Statistics of Diffeomorphisms

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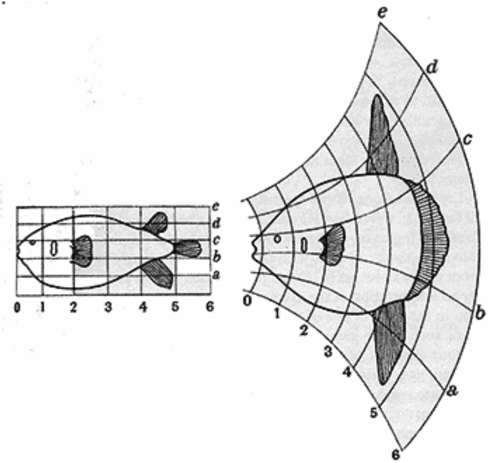
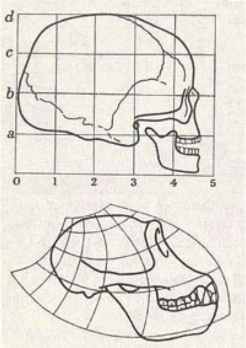
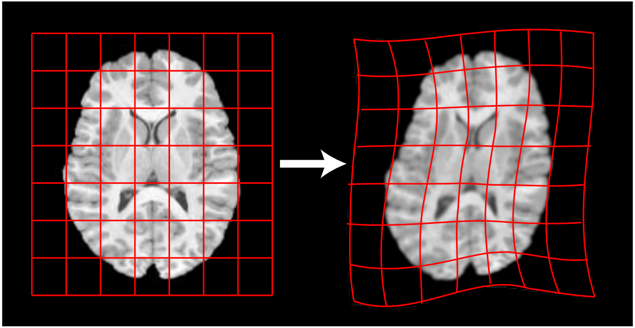
Transformation Models





From D’Arcy Thompson, *On Growth and Form*, 1917.

Deformation-Based Morphometry



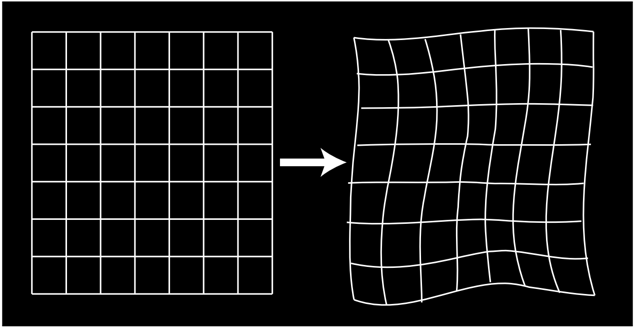
Step 1: Use deformable registration to align images.

Step 2: Throw away the images!

Step 3: Analyze the resulting deformation fields.

Shape differences are encoded in the transformation

Deformation-Based Morphometry



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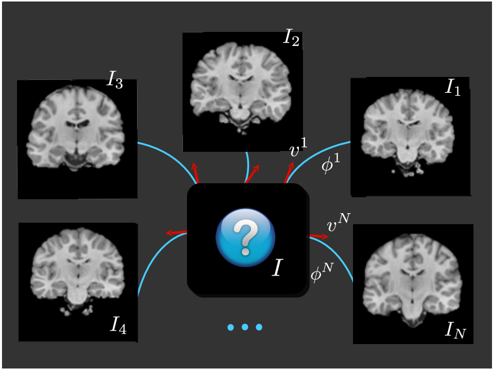
Population Analysis



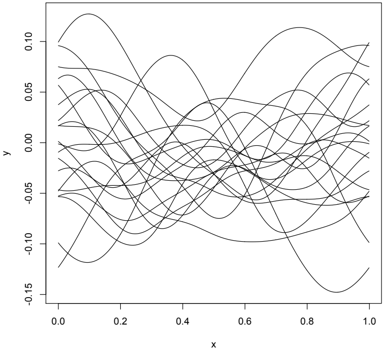
- Register each image to an *atlas* (a.k.a. *template*)
- Simultaneously estimate the atlas
- Statistical analysis of deformation fields
- Atlas provides common coordinates for other data (functional, diffusion, etc.)

