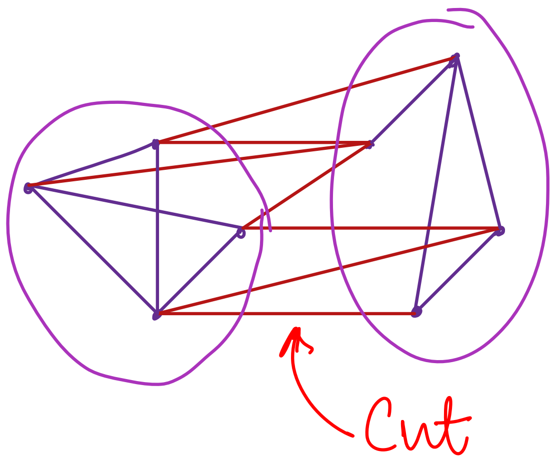
MAXCUT : Finding a large cut.

Recall that a cut is a partition of vertices into 2 disjoint sets and value of the cut is the weight of all edges crossing the cut.



If wt of each edge is equal to 1, then wt. of this cut is 7.

Maximum cut: Given a graph $G=(V,E)$ such that all (MAXCUT) the wt of edges in $E = 1$, find the maximum value of a cut attainable in G.

$OPT = e^*$

α-approx $\Rightarrow$ algo should output $\geq \alpha \cdot e^*$

Remark: Similar to the case of MAXSAT, MAXCUT is NP hard.

Question: Can we approach this question using randomness?

Attempt 1: Let a set A be a random set constructed by sampling each element of V and adding it to the set with prob $1/2$ and left out otherwise.

Let $B = V \setminus A$.

Theorem: Let $G=(V,E)$ be an undirected graph on n vertices and m edges. There exists a partition of V into disjoint sets A and B s.t at least $\frac{m}{2}$ edges connect a vertex in A to a vertex in B. That is, there is a cut of value at least $\frac{m}{2}$.

Proof: Let sets A and B be as constructed above in Attempt 1. Let $e_1, \dots, e_m$ be the enumeration of edges in G.

For all $i \in [m]$, let X_i be a Bernoulli r.v s.t

$$X_i = \begin{cases} 1 & \text{if edge } i \text{ connects } A \text{ to } B \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}
\Pr[X_i = 1] &= \Pr[\text{End points of edge } e_i \text{ are in distinct sets}] \\
&= \Pr[(u_i \in A \wedge v_i \in B) \vee (u_i \in B \wedge v_i \in A)] \\
&= \Pr[(u_i \in A \wedge v_i \in B)] + \Pr[(u_i \in B \wedge v_i \in A)] \\
&= \Pr[u_i \in A] \cdot \Pr[v_i \in B] + \Pr[u_i \in B] \cdot \Pr[v_i \in A] \\
&= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \\
&= \frac{1}{2}.
\end{aligned}$$

Since X_i is a Bernoulli r.v; $\mathbb{E}[X_i] = \frac{1}{2}$.

Note that the size of the cut $(A,B) = \sum_{i=1}^{m} X_i$.

Expected size of the cut $(A,B) = \mathbb{E}\left[\sum_{i=1}^{m} X_i\right] = \sum_{i=1}^{m} \mathbb{E}[X_i]$

$$= \frac{m}{2}.$$

Since the expectation of the r.v is $\frac{m}{2}$, then there exists a partition A, B s.t at least $\frac{m}{2}$ edges connect A to B.

Recall the reverse markov type of argument in Lecture 9.

Question: What is the expected no. of ~~samples~~ trials are needed to find a cut of value at least $\frac{m}{2}$?

Let p be the prob that a random cut (A,B) has at least $\frac{m}{2}$ edges.

$$\frac{m}{2} = \mathbb{E}[|cut(A,B)|]$$

$$= \sum_{i \le \frac{m}{2}-1} i \cdot \Pr[|cut(A,B)| = i] + \underbrace{\sum_{i \ge \frac{m}{2}} i \cdot \Pr[|cut(A,B)| = i]}$$

$$\le m \cdot \underbrace{\sum_{i \ge \frac{m}{2}} \Pr[|cut(A,B)| = i]}$$

$$\le \left(\frac{m}{2}-1\right) \cdot (1-p) + m \cdot p$$

$$= \frac{m}{2} - 1 - \frac{mp}{2} + p + mp$$

$$\Rightarrow \quad p \geq \frac{1}{\frac{m}{2}+1}.$$

We obtained the probability of success in a trial to be $\frac{1}{\frac{m}{2}+1}$ and we are now asking what is the expected no. of trials needed to succeed.

