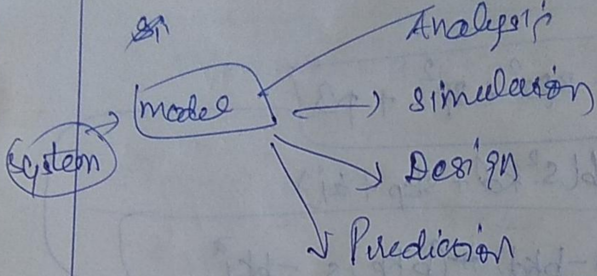
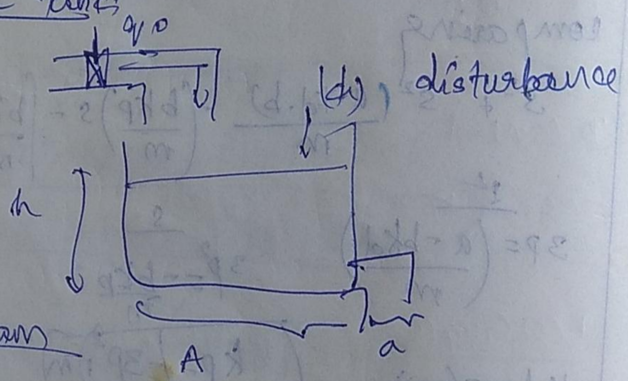
Section 2

Intro to modelling:

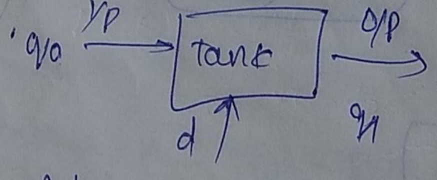


eg:- water tank:

Diagram



Block diagram



Eqns: $A\frac{dh}{dt} = q_0 + d - a\sqrt{2gh}$

$$\boxed{q_1 = a\sqrt{2gh}}$$

Structuring
- subsystems
- i/p, o/p
- internal variables

Basic eqns
- conservation
- constitutive eq:- V=IR, Hookes law

state space
choose state variable
$\hat{x}$

What is it that changes? What is causing the change?

$$\frac{dv}{dt} = \dot{V} = (q_0 + d - q_1)$$

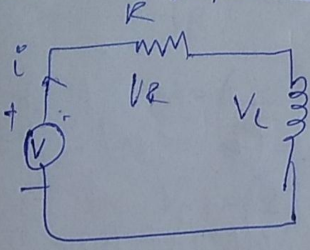
$q_1 \rightarrow a\sqrt{2gh}$

$$= q_0 + d - a\sqrt{2gh}$$

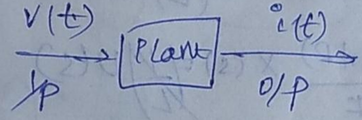
$$= q_0 + d - a\sqrt{2g(V/A)} \qquad q_1 = a\sqrt{2g(V/A)}$$

$x = V$ $u = q_0$ $y = q_1$

Electrical systems:



Structuring



Conservation laws:

$\sum V_i \Rightarrow$ (Kirchoff Law

$V - V_R - V_L = 0$.

constitutive $V_R = iR$

$V_L = L\frac{di}{dt}$.

SS:

What is changing?

i

$$\frac{di}{dt} = \left(\frac{1}{L}\right)V_L \rightarrow (V - V_R) \rightarrow (V - iR)$$

$$\therefore \frac{di}{dt} = \left(\frac{1}{L}\right)(V - iR)$$

$\dot{x} = \left(-\frac{R}{L}\right) x + \left(\frac{1}{L}\right) u$ $x = i =$ current

$u = V = V_0$ ltage.

$y = x$

eg: $V_L =$ o/p signal

$V_L = h(x, u)$

$V_L = (V - V_R) = V - iR$

$= (x)(-R) + (u)$ $\downarrow$ voltage

$s.X(s) = \left(-\frac{R}{L}\right) X(s) + \left(\frac{1}{L}\right) U(s)$

$\left(s + R/L\right) X(s) = \left(\frac{1}{L}\right) U(s).$

$\frac{(Ls+R)}{\cancel{L}} X(s) = \left(\frac{1}{\cancel{L}}\right) U(s)$

$\frac{X(s)}{U(s)} = \frac{1}{(Ls+R)} = \frac{R}{R\left(\left(\frac{L}{R}\right)s+1\right)}$

$\downarrow \tau =$ time constant

$= \frac{50\,mH}{1000} = 50 \times 10^{-3} \times 10^{-3}$

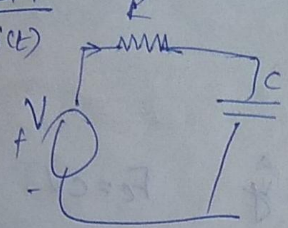
Soln of $= 5 \times 10^{-5}.$

$\tau \frac{dv}{dt} + v = f(t)$

$V(t) = V_0 e^{-t/\tau} = V_0 e^{-\frac{t}{L}(R)}$

$$\boxed{V(t) = \frac{V_0}{e^{\frac{t}{(L)}(R)}}}$$

R-C ckt:



