THE ABELIANISATION OF THE REAL CREMONA GROUP

SUSANNA ZIMMERMANN

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Abstract. We present the Abelianisation of the birational transformations of $\mathbb{P}^2_{\mathbb{R}}$. Its kernel is equal to the normal subgroup generated by $\mathrm{PGL}_3(\mathbb{R})$, and contains all elements of degree ≤ 4. The description of the quotient yields the existence of normal subgroups of index 2^n for any n and implies that any normal subgroup generated by a countable set of elements is a proper subgroup. This also holds for the group of birational diffeomorphisms respectively of $\mathbb{P}^2_{\mathbb{R}}$, $\mathbb{A}^2_{\mathbb{R}}$ and the sphere.

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1. Introduction

Let $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2) \subset \mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^2)$ be the groups of birational transformations of the projective plane defined over the respective fields of real and complex numbers, and $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) \simeq \mathrm{PGL}_3(\mathbb{R})$, $\mathrm{Aut}_{\mathbb{C}}(\mathbb{P}^2) \simeq \mathrm{PGL}_3(\mathbb{C})$ the respective subgroups of linear transformations.

According to the Noether-Castelnuovo Theorem [Cas1901], the group $\mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^2)$ is generated by $\mathrm{Aut}_{\mathbb{C}}(\mathbb{P}^2)$ and the standard quadratic transformation $\sigma_0\colon [x:y:z] \dashrightarrow [yz:xz:xy]$. As an abstract group, it is not simple [CL2013], i.e. there exist non-trivial, proper normal subgroups $N \subset \mathrm{Bir}(\mathbb{P}^2)$. However, all such groups have uncountable index (see Remark 5.12) and the isomorphism class of the corresponding quotients $\mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^2)/N$ is quite complicated (essentially as complicated as $\mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^2)$ itself). Moreover, the normal subgroup generated by any non-trivial element which preserves a pencil of lines or which has degree $d \leq 4$ is the whole group (see [Giz1994, Lemma 2] and Lemma 5.13) and the group is perfect [CD2013], which means that $\mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^2)$ is equal to its commutator subgroup.

As we will show, the situation for the group $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is quite different. First of all, the group generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) = \mathrm{PGL}_3(\mathbb{R})$ and σ_0 is certainly not the whole group, as all its elements have only real base-points. This is not the case for $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$; for instance the circle inversion $\sigma_1\colon [x:y:z] \dashrightarrow [xz:yz:x^2+y^2]$ of Appolonius has non-real base-points. The group $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ however is generated by $\mathrm{PGL}_3(\mathbb{R})$, σ_0, σ_1, and all standard quintic birational maps (see Definition 2.2, see also [BM2012]).

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Using these generators, we find an explicit presentation of the group $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ (see Theorem 2.9) and a natural quotient, which is our main result.

Theorem 1.1. *The group $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is not perfect: its Abelianisation is isomorphic to*

$$\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)/[\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2),\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)] \simeq \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}.$$

Moreover, the commutator subgroup $[\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2),\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)]$ is the normal subgroup generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) = \mathrm{PGL}_3(\mathbb{R})$, and contains all elements of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ of degree ≤ 4.

In particular, we obtain the following result, similar to the case of $\mathrm{Bir}(\mathbb{P}^n)$, $n \geq 3$ [Pan1999]:

Corollary 1.2. *The group $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is not generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ and a countable set of elements.*

Further, we determine the derived series of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$.

Corollary 1.3. *The sequence of iterated commutator subgroups of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is stationary. More precisely: Let $H := [\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2),\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)]$. Then $[H,H]=H$.*

Let X be a real variety. We denote by $X(\mathbb{R})$ its set of real points of and by $\mathrm{Aut}(X(\mathbb{R})) \subset \mathrm{Bir}(X)$ the subgroup of birational transformations defined at each point of $X(\mathbb{R})$. It is also called the group of *birational diffeomorphisms* of X, and is, in general, strictly larger than the group of automorphisms $\mathrm{Aut}_{\mathbb{R}}(X)$ of X defined over $\mathbb{R}$. The group $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$ is generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ and the standard quintic transformations (see Definition 2.2) [RV2005, BM2012]. In the following, let $\mathbb{P}^3 \supset \mathcal{Q}_{3,1} = \{[y : x : y : z] \in \mathbb{P}^3 \mid x^2+y^2+z^2 = w^2\}$.

Using real birational transformations $f\colon \mathbb{P}^2 \dashrightarrow X$ and the explicit construction of the quotient, we find the the following corollaries.

Corollary 1.4. *There exist surjective group homomorphisms*

$$\mathrm{Aut}(\mathbb{P}^2(\mathbb{R})) \to \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}, \quad \mathrm{Aut}(\mathbb{A}^2(\mathbb{R})) \to \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}, \quad \mathrm{Aut}(\mathcal{Q}_{3,1}(\mathbb{R})) \to \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$$

Corollary 1.5. *The statement in Theorem 1.2 also holds for $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$, $\mathrm{Aut}(\mathbb{A}^2(\mathbb{R}))$ and $\mathrm{Aut}(\mathcal{Q}_{3,1}(\mathbb{R}))$, replacing $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ for the latter two by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{A}^2)$, $\mathrm{Aut}_{\mathbb{R}}(\mathcal{Q}_{3,1})$ respectively..*

Corollary 1.6. *For any real birational map $\psi\colon \mathbb{F}_0 \dashrightarrow \mathbb{P}^2$, the group $\psi\,\mathrm{Aut}(\mathbb{F}_0(\mathbb{R}))\psi^{-1}$ is a subgroup of $\ker(\varphi)$.*

Corollary 1.7. *For any $n \in \mathbb{N}$ there is a normal subgroup of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ of index 2^n containing all elements of degree ≤ 4.*
The same statement holds for $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$, $\mathrm{Aut}(\mathbb{A}^2(\mathbb{R}))$ and $\mathrm{Aut}(\mathcal{Q}_{3,1}(\mathbb{R}))$.

Corollary 1.8. *The normal subgroup of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ generated by any countable set of elements of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is a proper subgroup of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$.*
The same statement holds for $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$, $\mathrm{Aut}(\mathbb{A}^2(\mathbb{R}))$ and $\mathrm{Aut}(\mathcal{Q}_{3,1}(\mathbb{R}))$.

The plan of the article is as follows: After giving the basic definitions and notations in Section 2, we define in Section 3 a surjective group homomorphism from the subgroup $\mathcal{J}_\circ \subset \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ of elements preserving a pencil of conics to the group $\bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$. Section 4 is entirely devoted to the proof of an explicit presentation of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ by generators and relations, given in Theorem 2.9, on which the subsequent section is based. The proof is rather long and technical, and one might skip this section for more comfortable reading and return to it at the end of the paper. In Section 5, we extend the homomorphism to a surjective group homomorphism $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2) \to \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$ and prove Theorem 1.2, Corollary 1.7 and Corollary 1.4. In Section 6, we prove that its kernel is the normal subgroup generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, which will turn out to be commutator subgroup of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$. We will finally be able to prove Theorem 1.1.

In [Pol2015] one can find a description of the elementary links between real rational surfaces and relations between them. However, this description was not used in the proof of Theorem 2.9.

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2. Basic notions

We now give some basic notations and definitions. Throughout the article, every variety and rational map is defined over $\mathbb{R}$, unless stated otherwise.

We recall that a real birational transformation f of $\mathbb{P}^2$ is given by

$$f\colon [x_0 : x_1 : x_2] \dashrightarrow [f_0(x_0,x_1,x_2) : \cdots : f_2(x_0,x_1,x_2)]$$

where $f_0, f_1, f_2 \in \mathbb{R}[x_0,x_1,x_2]$ are homogeneous of equal degree and without common divisor, and f has an inverse of the same form. We say that the pre-image by f of the linear system of lines in $\mathbb{P}^2$ is *the linear system of* f. It is the linear system of curves in $\mathbb{P}^2$ generated by the curves $\{f_0 = 0\}, \{f_1 = 0\}, \{f_2 = 0\}$ and has no fixed components. By abuse of terminology, we refer to its base-points by *base-points of* f. A simple base-points of f is a base-point of multiplicity 1. Some base-points of f might not be points on $\mathbb{P}^2$ but points on a blow-up of $\mathbb{P}^2$. A point lying on the exceptional divisor of a point $q \in \mathbb{P}^2$ is called *infinitely near* q.

Definition 2.1. We define two rational fibrations

$$\begin{aligned} &\pi_*\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \\ &[x:y:z] \mapsto [y:z] \end{aligned}$$

$$\begin{aligned} &\pi_\circ\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \\ &[x:y:z] \mapsto [y^2+(x+z)^2 : y^2+(x-z)^2] \end{aligned}$$

whose fibres are respectively the lines through $[1:0:0]$ and the conics through $p_1 := [1:i:0], p_2 := [0:1:i]$, and their conjugates $\bar{p}_1 = [1:-i:0], \bar{p}_2 = [0:1:-i]$.

We define by $\mathcal{J}_*$, $\mathcal{J}_\circ$ the subgroups of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ preserving the fibrations $\pi_*, \pi_\circ$:

$$\mathcal{J}_* = \{f \in \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2) \mid \exists \hat{f} \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^1)\colon \hat{f}\pi_* = \pi_* f\}$$
$$\mathcal{J}_\circ = \{f \in \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2) \mid \exists \hat{f} \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^1)\colon \hat{f}\pi_\circ = \pi_\circ f\}$$

Extending the scalars to $\mathbb{C}$, the analogues of these groups are conjugate in $\mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^2)$ and are called *de Jonquières groups*. In $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$, the groups $\mathcal{J}_\circ, \mathcal{J}_*$ are not conjugate. This can, for instance, be seen as consequence of Proposition 5.3 (see Remark 5.11).

