Module 1 | Lesson 3

# Least Squares and Maximum Likelihood

# Least Squares and Maximum Likelihood

By the end of this video, you will be able to…

- State the connection between the method of least squares and maximum likelihood with Gaussian random variables

# Revisiting the Least Squares Criterion

- Recall the least squares criterion

$$\mathscr{L}_{\mathrm{LS}}(x) = (y_1 - x)^2 + (y_2 - x)^2 + \ldots + (y_m - x)^2$$

- We've said that the optimal estimate, $\hat{x}$ , is the one that minimizes this 'loss':

$$\hat{x}_{\mathrm{LS}} = \operatorname{argmin}_x \mathscr{L}_{\mathrm{LS}}(x) = \operatorname{argmin}_x \left(e_1^2 + e_2^2 + \ldots + e_m^2\right)$$

*Why squared errors?*

# Least Squares and Maximum Likelihood

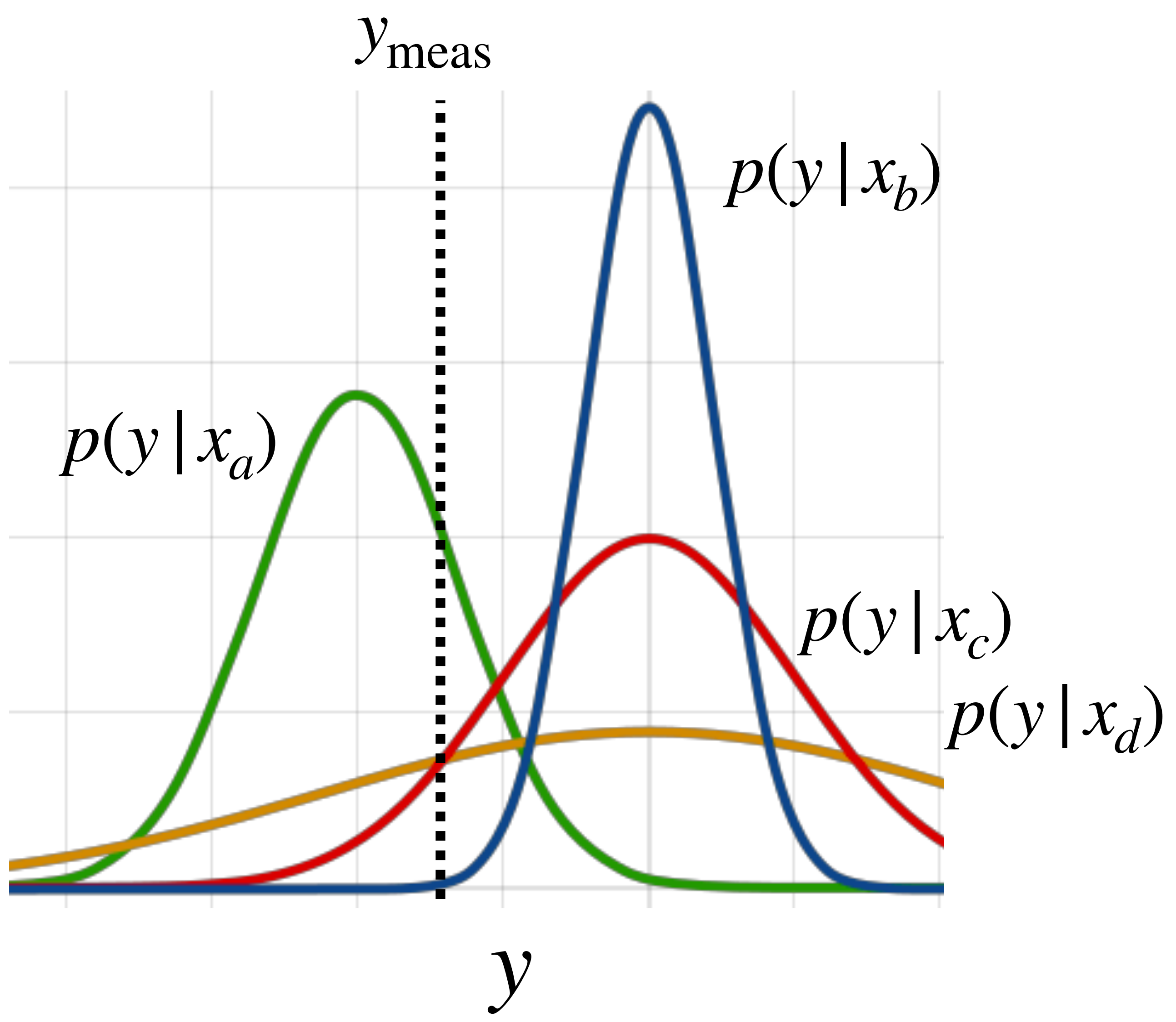
Carl Friedrich Gauss
'*Princeps mathematicorum*'

**Gauss** showed the connection between his method of least squares and maximum likelihood with Gaussian measurement models

# The Method of Maximum Likelihood

- We can ask which $x$ makes our measurement *most likely*. Or, in other words, which $x$ maximizes the conditional probability of $y$:

$$\hat{x} = \text{argmax}_{\,x}\; p(y\,|\,x)$$




Which $x$ is the most likely given the measurement?

