3-SAT

Let us repeat what we had done for 2-SAT to 3-SAT.

3-SAT Algorithm:

1. Start with an arbitrary truth assignment. ← Say S
2. Repeat up to m times, terminating if all clauses are satisfied:
   (a) Choose an arbitrary clause that is not satisfied.
   (b) Choose one of the literals uniformly at random, and change the value of the variable in the current truth assignment.
3. If a valid truth assignment has been found, return it.
4. Otherwise, return that the formula is unsatisfiable.

A_i := Truth assignment at time i/ step i.

As before let X_i be the r.v that denotes the no. of vars having the same value in A_i and S.

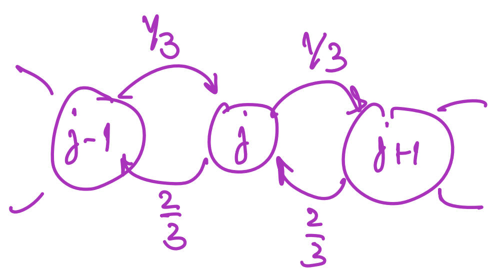
$$\Pr[X_{i+1} = 1 \mid X_i = 0] = 1.$$

$$1 \le j \le n-1 \quad \begin{cases} \Pr[X_{i+1} = j+1 \mid X_i = j] \geq \frac{1}{3} \\ \Pr[X_{i+1} = j-1 \mid X_i = j] \leq \frac{2}{3} \end{cases}$$

Only var. disagrees
2 var
3 vars

Let $Y_0 = X_0, Y_1, \ldots$ be the "pessimistic version" of r.vs $X_0, X_1, \ldots$ such that

$$1 \le j \le n-1 \quad \begin{cases} \Pr[Y_{i+1} = j+1 \mid Y_i = j] = \frac{1}{3} \\ \Pr[Y_{i+1} = j-1 \mid Y_i = j] = \frac{2}{3} \end{cases}$$



$h_j = \mathbb{E}[Z_j]$.

Recall that h_j is the expected no. of steps to reach n starting from j. Z_j: r.v denoting no. of steps from j

start with an assignment that agrees with S in j locations.

Recall that $h_n = 0$ and $h_0 = h_1 + 1$. Further, $1 \le j \le n-1$

$$\mathbb{E}[Z_j] = \frac{1}{3}\left(1 + \mathbb{E}[Z_{j+1}]\right) + \frac{2}{3}\left(1 + \mathbb{E}[Z_{j-1}]\right) \qquad Z_j \xrightarrow{1/3} 1 + Z_{j+1}, \quad Z_j \xrightarrow{2/3} 1 + Z_{j-1}$$

$$\Rightarrow \quad h_j = \frac{h_{j+1}}{3} + \frac{2 \cdot h_{j-1}}{3} + 1.$$

$$3 h_j = h_{j+1} + 2 h_{j-1} + 3$$

$h_j = 2^{n+2} - 2^{j+2} - 3(n-j)$ is a unique solution.

$2^{n+2} \quad 2^{n-1+2} \quad 2^{n+1}$

Note that $h_j = \Theta(2^n)$. "Almost searching the whole solution space".

$\left(\frac{4}{3}\right)^n \cdot n^{O(1)}$?

Qn: What if we start w/ an assignment chosen uniformly at random?

Modified 3-SAT Algorithm:

Yet to fix.

1. Repeat up to m times, terminating if all clauses are satisfied:
   - **(a)** Start with a truth assignment chosen uniformly at random.
   - **(b)** Repeat the following up to $3n$ times, terminating if a satisfying assignment is found:
     - **i.** Choose an arbitrary clause that is not satisfied.
     - **ii.** Choose one of the literals uniformly at random, and change the value of the variable in the current truth assignment.
2. If a valid truth assignment has been found, return it.
3. Otherwise, return that the formula is unsatisfiable.

Let q, be the prob that we reach the satisfying assign-ment in $\leq 3n$ steps starting w/ an assignment chosen uniformly at random.

q_j is the lower bound on prob that we reach S starting from an assignment that disagrees w/ S in j locations.