We define a type of real birational transformation called *standard quintic tranformation*, which has been described often, for instance in [BM2012, §3] [CS2016, Lemma 6.3.10], [Hud1927, §VI.23], [RV2005, §1].

Definition 2.2. Let $q_1, \bar{q}_1, q_2, \bar{q}_2, q_3, \bar{q}_3 \in \mathbb{P}^2$ be three pairs of non-real conjugate points of $\mathbb{P}^2$, not lying on the same conic. Denote by $\pi\colon X \to \mathbb{P}^2$ the blow-up of these points. The strict transforms of the six conics passing through exactly five of the six points are three pairs of non-real conjugate (-1)-curves. Their contraction yields a birational morphism $\eta\colon X \to \mathbb{P}^2$ which contracts the curves onto three pairs of non-real points $r_1, \bar{r}_1, r_2, \bar{r}_2, r_3, \bar{r}_3 \in \mathbb{P}^2$. We choose the order so that r_i is the image of the conic not passing through q_i. The birational map $\varphi := \eta\pi^{-1}$

$$\begin{tikzcd} & X \arrow[dl, "\pi"'] \arrow[dr, "\eta"] & \\ \mathbb{P}^2 \arrow[rr, dashed, "\varphi"] & & \mathbb{P}^2 \end{tikzcd}$$

is contained in $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$, is of degree 5 and is called *standard quintic transformation*.

Standard quintic transformations have the following properties, which can be checked straight forwardly (see also [BM2012, Example 3.1])

Lemma 2.3. *Let $\theta \in \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ be a standard quintic transformation. Then:*

(1) *The points $q_1, \bar{q}_1, q_2, \bar{q}_2, q_3, \bar{q}_3$ are the base-points of θ and $r_1, \bar{r}_1, r_2, \bar{r}_2, r_3, \bar{r}_3$ are the base-points of θ^{-1}, and they are all of multiplicity 2.*
(2) *For $i, j = 1, 2, 3$, $i \neq j$, θ sends the pencil of conics through $q_i, \bar{q}_i, q_j, \bar{q}_j$ onto the pencil of conics through $r_i, \bar{r}_i, r_j, \bar{r}_j$.*
(3) *We have $\theta \in \mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$.*

The family of standard quintic transformations plays an important role in $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$: Let

$$\sigma_0 \colon [x:y:z] \dashrightarrow [yz:xz:xy]$$
$$\sigma_1 \colon [x:y:z] \dashrightarrow [xz:yz:x^2+y^2]$$

Theorem 2.4 ([RV2005],[BM2012]). *The group $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is generated by σ_0, σ_1, $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ and all standard quintic transformations.*

Lemma 2.5. *For any standard quintic transformation θ there exists $\alpha, \beta \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ such that $\beta\theta\alpha \in \mathcal{J}_\circ$.*

Proof. For any two non-collinear non-real pairs of conjugate points there exists $\alpha \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ that sends the two pairs onto $p_1 := [1:i:0], p_2 := [0:1:i]$ and their conjugates $\bar{p}_1 = [1:-i:0], \bar{p}_2 = [0:1:-i]$. Let θ be a standard quintic transformation. Then there exists $\alpha, \beta \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ that send q_1, q_2 (resp. r_1, r_2) onto p_1, p_2. The transformation $\beta\theta\alpha^{-1}$ preserves the pencil of conics through $p_1, \bar{p}_1, p_2, \bar{p}_2$ (Lemma 2.3) and is thus contained in $\mathcal{J}_\circ$. □

Note that $\sigma_0 \in \mathcal{J}_*$ and $\sigma_1 \in \mathcal{J}_\circ$. Thus, Theorem 2.4 implies the following corollary:

Corollary 2.6. *The group $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, $\mathcal{J}_*$, $\mathcal{J}_\circ$.*

Using these generating groups, we can give a representation of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ in terms of generating sets and relations:

Define $S := \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) \cup \mathcal{J}_* \cup \mathcal{J}_\circ$ and let F_S be the free group generated by S. Let $w \colon S \to F_S$ be the canonical word map.

Definition 2.7. We denote by $\mathcal{G}$ be the following group:

$$F_S / \left\langle \begin{array}{ll} w(f)w(g)w(h), & f,g,h \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2),\ fgh = 1 \text{ in } \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) \\ w(f)w(g)w(h), & f,g,h \in \mathcal{J}_*,\ fgh = 1 \text{ in } \mathcal{J}_* \\ w(f)w(g)w(h), & f,g,h \in \mathcal{J}_\circ,\ fgh = 1 \text{ in } \mathcal{J}_\circ \\ \text{the relations in the list below} & \end{array} \right\rangle$$

(rel. 1) Let $\theta_1, \theta_2 \in \mathcal{J}_\circ$ be standard quintic transformations and $\alpha_1, \alpha_2 \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$.

$$w(\alpha_2)w(\theta_1)w(\alpha_1) = w(\theta_2) \text{ in } \mathcal{G} \quad \text{if} \quad \alpha_2\theta_1\alpha_1 = \theta_2.$$

(rel. 2) Let $\tau_1, \tau_2 \in \mathcal{J}_* \cup \mathcal{J}_\circ$ both of degree 2 or of degree 3 and $\alpha_1, \alpha_2 \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$.

$$w(\tau_1)w(\alpha_1) = w(\alpha_2)w(\tau_2) \text{ in } \mathcal{G} \quad \text{if} \quad \tau_1\alpha_1 = \alpha_2\tau_2.$$

(rel. 3) Let $\tau_1, \tau_2, \tau_3 \in \mathcal{J}_*$ all of degree 2, or τ_1, τ_2 of degree 2 and τ_3 of degree 3, and $\alpha_1, \alpha_2, \alpha_3 \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$.

$$w(\tau_2)w(\alpha_1)w(\tau_1) = w(\alpha_3)w(\tau_3)w(\alpha_2) \quad \text{in} \quad \mathcal{G} \text{ if } \tau_2\alpha_1\tau_1 = \alpha_3\tau_3\alpha_2.$$

Remark 2.8. The generalised amalgamated product of $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, $\mathcal{J}_*$, $\mathcal{J}_\circ$ along all pairwise intersections is the quotient of the free product of the three groups modulo the relations given by the pairwise intersections.

Note that the group $\mathcal{G}$ is isomorphic to the quotient of the generalised amalgamated product of $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, $\mathcal{J}_*$, $\mathcal{J}_\circ$ along all intersections by the relations in the above list.

Since $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2), \mathcal{J}_*, \mathcal{J}_\circ$ (Corollary 2.6), there exists a natural surjective group homomorphism $F_S \to \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ which gives rise to a group homomorphism $\mathcal{G} \to \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$, since all relations above hold in $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$.

Theorem 2.9. *The natural surjective group hommormophism $\mathcal{G} \to \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is an isomorphism.*

The proof of Theorem 2.9 is quite long and technical, so that the whole Section 4 is devoted to it. For more comfortable reading, one may skip it and return to it at the end of the paper. The method used in the proof has been described in [Bla2012], [Isk1985] and [Zim2015], and is to study linear systems of birational transformations of $\mathbb{P}^2$ and their base-points.

We now give some further notation used throughout the article.

Definition 2.10. Let $f \in \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ and p be a point that belongs to $\mathbb{P}^2$ as a proper or infinitely near point. Assume moreover that p is not a base-point of f. We define a point $f_\bullet(p)$, which will also be in $\mathbb{P}^2$ or infinitely near. For this, take a minimal resolution of f

$$\begin{tikzcd} & S \arrow[dl, "\nu_1"'] \arrow[dr, "\nu_2"] & \\ \mathbb{P}^2 \arrow[rr, dashed, "f"] & & \mathbb{P}^2 \end{tikzcd}$$

where ν_1, ν_2 are sequences of blow-ups. Since p is not a base-point of f it corresponds via ν_1 to a point of S or infinitely near. Using ν_2 we view this point on $\mathbb{P}^2$, again maybe infinitely near, and call it $f_\bullet(p)$.

Remark 2.11. Note that $f_\bullet$ is a one-to-one correspondence between the sets

$$(\mathbb{P}^2 \cup \{\text{infinitely near points}\}) \setminus \{\text{base-points of } f\} \quad \text{and}$$
$$(\mathbb{P}^2 \cup \{\text{infinitely near points}\}) \setminus \{\text{base-points of } f^{-1}\}$$

Furthermore, if p is a base-point of a linear system Λ of multiplicity m that is not base-point of f, then $f_\bullet(p)$ is a base-point of $f(\Lambda)$ of multiplicity m.

Lemma 2.12 ([AC2002, Corollary 4.1.14])**.** *Let $f, g \in \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ and let p be a base-point of f of multiplicity m that is not a base-point of g^{-1}. Then $(g^{-1})_\bullet(p)$ is a base-point of fg of multiplicity m.*

Definition 2.13.

(1) Let $C \subset \mathbb{P}^2$ be an irreducible (closed) curve, $f \in \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ and $\mathrm{Bp}(f)$ the set of base-points of f. We denote by
$$f(C) := \overline{f(C \setminus \mathrm{Bp}(f))}$$
the (Zariski-) closure of the image by f of C minus the base-points of f, and call it *the image of C by f*.

(2) Throughout the article, we fix the notation
$$p_1 := [1 : i : 0], \quad p_2 := [0 : 1 : i]$$
for these two specific points of $\mathbb{P}^2$, because we will use them extremely often.

