2-SAT

Recall that 2-SAT is a 2-CNF over n-variables s.t

$$F = C_1 \wedge C_2 \wedge \cdots \wedge C_m \quad \text{and} \quad C_i = L_{i_1} \vee L_{i_2} \quad \forall\, i \in [m].$$

We want to assign $\{T, F\}$ assignments to the variables s.t F is satisfied. That is, all clauses are satisfied.

Remark: Not every 2-SAT instance is satisfiable.

Ex: $$(x_1 \vee x_2) \wedge (\bar{x}_1 \vee x_2) \wedge (\bar{x}_2 \vee x_1) \wedge (\bar{x}_1 \vee \bar{x}_2).$$

Algo for 2-SAT:

1. Start with an arbitrary truth assignment.
2. Repeat until all clauses are satisfied or at most $2mn^2$ many iterations:
   - Choose an arbitrary clause that is not sat.
   - Choose a literal at random and flip the value of the variable corresponding to the literal.
3. If a valid truth assignment was found, return that assignment, else return UNSAT.

Question: What is the max-value of $\underline{m}$?

$\underline{O(n^2)}$

$A_0 \quad (a_1, a_2, \ldots, a_n)$

$\swarrow \tfrac{1}{2} \qquad \searrow \tfrac{1}{2}$

$A_1 \quad (\bar{a}_1, a_2, \ldots, a_n) \quad (a_1, \bar{a}_2, \ldots, a_n)$

Correctness and guarantees of the algo.

$S = (S_1, \dots, S_n)$.

Let S be a satisfying assignment for the n-variables.

A_i := Assignment to the variables after step i.

X_i := No. of variables having the same values in A_i and S.

$$X_i := \left| \{ j \mid A_i(x_j) = S(x_j) \} \right|.$$

If $X_i = n$ then $A_i = S$ and that is a satisfying assgn.

Question: How long does it take to reach S? (does it depend on where you start?).

Obs: $\Pr[X_{i+1} = 1 \mid X_i = 0] = 1.$

$(A_i)_j \neq S_j$.

A_i and S disagree everywhere.

j^{th}

A_{i+1} $\quad (A_{i+1})_j = S_j$

Suppose $1 \leq X_i \leq n-1$.

Let C be a clause that is not satisfied by A_i.

$A_i = \begin{matrix} x_1 \to F \\ x_2 \to T \end{matrix}$

Claim: At least one of the two variable assignments is inconsistent with S.

$\begin{cases} A_i(x_1) \neq S(x_1) \\ A_i(x_2) = S(x_2) \end{cases}$ $\quad L_1 \vee L_2 \quad S(x_2) = T$

$x_1 \vee \bar{x}_2$

Pf: If both are consistent then C would have been satisfied as S satisfies all the clauses. $\square$

$\to A_i(x_1) \neq S(x_1)$

$\to A_i(x_2) \neq S(x_2)$

Now, if we randomly pick one of the two vars and flip, prob that X_{i+1} increases is $\geq \frac{1}{2}$.

x_1

$A_{i+1}(x_1) = S(x_1)$

$A_{i+1}(x_2) = S(x_2)$

process

$X_{(?)} = n$

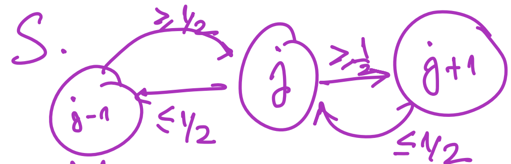
Random

$$\Pr[X_{i+1} = j+1 \mid X_i = j] \geq \tfrac{1}{2}.$$

$$\Rightarrow \Pr[X_{i+1} = j-1 \mid X_i = j] \leq \tfrac{1}{2}.$$

$\frac{1}{2}$ 1 $\frac{1}{2}, 0$

It depends whether two variables or one variable and is inconsistent w/ S.



Remark: $X_0, X_1, \dots$ need not be a Markov chain!

Let us now look at a related Markov chain over the r.vs $Y_0, Y_1, \dots$.

$$Y_0 = X_0$$

$$\to \Pr[Y_{i+1} = 1 \mid Y_i = 0] = 1.$$

RP2

$$\begin{cases} \Pr[Y_{i+1} = j+1 \mid Y_i = j] = \frac{1}{2}. \\ \Pr[Y_{i+1} = j-1 \mid Y_i = j] = \frac{1}{2}. \end{cases}$$

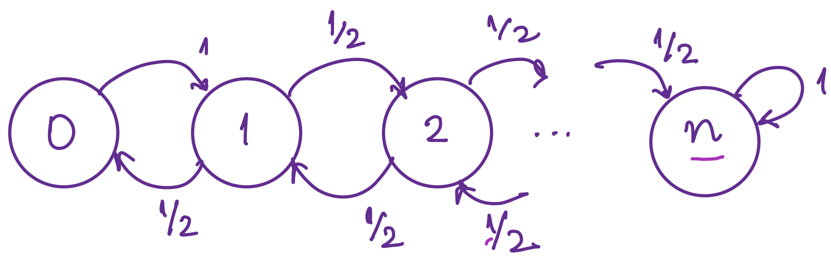
$Y_i \simeq n$

Expected time to reach n is larger in Y than X

$\Downarrow$

E[Steps to reach n in rp1] $\leq$ E[Steps to reach n in rp2]

$j \xrightarrow{\geq 1/2} j+1$



Vertices: $\{0, 1, \dots, n\}$.

