,0)} = 0 \qquad \left.\frac{\partial f_1}{\partial x_2}\right|_{(0,0)} = 1$

$\left.\frac{\partial f_2}{\partial x_1}\right|_{(0,0)} = -\frac{k_1}{m} - 3\left(\frac{k_2}{m}\right) x_1^2$

$= -k_1/m \qquad \left.\frac{\partial f_2}{\partial x_2}\right|_{(0,0)} = 0$

$A = \begin{bmatrix} 0 & 1 \\ -\frac{k_1}{m} & 0 \end{bmatrix} \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix}$

Reachability :

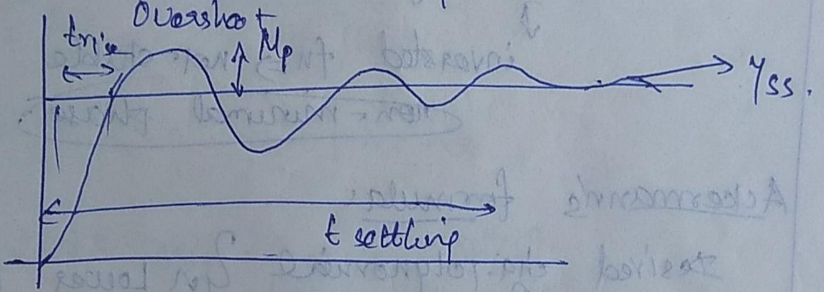
$$\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} \Rightarrow \text{full rank}$$

square matrix has full rank only if $\det(A) \neq 0$

Tuning:

Time-response

$$G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$



arccos (ξ)

Risetime real value of poles

system response

as u go left, system response is faster

	Value	30	45°	90°
		$\xi = 0.5$	$1/\sqrt{2}$	1
Rise time	$T_r = \frac{1}{\omega_n} \cdot e^{\arccos\xi / \tan(\arccos\xi)}$	$1.8/\omega_n$	$2.2/\omega_n$	$2.7/\omega_n$
Overshoot	$M_p = e^{-\pi\xi/\sqrt{1-\xi^2}}$	16 %	4 %	0 %
t settling (2%)	$T_s = 4/\xi\omega_n$	$8.0/\omega_n$	$5.7/\omega_n$	$4.0/\omega_n$

Idea: place two dominant poles accordingly to 2nd order systems

Rest of poles:

2-3 times → faster / means further to left

Zeros: Zeros in LHP give additional overshoot
Zeros in RHP give -ve undershoot @ start
↓
inverted fn ⇒ not stable
non-minimal phase

Ackermann's formula:

Desired cha. polynomial
⇕
CL polynomial
} in Lower order system

for H.O. systems:

$$K = [(p_1 - a_1) \quad p_2 - a_2 \quad p_n - a_n] \tilde{W}_r W_r^{-1}$$

where $W_r = [B \quad AB \quad \dots \quad A^{n-1}B]$

$$\tilde{W}_r = \begin{bmatrix} 1 & a_1 & a_2 & \dots & a_{n-1} \\ & 1 & a_1 & \dots & a_{n-2} \\ & & & 1 & a_1 \\ & & & & 1 \end{bmatrix}^{-1}$$

Revisit example

determine ch. polynomial for system

$$A = \begin{bmatrix} 0 & V_0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} aV_0/b \\ V_0/b \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$|sI - A| = \begin{vmatrix} s & -V_0 \\ 0 & s \end{vmatrix} = s^2 = s^2 + 0s + 0$$

$\cancel{a_1=1}$ $a_0 = 1$ $a_1 = 0$ $a_2 = 0$

$$p(s) = s^2 + 2\xi\omega_n s + \omega_n^2$$

$$K = [p_1 - a_1 \quad p_2 - a_2]\tilde{W}_r W_r^{-1}$$

$$= [2\xi\omega_n - 0 \quad \omega_n^2 - 0]\tilde{W}_r W_r^{-1}$$

$$W_r = [B \quad AB] = \begin{bmatrix} aV_0/b & V_0^2/b \\ V_0/b & 0 \end{bmatrix}$$

