

LAB 11 : Hidden Markov Model

In [1]:

```
import numpy as np
import matplotlib.pyplot as plt
import pandas as pd
```

Please refer to the following [article](#) to understand Hidden Markov Model

Here we will be dealing with 3 major problems :

1. Evaluation Problem
2. Learning Problem
3. Decoding Problem

1. Evaluation Problem : Implementation of Forward and Backward Algorithm

The generalised forward algorithm equation is given as:

$$\begin{aligned}\alpha_j(t+1) &= p(v_k(1) \dots v_k(t+1), s(t+1) = j) \\ &= b_{jv(t+1)} \sum_{i=1}^M a_{ij} \alpha_i(t)\end{aligned}$$

where $\alpha_i(t)$ = Forward probability at $t = t$

a_{ij} = Transition Probability from i to j

$b_{jv(t+1)}$ = Emission Probability of observed state v(t+1) given hidden state j at $t = t$

$$\alpha_j(0) = p(v_k(0), s(0) = j) \quad (7)$$

$$= \pi_j b_{jk} \quad (8)$$

where π = initial distribution, (9)

$$b_{jv(0)} = \text{Emission Probability of observed state v(t+1) given hidden state j at } t = 0 \quad (10)$$

The generalised backward algorithm equation is given as:

$$\begin{aligned}\beta_i(t) &= p(v_k(t+1) \dots v_k(T) | s(t) = i) \\ &= \sum_{j=0}^M \beta_j(t+1) b_{jv(t+1)} a_{ij}\end{aligned}$$

where $\beta_i(t)$ = Backward probability at $t = t$

a_{ij} = Transition Probability from i to j

$b_{jv(t+1)}$ = Emission Probability of observed state v(t+1) given hidden state j at

By Intuition using Trellis: $\beta_i(T) = 1$

In [2]:

```
data = pd.read_csv('data_python.csv') ## Read the data, change the path accordingly

V = data['Visible'].values

# Transition Probabilities
a = np.array(((0.54, 0.46), (0.49, 0.51)))

# Emission Probabilities
b = np.array(((0.16, 0.26, 0.58), (0.25, 0.28, 0.47)))

# Equal Probabilities for the initial distribution
initial_distribution = np.array((0.5, 0.5))## Write your code here

def forward(V, a, b, initial_distribution):
    alpha = np.zeros((V.shape[0], a.shape[0]))
    ## Write your code here
    alpha[0,:] = initial_distribution*b[:,V[0]]
    for t in range(1, V.shape[0]):
        for j in range(a.shape[0]):
            alpha[t, j] = alpha[t - 1].dot(a[:, j]) * b[j, V[t]]
    return alpha

alpha = forward(V, a, b, initial_distribution)
print("alpha values:")
print(alpha)
print("=====")

def backward(V, a, b):
    beta = np.zeros((V.shape[0], a.shape[0]))
    ## Write your code here
    beta[V.shape[0] - 1] = np.ones((a.shape[0]))
    for t in range(V.shape[0] - 2, -1, -1):
        for j in range(a.shape[0]):
            beta[t, j] = (beta[t + 1] * b[:, V[t + 1]]).dot(a[j, :])
    return beta

beta = backward(V, a, b)
print("beta values:")
print(beta)
```

```
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```

1. Learning Problem : Implementation of Baum Welch Algorithm

The EM algorithm is given as:

- initialize a and b
- iterate until convergence or for n iterations
- ■ E-step:
 - ○ $x_{ij}(t) = \frac{\alpha_i(t)a_{ij}b_{jk}v(t+1)\beta_j(t+1)}{\sum_{i=1}^M \sum_{j=1}^M \alpha_i(t)a_{ij}b_{jk}v(t+1)\beta_j(t+1)}$
 - ○ $\gamma_i(t) = \sum_{j=1}^M x_{ij}(t)$
- ■ M-step:
 - ○ $\hat{a}_{ij} = \frac{\sum_{t=1}^{T-1} x_{ij}(t)}{\sum_{t=1}^{T-1} \sum_{j=1}^M x_{ij}(t)}$
 - ○ $\hat{b}_{jk} = \frac{\sum_{t=1}^T \gamma_j(t) \mathbf{1}(v(t)=k)}{\sum_{t=1}^T \gamma_j(t)}$
- return a b

$$\text{where } x_{ij} = p(s(t) = i, s(t+1) = j | V^T, \theta) = \frac{p(s(t) = i, s(t+1) = j, V^T | \theta)}{p(V^T | \theta)} \quad (17)$$

$$\gamma_i(t) = p(s(t) = i | V^T, \theta) = \frac{p(s(t) = i, V^T | \theta)}{p(V^T | \theta)} \quad (18)$$

In [3]:

