Three-wave mixing equations of motion

For analysis of a two mode coupling system (SHG, or degenerate three-wave mixing)

$\dot{a} = (-i\delta_a - \kappa_a)a - i2g_0 ba^* - i\sqrt{2\kappa_{a,1}}a_{pump}$

$\dot{b} = (-i\delta_b - \kappa_b)b - ig_0 a^2$

In the general case for any three wave mixing problem with two pump tones, we can write the Hamiltonian as

$$\begin{aligned}\frac{H}{\hbar} &= \omega_a \hat{a}^\dagger a + \omega_b \hat{b}^\dagger \hat{b} + \omega_c \hat{c}^\dagger c + g\left(\hat{a}\hat{b}^\dagger \hat{c} e^{it(\omega_a - \omega_{fb} + (\omega_{fb} - \omega_a))} + \hat{a}^\dagger \hat{b} \hat{c}^\dagger e^{-it(\omega_a - \omega_{fb} + (\omega_{fb} - \omega_a))}\right) \\ &+ i\sqrt{2\kappa_{a,1}\frac{P_a}{\hbar\omega_a}}\left(-\hat{a}e^{i\omega_a t} + \hat{a}^\dagger e^{-i\omega_a t}\right) \\ &+ i\sqrt{2\kappa_{b,1}\frac{P_b}{\hbar\omega_{fb}}}\left(-\hat{b}e^{i\omega_{fb} t} + \hat{b}^\dagger e^{-i\omega_{fb} t}\right)\end{aligned} \tag{1}$$

for the sake of simplicity, we can assume that g is the same for each case. We can then derive the equations of motion using the Heisenberg picture. The above equation is difficult to solve, so we apply a frame change using a unitary transformation. By definition, a unitary matrix is defined as $U^\dagger U = 1$. So we can insert it whereever we please and choose U in order to simplify the equation. Under this transformation, the Hamiltonian is written as

$$H \to UHU^\dagger + i\hbar\dot{U}U^\dagger = \breve{H} \tag{2}$$

where a useful identity is: $e^{\alpha\hat{x}}\hat{y}e^{-\alpha\hat{x}} = \hat{y} + \alpha[\hat{x},\hat{y}] + \frac{\alpha^2}{2!}[\hat{x},[\hat{x},\hat{y}]]$

we can define $\alpha = it,\ \hat{x} = \omega_a \hat{a}^\dagger \hat{a} + \omega_{fb}\hat{b}^\dagger \hat{b} + (\omega_{fb} - \omega_a)\hat{c}^\dagger c,\ \hat{y} = H,\ U = e^{it\left(\omega_a \hat{a}^\dagger \hat{a} + \omega_{fb}\hat{b}^\dagger \hat{b} + (\omega_{fb} - \omega_a)\hat{c}^\dagger c\right)}$

from the above definitions, we arrive at

$$\begin{aligned}UHU^\dagger = &\quad H + it\left[\omega_a \hat{a}^\dagger \hat{a} + \omega_{fb}\hat{b}^\dagger \hat{b} + (\omega_{fb} - \omega_a)\hat{c}^\dagger c, H\right] + ...\left(O^2\right) \\ i\dot{U}U^\dagger = &\quad i(i)\left(\omega_{fa}\hat{a}^\dagger \hat{a} + \omega_{fb}\hat{b}^\dagger \hat{b} + ...\right)e^{it\left(\omega_a \hat{a}^\dagger \hat{a} + \omega_{fb}\hat{b}^\dagger \hat{b} + ...\right)}e^{-it\left(\omega_a \hat{a}^\dagger \hat{a} + \omega_{fb}\hat{b}^\dagger \hat{b} + ...\right)} \\ i\dot{U}U^\dagger = &\quad -\left(\omega_a \hat{a}^\dagger \hat{a} + \omega_{fb}\hat{b}^\dagger \hat{b} + (\omega_{fb} - \omega_a)\hat{c}^\dagger c\right)\end{aligned} \tag{3}$$

number operators will always commute with each other, i.e. $\left[\hat{a}^\dagger \hat{a}, \hat{a}^\dagger \hat{a}\right] = 0$, and the commutator of number operators with any ladder operator is the ladder operator itself. $\left[\hat{a}^\dagger \hat{a}, \hat{a}^\dagger\right] = \hat{a}^\dagger$, $\left[\hat{a}^\dagger \hat{a}, \hat{a}\right] = -\hat{a}$