$$\tilde{W}_r = \begin{bmatrix} 1 & a_1 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$K = \underset{1\times 2}{[2\xi\omega_n \quad \omega_n^2]} \underset{2\times 2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} aV_0/b & V_0^2/b \\ V_0/b & 0 \end{bmatrix}^{-1}$$

$$= [2\xi\omega_n \quad \omega_n^2] \begin{bmatrix} aV_0/b & V_0^2/b \\ V_0/b & 0 \end{bmatrix}^{-1}$$

$$= \frac{1}{\left(\frac{-V_0}{b}\right)\left(\frac{V_0^2}{b}\right)} \begin{bmatrix} \cancel{a/b}\,0 & V_0/b \\ \frac{V_0^2}{b} & \frac{-aV_0}{b} \end{bmatrix} [2\xi\omega_n \quad \omega_n^2]$$

$$\begin{array}{ccc} k_1 & k_2 & \end{array}$$

$$K = \begin{bmatrix} \frac{b\omega n^2}{V_0^2} & \frac{2\xi\omega n b}{V_0} - \frac{a b \omega n^2}{V_0^2} \end{bmatrix}$$

LQR: optimises cost function:

$$J = \int_0^\infty (\underbrace{x^T Q_x x}_{\text{state cost}} + \underbrace{u^T Q_u u}_{\text{control cost.}}) \, dT$$

symmetric, positive semi-definite

$u = -Kx \qquad K = Q_u^{-1} B S$. where

S is soln to $A^T S + SA - SBQ_u^{-1}B^T S + Q_x = 0$

Ricatti eqn

$Q_x \geq 0$ $Q_u > 0$ system must be reachable.

1) simplest choice:

$Q_x = I$, $Q_u = \rho I$.

$\therefore J = \int x^T x + (\rho) u^T u \cdot dt$. trade-off $\|x\|^2$ vs $\rho\|u\|^2$

ρ = large. $\Rightarrow$ control action small

2) $z = C_z x$. be the output you want to keep small

choose $Q_x = C_z^T C_z$ and $Q_u = \rho I$

Output weighting

3) Diagonal weighting

$$Q_x = \begin{bmatrix} q_1 & & \\ & \ddots & \\ & & q_n \end{bmatrix} \quad Q_u = \begin{bmatrix} \rho_1 & & \\ & \ddots & \\ & & \rho_p \end{bmatrix}$$

Bryson's rule:

$$q_i = \frac{\alpha_i^2}{x_{i,max}^2} \quad \text{and} \quad \rho_i = \frac{\beta_i^2}{u_{i,max}^2}$$

$x_{i,max}$ = largest response
$u_{i,max}$

α, β = additional individual weighting

$$\sum_{i=1}^{n} \alpha_i^2 = 1 \qquad \sum_{i=1}^{p} \beta_i^2 = 1$$

4. Trial & error

eg:

$$A = \begin{bmatrix} 0 & 12 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

$$Q_x = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix} \qquad Q_u = \rho$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad Q_u = 10.$$

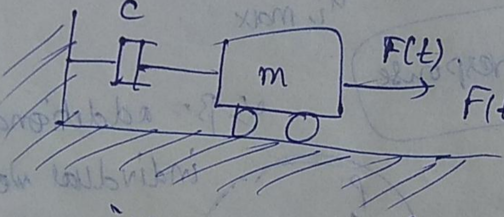
$$S = \begin{bmatrix} 0.292 & 0.470 \\ 0.470 & 2.754 \end{bmatrix} \Rightarrow \text{soln to +ve definite matrix}$$

$$u = -Kx \qquad K = Q_u^{-1} B^T S = [0.316 \ \ 1.108]$$

State fb control of mech. systems

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}0 & 1\\ 0 & -c/m\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}0\\ 1/m\end{pmatrix}u \quad (1)$$

$$y = (1 \quad 0)\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$



$\Sigma F = ma$

$F(t) - c\dot{x} = m\ddot{x}$

x_1 = position $= x$ $\quad \dot{x}_1 = x_2$

x_2 = velocity $= \dot{x}$ $\quad \dot{x}_2 = \ddot{x} = \left(\frac{-c}{m}\right)\dot{x} + F(t)$