$\Sigma V_i = 0 \Rightarrow$ $V - V_R - V_C = 0$ $\quad V_c = \frac{Q}{c}$

$V_R = iR \qquad V_C = c\int i.dt.$

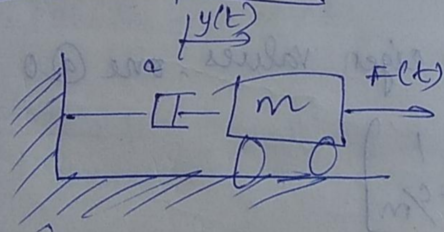
$\therefore V - \left(\frac{1}{c}\right)x - V_R = 0$

$\dot{x} = \left(\frac{1}{R}\right)(V_{in}) - \left(\frac{1}{CR}\right)x$

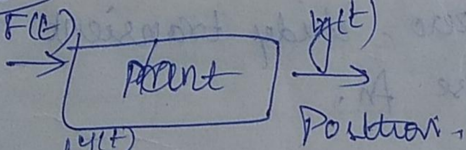
$A = \left(\frac{-1}{RC}\right) \qquad B = \left(\frac{1}{R}\right) \qquad U = V_{in}$

$V_C = \left(\frac{Q}{c}\right) = \left(\frac{x}{c}\right) - \left(\frac{1}{c}\right)x.$

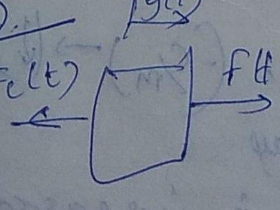
Mechanical systems:



Structuring:



FBD.



Laws:

$$ma = \sum F_i$$

$$ma = F - F_c$$

$$a = \frac{dv}{dt} = \dot{v} \qquad v = \frac{dy}{dt} = \dot{y} \qquad F_c = cv.$$

choose state variables
→ position, velocity

$$x = \begin{pmatrix} y \\ \dot{y} \end{pmatrix} \qquad \dot{x} = \begin{pmatrix} \dot{y} \\ \ddot{y} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} F$$

$$m\ddot{y} = F - cv$$

$$m\dot{x}_2 = -cx_2 + F$$

$$\dot{x}_2 = \left(-\frac{c}{m}\right) x_2 + \left(\frac{F}{m}\right)$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -c/m \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1/m \end{pmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (0) u$$

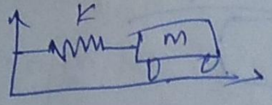
? system has 2 eigen values, one @ 0 other @ $-c/m$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -c/m \end{bmatrix}$$

one pole @ zero, study transient response using impulse fn.

if mass m is increased, $(-c/m) \to (\downarrow)$
∴ slower decay

same effect if (c) damper coeff is increased spring - mass :



$\Sigma F = ma.$

$\begin{pmatrix} x \\ \dot{x} \end{pmatrix}$ = state $\quad \dot{x} = \begin{pmatrix} \dot{x} \\ \ddot{x} \end{pmatrix} = \begin{pmatrix} x_2 \\ \dot{x}_2 \end{pmatrix}$

$\dot{x}_2 = \left(\frac{\Sigma F}{m}\right) = \left(\frac{1}{m}\right) F - kx = \boxed{-k(x_1) + \left(\frac{1}{m}\right) F}$

u

$\boxed{\dot{x}_2 = -kx_1 + \left(\frac{1}{m}\right) u}$

$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -k/m & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1/m \end{pmatrix} u.$

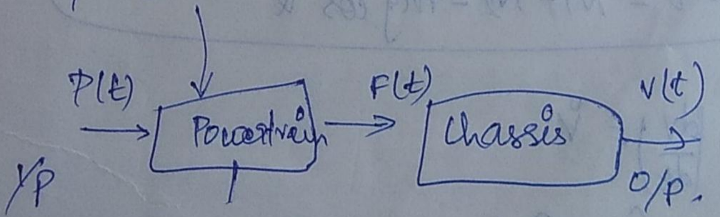
$A - \lambda I = 0 \Rightarrow \begin{vmatrix} -\lambda & 1 \\ -k/m & -\lambda \end{vmatrix} = 0$

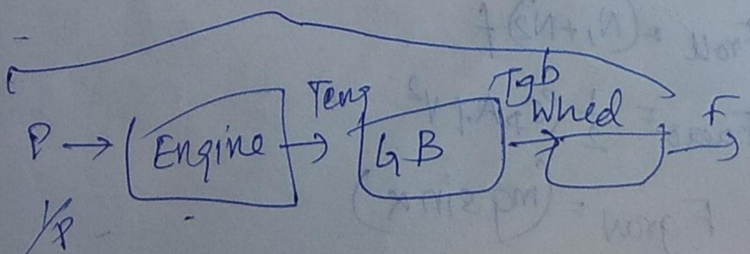
$\lambda^2 + k/m = 0 \quad \lambda^2 = -k/m \quad \boxed{\lambda = \pm j \sqrt{k/m}}$

Long. vehicle dynamics :

Structuring :

powertrain + chassis





Powertrain (Engine) , 1st order dynamics

$$\boxed{\dot{T}_{eng} = \left(\frac{-1}{\tau}\right) T_{eng} + \left(\frac{K}{\tau}\right) P}$$