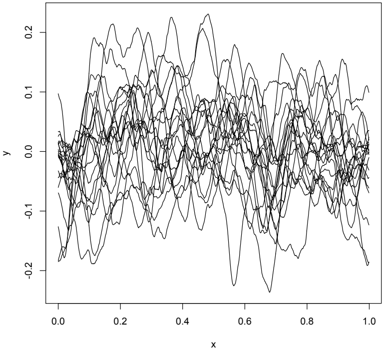
Riemannian Metric on Velocity Fields

Sobolev metric: $\langle v, w \rangle_V = \int_\Omega \langle Lv(x), w(x) \rangle dx$

L is a symmetric, positive-definite differential operator, e.g., $L = (I - \alpha\Delta)^s$.

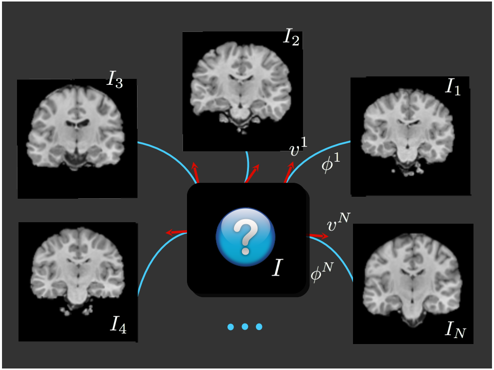
How Smooth Are These Functions?





$$L = (I - 0.02\Delta)^8 \qquad L = (I - 0.01\Delta)$$

Diffeomorphic Atlas Estimation



Minimize: image match + geodesic energy

$$\min_{I, v^i} \sum_{i=1}^{N} \frac{1}{2\sigma^2} \left\| I \circ (\phi^i)^{-1} - I_i \right\|^2 + \|v^i\|_L^2$$

Joshi et al. 2004, Vialard et al. 2011

Bayesian Atlas Estimation

Likelihood: iid Gaussian on each of the M voxels

$$p(I_i \,|\, v^i, I) = \frac{1}{(2\pi)^{M/2}\sigma^M} \exp\left(-\frac{\|I \circ (\phi^i)^{-1} - I_i\|^2}{2\sigma^2}\right)$$

Prior: multivariate Gaussian on discretized velocity v^i

$$p(v^i) = \frac{1}{(2\pi)^{\frac{M}{2}} |L^{-1}|^{\frac{1}{2}}} \exp\left(-\frac{(Lv^i, v^i)}{2}\right)$$

Zhang et al., IPMI 2013

Inference

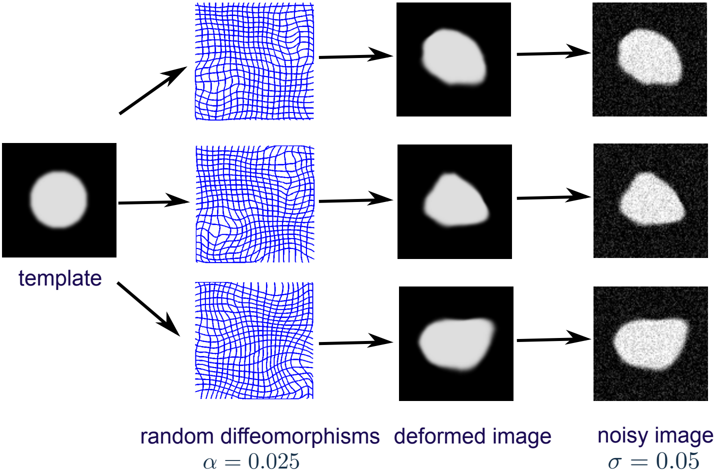
Log posterior:

$$\log \prod_{i=1}^{N} p\left(v^i \mid I_i; \theta\right) \;\propto\; \frac{N}{2} \log |L| - \frac{1}{2} \sum_{i=1}^{N} (Lv^i, v^i)$$