$= \left(\frac{-c}{m}\right)x_2 + F(t)$

$\dot{x}_1 = x_2$

$\dot{x}_2 = \left(\frac{-c}{m}\right)x_2 + \left(\frac{u}{m}\right)$

$s^2 + 2\xi\omega_n s + \omega_n^2$

$K = (K_{11} \quad K_{12})$

$\dot{x} = Ax + Bu.$

$= Ax - BKx = (A - BK)x$

$\dot{x} = (A - BK)x$

$B = \begin{pmatrix}0\\ 1/m\end{pmatrix}_{2\times 1} \qquad K = (K_{11} \quad K_{12})_{1\times 2}$

$BK = \begin{pmatrix}0\\ 1/m\end{pmatrix}(K_{11} \quad K_{12}) = \begin{pmatrix}0 & 0\\ \frac{K_{11}}{m} & \frac{K_{12}}{m}\end{pmatrix}$

$$(A-BK) = \begin{pmatrix} 0 & 1 \\ 0 & -c/m \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ k_{11}/m & \frac{k_{12}}{m} \end{pmatrix}$$

$$(A-BK) = \begin{pmatrix} 0 & 1 \\ \frac{-k_{11}}{m} & \frac{-c}{m} - \frac{k_{12}}{m} \end{pmatrix}$$

$$|\lambda - (A-BK)| = \begin{vmatrix} \lambda & -1 \\ \left(\frac{k_{11}}{m}\right) & \lambda + \left(\frac{c}{m}\right) + \left(\frac{k_{12}}{m}\right) \end{vmatrix}$$

$$= \lambda\left(\lambda + \frac{c+k_{12}}{m}\right) + \left(\frac{k_{11}}{m}\right)$$

$$\boxed{CL = \lambda^2 + \left(\frac{c+k_{12}}{m}\right)\lambda + \left(\frac{k_{11}}{m}\right)}$$

$$\text{Poles} = s^2 + 2\xi\omega_n s + \omega_n^2$$

$$\left(\frac{k_{11}}{m}\right) = \omega_n^2 \qquad \left(\frac{k_{12}+c}{m}\right) = 2\xi\omega_n.$$

$$\boxed{k_{11} = \omega_n^2(m)}$$

$$\boxed{k_{12} = 2\xi\omega_n(m) - c}$$

I

II $u = -k_x x + k_r r.$

Determine ss-reference gain k_r such that ss-level is one (1).

$$\cancel{Kr} = \frac{-1}{C(A-BK)^{-1}B}$$

$C = [1 \quad 0]_{1\times 2}$ $\quad (A-BK) = \begin{bmatrix} 0 & 1 \\ -\frac{K_{11}}{m} & \frac{-c-K_{12}}{m} \end{bmatrix}_{2\times 2}$

$B = \begin{pmatrix} 0 \\ 1/m \end{pmatrix}_{2\times 1}$

$$(A-BK)^{-1} = \frac{1}{|A-BK|} \begin{pmatrix} \frac{-c-K_{12}}{m} & -1 \\ K_{11}/m & 0 \end{pmatrix}$$

$$|A-BK| = 0 + \left(\frac{K_{11}}{m}\right) = \left(\frac{K_{11}}{m}\right)$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\therefore Kr = \frac{-1}{[1 \quad 0]_{1\times 2} \frac{1}{\left(\frac{K_{11}}{m}\right)} \begin{bmatrix} \frac{-c-K_{12}}{m} & -1 \\ K_{11}/m & 0 \end{bmatrix}_{2\times 2} \begin{pmatrix} 0 \\ 1/m \end{pmatrix}_{2\times 1}} =$$

$$Kr = \frac{\left(\frac{-m}{K_{11}}\right)}{\begin{bmatrix} \frac{-c-K_{12}}{m} & -1 \end{bmatrix}_{1\times 2} \begin{bmatrix} 0 \\ 1/m \end{bmatrix}_{2\times 1}} = \left(\frac{-m}{K_{11}}\right) \frac{1}{\left(0 - \frac{1}{m}\right)}$$

$$= \frac{-m}{(K_{11})\left(\frac{-1}{m}\right)} = \frac{m^2}{K_{11}}$$

$$Kr = \frac{m^2}{\cancel{K_{11}}(\omega_n^2 m)} = \frac{m}{(\omega_n^2)}$$

$\cancel{\dot{x} = A - BK}$

$\dot{x} = Ax + B(-Kx + Kr r)$

$\dot{x} = (A - BK)x + Kr r.$

$s.X(s) = (A - BK)X(s) + Kr.R(s).$

$X(s)[s - (A - BK)] = Kr \cdot R(s).$

$\frac{X(s)}{R(s)} = \frac{Kr}{[s - (A - BK)]}$

$\therefore \frac{Y(s)}{R(s)} = \frac{c.Kr}{s - (A - BK)} \Rightarrow 0 \ \boxed{\text{zero's}}$