GB $\quad T_{gb} = i\, T_{eng}$

$$F = \frac{T_{gb}}{r_w} = \frac{i\, T_{eng}}{r_w}$$

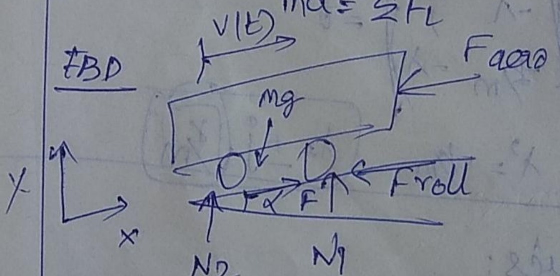
$$\dot{F} = \frac{i\, \dot{T}_{eng}}{r_w} = \left(\frac{-1}{\tau}\right) F + \left(\frac{Ki}{\tau r_w}\right) P$$

$$\boxed{\dot{F} = \left(\frac{-1}{\tau}\right) F + \left(\frac{Ki}{\tau r_w}\right) P}$$

chassis

$ma = \Sigma F_i$

FBD



$$x = ma = F - F_{rolling} - F_{aero} - F_{grav}$$
$$y = 0 = N_1 + N_2 - mg\cos\alpha$$

$$a = \left(\frac{dv}{dt}\right) = \dot{v}$$

$$F_{roll} = (N_1 + N_2) f$$

$$F_{aero} = \frac{1}{2} \rho C_D A_f V^2$$

$$F_{grav} = (mg \sin\alpha)$$

What's changing? Propulsion force, velocity

$\therefore x = \begin{pmatrix} F \\ V \end{pmatrix}$ $\dot{x} = \begin{pmatrix} \dot{F} \\ \dot{V} \end{pmatrix}$ $\dot{F} = \left(\frac{-1}{\tau}\right) F + \left(\frac{K_t i}{\tau r_w}\right) P$

$\dot{V} = a = \left(\frac{F}{m}\right) - \frac{F_{rolling}}{m} - \left(\frac{F_{aero}}{m}\right) - \left(\frac{F_{grav}}{m}\right)$

$= \cancel{\left(\frac{x_1}{m}\right)}$

$= \left(\frac{1}{m}\right) \left[F - mg\cos\alpha f - \frac{1}{2}\rho C_D A_f V^2 - mg\sin\alpha \right]$

$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \dot{F} \\ \dot{V} \end{pmatrix}$ $u = P$ $d_1 = \alpha$. $y = x_2$

$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} (-1/\tau) \\ \left(\frac{1}{m}\right)\left(x_1 - mg\cos\alpha f - \frac{1}{2}\rho C_d A_f x_2^2 - mg\sin\alpha\right) \end{pmatrix}$

$y = x_2$ $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Exercise.

$$\frac{dx_1}{dt} = \frac{-1}{\tau} x_1 + \frac{K i}{\tau r_w}(u)$$

$x_1 = F$
$x_2 = x_2$

$$\frac{dx_2}{dt} = \left(\frac{1}{m}\right)\left[x_1 - mgf.\cos d_1 - \frac{1}{2}\rho C_d A_f x_2^2 - mg\sin d_1\right)$$

$y = x_2$ ∵ d = disturbance slope

m = 2000 ; $K = 2000 \frac{Nm}{rad}$. i = 4 Gearratio

$a = 0.3\ m/s^2$

$q = 0$ deg. $V = 60\ km/hr = \frac{\overset{10}{\cancel{60}} \times 5}{\underset{3}{\cancel{18}}} = 16.66\ m/s$

$$\therefore 0.3 = \frac{1}{2000}\left(x_1 - (2000)(g)(0.015) - \left(\frac{1}{2}\right)(1.2)(\cancel{0.5})(4)(x_2)^2 - \cancel{200}\right)$$

$$0.3 = \frac{1}{2000}\left(x_1 - 2000g(0.015) - (1.2)x_2^2\right)$$

$$600 = x_1 - (294.3) - 1.2(x_2^2)$$

$$x_1 = (16.66)^2 1.2 + 294.3 + 600$$

$$x_1 = 1227.3\ N$$

Torque $= r_w(F) = 1227.3(0.5)$
$= 613.65\ Nm$ (@ wheels)

∴ GB torque = 613.65 Nm

Eq. torque $= \frac{T_{wheels}}{i} = \frac{613.65}{4} = 153.42$

T_{max} (eng) $= 2000$

acceleration in flat road $\Big\} = \frac{dx_2}{dt} = \left(\frac{1}{m}\right)\left(x_1 - mgf - \frac{1}{2}\rho C_d A_f x_2^2\right)$

$\omega_{eng} = (i)\,\omega_{gb}$.

60 km/hr $\Rightarrow$ 16.66 m/s.

1 revolution $\Rightarrow 2\pi(0.5) = \pi$ (metres).

$\frac{16.66 (m)}{\pi (m)} = 5.303$ $\frac{\text{revolutions}}{\text{second}}$

$\approx$ 318 rpm   21.2 rps

$\omega_{gb} = 318$ rpm   ~~1272 x 2π~~

$\omega_{eng} = 318 \times 4 = 1272$ rpm.

$= 133.348$ rad/s

$T_{eng.max}(\omega_{eng}) = 2000\left(1 - 0.6\left(\frac{\omega_{eng}}{150} - 1.2\right)^2\right)$

rad/s

~~$= 2000\left(1 - 0.6\,(7.28)^2\right)$~~

~~2000~~

$= 2000\left(1 - 0.6\,(-0.311)^2\right)$

$= 2000\,(1 - 0.058) = 0.941\,(2000)$

$= 1883.92$ Nm

$T_{engine} = 1883$ Nm.