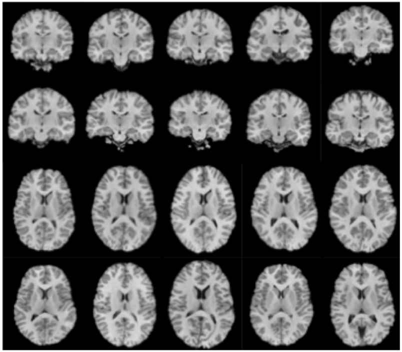
$$- \frac{MN}{2} \log \sigma - \frac{1}{2\sigma^2} \sum_{i=1}^{N} \| I \circ (\phi^i)^{-1} - I_i \|^2.$$

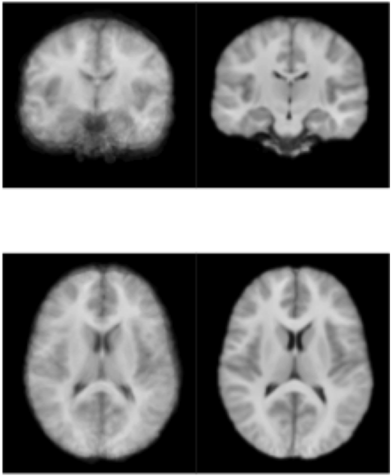
- Treat v^i as **latent random variables**
- Have to integrate the v^i out
- Approximated by Hamiltonian Monte Carlo

Synthetic Shape Generation



Bayesian Atlas Estimation





Input: 3D MR Images

Initialization Bayesian Atlas

What's the Best α?

$$\alpha = 28$$

Atlas

Deformed Atlas to Individual

What's the Best α?

$$\alpha = 2.8$$

Atlas

Deformed Atlas to Individual

What's the Best α?

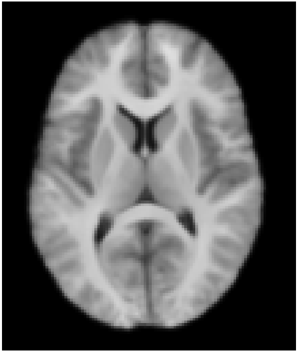
$$\alpha = 0.28$$

Atlas

Deformed Atlas to Individual

What's the Best α?

$$\boxed{\alpha = 0.028}$$

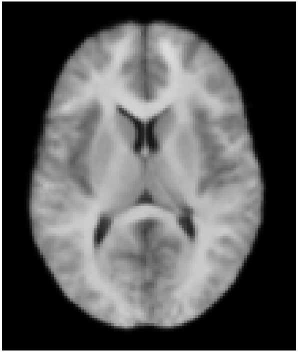


Atlas

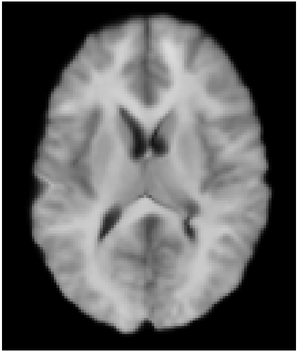
Deformed Atlas to Individual

What's the Best α?

$$\alpha = 0.0028$$



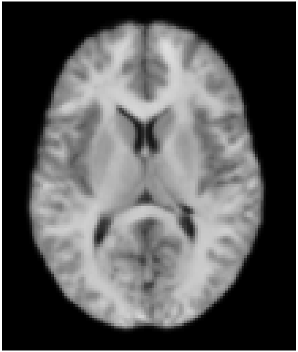
Atlas



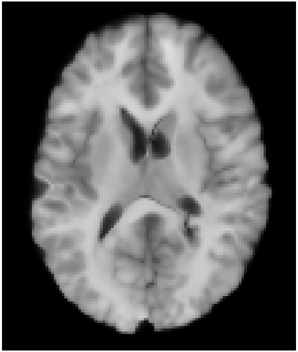
Deformed Atlas to Individual

What's the Best α?