We are interested in E[reach n in X] $\leq$ E[reach n in Y]

Transition matrix P:

$$P_{ij} = \begin{cases} 1/2 & \text{if } j = i+1 \text{ or } i-1 \ \forall\, 1 \leq i \leq n-1, \\ 1 & \text{if } i = 0 \text{ and } j = 1, \\ 1 & \text{if } i = j = n, \\ 0 & \text{otherwise.} \end{cases}$$

Let Z_j be a r.v representing the no. of steps to reach n from j. Let $h_j = \mathbb{E}[Z_j]$.

Expected no. of time steps needed to reach sat assgn if we start from an assgn that agrees w/ S at j loc.

From the above discussion,

$$h_n = 0, \text{ and}$$

$$h_0 = h_1 + 1.$$

Further $h_j \geq$ expected no of steps to fully agree with S starting from A_0 which agrees with S in j locations.

$$Z_j = \begin{cases} 1 + Z_{j-1} & \text{w.p } \frac{1}{2}, \text{ and} \\ 1 + Z_{j+1} & \text{w.p } \frac{1}{2}. \end{cases}$$

$$\Rightarrow \underbrace{\mathbb{E}[Z_j]}_{h_j} = \frac{1}{2}\left(1 + \underbrace{\mathbb{E}[Z_{j-1}]}_{h_{j-1}}\right) + \frac{1}{2}\left(1 + \underbrace{\mathbb{E}[Z_{j+1}]}_{h_{j+1}}\right).$$

$$\Rightarrow 2h_j = h_{j+1} + h_{j-1} + 2$$

$$\Rightarrow h_{j+1} - h_j = (h_j - h_{j-1}) - 2.$$

$$\sum_{j=1}^{k} (h_{j+1} - h_j) = \sum_{j=1}^{k}\left((h_j - h_{j-1}) - 2\right)$$

$$\left.\begin{aligned} h_{k+1} - h_1 &= h_k - h_0 - 2k \\ h_{k+1} &= h_k + (h_1 - h_0) - 2k \end{aligned}\right\} \Rightarrow \left.\begin{aligned} h_{k+1} &= h_k - 2k - 1. \\ h_k - h_{k+1} &= 2k+1. \end{aligned}\right\}$$

$$\sum_{k=0}^{n-1}(h_k - h_{k+1}) = \sum_{k=0}^{n-1}(2k+1)$$

$h_0 - h_n$

$= h_0$

$h_n = 0$

$h_{n-1} = (2(n-1)+1) + \cancel{h_n}^{0}$

$h_{n-1} = 2(n-1)+1$

$h_{n-2} = h_{n-1} + (2(n-2)+1)$

$$= 2\cdot\frac{n(n-1)}{2} + n = n^2 - n + n$$

$$= \underline{\underline{n^2}}.$$

$h_n \to h_0$

Expected no. of steps to find a sat. assignment is $\leq n^2$.

$\geq \frac{1}{2}$, $\leq 2n^2$ steps are needed.

$\hookrightarrow 1-\frac{1}{2m}$ $\leq 2mn^2$ steps.

By Markov's inequality,

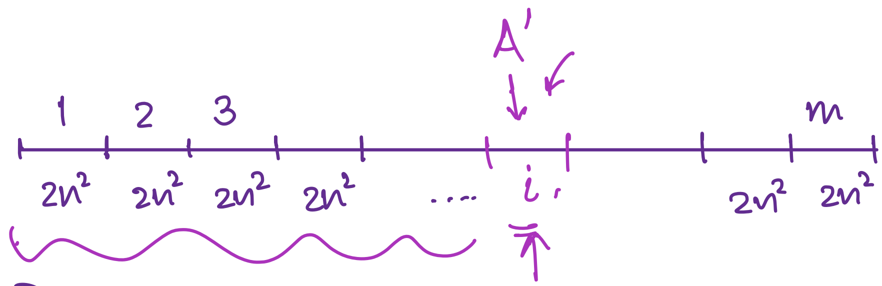
$$\Pr[Z_0 > 2n^2] \leq \frac{E[Z_0]}{2n^2} = \frac{h_0}{2n^2} = \frac{n^2}{2n^2} = \frac{1}{2}.$$

$> 2mn^2$ $\qquad < \frac{1}{2m}$

We can repeat this experiment a few times to reduce error.

$\hookrightarrow 1 - \frac{1}{2^m}$ instead of $1-\underline{\frac{1}{2m}}$

Let us split the total running time of $2\underline{m}n^2$ into m blocks of length $2n^2$ each.



Claim: Prob of failure in each segment is at most $\frac{1}{2}$ conditioned on no satisfying assignment was found in the previous segments.

For Markov chains, the history does not matter.

At time $(2n^2)\cdot i+1$ if no satisfying assignment was found before and it is at a state x_j, it is as if a new trial for segment $(i+1)$ is taking place.

$\Rightarrow$ Prob that we need $> 2n^2$ steps to reach a satisfying assignment is at most $\frac{1}{2}$.

$\Rightarrow$ W.p of at least $1 - \frac{1}{2^m}$, we can find a satisfying assignment in at most $2mn^2$ steps.

No. of steps needed $> 2mn^2$ if and only if, failure was seen in all m blocks before.

$\downarrow$

$\left(\frac{1}{2}\right)^m$.

Say $m = n$ then runtime $= 2n^3 = O(n^3)$. w.succ. prob $1 - \frac{1}{2^n}$.

$m = n^2$ $\quad O(n^4)$ $\quad 1 - \frac{1}{2^{n^2}}$.