$\cancel{Y(s) = c}$!

$Y = cx.$

$y = c$

$\frac{Y(s)}{X(s)} = c.$

$G_{ry}(s) = c(sI - A + BK)^{-1} BKr$

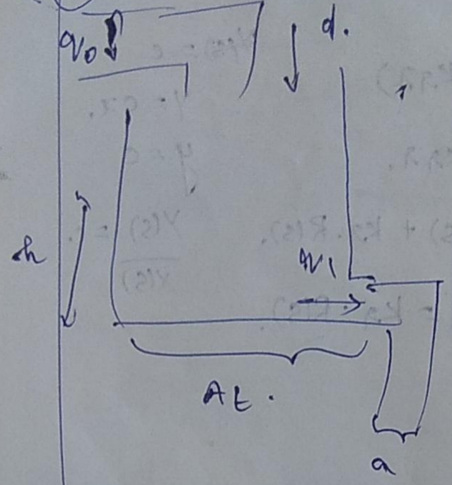
Actual answer

$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ \frac{K_{11}}{m} & s + \left(\frac{c + K_{12}}{m}\right) \end{bmatrix}^{-1} \begin{pmatrix} 0 \\ 1/m \end{pmatrix} Kr.$

$= \frac{[1 \ 0) \begin{pmatrix} s + \left(\frac{c+K_{12}}{m}\right) & 1 \\ \left(\frac{-K_{11}}{m}\right) & s \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} Kr}{K_{11}/m}$

$= \left(s + \left(\frac{c+K_{12}}{m}\right) \quad 1 \right) \begin{pmatrix} 0 \\ 1/m \end{pmatrix} (Kr) = \left(\frac{Kr}{m}\right)$

2



$$\dot{h} = \left(\frac{q_0}{A_t}\right) + \left(\frac{d}{A_t}\right) - \left(\frac{a}{A_t}\right)\sqrt{2gh}$$

$$q_1 = a\sqrt{2gh}$$

model.

h = level in tank

q_0 = inflow

q_1 = outflow

d = disturbance

$(h_0, q_0, 0)$

$$\Delta\dot{h} = A\Delta h + B\Delta q_0 + H\Delta d$$

$$\Delta q_1 = C\Delta h.$$

$$f_1 = \left(\frac{q_0}{A_t}\right) + d - \left(\frac{a}{A_t}\right)\sqrt{2g}\,(h)^{0.5}.$$

$$A = \left[\frac{\partial f_1}{\partial x_1}\right] = \frac{\partial f_1}{\partial h}.$$

$$= \cancel{0} + \cancel{0} - \left(\frac{a}{A_t}\right)\sqrt{2g}\,(0.5)(h)^{-0.5}.$$

<u>Linearising point</u>

$(h_e, q_{0e}, d_e) = (h_0, q_0, 0)$

$$B = \frac{\partial f_1}{\partial u} = \left(\frac{1}{A_t}\right)$$

$$C = \left(\frac{\partial h_1}{\partial x_1}\right) = (a\sqrt{2g})(0.5)(h)^{-0.5}$$

$$= \frac{a\sqrt{g}}{\sqrt{2h}}$$

leaf spring:

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix} x_2 \\ -\frac{k_1 x_1}{m} \quad -\frac{k_2 x_1^3}{m}\end{pmatrix}$$