Consider a particle moving on an integer line, w.p $\frac{1}{3}$ moving up by 1 and w/p $\frac{2}{3}$ moving down by 1.

$$\Pr[\text{Exactly } k \text{ moves down and } k+j \text{ moves up in } j+2k \text{ moves}]$$

$$= \binom{j+2k}{k} \left(\frac{2}{3}\right)^k \left(\frac{1}{3}\right)^{k+j}$$

Stirling's approx.

$$\sqrt{2\pi m}\left(\frac{m}{e}\right)^m \leq m! \leq 2\sqrt{2\pi m}\left(\frac{m}{e}\right)^m$$

Therefore, Prob of reaching a sat. assgn within $j+2k \leq 3n$ steps, starting from an assignment that disagrees with S in j locations is as follows.

$$q_j \geq \max_{k=0,\dots,j} \left\{ \binom{j+2k}{k} \left(\frac{2}{3}\right)^k \left(\frac{1}{3}\right)^{j+k} \right\}$$

$$\geq \binom{3j}{j} \left(\frac{2}{3}\right)^j \left(\frac{1}{3}\right)^{2j}$$

$$\geq \frac{\sqrt{3}}{8\sqrt{\pi j}} \left(\frac{27}{4}\right)^j \cdot \frac{2^j}{27^j}$$

$$\binom{3j}{j} = \frac{(3j)!}{(2j)!\, j!}$$

$$\geq \frac{\sqrt{2\pi(3j)} \cdot \left(\frac{3j}{e}\right)^{3j}}{4\sqrt{2\pi(2j)}\left(\frac{2j}{e}\right)^{2j} \cdot \sqrt{2\pi j}\left(\frac{j}{e}\right)^j}$$

$$\approx \frac{c}{\sqrt{j}} \cdot \frac{1}{2^j} \text{ where } c = \frac{\sqrt{3}}{8\sqrt{\pi}} \approx 0.1222..$$

$$\Rightarrow q \geq \sum_{j=0}^{n} q_j \cdot \Pr\left[\begin{array}{c}\text{a random assignment disagrees w/ } S \\ \text{in } j \text{ locations}\end{array}\right]$$

$$\geq \frac{1}{2^n} + \sum_{j=1}^{n} \binom{n}{j} \left(\frac{1}{2}\right)^n \cdot \frac{c}{\sqrt{j}} \cdot \frac{1}{2^j}$$

$$\geq \frac{c}{\sqrt{n}} \cdot \left(\frac{1}{2}\right)^n \cdot \underbrace{\sum_{j=0}^{n} \binom{n}{j} \cdot \frac{1}{2^j}}_{\left(1+\frac{1}{2}\right)^n}$$

$$= \frac{c}{\sqrt{n}} \left(\frac{1}{2}\right)^n \cdot \left(\frac{3}{2}\right)^n$$

$$= \frac{c}{\sqrt{n}} \cdot \left(\frac{3}{4}\right)^n .$$

Prob of finding a solution in $3n$ steps $\geq \frac{c}{\sqrt{n}} \left(\frac{3}{4}\right)^n$.

$\Rightarrow$ No. of trials needed $= \frac{1}{q} \leq \left(\frac{4}{3}\right)^n \cdot \frac{\sqrt{n}}{c}$.

and total no. of steps needed $\leq \left(\frac{4}{3}\right)^n \cdot \frac{\sqrt{n}}{c} \cdot 3n$.

$$= O\left(n^{3/2} \cdot \left(\frac{4}{3}\right)^n\right).$$

2^n
$\downarrow$
$1.3\cancel{4}3^n$

Repeat this r times to boost the probability to a value close to 1.

$$\Rightarrow m \approx O\left(n^{3/2} \cdot r \cdot \left(\frac{4}{3}\right)^n\right).$$

2^n $\quad (\frac{4}{3})^n \rightarrow 2^{n \log_2 \frac{4}{3}}$

Running $\quad 2^{O(n)}$

Fixed Parameter Tractable:
= Solvable.

Given a promise on the structure of the instance and parameter k that quantifies the structure, algorithms run in time

$$2^{O(k)} \cdot \text{poly}(n)$$

$$\binom{n}{k} \cdot \text{poly}(n)$$

$$n^k \, \text{poly}(n)$$