# Measurement Model

- Recall our simple measurement model:

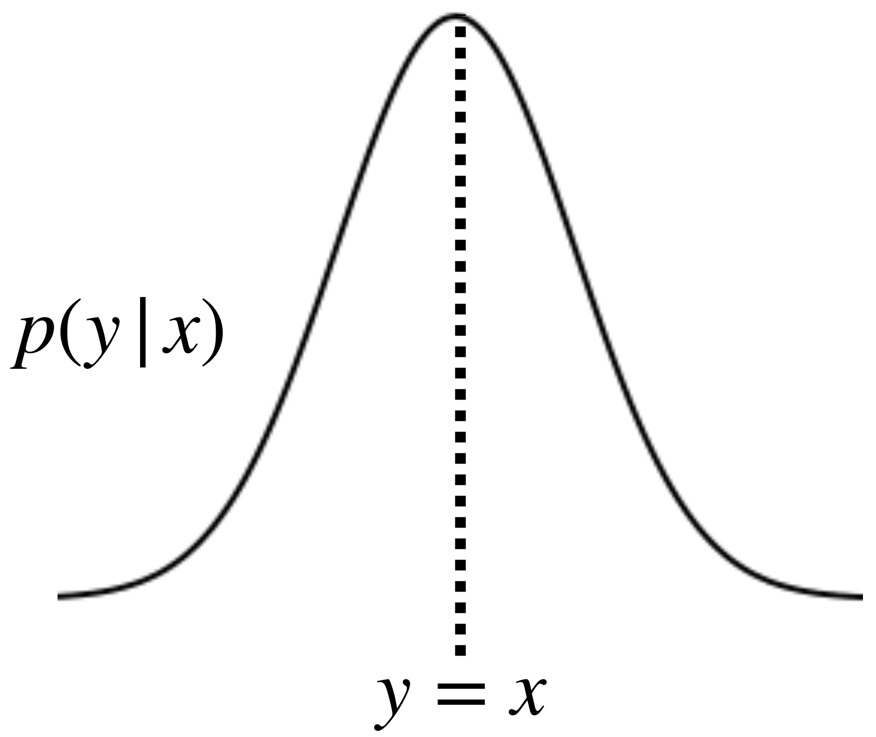
$$y = x + v$$

- We can convert this into a conditional probability on our measurement, by assuming some probability density for $v$. For example, if

$$v \sim \mathcal{N}(0,\ \sigma^2)$$

- Then:

$$p(y \mid x) = \mathcal{N}(x, \sigma^2)$$



# Least Squares and Maximum Likelihood

- Probability density function of a Gaussian is:

$$\mathcal{N}(z;\mu,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(z-\mu)^2}{2\sigma^2}}$$

- Our conditional measurement likelihood is

$$\begin{aligned} p(y \mid x) &= \mathcal{N}(y;x,\sigma^2) \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(y-x)^2}{2\sigma^2}} \end{aligned}$$

- If we have multiple independent measurements, then:

$$\begin{aligned} p(\mathbf{y} \mid x) &\propto \mathcal{N}(y_1;x,\sigma^2)\mathcal{N}(y_2;x,\sigma^2) \times \ldots \times \mathcal{N}(y_m;x,\sigma^2) \\ &= \frac{1}{\sqrt{(2\pi)^m \sigma^{2m}}} \exp\left( \frac{-\sum_{i=1}^{m}(y_i - x)^2}{2\sigma^2} \right) \end{aligned}$$

# Least Squares and Maximum Likelihood

- The maximal likelihood estimate (MLE) is given by

$$\hat{x}_{\text{MLE}} = \text{argmax}_{\,x}\; p(\mathbf{y} \,|\, x)$$

- Instead of trying to optimize the likelihood directly, we can take its logarithm:

$$\begin{aligned}\hat{x}_{\text{MLE}} &= \text{argmax}_{\,x}\; p(\mathbf{y} \,|\, x) \\ &= \text{argmax}_{\,x}\; \log p(\mathbf{y} \,|\, x)\end{aligned}$$

The logarithm is monotonically increasing!

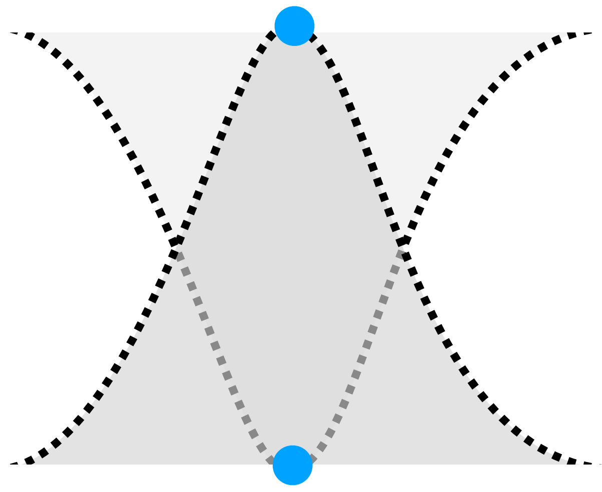
- Resulting in:

$$\log p(\mathbf{y} \,|\, x) = -\frac{1}{2\sigma^2}\left((y_1 - x)^2 + \ldots + (y_m - x)^2\right) + C$$

# Least Squares and Maximum Likelihood

- Since

$$\operatorname{argmin}_z f(z) = \operatorname{argmin}_z \left(-f(z)\right)$$



- The maximal likelihood problem can therefore be written as

$$\begin{aligned}\hat{x}_{\mathrm{MLE}} &= \operatorname{argmin}_x \; -\left(\log p(\mathbf{y} \,|\, x)\right) \\ &= \operatorname{argmin}_x \frac{1}{2\sigma^2}\left((y_1 - x)^2 + \ldots + (y_m - x)^2\right)\end{aligned}$$

# Least Squares and Maximum Likelihood

- So:

$$\hat{x}_{\mathrm{MLE}} = \operatorname{argmin}_x \frac{1}{2\sigma^2}\left((y_1 - x)^2 + \ldots + (y_m - x)^2\right)$$

- Finally, if we assume each measurement has a different variance, we can derive

$$\hat{x}_{\mathrm{MLE}} = \operatorname{argmin}_x \frac{1}{2}\left(\frac{(y_1 - x)^2}{\sigma_1^2} + \ldots + \frac{(y_m - x)^2}{\sigma_m^2}\right)$$

In both cases,

$$\hat{x}_{\mathrm{MLE}} = \hat{x}_{\mathrm{LS}} = \operatorname{argmin}_x \mathscr{L}_{\mathrm{LS}}(x) = \operatorname{argmin}_x \mathscr{L}_{\mathrm{MLE}}(x)$$

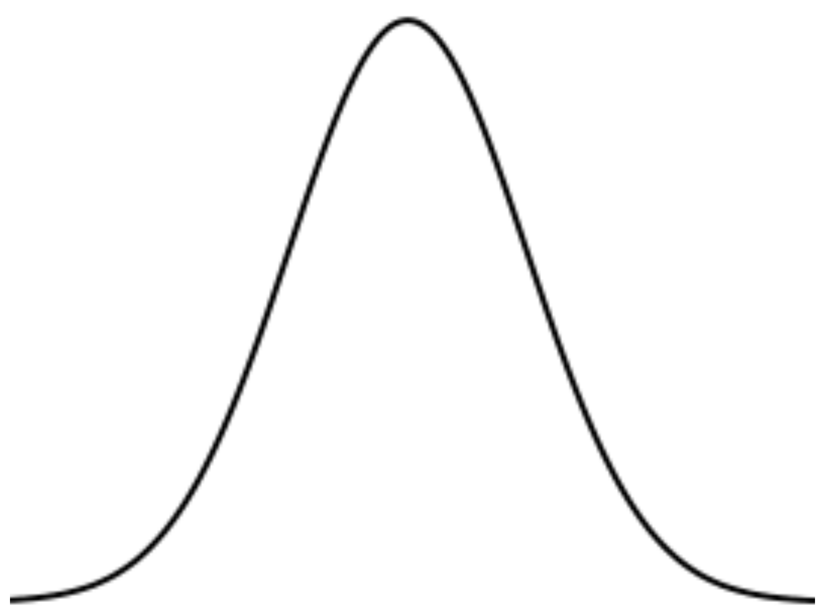
# The Central Limit Theorem

- In realistic systems like self driving cars, there are many sources of 'noise'

*Central Limit Theorem: When independent random variables are added, their normalized sum tends towards a normal distribution.*

- Why use the method of least squares?
  1. Central Limit Theorem: sum of different errors will tend be 'Gaussian'-ish
  2. Least squares is equivalent to maximum likelihood under Gaussian noise

# **Least Squares I** Some Caveats



Under the Gaussian PDF, samples ‘far away’ from the mean are ‘very improbable’

- ‘Poor’ measurements (e.g. outliers) have a significant effect on the method of least squares
- It’s important to check that the measurements roughly follow a Gaussian distribution

| # | Resistance (Ohms) |
|---|---|
| 1 | 1068 |
| 2 | 988 |
| 3 | 1002 |
| 4 | 996 |

$\hat{x} = 1013.5$

| # | Resistance |
|---|---|
| 1 | 1068 |
| 2 | 988 |
| 3 | 1002 |
| 4 | 996 |
| 5 (outlier) | 1430 |

$\hat{x} = 1096.8$

# **Summary** | Least Squares and Maximum Likelihood

- LS and WLS produce the same estimates as maximum likelihood assuming Gaussian noise
- Central Limit Theorem states that complex errors will tend towards a Gaussian distribution.
- Least squares estimates are significantly affected by *outliers*