$$\alpha = 0.00028$$

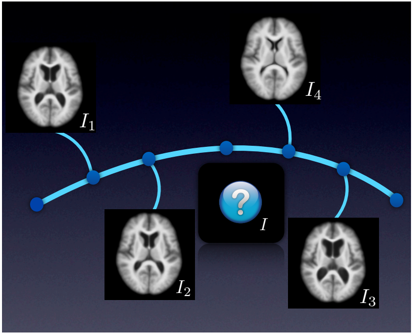


Atlas



Deformed Atlas to Individual

Bayesian Principal Geodesic Analysis



- Estimate *variability* of brain shape
- Sparsity prior determines the dimensionality

Zhang & Fletcher, MICCAI 2014, MedIA 2015

OASIS Data Experiment

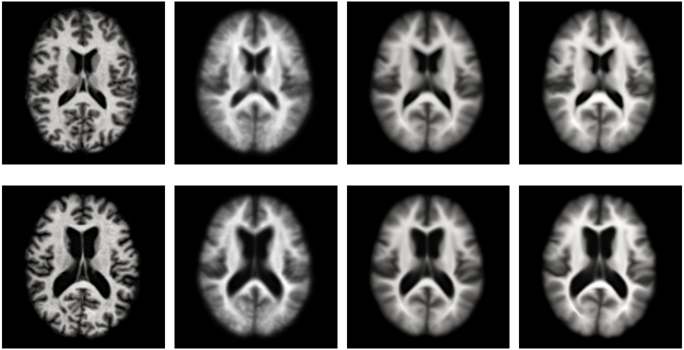
- Train BPGA on 130 MRI from OASIS (ages 60-95)
- How well can we reconstruct 20 unseen images?
- Compared to linear PCA on images (LPCA) and tangent space PGA (TPGA)

BPGA Modes of Variation

Principal Geodesic 1

Principal Geodesic 2

BPGA Result



(a) Observed (b) LPCA (c) TPCA (d) BPGA

	LPCA	TPCA	BPGA
Average MSE	4.2×10^{-2}	3.4×10^{-2}	2.8×10^{-2}
Std of MSE	1.25×10^{-2}	4.8×10^{-3}	4.2×10^{-3}

FLASH

Fourier-approximated

Lie

Algebra

SHooting

Zhang & Fletcher, IPMI 2015

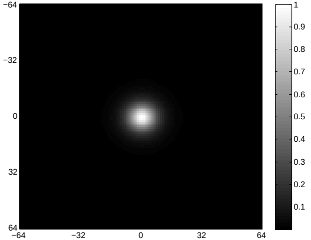
Fast Diffeomorphic Image Registration

Geodesic equation:

$$\frac{\partial v}{\partial t} = -K\left[(Dv)^T m + Dm\, v + m\,\mathrm{div}\, v\right]$$

$m = Lv$ is *momentum*

$K = L^{-1} = (\alpha\Delta + I)^{-s}$



Fourier coefficients of the discretized K operator on a 128×128 grid, with $\alpha = 3$, $s = 3$.

Use bandlimited velocity fields!

Fourier-Approximated Lie Algebra

Lie Algebra in Continuous Domain:
$[v, w] = Dvw - Dwv$

Lie Algebra in Discrete Domain:

$$[\tilde{v}, \tilde{w}] = (\tilde{D}\tilde{v}) \star \tilde{w} - (\tilde{D}\tilde{w}) \star \tilde{v},$$

$\tilde{v}, \tilde{w} \in \tilde{V}$: bandlimited velocity fields in Fourier domain.

Almost a Lie Algebra!

