Maximum Satisfiability

Satisfiability (SAT): Given a CNF formula with m clauses over n variables, decide if there exists an assignment to these variables that satisfies all the clauses.

NP-complete

Clause: OR of variables or their negations.

Ex: $(x_1 \vee x_2 \vee x_3)$; $(x_1 \vee \bar{x}_2 \vee x_3)$; ...

CNF: Conjunctive Normal Form (or) "Conjunction" of clauses (or) AND of ORs

Ex: $(x_1 \vee x_2) \wedge (\bar{x}_1 \vee \bar{x}_2)$; $(x_1 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee x_2)$; ...

Maximum Satisfiability (MAXSAT).: Given a set of m clauses over n-variables, determine the max number of clauses that can be satisfied.

NP-hard

Question: Can randomness help figure out the max number of clauses that can be satisfied.

CNF : $C_1 \wedge C_2 \wedge \cdots \wedge C_m$; $C_i : \bigvee_{j=1}^{k} x_{ij}$

Attempt 1: Set each variable to T or F with equal probability.

That is $\forall i \in [n]$; $x_i \rightarrow T$ w.p $1/2$ and $x_i \rightarrow F$ otherwise.

Theorem: Given any m clauses, there is a truth assignm-ent for the variables that satisfies at least $\frac{m}{2}$ clauses.

Proof: Let the truth values to the variables be assigned with equal probability (as in attempt 1). For all $i \in [m]$, let Z_i be a r.v s.t

$$Z_i = \begin{cases} 1 & \text{if clause } i \text{ is satisfied} \\ 0 & \text{otherwise.} \end{cases}$$

$\underline{x_1 \vee \bar{x}_2 \vee \cdots \vee \bar{x}_k} \}$

Let k be the no. of variables in clause i. Then

$Pr[Z_i = 1] = 1 - 2^{-k}$. // Only one of the 2^k possible assigments makes the clause not satisfiable.

Note that $k \geq 1$. Thus, $Pr[Z_i = 1] \geq \frac{1}{2}$.

From the above definition, we get that the expected no. of satisfied clauses is $\mathbb{E}[\sum_{i=1}^{m} Z_i] = \sum_{i=1}^{m} \mathbb{E}[Z_i] \geq \frac{m}{2}$.

We now claim that there is a ~~random~~ assignment that gives us at least $\frac{m}{2}$ satisfied clauses. Let $Z = \sum_{i=1}^{m} Z_i$.

Good event $Z \geq \mathbb{E}[Z]$

$$Pr[\underbrace{Z < \mathbb{E}[Z]}_{\text{Bad event}}] = Pr[m - Z > m - \mathbb{E}[Z]] < \frac{\mathbb{E}[m-Z]}{m - \mathbb{E}[Z]} = 1$$

Thus with prob that is strictly non-zero, there exists an assignment that satisfies at least $\frac{m}{2}$ clauses.

Remark: Proofs of the above form are generally refered to as Probabilistic Method.

"Showing existence of an object with a non-zero prob".

Approximation algorithms.

Given an instance I, let $m_*(I)$ be the maximum no. of clauses that can be satisfied. Let $m_A(I)$ be the no. of clauses satisfiable by algorithm A.

Let the performance ratio of the algorithm A be defined as follows:

$$\text{PerfRatio}(A) = \inf_I \frac{m_A(I)}{m_*(I)}.$$

If PerfRatio(A) is α, then A is an α-approximation algorithm. // α is also called approximation factor.

Ex: The above algorithm (Attempt 1) is a $\frac{1}{2}$-approx algo for MAX-SAT.

Question: Can we improve the approximation factor to $\frac{3}{4}$ instead of $\frac{1}{2}$ as was shown above?

Observation: If every clause had at least k literals, then each clause is satisfied with a prob of $1-2^{-k}$.

For $k=2$, we already have a $\frac{3}{4}$-approx algo.

Problematic instances: Clauses with single literals.

Attempt 2: LP relaxations and randomized rounding.

Step 1: Form an Integer program for MAX-SAT.

For all $j \in [m]$, z_j is a var associated w/ clause j s.t $z_j = 1$ if clause j is satisfied and $z_j = 0$ otherwise.

Similarly, for all $i \in [n]$, y_i be a var associated with x_i s.t $y_i = 1$ if x_i is set to True and $y_i = 0$ otherwise.

