3.2 (contd.) Martingales.

Recall that a sequence of r.vs $X_0, X_1, \dots$ is martingale if

$$\mathbb{E}[X_i \mid X_0, \dots, X_{i-1}] = X_{i-1}.$$

Let $\underbrace{Y_i = X_i - X_{i-1}}$. Then $\mathbb{E}[Y_i \mid X_0, \dots, X_{i-1}] = 0$.

Y_i is called a difference martingale.

Recall that $E[X|Y]$ is a r.v. Let $g(Y) = E[X|Y]$.

$$g(y) = \mathbb{E}[X \mid Y = y].$$

Let $\{X_i\}_{i=1}^n$ and $\{Y_i\}_{i=1}^n$ be two seq. of r.vs defined on common prob space s.t

$$\mathbb{E}[X_i \mid Y_1, \dots, Y_{i-1}] = X_{i-1}.$$

Then $\{X_i\}_{i=1}^n$ is a martingale with respect to $\{Y_i\}_{i=1}^n$.

Let A and $\{Z_i\}_{i=1}^n$ be r.vs on a common prob. space. Let $X_i = \mathbb{E}[A \mid Z_1, \dots, Z_i]$.

Claim: X_i is a martingale.

[Recall that it is sufficient to show that

$$\mathbb{E}[X_i \mid Z_1, \dots, Z_{i-1}] = X_{i-1}.$$]

$$\mathbb{E}[X_i \mid Z_1, \ldots, Z_{i-1}] = \mathbb{E}\big[\mathbb{E}[A \mid Z_1, \ldots, Z_i] \mid Z_1, \ldots, Z_{i-1}\big]$$

$$= \mathbb{E}[A \mid Z_1, \ldots, Z_{i-1}]$$

Since, $\mathbb{E}[X|Y] = \mathbb{E}[\mathbb{E}[X|Y,Z]|Y]$ for r.vs X, Y, Z.

$$= X_{i-1}$$

Question: Why is $\mathbb{E}[X|Y] = \mathbb{E}[\mathbb{E}[X|Y,Z]|Y]$?

This follows from "Law of iterated expectations".

Claim: $\mathbb{E}_Y[\mathbb{E}_X[X|Y]] = \mathbb{E}_X[X]$

Proof:

$$\mathbb{E}_Y[\mathbb{E}_X[X|Y]]$$

$$= \sum_y g(y) \cdot \Pr[Y=y] \quad \text{where } g(y) = \mathbb{E}_X[X|Y=y].$$

$$= \sum_y \left(\sum_x x \cdot \Pr[X=x \mid Y=y]\right) \Pr[Y=y]$$

$$= \sum_x x \left(\sum_y \Pr[X=x \mid Y=y] \cdot \Pr[Y=y]\right)$$

$$= \sum_x x \cdot \Pr[X=x]$$

$$= \mathbb{E}[X].$$

Claim: $\mathbb{E}[\mathbb{E}[X|Y,Z]|Y] = \mathbb{E}[X|Y]$.

Towards this, it is sufficient to show that for all values y over the range of Y,

$$\mathbb{E}[\mathbb{E}[X|Y,Z]|Y=y] = \mathbb{E}[X|Y=y].$$

$$\mathbb{E}[\mathbb{E}[X|Y,Z]|Y=y] = \sum_z \mathbb{E}[X|Y=y, Z=z] \cdot \Pr[Z=z|Y=y]$$

$$= \sum_z \sum_x x \cdot \Pr[X=x|Y=y, Z=z] \cdot \Pr[Z=z|Y=y]$$

$$= \sum_z \sum_x x \cdot \frac{\Pr[X=x, Y=y, Z=z]}{\Pr[Y=y, Z=z]} \cdot \frac{\Pr[Y=y, Z=z]}{\Pr[Y=y]}$$

$$= \sum_z \sum_x x \cdot \frac{\Pr[X=x, Y=y, Z=z]}{\Pr[Y=y]}$$

$$= \sum_x x \cdot \sum_z \frac{\Pr[X=x, Y=y, Z=z]}{\Pr[Y=y]}$$

$$= \sum_x x \cdot \frac{\Pr[X=x, Y=y]}{\Pr\{Y=y\}}$$

$$= \sum_x x \cdot \Pr[X=x|Y=y]$$

$$= \mathbb{E}[X|Y=y].$$

$\mathbb{E}[\mathbb{E}[X|Y,Z]|Y] = \mathbb{E}[X|Y]$ as both of these r.vs take same values over the range of Y.

For r.v's A and $\{Z_i\}_{i=1}^n$ defined over same probability space and X_i's defined to be $\mathbb{E}[A \mid Z_1, \dots, Z_i]$; X_i's are martingale. In particular they are called

Doob's martingale.

Back to balls and bins:

We defined Z to be a r.v which is equal to the no. of empty bins after m balls are thrown into n bins.

We also defined $X_1, \dots, X_m$ to be random variables that denote the choice of bin selections for m balls. That is, $X_v \in_R [1, n]$.

