Linear Programming Duality

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1 Introduction

One of the most important result in optimisation, both linear and non-linear, is the expression of a dual problem whereby constraints and cost function can be combined into a single objective function via the use of Lagrange multipliers. One consequence is that there exist several dual expressions in convex optimisation that make it possible to solve problems with contraints while using an unconstrained solver only, essentially allowing to solve constrained problems with gradient descent. This is useful when the problem is everywhere differentiable. When this is not the case, we have to deal with specialized tools like proximity operators, and specific duality expressions like Fenchel duality, which are considerably beyond the scope of this course.

In the case of Linear Programming (LP), the constraints make the problem non-differentiable and so not easily solved by standard gradient-descent solvers. Proximity and Fenchel duality can be applied, but are currently research-level problems, i.e. not well studied. However, LP has some specific symmetries that make it easy to express a dual problem to any LP, which can be used to improve algorithms. In this document, we will study these.

2 Primal and dual LP problems

Let us consider a maximization LP problem in standard form:

$$
\text{Primal problem} \quad
\begin{array}{lllllllllll}
\max & z & = & c_1x_1 & + & c_2x_2 & + & \dots & c_nx_n & & \\
\text{s.t.} & & & a_{11}x_1 & + & a_{12}x_2 & + & \dots & a_{1n}x_n & \leq & b_1 \\
& & & a_{21}x_1 & + & a_{22}x_2 & + & \dots & a_{2n}x_n & \leq & b_2 \\
& & & \vdots & & \vdots & & & \vdots & & \vdots \\
& & & a_{i1}x_1 & + & a_{i2}x_2 & + & \dots & a_{in}x_n & \leq & b_i \\
& & & \vdots & & \vdots & & & \vdots & & \vdots \\
& & & a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & a_{mn}x_n & \leq & b_m \\
\multicolumn{11}{l}{\text{with } \forall i \in \{1, \dots, n\}, x_i \geq 0}
\end{array}
\tag{1}
$$

then the dual problem is expressed in this way:

$$
\text{Dual problem} \qquad \begin{array}{llcccccccccc}
\min & w & = & b_1y_1 & + & b_2y_2 & + & \dots & b_my_m & & \\
\text{s.t.} & & & a_{11}y_1 & + & a_{21}y_2 & + & \dots & a_{m1}y_m & \geq & c_1 \\
& & & a_{12}y_1 & + & a_{22}y_2 & + & \dots & a_{m2}y_m & \geq & c_2 \\
& & & \vdots & & \vdots & & & \vdots & & \vdots \\
& & & a_{1j}y_1 & + & a_{2j}y_2 & + & \dots & a_{mj}y_m & \geq & c_j \\
& & & \vdots & & \vdots & & & \vdots & & \vdots \\
& & & a_{1n}y_1 & + & a_{2n}y_2 & + & \dots & a_{mn}y_m & \geq & c_n \\
\multicolumn{11}{l}{\text{with } \forall j \in \{1, \dots, m\}, y_i \geq 0}
\end{array} \qquad (2)
$$

2.1 Example

Consider the table, desk and chair problem from the course:

$$
\begin{array}{llcccccccccl}
\text{max.} & z & = & 60x_1 & + & 30x_2 & + & 20x_3 & & & \\
s.t. & & & 8x_1 & + & 6x_2 & + & x_3 & \leq & 48 & \text{(wood constraint)} \\
& & & 4x_1 & + & 2x_2 & + & 1.5x_3 & \leq & 20 & \text{(finishing constraint)} \\
& & & 2x_1 & + & 1.5x_2 & + & 0.5x_3 & \leq & 8 & \text{(carpentry constraint)} \\
\multicolumn{11}{l}{x_1, x_2, x_3 \geq 0}
\end{array} \qquad (3)
$$

where

$$
\begin{aligned}
x_1 &= \text{number of desks manufactured} \\
x_2 &= \text{number of tables} \\
x_3 &= \text{number of chairs}
\end{aligned}
$$

The dual of this problem is then:

$$
\begin{array}{llccccccccc}
\text{min.} & w & = & 48y_1 & + & 20y_2 & + & 8y_3 & & \\
s.t. & & & 8y_1 & + & 4y_2 & + & 2y_3 & \geq & 60 \\
& & & 6y_1 & + & 2y_2 & + & 1.5y_3 & \geq & 30 \\
& & & y_1 & + & 1.5y_2 & + & 0.5y_3 & \geq & 20 \\
\multicolumn{10}{l}{y_1, y_2, y_3 \geq 0,}
\end{array} \qquad (4)
$$

2.1.1 Exercise 1

- Solve both problems using for example the hand-written Python solver from the course, and verify that the optimum is numerically the same for both problems.

2.2 Non-normal examples

An LP where all the constraints are inequalities that are in the same direction is called a "normal" LP. Not all LPs are normal.

2.2.1 Exercise 2

- Normalize the primal and solve both the dual and the primal problems for the following example:

$$
\begin{array}{lcccccccc}
\text{max.} & z & = & 2x_1 & + & x_2 & & \\
s.t. & & & x_1 & + & x_2 & = & 4 \\
 & & & 2x_1 & - & x_2 & \geq & 3 \\
 & & & x_1 & - & x_2 & \leq & 1
\end{array}
\tag{5}
$$

$$x_1 \geq 0; \;\; x_2 \text{ n.n.p.},$$

  mind that x_2 is not necessarily positive (nnp).