Recall: Geometric distribution

$$\text{Expected no. of trials} = \frac{1}{p} \leq \frac{m}{2}+1.$$

Derandomization using Conditional Expectations

Instead of placing the vertices in A or B uniformly and independently as was done before, let us place vertices in a deterministic way one at a time in an arbitrary order (say $v_1, v_2, \ldots, v_n$).

Let x_i be the set where v_i is placed (where $x_i = A$ or $x_i = B$). Suppose first k vertices are already placed. Now consider the expected value of the cut conditioned on the placing of the first k vertices.

$$\mathbb{E}\left[|Cut(A,B)| \;\middle|\; \underbrace{x_1, \ldots, x_k}_{\text{placement of first } k \text{ vertices.}}\right]$$

Expectation over the random choice for the remaining vertices.

We would now place the next vertex s.t

Inductive argument.

$$\mathbb{E}\left[|cut(A,B)| \,\middle|\, x_1,\dots,x_k\right] \leq \mathbb{E}\left[|cut(A,B)| \,\middle|\, x_1,x_2,\dots,x_{k+1}\right].$$

$$\Rightarrow \quad \mathbb{E}\left[|cut(A,B)|\right] \leq \underbrace{\mathbb{E}\left[|cut(A,B)| \,\middle|\, x_1,\dots,x_n\right]}.$$

determined by our placement "algorithm".

Base case: $\mathbb{E}\left[|cut(A,B)|\right] \underset{\smile}{=} \mathbb{E}\left[|cut(A,B)| \,\middle|\, x_1\right]$

It does not matter where we place the first ball.

Induction step: We need to show that

$$\mathbb{E}\left[|cut(A,B)| \,\middle|\, x_1,\dots,x_k\right] \leq \mathbb{E}\left[|cut(A,B)| \,\middle|\, x_1,\dots,x_{k+1}\right].$$

If we were to place v_{k+1} randomly into A or B with uniform probability and let Y_{k+1} be the r.v correspon-ding to the choice of the set A or B. Then,

$$\begin{aligned}
\mathbb{E}\left[|cut(A,B)| \,\middle|\, x_1,\dots,x_k\right] &= \frac{1}{2}\cdot \mathbb{E}\left[|cut(A,B)| \,\middle|\, x_1,\dots,x_k, Y_{k+1}=A\right] \\
&\quad + \frac{1}{2}\cdot \mathbb{E}\left[|cut(A,B)| \,\middle|\, x_1,\dots,x_k, Y_{k+1}=B\right]. \\
&\leq \max\{\mathrm{I}, \mathrm{II}\}
\end{aligned}$$

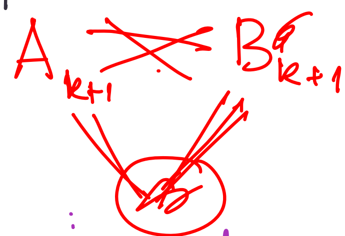
where $I = \mathbb{E}\left[|Cut(A,B)| \,\middle|\, x_1, \ldots, x_k, Y_{k+1} = A\right]$ and

$II = \mathbb{E}\left[|Cut(A,B)| \,\middle|\, x_1, \ldots, x_k, Y_{k+1} = B\right]$.

Observe that computing I and II will help us underst-and what would be a right choice for v_{k+1}.

For I, we know the value of cut just contributed by vertices $v_1, \ldots, v_{k+1}$ and the edges incident between them. For the rest of the edges run the random experiment as before and compute the expectation.

(Similarly for II).



Observation: Choice of set for v_{k+1} depends on maximizing the no. of edges crossing the cut.
For edges not containing v_{k+1} as an endpoint, their contribution to the value of cut in I and II is same.

Final strategy: Place a vertex in the side with fewer neighbours and break ties arbitrarily.

Entire discussion can be summed up as follows.

Theorem: A greedy algorithm for MAXCUT as described above always guarantees a cut with at least $\frac{m}{2}$ ~~steps~~ edges.

[GW] MAXCUT has a $\frac{7}{8}$-approx!

$$\mathbb{E}[\underline{|\text{clauses satisfied}|}]$$

enumerating the literals. $x_1, \dots, x_n$

$$\mathbb{E}\left[|\text{clauses sat.}| \;\middle|\; x_1 = v_1, x_2 = v_2, \dots, x_k = v_k\right]$$

$v_i \in \{T, F\}$. $(\underline{x_1} \vee \underline{x_2} \vee \underline{x_3})$

$$\max \begin{cases} \mathbb{E}\left[\underline{|\text{clauses sat}|} \;\middle|\; \underline{x_1 = v_1}, \underline{x_2 = v_2}, \dots, \underline{x_k = v_k}, \underline{x_{k+1} = T}\right] \\ \mathbb{E}\left[\underline{|\text{clauses sat}|} \;\middle|\; \quad '' \quad x_{k+1} = F\right]. \end{cases}$$