We also defined r.v.s Z_t as follows.

$$Z_t = \mathbb{E}[Z \mid X_1, \dots, X_t]$$

From inspection it is clear that $\{Z_i\}$'s are Doob's martingale

Further, we showed that

$$\mathbb{E}[Z_t \mid Y_{t-1}] = Z_{t-1}$$

where Y_{t-1} is the no. of empty bins at time $t-1$.

$$\mathbb{E}[\mathbb{E}[Z \mid Y_t] \mid Y_{t-1}] = \mathbb{E}[Z \mid Y_{t-1}]$$

(Let us go back to Lecture 6 notes and go over this again).

Random graphs

$G(n,p)$ or $G_{n,p}$: A graph on n vertices whose edges are picked independently w. prob of p each.

Let Z_i be an indicator r.v for the presence of i^{th} edge.

Let $A = f(Z_1, \dots, Z_{\binom{n}{2}})$ be a property defined on graphs.

Then $X_i = \mathbb{E}[A \mid Z_1, \dots, Z_i]$ is a Doob martingale.

In particular X_i's are "Edge Exposure Martingale".

Let Z_i' be a vector in $\{0,1\}^{n-i}$ indicating the presence of edges between vertex i and all vertices $j > i$.

Let $A' = f(Z_1', \dots, Z_n')$ be a graph property defined on graphs. Let $X_i' = \mathbb{E}[A' \mid Z_1', \dots, Z_i']$.

As discussed above, X_i's are also Doob's martingale.

In particular, X_i's are "Vertex Exposure Martingales".

Before we see some examples, we need the following.

A function $f(z_1, \dots, z_m)$ is c-Lipschitz if changing the value of any one of the coordinates causes the value of f to change by at most $\pm c$.

$$|f(z_1,\dots,z_i,\dots,z_m) - f(z_1,\dots,\hat{z}_i,\dots,z_m)| \le c.$$

Claim: If f is c-Lipschitz and z_i is independent of $z_{i+1},\dots,z_m$ conditioned on $z_1,\dots,z_{i-1}$, then the Doob martingale $X_i = \mathbb{E}[f(z_1,\dots,z_m) \mid z_1,\dots,z_i]$ satisfies

$$|X_i - X_{i-1}| \le c.$$

Proof: Let $\hat{z}_i$ be a r.v w. same distribution as z_i conditioned on $z_1,\dots,z_{i-1}$ but independent of $z_i, z_{i+1},\dots$.

$$\begin{aligned}
X_{i-1} &= \mathbb{E}[f(z_1,\dots,z_i,\dots,z_m) \mid z_1,\dots,z_{i-1}] \\
&= \mathbb{E}[f(z_1,\dots,\hat{z}_i,\dots,z_m) \mid z_1,\dots,z_{i-1}] && \text{// Since } z_i \text{ and } \hat{z}_i \text{ are i.d.} \\
&= \mathbb{E}[f(z_1,\dots,\hat{z}_i,\dots,z_m) \mid z_1,\dots,z_{i-1},z_i]. && \text{// } z_i \text{ and } \hat{z}_i \text{ are ind.}
\end{aligned}$$

And thus,

$$\begin{aligned}
|X_i - X_{i-1}| &= \Big|\mathbb{E}[f(z_1,\dots,z_i,\dots,z_m) \mid z_1,\dots,z_{i-1},z_i] \\
&\qquad - \mathbb{E}[f(z_1,\dots,\hat{z}_i,\dots,z_m) \mid z_1,\dots,z_{i-1},z_i]\Big| \\
&= \Big|\mathbb{E}[f(z_1,\dots,z_i,\dots,z_m) - f(z_1,\dots,\hat{z}_{i-1},\dots,z_i) \mid z_1,\dots,z_i]\Big| \\
&\le c.
\end{aligned}$$

Application: Chromatic no. of $G_{n,1/2}$.

Chromatic number of graph G (denoted by $\chi(G)$) : Min no. of colours required to colour the vertices of G s.t no two adjacent vertices receive the same colour.

We want to estimate $\chi(G_{n,1/2})$ with good guarantees.

Let X denote the chromatic no. of a random graph.

Note that the sequence $Z_1, \dots, Z_n$ expose vertices and they "reveal" new edges connected to "$j > i$".

Let $X = f(Z_1, \dots, Z_n)$ and $X_i = \mathbb{E}[X \mid Z_1, \dots, Z_i]$.

Claim: (a) f is 1-Lipschitz.

(b) X_i's are Doob's martingale.

Proof of (a): Modifying Z_i to $\hat{Z}_i$ by adding edges would force us to add at most one colour. Deleting edges does not decrease the chromatic number by more than 1. Thus,

$$|f(Z_1, \dots, Z_i, \dots, Z_m) - f(Z_1, \dots, \hat{Z}_i, \dots, Z_m)| \leq 1.$$

$$\Downarrow$$

$$|X_i - X_{i-1}| \leq 1.$$

Now we can apply Azuma's ineq.

Theorem [Shamir-Spencer (1987)]

Let X be the chromatic number of $G \in G_{n,1/2}$. Then

$$\Pr\left[|X - \mathbb{E}[X]| \geq \lambda\right] \leq 2 \cdot \exp\left(-\frac{\lambda^2}{2n}\right).$$