where

$$\begin{aligned}&e^{it\left(\omega_a \hat{a}^\dagger \hat{a} + \omega_{fb}\hat{b}^\dagger \hat{b} + (\omega_{fb} - \omega_a)\hat{c}^\dagger c\right)}\left(g\hat{a}\hat{b}^\dagger \hat{c} e^{it(\omega_a - \omega_{fb} + (\omega_{fb} - \omega_a))}\right)e^{-it\left(\omega_a \hat{a}^\dagger \hat{a} + \omega_{fb}\hat{b}^\dagger \hat{b} + (\omega_{fb} - \omega_a)\hat{c}^\dagger c\right)} \\ &= g\hat{a}\hat{b}^\dagger \hat{c}\end{aligned}$$

Since we can see that, $e^{\alpha\hat{c}^\dagger \hat{c}}\hat{c}e^{-\alpha\hat{c}^\dagger \hat{c}} = \hat{c} + \alpha\hat{c} + \frac{\alpha^2}{2!}\hat{c} + \frac{\alpha^3}{3!}\hat{c} + ... = \hat{c}e^\alpha$

The simplified Hamiltonian, with the unitary transformation where $U = e^{it\left(\omega_a \hat{a}^\dagger \hat{a} + \omega_{fb}\hat{b}^\dagger \hat{b} + (\omega_{fb} - \omega_a)\hat{c}^\dagger c\right)}$ can be written as,

$$\begin{aligned}\breve{H} = &\ (\omega_b - \omega_{fb})\hat{b}^\dagger \hat{b} + (\omega_c - (\omega_{fb} - \omega_a))\hat{c}^\dagger c + g\hat{a}\hat{b}^\dagger \hat{c} + g\hat{a}^\dagger \hat{b}\hat{c}^\dagger \\ &+ i\epsilon_a\left(-\hat{a} + \hat{a}^\dagger\right) + i\epsilon_{fb}\left(-\hat{b} + \hat{b}^\dagger\right)\end{aligned} \tag{4}$$

The time variation of an operator $p = \hat{a}, \hat{b}, \hat{c}$ can be written as $\frac{d}{dt}p = -i \cdot [p, H] - \kappa_p p$, therefore the coupled equations of motions of modes a, b, c are

$$\begin{aligned}\frac{d}{dt}\hat{a} &= -\kappa_a \hat{a} - ig\hat{b}\hat{c}^\dagger + \epsilon_a \\ \frac{d}{dt}\hat{b} &= -\left(i(\omega_b - \omega_{fb}) + \kappa_b\right)\hat{b} - ig\hat{a}\hat{c} + \epsilon_b \\ \frac{d}{dt}\hat{c} &= -\left(i(\omega_c - (\omega_{fb} - \omega_a)) + \kappa_c\right)\hat{c} - ig\hat{a}^\dagger \hat{b}\end{aligned} \tag{5}$$

To factor in temperature effects, we note that the resonant frequencies are dependent on temperature with a steady-state relation of,

$$\omega_{a,b,c} = \omega_{a,b,c}(T_0) + K_{a,b,c}(T_{ring} - T_0) = \omega_{a,b,c}(T_0) + K_{a,b,c}\Delta T \tag{6}$$

Where $K_{a,b,c}$ is the respective temperature coefficient of the various modes, and is approximately equal to $K_{a,b,c} = \varepsilon + \frac{\frac{dn}{dT}}{n_0}$, containing the effects of both thermal expansion (ε) and refractive index change (generally has a negative sign). The simplified differential equation (thermal diffusion, see C. Schmidt et. al. 2008 "Nonlinear thermal effects in optical microspheres at different wavelength sweeping speeds" for details) that describes the nonlinear dynamics of temperature effects can be written as

$$\frac{d}{dt}\Delta T = K_{th,a}\left|a\right|^2 + K_{th,b}\left|b\right|^2 + K_{th,c}\left|c\right|^2 - \gamma_{th}\Delta T \tag{7}$$

Where K_{th} is dependent on the absorption of mode a,b,c photons, specific heat, as well as material density. For simplification of the model, we can express ΔT as a time independent quantity. From experiments, the relaxation time is on the order of tens of microseconds.