$T_{gb} = 1883 (\cancel{1}) \cdot 4 = 7535$ Nm

$T_{wheels} = 7535$ Nm

$F_{wheels} = x_1 = 15071.39$ N

x_2 = acceleration

$$= \frac{1}{2000} \left(15071 - (2000)(g)(0.015) - (12)(x_2^2) \right)$$

$= 7.5535 - 0.14715 -$

Answers

$$T_{engine} = \frac{T_{wheels}}{i} = \frac{F_{wheels} \times r_w}{i} \quad \text{eqn } (1)$$

$$F_{wheels} = ma + mgf + 0.5 \rho C_d A_f x_2^2$$

$$= 2000 (0.3) + (2000)(9.81)(0.015)$$
$$+ (0.5)(1.2)(0.5)(4)(16.66)^2$$

$$= 6000 + 2943 + 333 = 9276 \text{ N}.$$

$$T_{engine} = \frac{9276 \times 0.5}{4} = 1160 \text{ Nm}$$

(2) $T_{eng,max}(\omega_{eng}) = 2000\left(1 - 0.6\left(\frac{\omega_{eng}}{150} - 1.2\right)^2\right)$

max. acceleration = ? @ 60 kmph on flat road.

60 kmph → 16.66 m/s → ω_{wheels} = 318 rpm

ω_{engine} = 1272 rpm = 21.2 rps

= 133.34 rad/s

$T_{eng.max} = 2000\left(1 - 0.6\left(\frac{133.34}{150} - 1.2\right)^2\right)$

= 1883 Nm

eqn (ii) ⇒ Total force

$= \left(\frac{1883 \times 4}{0.5}\right) = ma + mgf + 0.5\, C_d A_f \rho\, x_2^2$

$= 20000(a) + 2943 + 333$

$\frac{15066.72 - 2943 - 333}{20000} = 0.589\ m/s^2$

max acceleration = 0.589 m/s²

(3) V = 40 kmph = 11.11 m/s (i = 8).

~~ω_{eng} =~~ ω_{wheels} =

1 rev = 3.14 m/s
11.11 m/s² } = 3.538 rev/s

$= 22.22$ revolu/s.

$W_{engine} = 3.58 \times i = 22.22$

$W_{engine} = 177$ rad/s

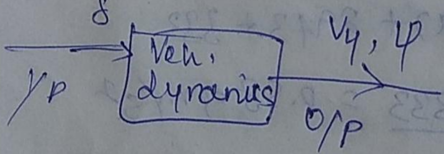
Torque = 1558 Nm.

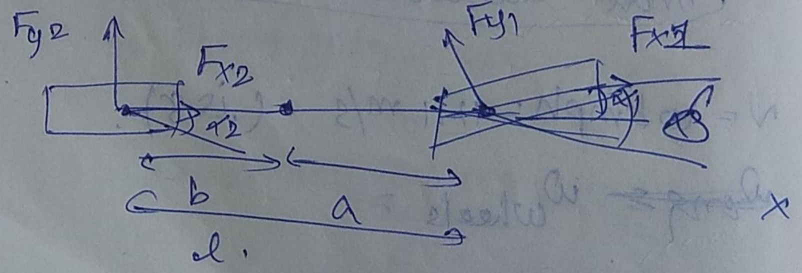
$$\frac{1558 \times 8}{0.5} = ma + mgf\cos\alpha_1 + 0.5\rho C_d A_f v^2 + mg\sin\alpha_1 \quad \text{eqn 2}$$

Solving eqn2, until $a = 0$, $\alpha_1 = 8.4$ deg

Single track model

Bicycle model - Linear.





Force balance

$ma = \sum_i F_i$

$ma_x = F_{x1}\cos\delta + F_{x2} - F_{y1}\sin\delta$

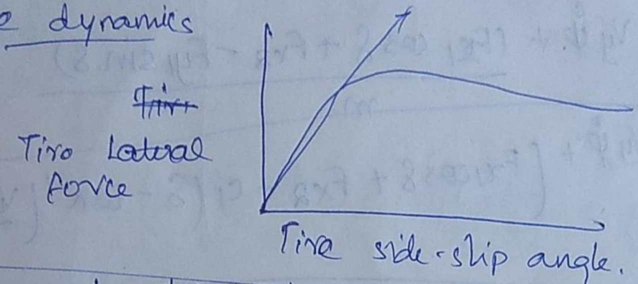
$ma_y = F_{y2} + F_{x1}\sin\delta + F_{y1}\cos\delta$

Torque balance:

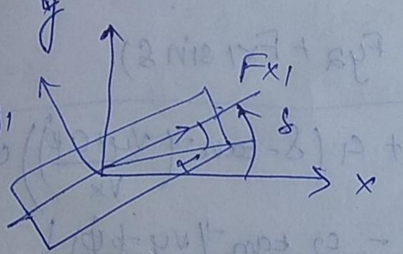
$$J\ddot{\psi} = \sum_i T_i = a\,(F_{x1}\sin\delta + F_{y1}\cos\delta) - b\,F_{y2}.$$

Constitutive:

Tire dynamics



$$\boxed{\alpha_i = -\tan^{-1}\left(\frac{V_{yi}}{V_{xi}}\right)} \rightarrow \text{general eqn}$$



Side-slip angle is angle b/w lateral & long velocity of tire in vehicle co-ordinate system.