$x_{1e} : 0 : \pm\sqrt{k_1 k_2}$

$$(x_{1e}, x_{2e}) = (\sqrt{k_1 k_2}, 0)$$

$$A = \frac{\partial f_1}{\partial x_1} \qquad \frac{\partial f_1}{\partial x_2} \qquad \frac{\partial f_2}{\partial x_1}$$

$$\Downarrow \qquad \Downarrow$$

$$0 \qquad 1 \qquad \left(\frac{-k_1}{m}\right) - \left(\frac{k_2}{m}\right)(3x_1^2)$$

$$= \left(\frac{-k_1}{m}\right) \frac{3k_2}{m}\left(\frac{k_1}{k_2}\right) = \left(\frac{-k_1}{m}\right) + \frac{3k_1}{m}$$

$$= \frac{2k_1}{m}$$

$$A = \begin{pmatrix} 0 & 1 \\ \frac{2k_1}{m} & 0 \end{pmatrix}$$

Long dynamics

$$\frac{dx_1}{dt} = -\frac{1}{\tau}\cdot x_1 + \frac{k_i}{\tau r_w}\,\omega$$

$$\frac{dx_2}{dt} = \frac{1}{m}\left(x_1 - mgf\cos d_1 - \frac{1}{2}\rho A_f C_d x_2^2 - mg\sin d_1\right).$$

x_1 = force @ wheels

x_2 = vehicle velocity

d_1 = disturbance.

$(x_{1e}, x_{2e}, u_e, d_{1e}) = (15\,m/s)$ flat road

$0 = -\frac{1}{\tau} x_1 + \frac{(k_i)}{\tau r_w} u$

$\frac{x_1}{\tau} = \frac{k_i}{\tau r_w}(u)$ — (1)

$x_1 = \frac{(k_i)(u)}{(r_w)}$

$0 = \left(\frac{1}{m}\right)(x_1 - (20000)(9.82)(0.015) - (0.5)(1.2)(0.5)(4)(15)^2 - 0$

$0 = \left(\frac{x_1}{m}\right) - (9.82)(0.015) - (1.2)(225)$.

$x_1 = [(1.2)(225) + (9.82)(0.015)][20{,}000]$

$x_1 = 270 \times 20{,}000$

$0 = \left(\frac{1}{m}\right)\left(x_1 - mgf - \frac{1}{2}\rho A C_d x_2^2\right)$.

$x_{1e} = mgf + \left(\frac{1}{2}\right)\rho A C_d x_2^2$

$x_{1e} = 2946 + 270 = 3216$

$x_{1e} = 3216$

$$(1) \Rightarrow (x_{1e}) = \frac{K \cdot i(u)}{(r_w)}$$

$$u_e = \frac{(x_{1e})(r_w)}{(K \cdot i)} = \frac{3216 \times 0.5}{(2000 \times 4)}$$

Partial derivatives.

$$\frac{dx_1}{dt} = \frac{-1}{\tau} x_1 + \frac{K i}{\tau \cdot r_w} u.$$

$$\frac{dx_2}{dt} = \left(\frac{1}{m}\right)(x_1 - mgf\cos d_1 - \frac{1}{2}\rho C_d A_f x_2^2 - mg \sin d_1).$$

$$\frac{\partial f_1}{\partial x_1} = \left(-\frac{1}{\tau}\right). \qquad \frac{\partial f_2}{\partial x_1} = \left(\frac{1}{m}\right)$$

$$\frac{\partial f_1}{\partial x_2} = 0 \qquad \frac{\partial f_2}{\partial x_2} = -2 \cdot \left(\frac{1}{2}\right)(\rho C_d A_f) x_2.$$

$$\boxed{= -\rho C_d A_f \cdot x_2}.$$

$$A = \begin{pmatrix} \frac{-1}{\tau} & 0 \\ 1/m & -\rho C_d \cdot A_f \cdot x_2 \end{pmatrix}$$

$$B = \begin{pmatrix} \frac{K \cdot i}{\tau \cdot r_w} \\ 0 \end{pmatrix} \qquad \frac{\partial f}{\partial d} \Big|_{stab} \begin{pmatrix} 0 \\ -g \cdot f \cdot (-\sin d_1) - g \cdot \cos d_1 \end{pmatrix}$$

$$C = (0 \quad 1) \qquad = \begin{pmatrix} 0 \\ gf \cdot \sin d_1 + g \cos d_1 \end{pmatrix}$$

$D = 0$

$= 0$