Markov Decision Processes

CPS 271

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Applications of MDPs

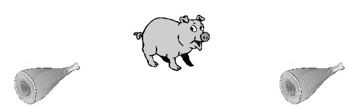
- Economics/Operations Research
  - Fleet maintenance (Howard, Rust)
  - Road maintenance (Golabi et al.)
  - Packet Retransmission (Feinberg et al.)
  - Nuclear plant management (Rothwell & Rust)

Applications of MDPs

- AI/Computer Science
  - Robotic control
    (Koenig & Simmons, Thrun et al., Kaelbling et al.)
  - Air Campaign Planning (Meuleau et al.)
  - Elevator Control (Barto & Crites)
  - Computation Scheduling (Zilberstein et al.)
  - Control and Automation (Moore et al.)
  - Spoken dialogue management (Singh et al.)
  - Cellular channel allocation (Singh & Bertsekas)

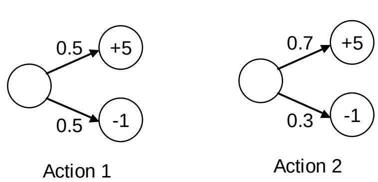
Applications of MDPs

- EE/Control
  - Missile defense (Bertsekas et al.)
  - Inventory management (Van Roy et al.)
  - Football play selection (Patek & Bertsekas)
- Agriculture
  - Herd management (Kristensen, Toft)



The MDP Framework

- We return to thinking at the level of *states*
- Probabilistic state transitions
- Multiple actions
- Reward function



How Do MDPs Differ From Search

- In search, we preferred to think about trees because graphs made things complicated
- Would like to use expectimax, but
- When we add probabilities, tree assumption becomes unrealistic
- Need to address loops head on

The MDP Framework

- State space: S
- Action space: A
- Transition function: P
- Reward function: R
- Discount factor: γ
- Policy: $\pi(s) \rightarrow a$

$ $ $ $

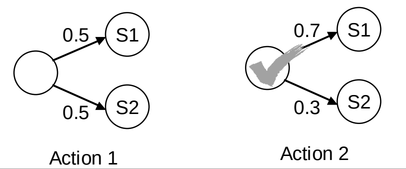
Objective: *Maximize expected, discounted return*

$$E\sum_{t=0}^{\infty}\gamma^t r_t$$

Finding Good Policies

Suppose an expert told you the “value” of each state:

V(S1) = 10 V(S2) = 5

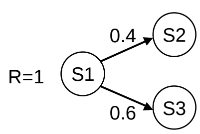


Value Determination

Determine the value of each state under policy p

$$V(s) = R(s) + \gamma \sum_{s'} P(s' \mid s, \pi(s)) V(s')$$

Bellman Equation



$$V(s_1) = 1 + \gamma(0.4V(s_2) + 0.6V(s_3))$$

Matrix Form

$$\mathbf{P} = \begin{pmatrix} P(s_1 \mid s_1, \pi(s_1)) & P(s_2 \mid s_1, \pi(s_1)) & P(s_3 \mid s_1, \pi(s_1)) \\ P(s_1 \mid s_2, \pi(s_2)) & P(s_2 \mid s_2, \pi(s_2)) & P(s_3 \mid s_2, \pi(s_2)) \\ P(s_1 \mid s_3, \pi(s_3)) & P(s_2 \mid s_3, \pi(s_3)) & P(s_3 \mid s_3, \pi(s_3)) \end{pmatrix}$$

$$\mathbf{V} = \gamma \mathbf{P}_\pi \mathbf{V} + \mathbf{R}$$

How do we solve this system?

Solving for Values

$$\mathbf{V} = \gamma \mathbf{P}_\pi \mathbf{V} + \mathbf{R}$$

For moderate numbers of states we can solve this system exacty:

$$\mathbf{V} = (\mathbf{I} - \underbrace{\gamma \mathbf{P}_\pi}_{})^{-1} \mathbf{R}$$

Guaranteed invertible because $\gamma \mathbf{P}_\pi$ has spectral radius <1

Solving for Values

$$\mathbf{V} = \gamma \mathbf{P}_\pi \mathbf{V} + \mathbf{R}$$

For larger numbers of states we can solve this system indirectly by Gauss iteration:

$$\mathbf{V}_{i+1} = \gamma \mathbf{P}_\pi \mathbf{V}_i + \mathbf{R}$$

Guaranteed convergent because $\gamma \mathbf{P}_\pi$ has spectral radius <1

We can also prove that this iteration is a *contraction* in max norm.

Improving Policies

- We can compute the value of a single policy
- How do we get the optimal policy?
- Need to ensure that we take the optimal action in every state:

$$V(s) = R(s) + \gamma \max_a \sum_{s'} P(s'|s,a)V(s')$$

Value Iteration

We can solve the system directly with a max in the equation
Can we solve it iteration?

$$V_{i+1}(s) = R(s) + \gamma \max_a \sum_{s'} P(s'|s,a)V_i(s')$$

•Called *value iteration* or simply *successive approximation*
•We can show that this is also a contraction in max norm
•Guaranteed to converge to optimal policy
•Converges exponentially quickly

Can we do better?