$$\alpha_1 = \delta - \tan^{-1}\left(\frac{V_y + a\dot{\psi}}{V_x}\right)$$

$$\alpha_2 = -\tan^{-1}\left(\frac{V_y - b\dot{\psi}}{V_x}\right)$$

$$F_i = C_\alpha \cdot \alpha_i$$

Long. & Lateral acc. in inertial frame

$$\boxed{\begin{aligned} a_x &= \dot{V}_x - V_y\dot{\psi} \\ a_y &= \dot{V}_y + V_x\dot{\psi} \end{aligned}}$$

$$\dot{x} = f(x, u)$$
$$y = h(x, u)$$

choosing state variables
3 time derivatives
$v_x, v_y, \dot{\psi}$

$$\dot{v}_x = a_x + v_y \dot{\psi}$$
$$= v_y \dot{\psi} + \frac{(F_{x1} \cos\delta + F_{x2} - F_{y1} \sin\delta)}{m}$$

$$\dot{v}_x = v_y \dot{\psi} + \left(F_{x1}\cos\delta + F_{x2} - c_1\left(\delta - \tan^{-1}\left(\frac{v_y + a\dot{\psi}}{v_x}\right)\right)\sin\delta\right)$$

$$\dot{v}_y = a_y - v_x \dot{\psi}$$
$$= -v_x \dot{\psi} + \frac{1}{m}(F_{y1}\cos\delta + F_{y2} + F_{x1}\sin\delta)$$

$$\dot{v}_y = -v_x \dot{\psi} + \left(\frac{1}{m}\right)\left(F_{x1}\cos\delta + c_1\left(\delta - \tan^{-1}\left(\frac{v_y + a\dot{\psi}}{v_x}\right)\right)\cos\delta - c_2 \tan^{-1}\left(\frac{v_y - b\dot{\psi}}{v_x}\right)\right)$$

$$\ddot{\psi} = \frac{1}{J}\left(\overset{a}{F_{x1}}\sin\delta + F_{y1}\cos\delta\right) - bF_{y2}$$

$$\ddot{\psi} = \frac{1}{J}\left(aF_{x1}\sin\delta + a\left(c_1\left(\delta - \tan^{-1}\left(\frac{v_y + a\dot{\psi}}{v_x}\right)\right)\cos\delta + bc_2\tan^{-1}\left(\frac{v_y - b\dot{\psi}}{v_x}\right)\right)\right)$$

~~Assumptions~~: (1) $\dot{v}_x = 0$ constant velocity

tan side-slip angle = side-slip angle

(2) $\tan^{-1}\delta = \delta$

(3) $F_x = 0 \Rightarrow$ no acceleration.

(4) $\cos\delta = 1$

$\sin\delta = \delta$.

$$\dot{v}_y = -v_x\dot{\psi} + \left(\frac{1}{m}\right)\left(c_1\delta - c_1\left(\frac{v_y + a\dot{\psi}}{v_x}\right) - c_2\left(\frac{v_y - b\dot{\psi}}{v_x}\right)\right)$$

$$\ddot{\psi} = \frac{1}{J}\left(ac_1\delta - ac_1\left(\frac{v_y + a\dot{\psi}}{v_x}\right) + bc_2\left(\frac{v_y - b\dot{\psi}}{v_x}\right)\right)$$

$x_1 = v_y$ $\quad u = \delta \quad y_1 = x_1 \quad y_2 = x_2$

$x_2 = \dot{\psi}$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{pmatrix} -v_x x_2 + \left(\frac{1}{m}\right)\left(c_1 u - c_1\left(\frac{x_1 + a x_2}{v_x}\right) - c_2\left(\frac{x_1 - b x_2}{v_x}\right)\right) \\ \frac{1}{J}\left(ac_1 u - ac_1\left(\frac{x_1 + a x_2}{v_x}\right) + bc_2\left(\frac{x_1 - b x_2}{v_x}\right)\right)\end{pmatrix}$$

$$A = \begin{bmatrix} \left(\frac{c_1 - c_2}{m v_x}\right) & -v_x - \frac{ac_1}{m v_x} + \frac{c_2 b}{m v_x} \\ \frac{-ac_1}{J v_x} + \frac{bc_2}{v_x J} & -\frac{a^2 c_1}{J v_x} - \frac{b^2 c_2}{J v_x} \end{bmatrix} \qquad B = \begin{pmatrix} c_1/m \\ ac_1/J \end{pmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} \frac{c_1 - c_2}{m v_x} & -v_x + \left(\frac{-ac_1 + bc_2}{m v_x}\right) \\ \frac{-ac_1 + bc_2}{J v_x} & \frac{-a^2 c_1 - b^2 c_2}{J v_x} \end{bmatrix} \qquad D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Alternate model

$$A = \begin{bmatrix} \dfrac{-c_1+c_2}{mV_x} & \dfrac{-ac_1+bc_2}{mV_x} - V_x \\ -\dfrac{ac_1-bc_2}{JV_x} & -\left(\dfrac{a^2c_1+b^2c_2}{JV_x}\right) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$V_x = 10$ $\quad eig(A) = -7.74, -20.8561$
$V_x = 20$ $\quad eig(A) = -1.8582, -12.44$ $\Big\}$ stable
$V_x = 30$ $\quad eig(A) = 0.2557, -9.7885$ $\}$ unstable

Alternate model :- $\beta = \tan^{-1}\left(\dfrac{V_y}{V_x}\right) = \dfrac{V_y}{V_x}$

$x_1 = V_y$ $\quad z_1 = \beta = \dfrac{V_y}{V_x}$ $\quad \dot{\psi} = z_2$

$x_2 = \dot{\psi}$

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} V_y/V_x \\ \dot{\psi} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{V_x} & 0 \\ 0 & 1 \end{bmatrix}}_{\text{Transformation matrix}} \begin{bmatrix} V_y \\ \dot{\psi} \end{bmatrix}$$

$z = Tx$

But u need

$x = TZ$

$$\begin{bmatrix} V_y \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} V_x & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_y/V_x \\ \dot{\psi} \end{bmatrix}$$

$$z_1 = \left(\frac{v_y}{v_x}\right) \qquad z_2 = \dot{\psi}$$

$\therefore$ A original

$$\dot{v}_y = -\left(\frac{c_1+c_2}{m v_x}\right)(v_y) + \left(\frac{-ac_1+bc_2}{m v_x}\right)\dot{\psi} - v_x\dot{\psi}$$

$$(v_x)\dot{\beta} = \dot{v}_y \Rightarrow \dot{\beta} = \left(\frac{\dot{v}_y}{v_x}\right)$$

$$\dot{\beta} = -\left(\frac{c_1+c_2}{m v_x}\right)\left(\frac{v_y}{v_x}\right) + \left(\frac{-ac_1+bc_2}{m v_x}\right)\frac{\dot{\psi}}{(v_x)} - \dot{\psi}$$

$$\boxed{\dot{\beta} = -\left(\frac{c_1+c_2}{m v_x}\right)\beta + \left(\left(\frac{bc_2-ac_1}{m v_x^2}\right) - 1\right)\dot{\psi}}$$

$$\ddot{\psi} = -\left(\frac{ac_1-bc_2}{J v_x}\right)(v_y) - \left(\frac{a^2c_1+b^2c_2}{J v_x}\right)(\dot{\psi})$$

$$\boxed{\ddot{\psi} = -\frac{(ac_1-bc_2)}{J}\beta - \left(\frac{a^2c_1+b^2c_2}{J v_x}\right)\dot{\psi}}$$

under-steering, over-steering.

$$|(\lambda I - A)| = 0$$

$$\begin{vmatrix} \lambda + \left(\frac{c_1+c_2}{mV_x}\right) & V_x + \left(\frac{bc_2 - ac_1}{mV_x}\right) \\ \frac{ac_1 - bc_2}{JV_x} & \lambda + \left(\frac{a^2c_1 + b^2c_2}{JV_x}\right) \end{vmatrix} = 0$$

$$\left(\lambda + \left(\frac{c_1+c_2}{mV_x}\right)\right)\left(\lambda + \left(\frac{a^2c_1+b^2c_2}{JV_x}\right)\right) - \left(V_x + \frac{bc_2 - ac_1}{mV_x}\right)\left(\frac{ac_1 - bc_2}{JV_x}\right) = 0$$

a_1

$$\lambda^2 + \left[\left(\frac{a^2c_1+b^2c_2}{JV_x}\right) + \left(\frac{c_1+c_2}{mV_x}\right)\right]\lambda + \left[\left(\frac{c_1+c_2}{mV_x}\right)\left(\frac{a^2c_1+b^2c_2}{JV_x}\right) - V_x\left(\frac{ac_1-bc_2}{JV_x}\right) + \left(\frac{bc_2-ac_1}{mV_x}\right)\left(\frac{ac_1-bc_2}{JV_x}\right)\right] = 0$$

a_2

$$(bc_2 - ac_1) - (bc_2 - ac_1)$$

$$-\frac{(bc_2 - ac_1)^2}{mJV_x^2}$$

$$a_2 = \left(\frac{1}{Jv_x}\right)\left\{\frac{(c_1+c_2)(a^2c_1+b^2c_2)}{mv_x} - v_x(ac_1-bc_2) - b^2c_2^2 + abc_1c_2 - a^2c_1^2 + abc_1c_2\right\}$$

$$a_2 = \left(\frac{1}{Jv_x}\right)\left(\frac{1}{mv_x}\right)\left((c_1+c_2)(a^2c_1+b^2c_2) - mv_x^2(ac_1-bc_2) - (bc_2-ac_1)^2\right)$$

$$a_2 = \frac{1}{mJv_x^2}\left[(c_1+c_2)(a^2c_1+b^2c_2) - mv_x^2(ac_1-bc_2) - [bc_2-ac_1]^2\right]$$

$\Downarrow$

$$a_2 = \frac{c_1c_2(a+b)^2}{JmV_x^2}\left(1 + \frac{bc_2-ac_1}{(c_1c_2(a+b))^2}\,mV_x^2\right)$$

when is $a_2 <$ if $bc_2 < ac_1$

$$\left(\frac{c_1c_2(a+b)^2}{JmV_x^2}\right)\left(1+\frac{bc_2-ac_1}{c_1c_2}\right)$$

$$a = \frac{c_1c_2(a+b)^2}{mJv_x^2} + \left(\frac{bc_2-ac_1}{J}\right) < 0$$

$$\frac{c_1c_2(a+b)^2}{mJv_x^2} < -\left(\frac{bc_2-ac_1}{J}\right)$$

$$\frac{10^5\times 8\times 10^4\times(2.8)^2}{17\times 10^2\times 2\times 10^3\times v_x^2}$$

$$a_2 = \frac{4c_2(a+b)^2}{mJv_x^2}\left[1 + \frac{bc_2 - ac_1}{4c_2(a+b)^2} mV_x^2\right).$$

$\lambda^2 + a_2\lambda + a_2 = 0$

if a_1, a_2 are +ve, -ve real roots stable system.

