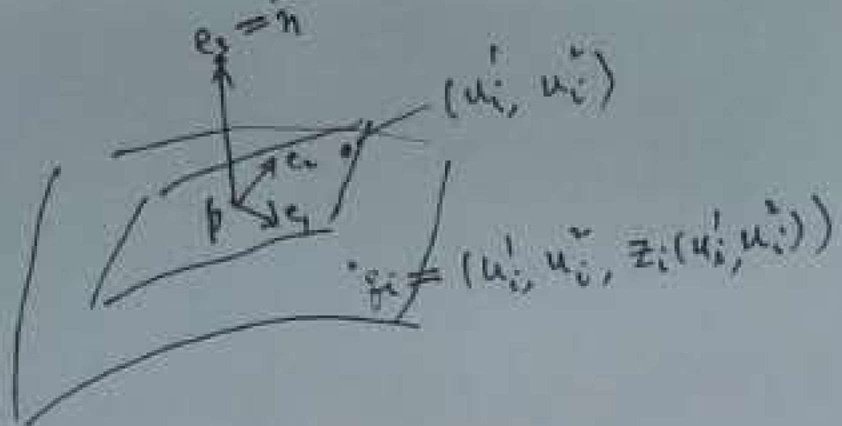

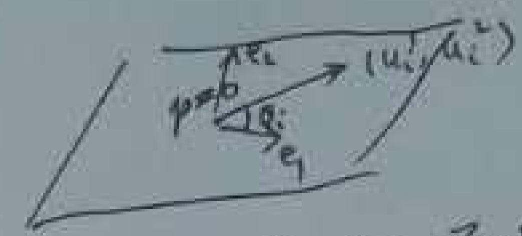

(u_i^1, u_i^2) 单位化:

$l_i = \sqrt{(u_i^1)^2 + (u_i^2)^2}$,

$(u_i^1, u_i^2) = l_i (v_i^1, v_i^2)$

$v_i^\alpha = \dfrac{u_i^\alpha}{l_i}$, $\alpha = 1, 2$

$(v_i^1, v_i^2) = (\cos\theta_i, \sin\theta_i)$

$f(g_i) = f(u_i^1, u_i^2)$

$f(p) = f(0)$

差商: $M_i = \dfrac{f(u_i^1, u_i^2) - f(0)}{l_i} \approx \dfrac{\partial f}{\partial u^1}\cos\theta_i + \dfrac{\partial f}{\partial u^2}\sin\theta_i$

$= \dfrac{\partial f}{\partial u^1} v_i^1 + \dfrac{\partial f}{\partial u^2} v_i^2 = \dfrac{\partial f}{\partial u^\alpha} v_i^\alpha$

现在要去求出梯度向量 $\begin{pmatrix} \frac{\partial f}{\partial u^1} \\ \frac{\partial f}{\partial u^2} \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$, 使得:

K是p的邻近点 g_i 的个数

$$\sum_{i=1}^{K} \left(X_1 v_i^1 + X_2 v_i^2 - M_i \right)^2 = \min, \quad 即 \quad \arg\min_X \sum_{i=1}^{K} \left(X_\alpha v_i^\alpha - M_i \right)^2$$

对 X_α 求导: $\sum_i (X_\beta v_i^\beta - M_i) v_i^\alpha = 0$, 即 $\sum_i v_i^\alpha v_i^\beta X_\beta = \sum_i M_i v_i^\alpha$

令 $A = (a^{\alpha\beta})$, 2×2阵

这里 $a^{\alpha\beta} = \sum_i v_i^\alpha v_i^\beta$

显然A是正定的. ∴有逆阵 A^{-1}.

令 $b^\alpha = \sum_i M_i v_i^\alpha$

于是 $AX = b$, 即 $a^{\alpha\beta} X_\beta = b^\alpha$

$$\therefore X = A^{-1} b = (a^{-1})_{\beta\alpha} b^\alpha$$

$$\begin{pmatrix} \frac{\partial f}{\partial u^1} \\ \frac{\partial f}{\partial u^2} \end{pmatrix} = \sum_i (a^{-1})_{\beta\alpha} v_i^\alpha M_i = \sum_i a^{-1}_{\beta\alpha} v_i^\alpha \cdot \frac{f(g_i) - f(0)}{l_i}$$

∴函数 f 的梯度向量可表为其邻近点的函数值 $f(g_i)$ 的带权平均值.

这是因为 $\forall$ 向量 $P = (P_\alpha)$

$$P^T A P = a^{\alpha\beta} P_\alpha P_\beta = \sum v_i^\alpha v_i^\beta P_\alpha P_\beta = \sum_i (v_i^\alpha P_\alpha) \cdot (v_i^\beta P_\beta) \geq 0,$$

等号成立 $\Longleftrightarrow$ $v_i^\alpha P_\alpha = 0$, $\forall i$

显然~~若~~ $v_i = (v_i^\alpha)$ ~~[illegible]~~ 中至少有两个线性无关向量 v_i, v_j, 则由

$P \perp v_i$, $P \perp v_j$ 可推出 $P \equiv 0$