- In particular, how do you propose to deal with equalities ?
- verify that the numerical solution is the same for both problems again.

3 Interpretation of the dual

In the dual of a problem, variables in the primal become constraints in the dual and vice-versa.

3.1 Exercise 3

- How do you interpret the constraints in the dual problem of the desks, tables and chairs problem (problem described in LP (4))?
- How do you interpret the variables in this dual problem ?

  Hints: consider the primal from the point of view of the seller of *finished products,* and the dual from point of view of the buyer of *resources.* The seller can use the resources to make product they can sell, the buyer can purchase the resources. How does the buyer set their prices ?

- Revisit the "famous diet" problem (the chocolate bar, ice-cream, soda and cake diet) from the second lecture. Write the dual of this problem, solve both the primal and the dual, and interpret the dual solution.

4 The Dual Theorem

The dual theorem is one of the most important result in linear programming. It states that if the primal has an optimal solution, then so does the dual, and both the dual and the primal have the same objective function value. In the following, we assume the primal is a normal max problem with m constraints and n variables. The dual is then a normal min problem with n constraints and m variables.

4.1 Weak duality

If we choose any feasible solution to the primal and any feasible solution to the dual, then the w value of the dual will be at least as large as the z solution for the primal.

Let $\mathbf{x} = [x_1, \ldots, x_n]^\top$ be any feasible solution to the primal problem and $\mathbf{y} = [y_1, \ldots, y_m]$ be any feasible solution to the dual, then $z(\mathbf{x}) = \mathbf{c}^\top \mathbf{x} \leq w(\mathbf{y}) = \mathbf{b}^\top \mathbf{y}$.

4.1.1 Exercise 4

- Prove this result.
- Hint:
  - consider the sum
$$\sum_{i=1}^{m} \sum_{j=1}^{n} y_i a_{ij} x_j$$
  - compare this sum to the expressions for z and w by using the sum of the normal inequalities in the primal and in the dual separately.

4.2 Optimal cost lemma

Let $\bar{\mathbf{x}} = [\bar{x}_1, \ldots, \bar{x}_n]^\top$ be a feasible solution to the primal and $\bar{\mathbf{y}} = [\bar{y}_1, \ldots, \bar{y}_m]^\top$ be a feasible solution to the dual. if $\mathbf{c}\bar{\mathbf{x}} = \bar{\mathbf{y}}\mathbf{b}$, then $\bar{\mathbf{x}}$ is optimal for the primal and $\bar{\mathbf{y}}$ is optimal for the dual.

4.2.1 Exercise 5

- Prove this result.
- Hint: use the weak duality result from previous section.

4.3 Infeasible and unbounded LPs lemmas

We have the following pair of lemmas

- If the primal is unbounded, then the dual is infeasible
- if the dual is unbounded, then the primal is infeasible

4.3.1 Exercise 6

- Prove these results using the weak duality lemma.
- Show with a counter-example that these implications are not equivalences, e.g. find a problem which is infeasible both in the primal and the dual.

Infeasible and unbounded LPs are not as important as the case where an optimal solution exists, which we consider now.

4.4 The Dual Theorem

Let $\mathbf{x}_b$ be an optimal solution for the primal problem with matrix $\mathbf{B}$. Recall that we have $\mathbf{x}_b = \mathbf{B}^{-1}\mathbf{b}$ and that the optimal cost is $\bar{z} = \mathbf{c}_b^\top \mathbf{x}_b$. Then $\mathbf{y}_b = (\mathbf{c}_b^\top \mathbf{B}^{-1})^\top$ is an optimal solution for the dual problem ; and $\bar{z} = \mathbf{c}_b^\top \mathbf{x}_b = \mathbf{b}_b^\top \mathbf{y}_b = \bar{w}$.

4.5 Exercise 7

- Use the fact that $\mathbf{B}$ is the matrix of the optimal basis of the primal to show that $\mathbf{c}_b^\top \mathbf{B}^{-1}$ is dual feasible.
- Show that the optimal primal objective value is the dual objective function for $\mathbf{c}_b^\top \mathbf{B}^{-1}$.
- Use the optimal cost lemma from above to prove the Dual Theorem.

4.6 Complementary slackness

As before, we assume the primal is a max normal problem and the dual is a min normal problem. Let $s_1, \dots, s_m$ be the slack variables for the primal and $e_1, \dots, e_n$ be the excess variables for the dual.

Let $\mathbf{x} = [x_1, \dots, x_n]$ be a feasible primal solution and $\mathbf{y} = [y_1, \dots, y_m]$ be a feasible dual solution. Then $\mathbf{x}$ is primal optimal and $\mathbf{y}$ is dual optimal if and only if

$$s_i y_i = 0 \;\; \forall i \in 1, \dots, m \tag{6}$$
$$e_j x_j = 0 \;\; \forall j \in 1, \dots, n \tag{7}$$

4.7 Exercise 8

- Verify the complementary slackness theorem for the Desks, Tables and Chairs problem of the course and its dual.