Modes a, b, c can be considered as being "coupled" to the environment, with the following equations of motions,

$$\begin{aligned}
\frac{d}{dt}\hat{a} &= -\left(i\left(\omega_a + K_a\Delta T\right)\kappa_a\right)\hat{a} - ig\hat{b}\hat{c}^\dagger + \epsilon_a \\
\frac{d}{dt}\hat{b} &= -\left(i\left(\left(\omega_b + K_b\Delta T\right) - \omega_{fb}\right) + \kappa_b\right)\hat{b} - ig\hat{a}\hat{c} + \epsilon_b \\
\frac{d}{dt}\hat{c} &= -\left(i\left(\left(\omega_c + K_c\Delta T\right) - \left(\omega_{fb} - \omega_a\right)\right) + \kappa_c\right)\hat{c} - ig\hat{a}^\dagger\hat{b} \\
\frac{d}{dt}\Delta T &= K_{th,a}\left|a\right|^2 + K_{th,b}\left|b\right|^2 + K_{th,c}\left|c\right|^2 - \gamma_{th}\Delta T
\end{aligned} \tag{8}$$

Steady state

From equation 5, at steady state we can equate the equations of motions to 0, in the case of the cavity photon number of b mode, we perform the following algebraic manipulations

$$\begin{aligned}
a &= \frac{-igbc^* + \epsilon_a}{\kappa_a} \\
b &= \frac{-igac + \epsilon_b}{i\delta_b + \kappa_b} \\
c &= \frac{-iga^*b}{i\delta_c + \kappa_c} \\
b &= \frac{igac - \epsilon_b}{X_b} \\
c &= \frac{iga^*b}{X_c} \\
a &= \frac{-igb}{\kappa_a}\left(\frac{-igb^*a}{X_c^*}\right) + \frac{\epsilon_a}{\kappa_a} = \frac{-g^2\left|b\right|^2 a}{\kappa_a X_c^*} + \frac{\epsilon_a}{\kappa_a} \\
&= \frac{\epsilon_a X_c^*}{\kappa_a X_c^* + g^2\left|b\right|^2} \\
c &= \frac{ig\left(igcb^* + \epsilon_a\right)}{\kappa_a X_c} b = \frac{-g^2 c\left|b^2\right|}{\kappa_a X_c} + \frac{ig\epsilon_a b}{\kappa_a X_c} \\
&= \frac{ig\epsilon_a b}{\kappa_a X_c + g^2\left|b^2\right|} \\
\hat{b} &= \frac{ig}{X_b}\frac{\epsilon_a X_c^*}{\kappa_a X_c^* + g^2\left|b\right|^2}\frac{ig\epsilon_a b}{\kappa_a X_c + g^2\left|b^2\right|} - \frac{\epsilon_b}{X_b}
\end{aligned} \tag{9}$$

$$= \frac{-g^2\epsilon_a^2 X_c^*\hat{b}}{X_b\left(\kappa_a^2 X_c^* X_c + g^2\left|b\right|^2\kappa_a X_c + g^2\left|b\right|^2\kappa_a X_c^* + g^2 g^2\left|b\right|^2\left|b\right|^2\right)} - \frac{\epsilon_b}{X_b} = \frac{g^2\epsilon_a^2\kappa_c\hat{b}}{-\kappa_b\left(\kappa_a^2\kappa_c^2 - 2g^2\left|b\right|^2\kappa_a\kappa_c + g^2 g^2\left|b\right|^2\left|b\right|^2\right)} + \frac{\epsilon_b}{\kappa_b}$$

$$=> b\left(\kappa_b\left(\kappa_a^2\kappa_c^2 - 2g^2\left|b\right|^2\kappa_a\kappa_c + g^2 g^2\left|b\right|^2\left|b\right|^2\right) + g^2\epsilon_a^2\kappa_c\right) = \epsilon_b\left(\kappa_a^2\kappa_c^2 - 2g^2\left|b\right|^2\kappa_a\kappa_c + g^2 g^2\left|b\right|^2\left|b\right|^2\right)$$

$$b = \frac{\epsilon_b\left(\kappa_a^2\kappa_c^2 - 2g^2\left|b\right|^2\kappa_a\kappa_c + g^2 g^2\left|b\right|^2\left|b\right|^2\right)}{\kappa_b\left(\kappa_a^2\kappa_c^2 - 2g^2\left|b\right|^2\kappa_a\kappa_c + g^2 g^2\left|b\right|^2\left|b\right|^2\right) + g^2\epsilon_a^2\kappa_c} = \frac{\epsilon_b\left(A - B\left|b\right|^2 + C\left(\left|b\right|^2\right)^2\right)}{\kappa_b\left(A - B\left|b\right|^2 + C\left(\left|b\right|^2\right)^2\right) + D}$$