```
def baum_welch(V, a, b, initial_distribution, n_iter=100):
    ## Write your code here
    M = a.shape[0]
    T = len(V)

    for n in range(n_iter):
        alpha = forward(V, a, b, initial_distribution)
        beta = backward(V, a, b)

        xi = np.zeros((M, M, T - 1))
        for t in range(T - 1):
            denominator = np.dot(np.dot(alpha[t, :, :].T, a) * b[:, V[t + 1]].T, beta[t + 1, :])
            for i in range(M):
                numerator = alpha[t, i] * a[i, :] * b[:, V[t + 1]].T * beta[t + 1, :].T
                xi[i, :, t] = numerator / denominator

        gamma = np.sum(xi, axis=1)
        a = np.sum(xi, 2) / np.sum(gamma, axis=1).reshape((-1, 1))

        gamma = np.hstack((gamma, np.sum(xi[:, :, T - 2], axis=0).reshape((-1, 1)))))

        K = b.shape[1]
        denominator = np.sum(gamma, axis=1)
        for l in range(K):
            b[:, l] = np.sum(gamma[:, V == l], axis=1)

        b = np.divide(b, denominator.reshape((-1, 1)))
    return (a,b)

data = pd.read_csv('data_python.csv')

V = data['Visible'].values

# Transition Probabilities
a = np.ones((2, 2))
a = a / np.sum(a, axis=1)

# Emission Probabilities
b = np.array(((1, 3, 5), (2, 4, 6)))
b = b / np.sum(b, axis=1).reshape((-1, 1))

# Equal Probabilities for the initial distribution
initial_distribution = np.array((0.5, 0.5))

a,b = baum_welch(V, a, b, initial_distribution, n_iter=100)

print("a:\n",a)
print("b:\n",b)
```

```
a:
[[0.53816345 0.46183655]
 [0.48664443 0.51335557]]
b:
[[0.16277513 0.26258073 0.57464414]
 [0.2514996 0.27780971 0.47069069]]
```

1. Decoding Problem : Implementation of Viterbi Algorithm

The following equation represents the highest probability along a single path for first t observations which ends at state i .

$$\omega_i(t) = \max_{s_1, \dots, s_{T-1}} p(s_1, s_2, \dots, s_T = i, v_1, v_2, \dots, v_T | \theta)$$

We can use the same approach as the Forward Algorithm to calculate $\omega_i(t+1)$

$$\omega_i(t+1) = \max_j (\omega_i(t)a_{ij}b_{jv(t+1)})$$

Now to find the sequence of hidden states we need to identify the state that maximizes $\omega_i(t)$ at each time step t

$$\arg \max_t \omega(t)$$

Once we complete the above steps for all the observations, we will first find the last hidden state by maximum likelihood, then using backpointer to backtrack the most likely hidden path.

In [4]:

```
def viterbi(V, a, b, initial_distribution):

    ## Write your code here
    T = V.shape[0]
    M = a.shape[0]

    omega = np.zeros((T, M))
    omega[0, :] = np.log(initial_distribution * b[:, V[0]])

    prev = np.zeros((T - 1, M))

    for t in range(1, T):
        for j in range(M):
            # Same as Forward Probability
            probability = omega[t - 1] + np.log(a[:, j]) + np.log(b[j, V[t]])

            # This is our most probable state given previous state at time t (1)
            prev[t - 1, j] = np.argmax(probability)

        # This is the probability of the most probable state (2)
        omega[t, j] = np.max(probability)

    # Path Array
    S = np.zeros(T)

    # Find the most probable last hidden state
    last_state = np.argmax(omega[T - 1, :])

    S[0] = last_state

    backtrack_index = 1
    for i in range(T - 2, -1, -1):
        S[backtrack_index] = prev[i, int(last_state)]
        last_state = prev[i, int(last_state)]
        backtrack_index += 1

    # Flip the path array since we were backtracking
    S = np.flip(S, axis=0)

    # Convert numeric values to actual hidden states
    result = []
    for s in S:
        if s == 0:
            result.append("A")
        else:
            result.append("B")
    return result

data = pd.read_csv('data_python.csv')

V = data['Visible'].values

# Transition Probabilities
a = np.ones((2, 2))
a = a / np.sum(a, axis=1)

# Emission Probabilities
b = np.array(((1, 3, 5), (2, 4, 6)))
,
```

```
a, b = baum_welch(V, a, b, initial_distribution, n_iter=100)

result = viterbi(V, a, b, initial_distribution)
print(result)
```

1. Use the built-in **hmmlearn** package to fit the data and generate the result using the decoder.

```
In [5]: # !pip install hmmlearn
```

```
In [6]: ## Write your code here
from hmmlearn import hmm
model = hmm.GaussianHMM(n_c)
V = np.reshape(V, (-1, 1))
model.fit(V);
print("Transition matrix:")
print(model.transmat_)
print("Predicted sequence:")
print(model.decode(V));
```

In []: