Previous lecture: Chernoff bound.

Remark: For Chernoff bound, X is a sum of independent Poisson trials.

Question: What about dependent r.vs?

Let us revisit Balls and Bins problem. Recall that the r.vs related to bin loads are dependent.

3.2 Martingales.

A sequence of random variables $X_0, X_1, \dots,$ is said to be a martingale sequence if for all $i > 0$,

$$E[X_i \mid X_0, \dots, X_{i-1}] = X_{i-1}. \qquad E[X_i \mid Y_0, Y_1, \dots, Y_{i-1}] = X_{i-1}.$$

$\hookrightarrow E[X_i \mid X_{i-1}] = X_{i-1}.$

Example: A gambler w/ initial capital X_0 and capital after i^{th} bet X_i play a fair game s.t expected gain/loss from each bet is zero. Then $X_0, X_1, \dots$ is a martingale.

Conditional distribution and expectations.

Let X be a r.v and E be any event that occurs w nonzero probability. The conditional density function of X given E is given by $Pr[X = x \mid E]$. Further E can be an event that another r.v Y takes a particular value.

We denote the joint density function of X and Y by $p(x,y)$ and

$$\Pr[X=x \mid Y=y] = \frac{p(x,y)}{\Pr[Y=y]} = \frac{p(x,y)}{\sum_x p(x,y)}$$

Conditional expectation of X given $Y=y$ is

$$\mathbb{E}[X \mid Y=y] = \frac{\sum_x x \cdot p(x,y)}{\sum_x p(x,y)} .$$

$$\mathbb{E}[X \mid Y=y] = \sum x \cdot \Pr[X=x \mid Y=y]$$

Remark: $\mathbb{E}[X|Y]$ is a r.v that is defined to be equal to $f(Y)$ and $f(y) = \mathbb{E}[X \mid Y=y]$.

Lemma: $\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X]$.

$$\mathbb{E}[X_i] = \mathbb{E}[\mathbb{E}[X_i \mid X_0, \dots, X_{i-1}]] = \mathbb{E}[X_{i-1}] = \dots = \mathbb{E}[X_0].$$

Azuma's inequality:

Let $X_0, X_1, \dots$ be a martingale sequence s.t for every k, $|X_k - X_{k-1}| \leq c_k$ where c_k may depend on k.

Then for all $t \geq 0$ and any $\lambda > 0$,

$$\Pr[|X_t - X_0| \geq \lambda] \leq 2 \cdot \exp\left(-\frac{\lambda^2}{2 \sum_{k=1}^{t} c_k^2}\right).$$

If all c_k are equal and are independent of k,

$$\Pr[|X_t - X_0| \geq \lambda c \sqrt{t}] \leq 2 \cdot \exp\left(-\frac{\lambda^2}{2}\right).$$

$\mathbb{E}[Z] = n \cdot \left(1-\frac{1}{n}\right)^m$.

Concentration bound for the no. of empty bins:

Let Z be the no. of empty bins after m balls are thrown into n bins. Let $X_1, X_2, \ldots$ denote the random choices made by balls $1, 2, \ldots$.

Let $Z_t = \mathbb{E}[Z \mid X_1, \ldots, X_t]$ denote the conditional expectation of r.v Z at time t.

$\mathbb{E}[Z] = n\left(1-\frac{1}{n}\right)^m \sim n \cdot e^{-m/n}$.

Observation: $Z_0 = \mathbb{E}[Z]$ and $Z_m = Z$.

Claim: $Z_0, Z_1, \ldots$ form a martingale.

That is $\mathbb{E}[Z_t \mid X_1, \ldots, X_t] = Z_{t-1}$ for all $t > 0$.

Let $Z(Y,t)$ be defined as the expectation of Z given that Y bins are empty at time t. ($X_1, \ldots, X_t$ give the value of Y at time t. Let us denote it by Y_t).

$$Z(Y,t) = \mathbb{E}[Z \mid Y \text{ bins are empty at } t]$$
$$= Y \left(1 - \frac{1}{n}\right)^{m-t}.$$

$$Z_{t-1} = Z(Y_{t-1}, t-1) = Y_{t-1} \cdot \left(1-\frac{1}{n}\right)^{m-t+1}.$$

Let us look at how Z_t can be computed. Firstly $Y_t = Y_{t-1}$ with a prob of $1 - \frac{Y_{t-1}}{n}$ and $Y_t = Y_{t-1} - 1$ w/p

$\frac{Y_{t-1}}{n}$.

t^{th} ball $\xrightarrow{w.p}$ $1-\frac{Y_{t-1}}{n}$ goes to nonempty bin

$\xrightarrow{w.p}$ $\frac{Y_{t-1}}{n}$ goes to empty bin.

Case when $Y_t = Y_{t-1}$:

$1-\frac{Y_{t-1}}{n}$ $$Z_t = z(Y_t, t) = Z(Y_{t-1}, t) = Y_{t-1}\left(1-\frac{1}{n}\right)^{m-t}.$$

$\hookrightarrow Z_t - Z_{t-1}$

Case when $Y_t = Y_{t-1} - 1$:

$\frac{Y_{t-1}}{n}$ $$Z_t = Z(Y_t, t) = (Y_{t-1} - 1)\left(1-\frac{1}{n}\right)^{m-t}.$$

$\hookrightarrow Z_t - Z_{t-1}$

$$\begin{aligned} E[Z_t \mid Y_{t-1}] &= \left(1-\frac{Y_{t-1}}{n}\right)\cdot Y_{t-1}\cdot\left(1-\frac{1}{n}\right)^{m-t} \\ &\quad + \left(\frac{Y_{t-1}}{n}\right)(Y_{t-1}-1)\left(1-\frac{1}{n}\right)^{m-t} \\ &= Y_{t-1}\left(1-\frac{1}{n}\right)^{m-t}\left[1-\frac{Y_{t-1}}{n}+\frac{Y_{t-1}}{n}-\frac{1}{n}\right] \\ &= Y_{t-1}\left(1-\frac{1}{n}\right)^{m-t+1} = Z_{t-1}. \end{aligned}$$

Thus, $Z_0, Z_1, \ldots$ form a martingale.

Claim: $|Z_t - Z_{t-1}| \le \left(1-\frac{1}{n}\right)^{m-t}$ ✓

From this, we get that $c_t = \left(1-\frac{1}{n}\right)^{m-t}$. Using Azuma's ineq. we get the following:

$$\Pr\left[|Z - \mathbb{E}[Z]| \geq \lambda\right] = \Pr\left[|Z_m - Z_0| \geq \lambda\right]$$

$$\leq 2 \cdot \exp\left(-\frac{\lambda^2}{2\sum_{t=1}^{m} C_t^2}\right) \cdot \Big\} *$$

w.p $(1 - *)$ r.v Z is concentrated in the λ- radius of $\mathbb{E}[Z]$.

Reference:

1. Kamath, Motwani, Palem, Spirakis "Tail Bounds for Occupancy and Sat. threshold conj". 1994.

2. Section 4.3 of Motwani - Raghavan.