stable $\Rightarrow$ $a_1, a_2 = +ve$ $\Rightarrow$ $bc_2 > ac_1$ n always

if $bc_2 < ac_1$

it depends on ~~force~~ V_x

$bc_2 = (1.3) \times 80 \times 10^3$

$ac_1 = (1.5) \times 100 \times 10^3$

$bc_2 = 104 \times 10^3$

$ac_1 = 150 \times 10^3$

$bc_2 < ac_1$

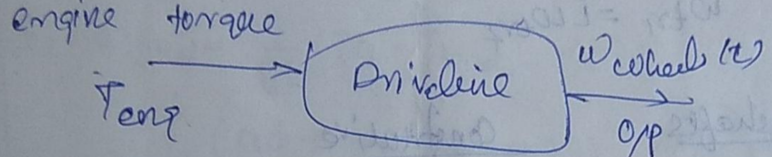
$\therefore$ @ some V_x open-loop system is unstable.

$$\Rightarrow a_2 = \frac{80 \times 100 \times 10^6 (2.8)^2}{1700 \times 2 \times 10^3 \times V_x^2}\left[1 + \left(\frac{(-46) \times 10^3}{80 \times 100 \times 10^6 \times (2.8)^2}\right)(1.7) \times 10\right)$$

Drivetrain modelling:

1st order system assumption valid for high velocities

@ low speeds, high gear-ratios, oscillation of gear shafts occur



$\boxed{J\dot{\omega} = \sum_i T_i}$

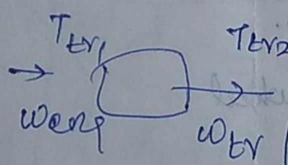
Engine

Inertia

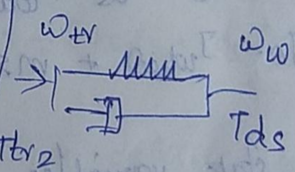
T_{eng}, ω_{eng}

T_{tr1}

Transmission



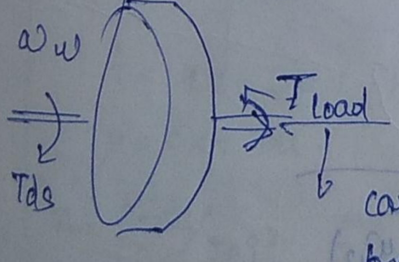
Driveshafts



T_{tr2}

spring & damper

Wheels & chassis



can be determined from Eq. vehicle dynamics but as disturbance here

$T_{eng} - T_{tr1} = J_f \cdot \dot{\omega}_{eng}$

conservation law for transmission

$0 = T_{tr2} - i T_{tr1}$

Constitutive

$\omega_{tr1} = i \omega_{tr2}$

<u>Driveshafts</u>

$0 = T_{ds} - T_{tr2}$

<u>Constitutive</u>

$T_{ds} = c_s \left(\int (\omega_{tr} - \omega_w) \, dt \right) + d_s (\omega_{tr} - \omega_w)$

Rotational stiffness

<u>Wheels & chassis</u>

$J_c \cdot \dot{\omega}_w = T_{ds} - T_{load}.$

$J_c = J_{wheel} + m r^2_{wheel}$

<u>choose state variables</u>

1) engine speed ω_{eng}

2) wheel speed ω_{wheel}

3) torsion $\phi = \int \left(\frac{\omega_{eng}}{i} - \omega_w \right)$

$$\boxed{T_{ds} = c_s \phi + d_s (\omega_{tr} - \omega_w)}$$

$$T_{ds} = c_s x_3 + d_s \left(x_1 / i - x_2 \right)$$

$$= \left(\frac{d_s}{i} \right) x_1 - d_s x_2 + c_s x_3$$

$$\dot{x}_1 = \dot{\omega}_{eng} = \frac{T_{eng} - T_{tri}}{J_f}$$

x_4

$$T_{tri} = \frac{T_{tr2}}{i} = \frac{T_{ds}}{i} = \frac{1}{i}\left(c_s \phi + d_s(\omega_{tr} - \omega_W)\right)$$

$$= \left(\frac{1}{i}\right)\left(c_s\phi + d_s(\omega_{eng}/i - \omega_W)\right)\left(\frac{-1}{J_f}\right)$$

$$\dot{x}_2 = \dot{\omega}_{whel} = \frac{1}{J_c}(T_{ds} - T_{load})$$

$$= \frac{1}{J_c}\left(c_s\phi + d_s(\omega_{eng}/i - \omega_\omega) - T_{load}\right)$$

disturbance

$$\dot{x}_3 = \dot{\phi} = \frac{\omega_{eng}}{i} - \omega_W$$

$$\begin{bmatrix} \dot{\omega}_{eng} \\ \dot{\omega}_{wheel} \\ \dot{\phi} \end{bmatrix} = $$

$$\dot{x}_1 = \frac{1}{J_f}\left(u - \frac{1}{i}\left(c_s x_3 + d_s\left(\frac{x_1}{i} - x_2\right)\right)\right)$$

$$\dot{x}_2 = \frac{1}{J_c}\left(c_s x_3 + d_s(x_1/i - x_2) - d\right)$$

$$\dot{x}_3 = x_1/i - x_2$$

$$\dot{x} = Ax + Bu + Hd$$

$$y = Cx + Du.$$

$$A = \begin{bmatrix} -d_s/J_f i^2 & \frac{d_s}{J_f i} & \frac{-c_s}{J_f i} \\ d_s/J_c i & -d_s/J_c & c_s/J_c \\ 1/i & -1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1/J_f \\ 0 \\ 0 \end{bmatrix} \quad H = \begin{bmatrix} 0 \\ -1/J_c \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