(3) The following definition will be used for base-points of elements of $\mathcal{J}_\circ$. Let $\eta\colon X \to \mathbb{P}^2$ be the blow-up of $p_1, \bar{p}_1, p_2, \bar{p}_2$. The morphism $\tilde{\pi}_\circ := \pi_\circ \eta\colon X \to \mathbb{P}^1$ is a real conic bundle with fibres being the strict transforms of the conics passing through $p_1, \ldots, \bar{p}_2$.

$$\begin{tikzcd} X \arrow[d, "\eta"'] \arrow[dr, "\tilde{\pi}_\circ"] & \\ \mathbb{P}^2 \arrow[r, dashed, "\pi_\circ"'] & \mathbb{P}^1 \end{tikzcd}$$

Let $\eta'\colon Y \to X$ be a birational morphism and $q \in Y$. We define
$$C_q := \pi_\circ^{-1}(\tilde{\pi}_\circ(\eta'(q))) \subset \mathbb{P}^2.$$
It is the conic passing through $p_1, \bar{p}_1, p_2, \bar{p}_2, \eta'(q)$, which is irreducible or the union of two lines. The latter case corresponds to $\tilde{\pi}_\circ(\eta'(q)) \in \{[1 : 0], [0 : 1], [1 : 1]\}$. We define
$$C_1 := \pi_\circ^{-1}([0 : 1]), \quad C_2 := \pi_\circ^{-1}([1 : 0]), \quad C_3 := \pi_\circ^{-1}([1 : 1]).$$
If we denote by $L_{p,q} \subset \mathbb{P}^2$ the line passing through $r, s \in \mathbb{P}^2$, then
$$C_1 = L_{p_1,p_2} \cup L_{\bar{p}_1,\bar{p}_2}, \quad C_2 = L_{p_1,\bar{p}_2} \cup L_{\bar{p}_1,p_2}, \quad C_3 = L_{p_1,\bar{p}_1} \cup L_{p_2,\bar{p}_2}.$$

3. A quotient of $\mathcal{J}_\circ$

We first construct a surjective group homomorphism $\varphi_\circ\colon \mathcal{J}_\circ \to \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$ and then (in Section 5) use the representation of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ by generators and relations (Theorem 2.9) to extend $\varphi_\circ$ to a homomorphism $\varphi\colon \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2) \to \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$. Both quotients are generated by classes of standard quintic transformations contained in $\mathcal{J}_\circ$, as we will see from the construction in Subsection 3.2.

In order to construct the surjective homomorphism $\mathcal{J}_\circ \to \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$, we first need some additional information about the elements of $\mathcal{J}_\circ$, such as their characteristic (Lemma 3.1) and their action on the pencil of conics passing through $p_1, \bar{p}_1, p_2, \bar{p}_2$ (Lemma 3.9).

3.1. **The group $\mathcal{J}_\circ$.** The next lemmata state the characteristic and some other properties of the elements of $\mathcal{J}_\circ$ (recall that for $f \in \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$, the characteristic of f is the sequence $(\deg(f); m_1^{e_1}, \ldots, m_k^{e_k})$ where $m_1, \ldots, m_k$ are the multiplicities of the base-points of f and e_i is the number of base-points of f which have multiplicity m_i (see [AC2002, Definition 2.1.7])). We will use these properties to obtain the action of $\mathcal{J}_\circ$ on the pencil of conics through $p_1, \ldots, \bar{p}_2$. The information will be used to construct the quotients. In Section 4 (proof of Theorem 2.9), we will use the properties to study linear systems and their base-points in connection with the relations given in Definition 2.7.

Lemma 3.1. *Any element of $\mathcal{J}_\circ$ of degree $d > 1$ has characteristic:*

$$\left(d;\ \frac{d-1}{2}^4,\ 2^{\frac{d-1}{2}}\right), \quad \text{if } \deg(f) \text{ is odd}$$
$$\left(d;\ \frac{d}{2}^2,\ \frac{d-2}{2}^2,\ 2^{\frac{d-2}{2}},\ 1\right), \quad \text{if } \deg(f) \text{ is even}$$

and $p_1, \ldots, \bar{p}_2$ are base-points of multiplicity $\frac{d}{2}, \frac{d-1}{2}$ or $\frac{d-2}{2}$.

Furthermore,

1. *no two double points are contained in the same conic through $p_1, \bar{p}_1, p_2, \bar{p}_2$,*
2. *any element of $\mathcal{J}_\circ$ exchanges or preserves the real reducible conics C_1 and C_2, and does not contract their components.*
3. *any element of $\mathcal{J}_\circ$ of even degree contracts one of the lines $L_{p_i,\bar{p}_i}$, $i \in \{1,2\}$ onto a point on a real conic different from C_1, C_2.*
4. *For every characteristic as above, there is an element of $\mathcal{J}_\circ$ having that characteristic.*

For example, the standard quintic transformations contained in $\mathcal{J}_\circ$ have characteristic $(5; 2^4, 2^2)$ by Definition 2.2 and Lemma 2.3.

Proof. Let $f \in \mathcal{J}_\circ$ be of degree $d > 1$. Let C be a general conic passing through $p_1, \bar{p}_1, p_2, \bar{p}_2$. By definition of $\mathcal{J}_\circ$, the curve $f(C)$ is a conic through $p_1, \bar{p}_1, p_2, \bar{p}_2$. Let $m(q)$ be the multiplicity of f at the point q. Computing the intersection of C on the blow-up of the base-points of f with the linear system of f gives the degree of $f(C)$:

$$2 = \deg(f(C)) = 2d - 2m(p_1) - 2m(p_2) = (d - 2m(p_1)) + (d - 2m(p_2)).$$

Computing the intersection of the linear system of f with the line $L_{p_i,\bar{p}_i}$, we obtain that $d \geq 2m(p_i)$, $i = 1, 2$.

If $d - 2m(p_1) = d - 2m(p_2) = 1$, then

$$m(p_1) = m(p_2) = \frac{d-1}{2}.$$

Else, $d - 2m(p_i) = 0$, $d - 2m(p_{3-i}) = 2$ for some $i \in \{1,2\}$, and so

$$m(p_i) = \frac{d}{2},\ m(p_{3-i}) = \frac{d-2}{2}, \quad i \in \{1,2\}.$$

Let q be a base-point of f not equal to $p_1, \bar{p}_1, p_2, \bar{p}_2$ and C_q its associated conic through $p_1, \bar{p}_1, p_2, \bar{p}_2$ (see Definition 2.13). Then $2 \geq \deg(f(C_q)) \geq 0$ and

$$0 \leq \deg(f(C_q)) \leq 2d - 2m(p_1) - 2m(p_2) - m(q) = 2 - m(q) \leq 2$$

In particular, $m(q) \in \{1, 2\}$. Let D be a general member of the linear system of f. The genus formula

$$0 = g(D) = \frac{(d-1)(d-2)}{2} - \sum_{q \text{ base-point of } f} \frac{m(q)(m(q)-1)}{2}$$

and $m(q) \in \{1, 2\}$ for all base-points q of f different from $p_1, \bar{p}_1, p_2, \bar{p}_2$ imply that

$$\frac{(d-1)(d-2)}{2} = 2\sum_{i=1}^{2} \frac{m(p_i)(m(p_i)-1)}{2} + |\{\text{base-points of multiplicity } 2\}|$$

and in particular that

$$|\{\text{base-points of multiplicity } 2\}| = \begin{cases} \frac{d-1}{2}, & d \text{ odd} \\ \frac{d-2}{2}, & d \text{ even} \end{cases}$$

Intersecting two general elements of the linear system of f, we get the classical equality

$$d^2 - 1 = \sum_{q \text{ base-point of } f} m(q)^2.$$

It yields that f has exactly one simple base-point if d is even and none otherwise. This yields the characteristics.

Calculating the intersection of a conic through $p_1, \bar{p}_1, p_2, \bar{p}_2$ with a general curve of the linear system of f implies that no two double points are contained in the same conic through $p_1, \bar{p}_1, p_2, \bar{p}_2$. The conics C_1, C_2, C_3 are the only reducible conics through $p_1, \dots, \bar{p}_2$, and C_1, C_2 each consist of two non-real lines while C_3 consists of two real lines.

If f has even degree, it has base-points $p_i, \bar{p}_i$ of multiplicity $m(p_i) = \frac{d}{2}$ and therefore contracts the line $L_{p_i,\bar{p}_i}$ onto the base-point of f^{-1} of multiplicity 1, and no other line is contracted (because f^{-1} has only one base-point of multiplicity 1). Because of this and the multiplicities of the base-points of f, f sends $L_{p_i,p_j}, L_{p_i,\bar{p}_j}, i \neq j$, onto non-real lines. This is also true if f has odd degree (simply because of the multiplicities of its base-points). Thus f preserves or exchanges C_1, C_2.

In particular, the induced automorphism $\hat{f}$ on $\mathbb{P}^1$ does not send $\pi_\circ(C_3)$ onto either of $\pi_\circ(C_1), \pi_\circ(C_2)$.

It follows that if f has even degree, the point $f(L_{p_i,\bar{p}_i})$ is contained in the conic $\pi_\circ^{-1}(\hat{f}(\pi_\circ(C_3))) \neq C_1, C_2$. In particular, the simple base-point of f^{-1} (which is $f(L_{p_i,\bar{p}_i})$) is not contained in C_1, C_2. By symmetry, the same holds for f.