Greedy Policy Construction

Pick action with highest expected future value:

$$\pi(s) = \arg\max_a R(s) + \gamma \underbrace{\sum_{s'} P(s'|s,a)V(s')}_{\text{Expectation over next-state values}}$$

$$\pi = \text{greedy}(V)$$

Bootstrapping: Policy Iteration

Idea: Greedy selection is useful even with suboptimal V

Guess V
p = greedy(V)
V = value of acting on p

Repeat until policy doesn't change

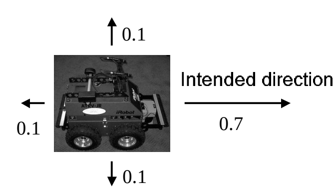
Guaranteed to find optimal policy
Usually takes very small number of iterations
Computing the value functions is the expensive part

Computational Complexity

- VI and PI are both contraction mappings w/rate γ
- VI costs less per iteration
- PI tends to take O(n) iterations in practice
- Open question: Subexponential bound on PI

- Is there a guaranteed poly time MDP algorithm???

MDP Example



Note: This is a **gross** oversimplification

If direction is blocked, stays in some location:

Simple Model

S			+1	→ Absorb
			-10	→ Absorb

Q: What is optimal policy?

Value Iteration (1)

0	0	0	1	→ Absorb
0		0	-10	→ Absorb
0	0	0	0	

Start with value 0 for all states
Discount = 0.99

Value Iteration (2)

0	0	0.69	1	→ Absorb
0		-0.99	-10	→ Absorb
0	0	0	-0.99	

Start with value 0 for all states
Discount = 0.99
Residual = 0.99

Value Iteration (5)

0.48	0.70	0.76	1	→ Absorb
0.23		-0.55	-10	→ Absorb
0	-0.20	-0.23	-1.40	

Start with value 0 for all states
Discount = 0.99
Residual = 0.23

Value Iteration (20)

0.78	0.80	0.81	1	→ Absorb
0.77		-0.44	-10	→ Absorb
0.75	0.69	-0.37	-0.92	

Start with value 0 for all states
Discount = 0.99
Residual = 0.008

Value Iteration (20)

0.78	0.80	0.81	1	→ Absorb
0.77		-0.44	-10	→ Absorb
0.75	0.69	-0.37	-0.92	

Start with value 0 for all states
Discount = 0.99
Residual = 0.008

Final Policy

0.78	0.80	0.81	1
0.77		-0.44	-10
0.75	0.69	-0.37	-0.92

→	→	→	1
↑		↑	-10
↑	←	←	←

Start with value 0 for all states
Discount = 0.99
Residual = 0.008

Policy Iteration (1)

↑	↑	↑	1
↑		↑	-10
↑	↑	↑	↑

-0.21	-0.22	-0.24	1
-0.22		-1.54	-10
-0.35	-1.27	-2.30	-8.93

Policy Iteration (2)

↑	←	→	1
↑		↑	-10
↑	←	←	←

0.37	0.41	0.76	1
0.36		-0.51	-10
0.35	0.31	0.05	-1.20

Policy Iteration (3)

→	→	→	1
↑		↑	-10
↑	←	←	←

0.78	0.80	0.81	1
0.77		-0.41	-10
0.75	0.70	0.39	-0.90

Done!

Compare with Value Iteration

0.78	0.80	0.81	1
0.77		-0.44	-10
0.75	0.69	-0.37	-0.92

Value Iteration
20 Iterations

0.78	0.80	0.81	1
0.77		-0.41	-10
0.75	0.70	0.39	-0.90

Policy Iteration
3 Iterations

Change Rewards

↑	←	→	1
→		↑	-500
↑	←	←	←

-2.60	-3.06	-6.37	1
-2.61		-61.9	-500
-2.85	-4.82	-19.1	-78.2

Strange, but optimal.

Linear Programming

$$V(s) = R(s) + \gamma \max_a \sum_{s'} P(s' \mid s, a) V(s')$$

Issue: Turn the non-linear max into a collection of linear constraints

$$\forall s, a : V(s) \geq \underbrace{R(s) + \gamma \sum_{s'} P(s' \mid s, a) V(s')}_{\text{Optimal action has tight constraints}}$$

MINIMIZE: $\sum_s V(s)$

Weakly polynomial; slower than PI in practice.

What's The Bad News?

- Works at the level of states = atomic events
- We usually have exponentially many of these
- Hot questions:
  - Combining learning/generalizations with MDPs
  - Solving MDPs when model is not known in advance (Reinforcement Learning)