$$D = (0)$$

Driveshaft torque T_{ds} as output.
with same state vector $\begin{pmatrix} x_1 & \text{eng spd} \\ x_2 & \text{whl spd} \\ x_3 & \text{torsion} \end{pmatrix}$

$$T_{ds} = c_s \phi + d_s (\omega_{tr} - \omega_{wheel})$$

$$= c_s (x_3) + d_s (\underbrace{\omega_{eng}}_{x_1}/i - \underbrace{\omega_{wheel}}_{x_2})$$

$$\boxed{T_{ds} = \frac{d_s (x_1)}{i} - d_s (x_2) + c_s (x_3)}$$

Alternate model

ϕ_{fw} · ϕ_{wheel} instead of ϕ

$A = 4 \times 4$ $\quad x = 4 \times 1$ $\quad u = 1 \times 1$

$B = 4 \times 1$ $\quad C = 1 \times 4$

Then

$$A = \begin{bmatrix} \frac{-d_s}{J_f i^2} & \frac{d_s}{J_f i} & \frac{-c_s}{J_f i^2} & \frac{c_s}{J_f i} \\ \frac{d_s}{J_c i} & \frac{-d_s}{J_c} & \frac{c_s}{J_c i} & \frac{-c_s}{J_c} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad B = \begin{pmatrix} 1/J_f \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$H = \begin{pmatrix} 0 \\ -1/J_c \\ 0 \\ 0 \end{pmatrix} \qquad C = [0 \ 1 \ 0 \ 0] \qquad D = [0]$$

$$a_2 = \frac{c_1 c_2 (a+b)^2}{mJV_x^2}\left(1 + \frac{bc_2 - ac_1}{c_1 c_2 (a+b)^2} mV_x^2\right)$$

$$a_2 = \frac{c_1 c_2 (a+b)^2}{mJV_x^2} + \frac{c_1 c_2 \cancel{(a+b)^2}}{\cancel{m}JV_x^2}\left(\frac{bc_2 - ac_1}{c_1 c_2 \cancel{(a+b)^2}}\right)\cancel{(mV_x^2)}$$

$$\frac{c_1 c_2 (a+b)^2}{mJV_x^2} + \left(\frac{1}{J}\right)(bc_2 - ac_1) < 0$$

$= 0$

$$\frac{80\times 10^3 \times 10^5 \times (2.8)^2}{1.7\times 10^3 \times 2\times 10^3 \times V_x^2} + \left(\frac{1}{2\times 1000}\right)(-46\times 10^3) = 0.$$

$$\Rightarrow \left(\frac{80\times 10^8 \times (2.8)^2}{1.7\times 2\times 10^6}\right)\left(\frac{1}{V_x^2}\right) = \left(\frac{46}{2}\right) = 23.$$

$$V_x^2 = \frac{80\times 10^8 \times (2.8)^2}{3.4\times 10^6 \times 23} = \frac{80\times (2.8)^2 \times 10^2}{3.4\times 23}$$

$$V_x^2 = 802.04\ m^2/s^2$$

$$V_x = 28.32\ m/s$$

Understeer coefficient:

$$K_u = \frac{m}{(a+b)}\frac{(bc_2 - ac_1)}{(c_1 c_2)}$$

$K_u > 0$ understeer

$K_u = 0$ neutral steer

$K_u < 0$ oversteer.

$$K_u = \frac{1700 \times (-46) \times 10^3}{(2.8) \times (180) \times 10^3} = -155.15$$

Critical velocity expressed as fn (K_u)

$$\cancel{\frac{1}{K_u} = \frac{(a+b)}{(m)}\left(\frac{bc_2 - ac_1}{c_1c_2}\right)}$$

$$\frac{1}{K_u} = \frac{(a+b)(c_1c_2)}{(m)(bc_2 - ac_1)}$$

$a_2 = 0$

$$\Rightarrow \frac{c_1c_2(a+b)^2}{mJV_x^2} + \left(\frac{1}{J}\right)(bc_2 - ac_1) = 0.$$

$$\frac{c_1c_2(a+b)^2}{mJV_x^2} = -\left(\frac{bc_2 - ac_1}{J}\right)$$

$$-V_x^2 = \frac{c_1c_2(a+b)^2}{(m)(bc_2 - ac_1)} = K_u(a+b)$$

$$-V_x^2 = \frac{(c_1c_2)(a+b)}{m(bc_2 - ac_1)}(a+b)$$

$$\boxed{V_x^2 = -1/K_u \cdot (a+b)}$$

$$\therefore V_x^2 = \frac{-(a+b)}{K_u}$$

$$V_x' = \sqrt{\frac{-(a+b)}{K_u}}$$

~~$K_u = ?$~~

$V_x = 28.32$ m/s $\quad (a+b) = 2.8$

$K_u = ?$ $\quad K_u = \frac{-(a+b)}{V_x^2}$

$$\boxed{K_u = \frac{-2.8}{802.0224}}$$

$$V_x \propto \frac{1}{\sqrt{K_u}}$$

To increase stability $V_x \uparrow$

$V_x \uparrow \Rightarrow K_u \downarrow$

$$K_u = m \frac{(bc_2 - ac_1)}{(a+b)(c_1+c_2)}$$

$m \downarrow$ $\quad ac_1 \uparrow$

$c_1 \uparrow \quad a \uparrow$