- Closure:
  $\forall \tilde{u}, \tilde{v} \in \tilde{V}, [\tilde{u}, \tilde{v}] \in \tilde{V}$
- Bilinearity:
  $[a\tilde{u} + b\tilde{v}, \tilde{w}] = a[\tilde{u}, \tilde{w}] + b[\tilde{v}, \tilde{w}]$
- Anticommutativity:
  $[\tilde{v}, \tilde{w}] = -[\tilde{w}, \tilde{v}]$
- Jacobi identity: **Doesn't Hold!**
  $[\tilde{u}, [\tilde{v}, \tilde{w}]] + [\tilde{w}, [\tilde{u}, \tilde{v}]] + [\tilde{v}, [\tilde{u}, \tilde{w}]] \neq 0$

EPDiff Equation in $\tilde{V}$

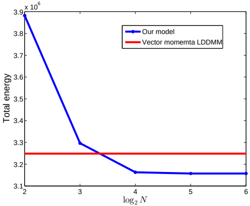
$$\begin{aligned}\frac{\partial \tilde{v}}{\partial t} &= -\operatorname{ad}^T_{\tilde{v}} \tilde{v} \\ &= -\tilde{K}\left[(\tilde{D}\tilde{v})^T \star \tilde{m} + \tilde{D}\tilde{m} \star \tilde{v} + \tilde{m} \star \tilde{\operatorname{div}}(\tilde{v})\right]\end{aligned}$$

$\tilde{L}$: band-limited L operator in Fourier domain

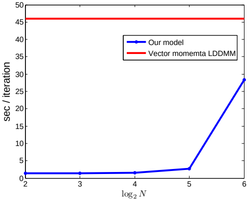
$\tilde{\operatorname{div}}$: Fourier transform of discrete divergence operator

Results on OASIS Brain Data

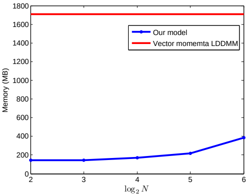
Comparison of pairwise image registration:
LDDMM vs. FLASH



(a) Total energy



(b) Run time



(c) Memory usage

Results on Atlas Building

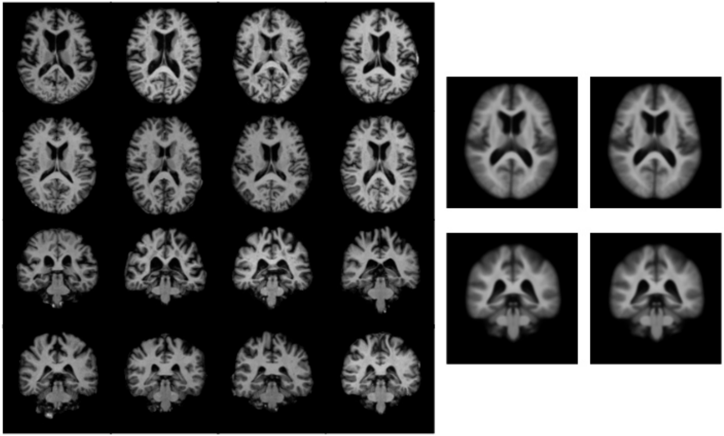


Figure: Left: axial and coronal slices from the 60 OASIS images.
Right: atlas estimated by our model and LDDMM.

Results on Atlas Building

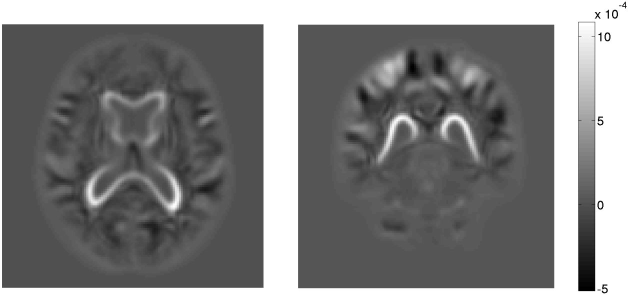


Figure: Atlas intensity difference between LDDMM and our method.

- LDDMM: ~ 2 hours
- Our method: 7.5mins

Open Source Software

Manifold statistics:

`bitbucket.org/vakra/manifoldstatistics`

FLASH:

`bitbucket.org/FlashC/flashc`