$$|b|^2 = \frac{\epsilon_b^2\left(A^2 - 2AB\,|b|^2 + \left(2AC + B^2\right)\left(|b|^2\right)^2 - 2BC\left(|b|^2\right)^3 + C^2\left(|b|^2\right)^4\right)}{D^2 + 2\kappa_b D\left(A - B\,|b|^2 + C\left(|b|^2\right)^2\right) + \kappa_b^2\left(A^2 - 2AB\,|b|^2 + (2AC + B^2)\left(|b|^2\right)^2 - 2BC\left(|b|^2\right)^3 + C^2\left(|b|^2\right)^4\right)}$$

$$= \frac{\epsilon_b^2\left(A^2 - 2AB\,|b|^2 + \left(2AC + B^2\right)\left(|b|^2\right)^2 - 2BC\left(|b|^2\right)^3 + C^2\left(|b|^2\right)^4\right)}{(D^2 + 2\kappa_b DA + \kappa_b^2 A^2) - (2\kappa_b DB + 2\kappa_b^2 AB)\,|b|^2 + (\kappa_b^2\,(2AC + B^2) + 2\kappa_b DC)\left(|b|^2\right)^2 - 2\kappa_b^2 BC\left(|b|^2\right)^3 + \kappa_b^2 C^2\left(|b|^2\right)^4}$$

$$=> 0 = \left(\epsilon_b^2 A^2 - 2\epsilon_b^2 AB\,|b|^2 + \epsilon_b^2\left(2AC + B^2\right)\left(|b|^2\right)^2 - 2\epsilon_b^2 BC\left(|b|^2\right)^3 + \epsilon_b^2 C^2\left(|b|^2\right)^4\right)$$
$$- \left(D^2 + 2\kappa_b DA + \kappa_b^2 A^2\right)|b|^2 + \left(2\kappa_b DB + 2\kappa_b^2 AB\right)\left(|b|^2\right)^2 - \left(\kappa_b^2\left(2AC + B^2\right) + 2\kappa_b DC\right)\left(|b|^2\right)^3 + 2\kappa_b^2 BC\left(|b|^2\right)^4 - \kappa_b^2 C^2\left(|b|^2\right)^5$$

From here we can gather the coefficients of the 5th order polynomial describing the steady-state solutions of the input cavity field.

$$\begin{aligned}
0 :&\epsilon_b^2 A^2 \\
1 :& - 2\epsilon_b^2 AB - \left(D^2 + 2\kappa_b DA + \kappa_b^2 A^2\right) \\
2 :&\epsilon_b^2\left(2AC + B^2\right) + \left(2\kappa_b DB + 2\kappa_b^2 AB\right) \\
3 :& - 2\epsilon_b^2 BC - \left(\kappa_b^2\left(2AC + B^2\right) + 2\kappa_b DC\right) \\
4 :&\epsilon_b^2 C^2 + 2\kappa_b^2 BC \\
5 :& - \kappa_b^2 C^2
\end{aligned}$$

where $A = \kappa_a^2\kappa_c^2, B = 2g^2\kappa_a\kappa_c, C = g^4, D = g^2\epsilon_a^2\kappa_c$

We can solve for modes c and a given the photon number of mode b

$$\begin{aligned}
|c|^2 =& \frac{g^2\epsilon_a^2\,|b|^2}{A - B\,|b|^2 + C\left(|b|^2\right)^2} \\
|a|^2 =& \frac{\kappa_c^2\epsilon_a^2}{A - B\,|b|^2 + C\left(|b|^2\right)^2}
\end{aligned} \tag{10}$$

Transmission

Due to the unitary matrix of the input-output relations, the transmitted normalized amplitiude of mode b can be written as

$$\begin{aligned}
\hat{b}_{out} =& \hat{b}_{in} - \sqrt{2\kappa_{b,1}}\hat{b} \\
=& \frac{\epsilon_b}{\sqrt{2\kappa_{b,1}}} + \sqrt{2\kappa_{b,1}}\frac{\epsilon_b\left(A - B\,|b|^2 + C\left(|b|^2\right)^2\right)}{\kappa_b\left(A - B\,|b|^2 + C\left(|b|^2\right)^2\right) + D}
\end{aligned} \tag{11}$$

with detuning,

$$b = \frac{-g^2\epsilon_a^2 X_c^* b}{X_b\left(\kappa_a^2 X_c^* X_c + g^2\,|b|^2\,\kappa_a X_c + g^2\,|b|^2\,\kappa_a X_c^* + g^2 g^2\,|b|^2\,|b|^2\right)} - \frac{\epsilon_b}{X_b}$$