Last but not least, we obtain every characteristic by composing quadratic transformations and standard quintic transformations contained in $\mathcal{J}_\circ$ □

Remark 3.2. The group $\mathcal{J}_\circ$ contains standard quintic transformations (Lemma 2.5). Remark that $\sigma_1 \colon [x:y:z] \dashrightarrow [xz : yz : x^2+y^2]$ is contained in $\mathcal{J}_\circ$.

The linear map $[x:y:z] \mapsto [z:-y:x]$ exchanges p_1 and p_2 (and $\bar{p}_1$ and $\bar{p}_2$), and the linear map $[x:y:z] \mapsto [-x:y:z]$ exchanges p_1 and $\bar{p}_1$ and fixes p_2. Both are contained in $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) \cap \mathcal{J}_\circ$.

Lemma 3.3. *For any $q \in \mathbb{P}^2(\mathbb{R})$ not collinear with any two of $\{p_1, \bar{p}_1, p_2, \bar{p}_2\}$ except the pair $(p_2, \bar{p}_2)$, there exists $f \in \mathcal{J}_\circ$ of degree 2 with base-points $p_1, \bar{p}_1, q$.*

In particular: Let $f \in \mathcal{J}_\circ$ of even degree d, the points $p_i, \bar{p}_i$ its base-points of multiplicity $\frac{d}{2}$ and r its simple base-point or the proper point of $\mathbb{P}^2$ to which the simple base-point is infinitely near.

Then there exists $\tau \in \mathcal{J}_\circ$ of degree 2 with base-points $p_i, \bar{p}_i, r$.

Proof. Since q is not collinear with $p_1, \bar{p}_1$, there exists $\alpha \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ that sends $p_1, \bar{p}_1, q$ onto $p_1, \bar{p}_1, [0:0:1]$. Let $t := (\sigma_1\alpha)^\bullet(p_2)$. The quadratic transformation $\sigma_1\alpha$ has base-points $p_1, \bar{p}_1, q$ and sends the pencil of conics through $p_1, \bar{p}_1, p_2, \bar{p}_2$ onto the pencil of conics through $p_1, \bar{p}_1, t, \bar{t}$. By assumption, the point p_2 is not on the lines $L_{q,p_1}, L_{q,\bar{p}_1}$ and thus $t, \bar{t}$ are proper points of $\mathbb{P}^2$ that are not collinear with $p_1, \bar{p}_1$. There exists $\beta \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ that fixes $p_1, \bar{p}_1$ and sends $t, \bar{t}$ onto $p_2, \bar{p}_2$. The quadratic transformation $\beta\sigma_1\alpha$ has base-points $p_1, \bar{p}_1, q$ and sends the pencil of conics through $p_1, \bar{p}_1, p_2, \bar{p}_2$ onto itself, i.e. is contained in $\mathcal{J}_\circ$.

Let $f \in \mathcal{J}_\circ$ of even degree d, $p_i, \bar{p}_i$ its base-points of multiplicity $\frac{d}{2}$ and r its simple base-point or the proper point of $\mathbb{P}^2$ to which the simple base-point is infintely near. By Bézout, $r, p_i, \bar{p}_i$ are not collinear

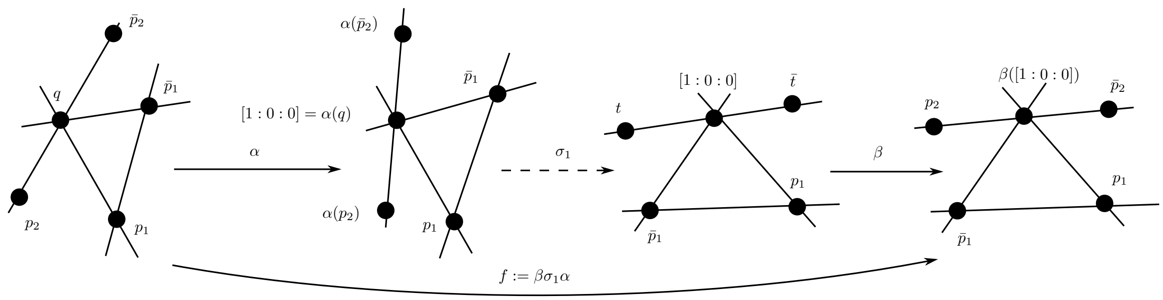

FIGURE 1. The construction of the quadratic map f with base-points $p_1, \bar{p}_1, q$.

and by Lemma 3.1 the points r, p_i, p_{3-i} and $r, \bar{p}_i, p_{3-i}$ are not collinear. Hence there exists $\tau \in \mathcal{J}_\circ$ of degree 2 with base-points $r, p_i, \bar{p}_i$. □

A similar statement holds for transformations of degree 3, which we will use in Section 4 and Section 6.

Lemma 3.4. *For every $r \in \mathbb{P}^2(\mathbb{R})$ not collinear with any two of $p_1, \bar{p}_1, p_2, \bar{p}_2$ there exists $f \in \mathcal{J}_\circ$ of degree 3 with base-points $r, p_1, \bar{p}_1, p_2, \bar{p}_2$ (with double point r).*

Proof. Since r is not collinear with any two of $p_1, \bar{p}_1, p_2, \bar{p}_2$, there exists $\tau_1 \in \mathcal{J}_\circ$ quadratic with base-points $r, p_1, \bar{p}_1$ (Lemma 3.3). The base-points of its inverse are $s, p_i, \bar{p}_i$ for some $s \in \mathbb{P}^2(\mathbb{R})$ and $i \in \{1, 2\}$. We can assume that $i = 1$ by exchanging p_1, p_2 if necessary (Remark 3.2). Since $r, p_2, \bar{p}_2$ are not collinear, also $s, p_2, \bar{p}_2$ are not collinear because τ_1 sends the lines through r onto the lines through s and preserves $\{p_2, \bar{p}_2\}$. Moreover, s is not collinear with p_1, p_2 because $(\tau_1^{-1})_\bullet(p_2) \in \{p_2, \bar{p}_2\}$ is a proper point of $\mathbb{P}^2$. Hence there exists $\tau_2 \in \mathcal{J}_\circ$ of degree 2 with base-point $s, p_2, \bar{p}_2$ (Lemma 3.3). The map $\tau_2\tau_1 \in \mathcal{J}_\circ$ is of degree 3 with base-points $r, p_1, \bar{p}_1, p_2, \bar{p}_2$. □

To prove the next lemma (Lemma 3.7), we are forced to introduce another kind of quintic transformation, which is just a degeneration of standard quintic transformations. They will pop up again in Section 6, where we look at relations between quadratic and standard quintic transformations in order to prove that the kernel of the Abelianisation map is equal to the normal subgroup generated by $\mathrm{Aut}_\mathbb{R}(\mathbb{P}^2)$.

We define a type of real birational transformation called *special quintic transformation.*

Definition 3.5. Let $q_1, \bar{q}_1, q_2, \bar{q}_2 \in \mathbb{P}^2$ be two pairs of non-real points of $\mathbb{P}^2$, not on the same line. Denote by $\pi_1 : X_1 \to \mathbb{P}^2$ the blow-up of the four points, and by $E_1, \bar{E}_1 \subset X_1$ the curves contracted onto $q_1, \bar{q}_1$ respectively. Let $q_3 \in E_1$ be a point, and $\bar{q}_3 \in \bar{E}_1$ its conjugate. We assume that there is no conic of $\mathbb{P}^2$ passing through $q_1, \bar{q}_1, q_2, \bar{q}_2, q_3, \bar{q}_3$ and let $\pi_2 \colon X_2 \to X_1$ be the blow-up of $q_3, \bar{q}_3$.

On X_2 the strict transforms of the two conics $C, \bar{C}$ of $\mathbb{P}^2$ passing through $q_1, \bar{q}_1, q_2, \bar{q}_2, q_3$ and $q_1, \bar{q}_1, q_2, \bar{q}_2, \bar{q}_3$ respectively, are non-real conjugate disjoint (-1) curves. The contraction of these two curves gives a birational morphism $\eta_2 \colon X_2 \to Y_1$, contracting $C, \bar{C}$ onto two points $r_3, \bar{r}_3$. On Y_1 we find two pairs of non-real (-1) curves, all four curves being disjoint: These are the strict transforms of the exceptional curves associated to $q_1, \bar{q}_1$, and of the conics passing through $q_1, \bar{q}_1, q_2, q_3, \bar{q}_3$ and $q_1, \bar{q}_1, \bar{q}_2, q_3, \bar{q}_3$ respectively. The contraction of these curves gives a birational morphism $\eta_1 : Y_1 \to \mathbb{P}^2$ and the images of the four curves are points $r_1, \bar{r}_1, r_2, \bar{r}_2$ respectively. The real birational map $\psi = \eta_1\eta_2(\pi_1\pi_2)^{-1} \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is of degree 5 and called *special quintic transformation.*

Remark 3.6. Let θ be a special quintic transformation and keep the notation of its definition. With similar argument as for the standard quintic transformations (Lemma 2.3) one shows that $q_1, \dots, \bar{q}_3$ are the base-points of θ, and are of multiplicity 2. Furthermore, θ sends the pencil of conics through $q_1, \bar{q}_1, q_2, \bar{q}_2$ onto the pencil of conics through $r_1, \bar{r}_1, r_2, \bar{r}_2$ and $\theta \in \mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$.

Lemma 3.7. *The group $\mathcal{J}_\circ$ is generated by its linear, quadratic and standard quintic elements.*

Proof. Let $f \in \mathcal{J}_\circ$. We use induction on the degree d of f. We can assume that $d > 2$.

