ASSIGNMENT-I
MFML

YUVARAJAN M
ID: 2023AD05092
Sec: 8

1) Given: $C = AB$ where A, B, C are $n \times n$

i^{th} row of C = linear Combination of B Combining

Coefficient from ith row of A

To prove: If A is lower triangle matrix, prove A^{-1} is lower triangle

Proof:
Let's Assume A is lower triangle matrix, in this case the process of linear combination preserves the result of multiplication as lower triangle.

i.e, if A is lower triangle then

$$C = AB$$

Resulting in C being lower triangle since A being lower triangular matrix it preserves the structure.

With this we can also understand that product of two lower triangular matrix will also result in lower triangular matrix.

Hence A & B could be both assumed as lower triangular matrix.

With this, if we can express multiplication of two lower triangular matrix is an Identity matrix, then we can say that inverse of lower triangle matrix is also lower triangular.

$$C = AB = I$$
$$A = B^{-1} I$$
$$A = B^{-1}$$

This proves the B^{-1} is a lower triangle since multiplication of lower triangle preserves the structure.

Heance inverse of lower triangle matrix is also lower triangular

YUVARAJAN M
ID: 2023AD05092
Sec: 8

2) Let say A is a matrix of order $m \times m$

Given:

Eigen value of A is n

n is an integer

All elements of A is an integer

To prove:

$$\det(A) = nK, \quad K \text{ is an integer}$$

Proof:

By definition we know that for any $n \times n$ matrix we can represent determinant of matrix as

$$\det(A_{m \times m}) = \lambda_1 \lambda_2 \cdots \lambda_m$$

specification of eigen value is given as n

$$\text{i.e. } \lambda_1 \lambda_2 \cdots \lambda_m = n$$

Given that all elements of A is an integer, & the product of all eigen value is also integer

$$\therefore \det(A) = nK, \text{ where } K \text{ is an integer}$$

For diagonal or upper triangular matrix the value of $K = 1$ $\det(A) = n$

But in this case since all elements of A is an integer the value of K may greater than or equal to 1 to adjust the difference in eigen value multiplication.

Hence, $d(A_{m \times m}) = nK$; for K is an integer and all elements of A is an integer

YUVARAJAN M
ID: 2023AD05092
Sec: 8

3). Given:

A is invertable matrix of order $n \times n$

To prove/disprove:

$$rank(A) = rank(AB)$$

Validation:

$A_{n\times n} \Rightarrow$ Let's understand the property

for invertable matrix we know the below statement is true.

$$\det(A) \neq 0$$
$$Rank(A_{n\times n}) = n$$

$\therefore$ We can conclude $Rank(A) = n$ for order $n\times n$

There is no definition to B, but for matrix multiplication we know the order of B

$A_{n\times n}$ can be multiplied with $B_{n\times m}$

where we know $rank(AB) \leq \min(rank(A), rank(B))$

$\therefore$ if $rank(B) < n$

then $rank(AB) \neq rank(A)$

if $rank(B) = n$

then $rank(AB) = rank(A)$

The condition can only proved if $rank(B) = n$, else it can't be proved.

With this we can conclude without the definition of B we cannot prove $rank(A) = rank(AB)$

YUVARAJAN M
ID: 2023AD05092
Sec: 8

4). Given: A matrix $a, b \in A$, $a=1, b=-1$

$$A^2 = 8I$$

(i) Eigenvalues of the matrix

we know that

$$A = \lambda I \quad , \quad \lambda = \text{Eigen values}$$

Squaring on both side we get

$$A^2 = \lambda^2 I$$

given $A^2 = 8I$

$$\lambda^2 = 8$$

$$\lambda = \pm\sqrt{8}$$

$$\lambda = \pm 2\sqrt{2}$$

(ii) Find ratio of $\frac{\lambda_{(max)}}{\lambda_{(min)}}$

$$\lambda = \pm 2\sqrt{2}$$

$$\therefore \lambda_{(max)} = 2\sqrt{2}$$

$$\lambda_{(min)} = -2\sqrt{2}$$

$$\frac{\lambda_{(max)}}{\lambda_{(min)}} = \frac{2\sqrt{2}}{-2\sqrt{2}} = -1$$

$$\therefore \frac{\lambda_{(max)}}{\lambda_{(min)}} = -1$$

YUVARAJAN M
ID: 2023AD05092
Sec 8

5. a) Given x, y, z are the vectors in a space V

$$x + y + z = 0$$

prove $\text{Span}(x, y) = \text{Span}(x, z) = \text{Span}(y, z)$

Proof:

Let take any vector V_1, the span of V_1 can be represented as $V_1 = ax + by$, $a, b \in R$

Let's take

$V_1 = \text{Span}(x, y)$

$V_2 = \text{Span}(x, z)$

$V_3 = \text{Span}(y, z)$

$\therefore V_1 = ax + by \quad V_2 = cx + dz \quad V_3 = ey + fz$

Given $x + y + z = 0$

we can represent $y = -x - z$

$z = -x - y$ ~~* -x + (xxx~~

$V_1 = ax + b(-x - z) \quad V_2 = cx + dz \quad V_3 = e(-x - z) + fz$

$V_1 = (a - b)x - bz \quad V_2 = cx + dz \quad V_3 = -ex + (f - e)z$

$\therefore V_1 \in \text{Span}(x, z) \quad V_2 \in \text{Span}(x, z) \quad V_3 \in \text{Span}(x, z)$

$\therefore \text{Span}(x, y) = \text{Span}(x, z) = \text{Span}(y, z)$

YUVARAJAN M
ID: 2023AD05092
Sec: 8

5) b. (i) Given $\text{Span}(v_1, v_2, \dots v_n) = V$

Prove $\text{Span}(v_1, v_2 - v_1, v_3 - v_1 \dots v_n - v_1) = V$

Proof:

$\text{Span}(v_1, v_2, v_3 \dots v_n) = a_1 v_1 + a_2 v_2 + a_3 v_3 + \dots + a_n v_n$

Lets take $v_n = v_1 + (v_n - v_1)$

$$V = a_1 v_1 + a_2(v_1 + (v_2 - v_1)) + a_3(v_1 + (v_3 - v_1)) + \dots + a_n(v_1 + (v_n - v_1))$$

$$V = (a_1 + a_2 + a_3 + \dots + a_n) v_1 + a_2(v_2 - v_1) + a_3(v_3 - v_1) + \dots + a_n(v_n - v_1) \quad \text{—(i)}$$

$\therefore \text{Span}(v_1, v_2, v_3 \dots v_n) = \text{Span}(v_1, v_2 - v_1, v_3 - v_1 \dots v_n - v_1) = V$

(ii) Given $v_1, v_2, v_3 \dots v_n$ are linearly independent

Prove $v_1, v_2 - v_1, v_3 - v_1 \dots v_n - v_1$ is also independent

Proof: if $v_1, v_2, v_3 \dots v_n$ is linearly independent

then $c_1 v_1 + c_2 v_2 + c_3 v_3 + \dots + c_n v_n = 0$

this implies that $c_1 = c_2 = c_3 = \dots = c_n = 0$

Thus based on (i) we know that $c_1 v_1 + c_2(v_2 - v_1) + \dots + c_n(v_n - v_1)$ can be expressed as

$$(c_1 - c_2 - c_3 - \dots c_n) v_1 + c_2 v_2 + c_3 v_3 + \dots + c_n v_n = 0$$

The above equation is expressed as 0 since $v_1, v_2, \dots v_n$ are linearly independent.

Heance $v_1, v_2 - v_1, v_3 - v_1, \dots v_n - v_1$ is also linearly independent