$$\hat{b} = \frac{-\epsilon_b\left(\kappa_a^2 X_c^* X_c + g^2\,|b|^2\,\kappa_a X_c + g^2\,|b|^2\,\kappa_a X_c^* + g^2 g^2\,|b|^2\,|b|^2\right)}{\left(X_b\left(\kappa_a^2 X_c^* X_c + g^2\,|b|^2\,\kappa_a X_c + g^2\,|b|^2\,\kappa_a X_c^* + g^2 g^2\,|b|^2\,|b|^2\right) + g^2\epsilon_a^2 X_c^*\right)} = \frac{-\epsilon_b\left(A' + B'\,|b|^2 + C'\left(|b|^2\right)^2\right)}{\left(X_b\left(A' + B'\,|b|^2 + C'\left(|b|^2\right)^2\right) + D'\right)}$$

$$|b|^2 = \frac{\epsilon_b^2\left(A'^2 + 2A' Re\{B'\}|b|^2 + \left(2A'C' + |B'|^2\right)\left(|b|^2\right)^2 + 2Re\{B'\}C'\left(|b|^2\right)^3 + C'^2\left(|b|^2\right)^4\right)}{\alpha+\beta}$$

$$\alpha = \left(|D|^2 + 2Re\{X_b D'^*\}A' + |X_b|^2 A'^2\right) + \left(2Re\{X_b B' D'^*\} + 2|X_b|^2 A' Re\{B'\}\right)|b|^2$$

$$\beta = \left(2Re\{X_b D'^*\}C' + |X_b|^2\left(2A'C' + |B'|^2\right)\right)\left(|b|^2\right)^2 + 2|X_b|^2 Re\{B'\}C'\left(|b|^2\right)^3 + |X_b|^2 C'^2\left(|b|^2\right)^4$$

$$\begin{aligned}
0 :& \epsilon_b^2 A'^2\\
1 :& -|D|^2 - 2Re\{X_b D'^*\}A' - |X_b|^2 A'^2 + 2\epsilon_b^2 A' Re\{B'\}\\
2 :& 2\epsilon_b^2 A'C' + \epsilon_b^2|B'|^2 - 2Re\{X_b B' D'^*\} - 2|X_b|^2 A' Re\{B'\}\\
3 :& 2\epsilon_b^2 Re\{B'\}C' - 2Re\{X_b D'^*\}C' - |X_b|^2\left(2A'C' + |B'|^2\right)\\
4 :& \epsilon_b^2 C'^2 - 2|X_b|^2 Re\{B'\}C'\\
5 :& -|X_b|^2 C'^2
\end{aligned}$$

$$\hat{b}_{out} = \frac{\epsilon_b}{\sqrt{2\kappa_{b,1}}} - \sqrt{2\kappa_{b,1}}\frac{-\epsilon_b\left(A' + B'|b|^2 + C'\left(|b|^2\right)^2\right)}{\left(X_b\left(A' + B'|b|^2 + C'\left(|b|^2\right)^2\right) + D'\right)} \tag{12}$$

Where $X_c = -i\left(\omega_c - (\omega_{fb} - \omega_a)\right) - \kappa_c,\ X_b = -i\left(\omega_b - \omega_{fb}\right) - \kappa_b,\ A' = \kappa_a^2 X_c^* X_c,\ B' = 2g^2\kappa_a X_c,\ C' = g^4,\ D' = g^2\epsilon_a^2 X_c^*,\ \epsilon_b = \sqrt{2\kappa_{b,1}\frac{P_b}{\hbar\omega_b}}$. Therefore, the transmitted power is

$$\begin{aligned}
P_{b,out} &= \left|\hat{b}_{out}\right|^2\hbar\omega_b = P_b\left|1 + \frac{2\kappa_{b,1}\left(A' + B'|b|^2 + C'\left(|b|^2\right)^2\right)}{\left(X_b\left(A' + B'|b|^2 + C'\left(|b|^2\right)^2\right) + D'\right)}\right|^2\\
T &= \left|1 + \frac{2\kappa_{b,1}\left(A' + B'|b|^2 + C'\left(|b|^2\right)^2\right)}{\left(X_b\left(A' + B'|b|^2 + C'\left(|b|^2\right)^2\right) + D'\right)}\right|^2
\end{aligned} \tag{13}$$