• If d is even, it has a (real) simple base-point. Denote by r the simple base-point of f or, if the simple base-points is not a proper point of $\mathbb{P}^2$, the proper point of $\mathbb{P}^2$ to which the simple base-point is infinitely near to. Let $p_i, \bar{p}_i$, $i \in \{1,2\}$ be the points of multiplicity $\frac{d}{2}$ (Lemma 3.1). By Lemma 3.3 there exists a quadratic transformation $\tau \in \mathcal{J}_\circ$ with base-points $p_i, \bar{p}_i, r$. The map $f\tau^{-1} \in \mathcal{J}_\circ$ is of degree $\leq d-1$.

• Suppose that d is odd and has a real base-point q. By Lemma 3.1, the points q, p_1, p_2 are of multiplicity 2, $\frac{d-1}{2}$, $\frac{d-1}{2}$ respectively. We can assume that q is a proper point of $\mathbb{P}^2$ (since no real point is infinitely near $p_1, \ldots, \bar{p}_2$). By Bézout, q is not collinear with p_i, p_j, $i, j \in \{1,2\}$, and so there exists $\tau \in \mathcal{J}_\circ$ of degree 2 with base-points $q, p_1, \bar{p}_1$ (Lemma 3.3). The map $f\tau^{-1} \in \mathcal{J}_\circ$ is of degree $d-1$.

• Suppose that d is odd and has no real base-points. Then $d \geq 5$ by Lemma 3.1. If it has a double point q different from $p_1, \ldots, \bar{p}_2$ which is a proper point of $\mathbb{P}^2$ then $p_1, \bar{p}_1, p_2, \bar{p}_2, q, \bar{q}$ are not on the same conic (Lemma 3.1). In particular, there exists a standard quintic transformation $\theta \in \mathcal{J}_\circ$ with those points its base-points (Definition 2.2, Lemma 2.5). The map $f\theta^{-1} \in \mathcal{J}_\circ$ is of degree $d-4$.

If it has no double points that are proper points of $\mathbb{P}^2$, there exists a double point q infinitely near one of the p_i. By Lemma 3.1, $p_1, \bar{p}_1, p_2, \bar{p}_2, q, \bar{q}$ are not contained on one conic, hence there exists a special quintic transformation $\theta \in \mathcal{J}_\circ$ with base-points $p_1, \bar{p}_1, p_2, \bar{p}_2, q, \bar{q}$ (Definition 3.5). The map $f\theta^{-1} \in \mathcal{J}_\circ$ is of degree $d-4$. By [BM2012, Lemma 3.7] and Remark 3.2, θ is the composition of standard quintic and linear transformations contained in $\mathcal{J}_\circ$. □

Recall that for each element $f \in \mathcal{J}_\circ$ there exists $\hat{f} \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^1)$ such that $\hat{f} \circ \pi_\circ = \pi_\circ \circ f$ (Definition 2.1). This induces a group homomorphism $\mathcal{J}_\circ \to \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^1)$ given by $f \mapsto \hat{f}$ (see Definition 2.1). Lemma 3.9 states that this action corresponds to a real scaling and that every scaling can be realised by a quadratic transformation. The cubic and standard quintic transformations scale by ± 1.

By $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^1, [0,\infty])$ we denote the subgroup of $\mathrm{PGL}_2(\mathbb{R})$ that preserves the real interval $[0,\infty]$ in $\mathbb{P}^1(\mathbb{R})$. The following short lemma will be used in the proof Lemma 3.9.

Lemma 3.8. *The rational map $\pi_\circ$ maps the set $\mathbb{P}^2(\mathbb{R}) \setminus \{[1:0:1],[1:0:-1]\}$ onto the set $\{[a:1] \in \mathbb{P}^1(\mathbb{R}) \mid a > 0\} \simeq \mathbb{R}_{>0}$.*

Proof. The set of points where $\pi_\circ \colon [x:y:z] \dashrightarrow [y^2+(x+z)^2 : y^2+(x-z)^2]$ is not defined is $\{p_1, \bar{p}_1, p_2, \bar{p}_2\}$, hence $\pi_\circ$ is defined on $\mathbb{P}^2(\mathbb{R})$ and continuous on it. Thus $\pi_\circ(\mathbb{P}^2(\mathbb{R}))$ is a connected subset of $\{[a:1] \in \mathbb{P}^1(\mathbb{R}) \mid a \geq 0\} \cup \{[1:0]\} \subset \mathbb{P}^1(\mathbb{R})$. The claim now follows with $\pi_\circ([1:0:1]) = [1:0]$ and $\pi_\circ([1:0:-1]) = [0:1]$. □

Lemma 3.9. *The action of $\mathcal{J}_\circ$ on $\mathbb{P}^1$ gives rise to a surjective homomorphism*

$$\mathcal{J}_\circ \to \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^1, [0,\infty]) \simeq \mathbb{R}_{>0} \ltimes \mathbb{Z}/2\mathbb{Z}$$

where $\mathbb{R}_{>0} \subset \mathrm{PGL}_2(\mathbb{R})$ is given by diagonal maps $[x:y] \mapsto [ax:by]$, $a, b \in \mathbb{R}_{>0}$ and $\mathbb{Z}/2\mathbb{Z}$ is generated by $[x:y] \mapsto [y:x]$.

Moreover, any element of $\mathbb{R}_{>0}$ is the image of a quadratic element of $\mathcal{J}_\circ$ and $\mathbb{Z}/2\mathbb{Z}$ is the image of a linear element.

Furthermore:

• *The cubic transformations are sent onto $(1,0)$ if they contract $L_{p_i,q}$ onto p_i or $\bar{p}_i$, $i = 1,2$, where q is the double point, and onto $(1,1)$ otherwise.*

• *The standard quintic transformations are sent onto $(1,0)$ or $(1,1)$.*

Proof. There are exactly three real reducible conics passing through $p_1, \bar{p}_1, p_2, \bar{p}_2$, namely C_1, C_2 and C_3, and their images by $\pi_\circ \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ are

$$\pi_\circ(C_1) = [0:1], \quad \pi_\circ(C_2) = [1:0], \quad \pi_\circ(C_3) = [1:1].$$

Let $f \in \mathcal{J}_\circ$ and $\hat{f}$ be the induced automorphism on $\mathbb{P}^1$. By Lemma 3.1, f preserves or exchanges C_1, C_2, which yields that $\hat{f}$ is of the form $\hat{f} \colon [u:v] \mapsto [au:bv]$ or $\hat{f} \colon [u:v] \mapsto [av:bu]$, $a, b \in \mathbb{R}^*$, where $[a:b] = \hat{f}(\pi_\circ(C_3)) = \pi_\circ(f(C_3))$. This yields a homomorphism

$$\psi \colon \mathcal{J}_\circ \to \mathbb{R}^* \rtimes \mathbb{Z}/2\mathbb{Z}.$$

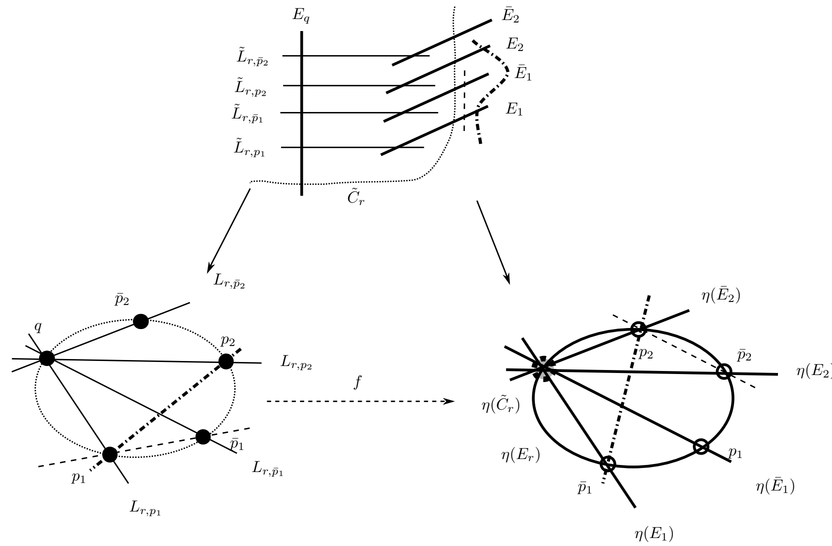

FIGURE 2. A cubic transformation that contracts $L_{p_i,q}$ onto p_i.

Lets show that the image of ψ is $\mathbb{R}_{>0} \rtimes \mathbb{Z}/2\mathbb{Z}$ and that any element of $\mathbb{R}_{>0}$ is the image of a quadratic transformation.

By Lemma 3.7, the group $\mathcal{J}_\circ$ is generated by its linear, quadratic and standard quintic elements. The map $\gamma\colon [x:y:z] \mapsto [-x:y:z]$ induces $\hat{\gamma}\colon [u:v] \mapsto [v:u]$, i.e. $\psi(\gamma) = (0,1)$. The linear and standard quintic transformations preserve C_3 and hence induce Id or $[x:y] \mapsto [y:x]$ on $\mathbb{P}^1$, hence are sent onto $(1,0)$ or $(1,1)$. Let $\tau \in \mathcal{J}_\circ$ be a quadratic transformation. It has base-points $p_i, \bar{p}_i, q$, for some $i \in \{1,2\}$, and sends $p_{3-i}, \bar{p}_{3-i}$ onto proper points of $\mathbb{P}^2$. In particular, q is not collinear with any two of $p_1, \bar{p}_1, p_2, \bar{p}_2$ except maybe $p_{3-i}, \bar{p}_{3-i}$. It follows that $q \in \mathbb{P}^2(\mathbb{R}) \setminus \{[1:0:1],[1:0:-1]\}$. On the other hand, take $q = [a:b:1] \in \mathbb{P}^2(\mathbb{R}) \setminus \{[1:0:1],[1:0:-1]\}$. Then q is not collinear with any two of $p_1, \bar{p}_1, p_2, \bar{p}_2$, except maybe $p_2, \bar{p}_2$. By Lemma 3.3 there exists a quadratic transformation $\tau \in \mathcal{J}_\circ$ with base-points $q, p_1, \bar{p}_1$.