Let C_j^+ be the set of literals that appear unnegated in C_j and C_j^- be the set of literals that appear neg-ated.

$$\text{maximize} \quad \sum_{j=1}^{m} z_j$$

Subject to

[0,1]

1. y_i and $z_j \in \{0,1\}$ $(\forall\, i \in [n]$ and $j \in [m])$

2. $\sum_{i \in C_j^+} y_i + \sum_{i' \in C_j^-} (1 - y_{i'}) \geq z_j \quad (\forall\, j \in [m])$.

$y_j \sim 0.56$

Step 2: Relax the Integer program to a Linear program.

Constraint 1': y_i and $z_j \in [0,1]$ $(\forall\, i \in [n]$ and $j \in [m])$

Let $\hat{y}_i$ ($\forall\, i \in [n]$) and $\hat{z}_j$ ($\forall\, j \in [m]$) be the solutions obtained from LP. Note that $\sum_{j=1}^{m} z_j \leq \sum_{j=1}^{m} \hat{z}_j$.

Step 3: Randomized rounding.

$\hat{y}_i = 0.75$

$y_i \to 1$ w.p

$$\forall\, i \in [n]; \quad y_i = \begin{cases} 1 & \text{w.p } \hat{y}_i \Rightarrow x_i = \text{True} \\ 0 & \text{otherwise} \Rightarrow x_i = \text{False} \end{cases}$$

and

$$\forall\, j \in [m]; \quad z_j = \begin{cases} 1 & \text{w.p } \hat{z}_j \Rightarrow \text{Clause } j \text{ is satisfied} \\ 0 & \text{otherwise} \Rightarrow \text{clause } j \text{ is not sat.} \end{cases}$$

Lemma: Let the clause C_j have k literals. The prob that C_j is satisfied by randomized rounding is at least $\beta_k \hat{z}_j$ where $\beta_k = 1 - \left(1 - \frac{1}{k}\right)^k$. $\approx 1 - e^{-1}$

By assuming Lemma, we can show the following.
Recall that Z_j is an Indicator r.v for the event that the clause C_j is satisfied. From the lemma we get that

$$\mathbb{E}[Z_j] = \Pr[Z_j = 1] \geq \beta_k \cdot \hat{z}_j .$$

Claim: $\sum_{j=1}^{m} \hat{z}_j \geq \sum_{j=1}^{m} z_j$

By linearity, $\mathbb{E}\left[\sum_{j=1}^{m} Z_j\right] \geq \sum_{j=1}^{m} \beta_k \hat{z}_j \geq (1 - e^{-1}) \sum_{j=1}^{m} \hat{z}_j$.

$$\geq (1 - e^{-1}) \sum_{j=1}^{m} z_j$$

Proof of Lemma: W.L.O.G let $C_j = x_1 \vee x_2 \vee \dots \vee x_k$.

From the linear program, we have $\sum_{i=1}^{k} \hat{y}_i \geq \hat{z}_j$.

Note that C_j is not satisfied if all variables y_i are rounded to zero.

$\Pr[\text{all vars } y_i \text{ are rounded to zero}]$

$$= \prod_{i=1}^{k} (1-\hat{y}_i)$$

$$\prod_{i=1}^{k} (1-\hat{y}_i) \leq \left(\frac{\sum_{i=1}^{k} (1-\hat{y}_i)}{k} \right)^k$$

$$\Rightarrow \Pr[C_j \text{ is satisfied}] = 1 - \prod_{i=1}^{k} (1-\hat{y}_i)$$

$$\geq 1 - \left[\frac{\sum_{i=1}^{k} (1-\hat{y}_i)}{k} \right]^k \qquad // \text{ AM-GM inequality.}$$

$$\left(\frac{\sum_{i=1}^{k} x_i}{k} \right)^k \geq x_1 \cdots x_k$$

$$\geq 1 - \left[\frac{k-\hat{z}_j}{k} \right]^k \qquad // \left\{ \sum_{i=1}^{k} \hat{y}_i \geq \hat{z}_j \right\}$$

$$= 1 - \left(1 - \frac{\hat{z}_j}{k}\right)^k$$

$$\geq \left[1 - \left(1-\frac{1}{k}\right)^k \right] \hat{z}_j \qquad // \text{ From the claim below.}$$

$$= \beta_k \cdot \hat{z}_j .$$

Claim: $f(x) = 1 - \left(1 - \frac{x}{k}\right)^k$ and $g(x) = \beta_k x$. Then $f(x) \geq g(x)$ $\forall\, x \in [0,1]$.

$$1 - \left(1 - \frac{1}{k}\right)^k$$