We have $\pi_\circ(\tau^{-1}(C_3)) = \pi_\circ(q) = [b^2 + (a+1)^2 : b^2 + (a-1)^2]$, which is not equal to $[0:1],[1:0]$. In particular, $\psi(\tau^{-1}) \in \mathbb{R}_{>0} \rtimes \mathbb{Z}/2\mathbb{Z}$, and it follows that $\psi(\mathcal{J}_\circ) \subset \mathbb{R}_{>0} \rtimes \mathbb{Z}/2\mathbb{Z}$.

Note that $\mathrm{pr}_1(\psi(\tau)) = \pi_\circ(q)$, so the image by $(\mathrm{pr}_1 \circ \psi)$ of the set of quadratic elements of $\mathcal{J}_\circ$ is equal to the image by $\pi_\circ$ of the set $\mathbb{P}^2(\mathbb{R}) \setminus \{[1:0:1],[1:0:-1]\}$. By Lemma 3.8, the image is equal to $\{[a:1] \in \mathbb{P}^1(\mathbb{R}) \mid a > 0\} \simeq \mathbb{R}_{>0}$.

In conclusion, every element of $\mathbb{R}_{>0}$ is the image of a quadratic element of $\mathcal{J}_\circ$, and ψ has image $\mathbb{R}_{>0} \rtimes \mathbb{Z}/2\mathbb{Z}$.

To complete the proof of the lemma, we calculate the image of C_1, C_2, C_3 under a cubic transformation f that contracts $L_{p_i,q}$ onto p_i or $\bar{p}_i$, $i = 1,2$, where q is the base-point of f of multiplicity 2. This way, we obtain that f preserves C_3 and preserves or exchanges C_1, C_2. $\square$

3.2. **The quotient.** Using Lemma 3.9, we now construct a surjective group homomorphism $\mathcal{J}_\circ \to \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$.

There are two constructions of the quotient - one geometrical and the other using the spinor norm on $\mathrm{SO}(x^2+y^2-tz^2, \mathbb{R}(t))$. We first give the geometrical construction and then the one via the spinor norm.

Definition 3.10. Let $f \in \mathcal{J}_\circ$. For any *non-real* base-point q of f, we have $\pi_\circ(C_q) = [a+ib:1]$ and $\pi_\circ(C_{\bar{q}}) = [a-ib:1]$ for some $a,b \in \mathbb{R}$, $b \neq 0$ (see Definition 2.13 for the definition of C_q). We define

$$\nu(C_q) := \frac{a}{\mid b \mid} \in \mathbb{R}.$$

Note that $\nu(C_q) = \nu(C_{\bar{q}})$. Moreover, $\nu(C_{q'}) = \nu(C_q)$ if and only if $\pi_\circ(C_q) = \lambda \pi_\circ(C_{q'})$ or $\pi_\circ(C_q) = \lambda \pi_\circ(C_{\bar{q}'})$ for some $\lambda \in \mathbb{R}_{>0}$.

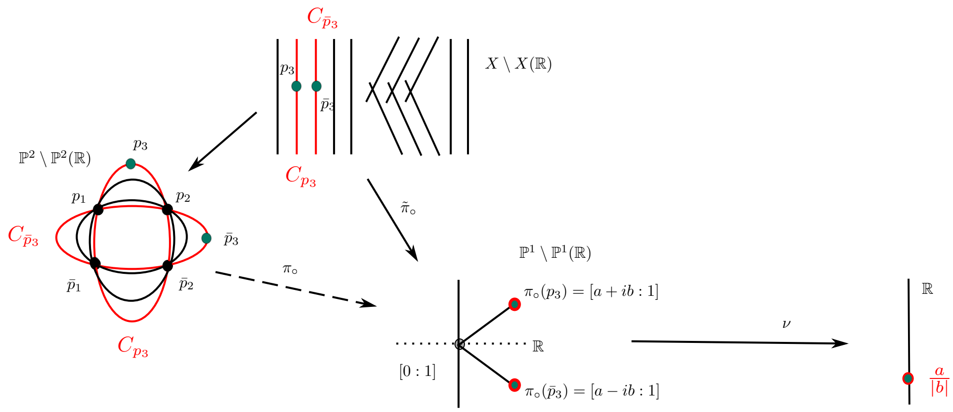

FIGURE 3. The map ν (Definition 3.10)

Definition 3.11. We define $e_\delta \in \oplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$ to be the "standard vector" given by

$$(e_\delta)_\varepsilon = \begin{cases} 1, & \delta = \varepsilon \\ 0, & \text{else} \end{cases}$$

Definition 3.12. Let $f \in \mathcal{J}_\circ$ and $S(f)$ be the set of non-real conjugate *pairs* of base-points of f different from $p_1, \ldots, \bar{p}_2$. We define

$$\varphi_\circ \colon \mathcal{J}_\circ \longrightarrow \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}, \qquad f \longmapsto \sum_{(q,\bar{q}) \in S(f)} e_{\nu(C_q)}$$

which is a well defined map according to Definition 3.10.

Remark 3.13. The following remarks directly follow from the definition of $\varphi_\circ$.

(1) If $S(f) = \emptyset$, then $\varphi_0(f) = 0$.
(2) For every $f \in \mathcal{J}_\circ$ of degree ≤ 4 the set $S(f)$ is empty (follows from its characteristic; Lemma 3.1), hence in particular $\varphi_0(f) = 0$.
(3) Let $\theta \in \mathcal{J}_\circ$ be a standard quintic transformation. Then $|S(f)| = 1$ and therefore $\varphi_\circ(\theta)$ is a "standard vector" by Definition 3.12.
(4) It follows from the definition of standard quintic transformations (Definition 2.2) that for every $\delta \in \mathbb{R}$ there exists a standard quintic transformation $\theta \in \mathcal{J}_\circ$ such that $\varphi_\circ(\theta) = e_\delta$.
(5) Let $\theta_1, \theta_2 \in \mathcal{J}_\circ$ be standard quintic transformations and $S(\theta_i) = \{(q_i, \bar{q}_i)\}$, $i = 1, 2$. If $C_{q_1} = C_{q_2}$ (or $C_{q_1} = C_{\bar{q}_2}$), then $\varphi_\circ(\theta_1) = \varphi_\circ(\theta_2)$.
(6) Let $\theta \in \mathcal{J}_\circ$ be a standard quintic transformation. Let $S(\theta) = \{(q_1, \bar{q}_1)\}$ and $S(\theta^{-1}) = \{(q_2, \bar{q}_2)\}$. Since θ induces Id or $[x : y] \mapsto [y : x]$ on $\mathbb{P}^1$ (Lemma 3.9), it follows that $\nu(C_{q_1}) = \nu(C_{q_2})$ and in particular $\varphi_\circ(\theta) = \varphi_\circ(\theta^{-1})$.
(7) Let $f \in \mathcal{J}_\circ$ and C be any non-real conic passing through $p_1, \ldots, \bar{p}_2$. The automorphism $\hat{f}$ on $\mathbb{P}^1$ induced by f is a scaling by a positive real number (Lemma 3.9), thus $\nu \circ \hat{f} = \nu$. In particular,
$$e_{\nu(f(C))} = e_{\nu(\hat{f}(C))} = e_{\nu(C)}.$$

Let us finally prove that $\varphi_\circ$ is a homomorphism of groups.

Lemma 3.14. *The map $\varphi_\circ \colon \mathcal{J}_\circ \to \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$ is a surjective group homomorphism and its kernel contains all elements of degree ≤ 4.*

Proof. It suffices to show that $\varphi_\circ$ is a group homomorphism: the surjectivity and the assertion on the kernel then follow from Remark 3.13 (2) and (4).

Let $f, g \in \mathcal{J}_\circ$. We want to show that $\varphi_\circ(fg) = \varphi_\circ(f) + \varphi_\circ(g)$. The group $\mathcal{J}_\circ$ is generated by its linear, quadratic and standard quintic elements (Lemma 3.7), so we can assume that f is a linear, quadratic or standard quintic element of $\mathcal{J}_\circ$. In particular, $S(f)$ is empty if f is linear or quadratic (Remark 3.13 (2)), and $|S(f)| = 1$ if f is a standard quintic transformation.

Suppose that $S(f)\cap S(g^{-1})=\emptyset$, then $S(fg)=S(g)\cup(g^{-1})^{\bullet}(S(f))$ (Lemma 2.12). If $S(f)=\emptyset$, we have $\varphi_\circ(f)=0$ (Remark 3.13 (1)), $S(fg)=S(g)$, and in particular $\varphi_\circ(fg)=\varphi_\circ(f)+\varphi_\circ(g)$. If $S(f)\neq\emptyset$, then $S(f)=\{(q,\bar{q})\}$. By Remark 3.13 (7), we have

$$e_{\nu(C_{(g^{-1})^{\bullet}(q)})}=e_{\nu(g^{-1}(C_q))}=e_{\nu(C_q)}$$

In particular,

$$\begin{aligned}\varphi_\circ(fg)&=\sum_{(p,\bar{p})\in S(fg)}e_{\nu(C_p)}=e_{\nu(C_{(g^{-1})^{\bullet}(q)})}+\sum_{(p,\bar{p})\in S(g)}e_{\nu(C_p)}\\&=e_{\nu(C_q)}+\sum_{(p,\bar{p})\in S(g)}e_{\nu(C_p)}=\varphi_\circ(f)+\varphi_\circ(g)\end{aligned}$$

Suppose that $\emptyset\neq S(f)\subset S(g^{-1})$. Then f is a standard quintic transformation. In order to make the argument a bit more simple, lets prove that $\varphi_\circ(g^{-1}f^{-1})=\varphi_\circ(g^{-1})+\varphi_\circ(f^{-1})$, which will yield the claim (since $\varphi_\circ(h)=\varphi_\circ(h^{-1})$ by Remark 3.13 (6)). Let $S(f)=\{(q,\bar{q})\}$, $S(f^{-1})=\{(q',\bar{q}')\}$.

We claim that $S((fg)^{-1})=f^{\bullet}\big(\ S(g^{-1})\setminus\{(q,\bar{q})\}\big)$: Let $X\to\mathbb{P}^2$ be the blow-up of the base-points $p_1,\bar{p}_1,p_2,\bar{p}_2,q,\bar{q}$ of f and remember f contracts C_q onto q'. The multiplicity $m_{(fg)^{-1}}(q')$ of $(fg)^{-1}$ in q' is equal to the intersection in X of the strict transform $\tilde{C}_q$ of C_q with the strict transform $\tilde{D}$ of a general member D of the linear system of g^{-1}.

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & & \\ \mathbb{P}^2 \arrow[rr, dashed, "f^{-1}"] & & \mathbb{P}^2 \arrow[r, dashed, "g^{-1}"] & \mathbb{P}^2 \end{tikzcd}$$

Since C_q contains exactly one base-point of g^{-1} (Lemma 3.1), namely q, the intersection is precisely

$$\begin{aligned}m_{(fg)^{-1}}(q')=\tilde{C}_q\cdot\tilde{D}&=2\deg(g^{-1})-2m_{g^{-1}}(p_1)-2m_{g^{-1}}(p_2)-\sum_{r\in C_q}m_{g^{-1}}(r)\\&=2\deg(g)-2(\deg(g)-1)-m_{g^{-1}}(q)=0.\end{aligned}$$

On the other hand, f does not touch the base-points of g^{-1} different from $q,\bar{q},p_1,\bar{p}_1,p_2,\bar{p}_2$. It follows that $S(g^{-1}f^{-1})=f^{\bullet}\big(\ S(g^{-1})\setminus\{(q,\bar{q})\}\big)$ (Lemma 2.12). In particular, we have by Remark 3.13 (6), (7)

$$\begin{aligned}\varphi_\circ(g^{-1}f^{-1})&=\sum_{(p,\bar{p})\in S(g^{-1}f^{-1})}e_{\nu(C_p)}=\sum_{(p,\bar{p})\in f^{\bullet}(\ S(g^{-1})\setminus\{(q,\bar{q})\})}e_{\nu_\circ(C_p)}\\&\overset{(7)}{=}\sum_{(p,\bar{p})\in S(g^{-1})\setminus\{(q,\bar{q})\}}e_{\nu(C_p)}=\varphi_\circ(g^{-1})-e_{\nu(C_q)}\\&=\varphi_\circ(g^{-1})-\varphi_\circ(f)\overset{(6)}{=}\varphi_\circ(g^{-1})+\varphi_\circ(f^{-1})\end{aligned}$$

□

3.3. **Construction of quotient using the spinor norm.** The quotient $\varphi\colon\mathcal{J}_\circ\to\bigoplus_{\mathbb{R}}\mathbb{Z}/2\mathbb{Z}$ is in fact given by the spinor norm, as explained in the following.

Blowing up the four base-points $p_1,\bar{p}_1,p_2,\bar{p}_2$ of the rational map $\pi_\circ\colon\mathbb{P}^2\dashrightarrow\mathbb{P}^1$ and contracting the strict transform of $L_{p_1,\bar{p}_1}$ (or $L_{p_2,\bar{p}_2}$) yields a del Pezzo surface Z of degree 6. The fibration $\pi_\circ$ becomes a morphism $\pi'_\circ\colon Z\to\mathbb{P}^1$, which is a conic bundle with two singular fibres, both having only one real point. The group $\mathcal{J}_\circ$ is the group of birational maps of Z preserving this conic bundle structure.

The contraction the two (-1)-sections on Z is a morphism

$$Z\to S=\{wz=x^2+y^2\}\subset\mathbb{P}^3,$$

onto the quadric in $\mathbb{P}^3$ whose real part is diffeomorphic to the sphere. We can choose the images of the sections to be the points $[0:1:i:0],[0:1:-i:0]\in\mathbb{P}^3$ and obtain that $Z=\{([w:x:y:z],[u:v])\in$

$\mathbb{P}^3 \times \mathbb{P}^1 \mid uz = vw, wz = x^2+y^2\}$ and

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "\pi'_\circ"] & S \arrow[d, dashed] \\ \mathbb{P}^1 \arrow[r] & \mathbb{P}^1 \end{tikzcd} \qquad \begin{tikzcd} ([w:x:y:z],[u:v]) \arrow[r] \arrow[d] & {[w:x:y:z]} \arrow[d, dashed] \\ {[u:v]} \arrow[r, equal] & {[w:z]} \end{tikzcd}$$

The generic fibre of $\pi'_\circ$ is the conic C in $\mathbb{P}^2_{\mathbb{R}(t)}$ given by $x^2+y^2-tz^2=0$. By Lemma 3.9, the projection $\pi'_\circ$ induces an exact sequence

$$1 \to \mathrm{Aut}_{\mathbb{R}(t)}(C) \longrightarrow \mathcal{J}_\circ \longrightarrow \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^1,[0,\infty]) \simeq \mathbb{R}_{>0} \rtimes \mathbb{Z}/2\mathbb{Z} \to 1,$$

which in fact is split by

$$\begin{aligned}(\lambda,0) &\mapsto \Big([w:x:y:z] \mapsto [\lambda w : \sqrt{\lambda}x : \sqrt{\lambda}y : z]\Big)\\ (0,1) &\mapsto ([w:x:y:z] \mapsto [z:x:y:w]).\end{aligned}$$

Furthermore, $\mathrm{Aut}_{\mathbb{R}(t)}(C)$ is isomorphic to the subgroup of $\mathrm{PGL}_3(\mathbb{R}(t))$ preserving the quadratic form $x^2+y^2-tz^2$, and is therefore isomorphic to $\mathrm{SO}(x^2+y^2-tz^2,\mathbb{R}(t))$.

The spinor norm θ is given by the exact sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Spin}(x^2+y^2-tz^2,\mathbb{R}(t)) \to \mathrm{SO}(x^2+y^2-tz^2,\mathbb{R}(t)) \xrightarrow{\theta} \mathbb{R}(t)^*/(\mathbb{R}(t)^*)^2.$$

More precisely, for a reflection f at a vector $v=(a(t),b(t),c(t))$, the spinor norm is the the length of v squared, i.e.

$$\theta(f) = a(t)^2+b(t)^2-tc(t)^2.$$

More information about the spinor norm may be found in [O'Me1973, §55]. As squares are moded out, we may assume that $a(t),b(t),c(t) \in \mathbb{R}[t]$. An element $g \in \mathbb{R}[t]$ is a square if and only if every root of g appears with even multiplicity. Thus we can identify $\mathbb{R}(t)^*/(\mathbb{R}(t)^*)^2$ with polynomials in $\mathbb{R}[t]$ having only simple roots, i.e. with $\mathbb{Z}/2\mathbb{Z} \oplus \bigoplus_{\overline{\mathbb{H}}} \mathbb{Z}/2\mathbb{Z}$, where $\overline{\mathbb{H}} \subset \mathbb{C}$ is the closed upper half plane and the first factor is the sign of the polynomial. A non-real root $a \pm ib$ is the root of $(t-a)^2+b^2$. In particular, the spinor norm induces a surjective homomorphism

$$\bar{\theta}\colon \mathrm{SO}(x^2+y^2-tz^2,\mathbb{R}(t)) \to \bigoplus_{\mathbb{H}} \mathbb{Z}/2\mathbb{Z}.$$

Let's look at it geometrically. Extending the scalars to $\mathbb{C}(t)$, the isomorphism $\mathrm{Aut}_{\mathbb{R}(t)}(C) \simeq \mathrm{SO}(x^2+y^2-tz^2,\mathbb{R}(t))$ extends to

$$\mathrm{Aut}_{\mathbb{R}(t)}(C) \subset \mathrm{Aut}_{\mathbb{C}(t)}(C) \simeq \mathrm{PGL}_2(\mathbb{C}(t)) \simeq \mathrm{SO}(x^2+y^2-tz^2,\mathbb{C}(t)) \simeq \mathrm{SO}(tx^2-yz,\mathbb{C}(t)),$$

where $\alpha\colon \mathrm{PGL}_2(\mathbb{C}(t)) \xrightarrow{\simeq} \mathrm{SO}(tx^2-yz,\mathbb{C}(t))$ is given as follows: every automorphism of $\mathbb{P}^1_{\mathbb{C}(t)}$ extends to an automorphism of $\mathbb{P}^2_{\mathbb{C}(t)}$ preserving the image of $\beta\colon \mathbb{P}^1_{\mathbb{C}(t)} \hookrightarrow \mathbb{P}^2_{\mathbb{C}(t)}$, $[u:v] \overset{\beta}{\mapsto} [uv:tu^2:v^2]$. Then α is given by

$$\alpha\colon \begin{pmatrix} a & b \\ c & d\end{pmatrix} \mapsto \begin{pmatrix} ad+bc & ac & bd \\ 2ab & a^2 & b^2 \\ 2cd & c^2 & d^2\end{pmatrix}$$

The group $\mathrm{PGL}_2(\mathbb{C}(t))$ is generated by its involutions: any element of $\mathrm{PGL}_3(\mathbb{C}(t))$ is conjugate to a matrix of the form

$$\begin{pmatrix} 0 & a \\ 1 & b\end{pmatrix},$$

which is a composition of involutions:

$$\begin{pmatrix} 0 & a \\ 1 & b\end{pmatrix} = \begin{pmatrix} 0 & a \\ 1 & 0\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix} 0 & a \\ 1 & b\end{pmatrix}.$$

All involutions in $\mathrm{PGL}_2(\mathbb{C}(t))$ are conjugate to matrices of the form

$$P := \begin{pmatrix} 0 & p \\ 1 & 0\end{pmatrix}, \quad p \in \mathbb{C}[t].$$

The image of P via α is

$$\alpha(P)=\begin{pmatrix}-1 & 0 & 0\\ 0 & 0 & -tp\\ 0 & -1/tp & 0\end{pmatrix}\in \mathrm{SO}(tx^2-yz,\mathbb{C}(t)),$$

which is a reflection at its eigenvector $v=(0,-tp,1)$ of eigenvalue 1. In particular, $\theta(P)$ is equal to the length of v squared, i.e. $\theta(P)=tp$, and so

$$\bar{\theta}(P)=p=-\det(P)\in \mathbb{C}(t)^*/(\mathbb{C}(t)^*)^2.$$

The isomorphism $\alpha\colon \mathrm{PGL}_2(\mathbb{C}(t))\simeq \mathrm{SO}(tx^2-yz,\mathbb{C}(t))$ is induced by a birational map $Z\to\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}$ that contracts one component in each singular fibre. The zeros of $\bar{\theta}(P)$ correspond to the fibres contracted by $f_P\colon \mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}\dashrightarrow\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}$, $(x,y)\mapsto(p/x,y)$ that have an odd number of base-points on them. So, for $f\in\mathrm{Aut}_{\mathbb{R}(t)}(C)$ the spinor norm $\bar{\theta}(f)$ corresponds to the non-real conics contracted by f "an odd number of times". Lemma 3.9 implies that $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^1,[0,\infty])$ acts on the image of these conics in $\mathbb{P}^1$ by real positive scaling. Observe that the quotient of $\mathbb{H}$ by $\mathbb{R}_{>0}$ is bijective to $\mathbb{R}$ via the map ν given in Definition 3.10 (see also Figure 3). We obtain a group homomorphism

$$\mathcal{J}_\circ\simeq \mathrm{SO}(x^2+y^2-tz^2,\mathbb{R}(t))\rtimes\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^1,[0,\infty])\to\bigoplus_{\mathbb{R}}\mathbb{Z}/2\mathbb{Z},$$

$$f\mapsto\begin{cases}\bar{\theta}(f), & f\in\mathrm{SO}(x^2+y^2-tz^2,\mathbb{R}(t))\\ 0, & f\in\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^1,[0,\infty])\end{cases}$$

which is exactly the quotient $\varphi_\circ$.

4. Presentation of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ by generating sets and relations

This section is devoted to the rather technical proof of Theorem 2.9. It is rather technical and in Chapters 5 and Chapter 6 we will only use Theorem 2.9 and no other results presented in this chapter. We therefore suggest to the reader to skip this chapter and return to it at the end of the paper.

We remind of the notation $p_1:=[1:i:0],p_2:=[0:1:i]$. Recall that $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2),\mathcal{J}_*,\mathcal{J}_\circ$ (Corollary 2.6).
Consider F_S, the free group generated by the set

$$S=\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)\cup\mathcal{J}_*\cup\mathcal{J}_\circ.$$

There is a natural word map $w\colon S\to F_S$, sending an element to its corresponding word.

Remark 4.1. Let $\mathcal{G}$ as in Definition 2.7. There exists a natural surjective homomorphism $\mathcal{G}\to\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$. By abuse of notation, we also denote by

$$w\colon\ \mathrm{Aut}_{\mathbb{R}}\cup\mathcal{J}_*\cup\mathcal{J}_\circ\to\mathcal{G}$$

the composition of $S\to F_S$ with the canonical projection $F_S\to\mathcal{G}$.

Remark 4.2. Suppose $\theta_1,\theta_2\in\mathcal{J}_\circ$ are special quintic transformations (see Definition 3.5). If there exist $\alpha_1,\alpha_2\in\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ such that $\theta_2=\alpha_2\theta_1\alpha_1$ then α_1,α_2 permute $p_1,\bar{p}_1,p_2,\bar{p}_2$ and are thus contained in $\mathcal{J}_\circ$. So, the relation

$$\textbf{(rel. 4)}\qquad w(\theta_2)=w(\alpha_2)w(\theta_1)w(\alpha_2)\quad\Longleftrightarrow\quad\theta_2=\alpha_2\theta_1\alpha_1\ \text{in}\ \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$$

is true in $\mathcal{G}$ and even in the generalised amalgamated product of $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)\ \mathcal{J}_*,\mathcal{J}_\circ$ along all the pairwise intersections. Therefore, we need not list this relation in Definition 2.7.

Remark 4.3. In the proof that $\mathcal{G}\simeq\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ (Theorem 2.9) the relations given in the definition of $\mathcal{G}$ (list in Definition 2.7) mostly turn up in the form of the following examples:

(1) Let $\theta\in\mathcal{J}_\circ$ be a standard quintic transformation (see Definition 2.2). Call its base-points $p_1,\bar{p}_1,p_2,\bar{p}_2,p_3,\bar{p}_3$, and the base-points of its inverse $p_1,\bar{p}_1,p_2,\bar{p}_2,p_4,\bar{p}_4$ where p_3,p_4 are non-real proper points of $\mathbb{P}^2$. There exist $i,j\in\{1,2\}$ such that θ sends the pencil of conics passing through $p_i,\bar{p}_i,p_3,\bar{p}_3$ onto the pencil of conics passing through $p_j,\bar{p}_j,p_4,\bar{p}_4$. Let $\alpha_1,\alpha_2\in\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ such that α_1 sends the set $\{p_1,\bar{p}_1,p_2,\bar{p}_2\}$ onto $\{p_i,\bar{p}_i,p_3,\bar{p}_3\}$, and α_2 sends the set

$\{p_j,\bar{p}_j,p_4,\bar{p}_4\}$ onto the set $\{p_1,\bar{p}_1,p_2,\bar{p}_2\}$, $i\in\{1,2\}$. Then $\alpha_2\theta\alpha_1\in\mathcal{J}_\circ$ is a standard quintic transformation. The relation $w(\alpha_2)w(\theta)w(\alpha_1)=w(\alpha_2\theta\alpha_1)$ holds in $\mathcal{G}$ (Definition 2.7 **(rel. 1)**).

(2) Let $\tau\in\mathcal{J}_\circ$ be of degree 2 or 3. By Lemma 3.1, τ has exactly one real base-point. Let r be the real base-point of τ and s the real base-point of τ^{-1}. Observe that τ sends the pencil of lines through r onto the pencil of lines through s. There exist $\alpha_1,\alpha_2\in\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ such that $(\alpha_1)^{-1}(r)=[1:0:0]=\alpha_2(s)$. Then $\alpha_2\tau\alpha_1$ is an element of $\mathcal{J}_*$ and the relation $w(\alpha_2)w(\tau)w(\alpha_1)=w(\alpha_2\tau\alpha_1)$ holds in $\mathcal{G}$ (Definition 2.7 **(rel. 2)**) .

(3) Let $\tau_1,\tau_2\in\mathcal{J}_\circ$ of degree 2 with base-points $p_i,\bar{p}_i,r$ and $p_j,\bar{p}_j,s$ respectively, and $\alpha\in\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ such that $\alpha(p_i)=p_j$ and $\alpha(r)=s$. Then $\tau_2\alpha(\tau_1)^{-1}$ is linear. The relation $w(\tau_2)w(\alpha)w((\tau_1)^{-1})=w(\tau_2\alpha(\tau_1)^{-1})$ holds in $\mathcal{G}$ (Definition 2.7 **(rel. 2)**).

(4) Let $\tau_1,\tau_2\in\mathcal{J}_*$ be of degree 2 with base-points $p:=[1:0:0],r_1,r_2$ and p,s_1,s_2 respectively, and $\alpha\in\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ with $\alpha(r_i)=s_i$ but $\alpha(p)\neq p$ (i.e. $\alpha\notin\mathcal{J}_*$). Suppose that the base-points of $(\tau_1)^{-1},(\tau_2)^{-1}$ are p,r_1',r_2' and p,s_1',s_2' respectively. Then $\tau_3:=\tau_2\alpha(\tau_1)^{-1}$ is quadratic with base-points $r_1',r_2',\tau_1(\alpha^{-1}(p))$ and its inverse has base-points $s_1',s_2',\tau_2(\alpha(p))$. There exist $\beta_1,\beta_2\in\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, $\tilde{\tau}_3\in\mathcal{J}_*$ of degree 2 such that $\tau_3=\beta_2\tilde{\tau}_3\beta_3$. The relation $w(\beta_2)w(\tilde{\tau}_3)w(\beta_2)=w(\tau_2)w(\alpha)w(\tau_1)$ holds in $\mathcal{G}$ (Definition 2.7 **(rel. 3)**).

Lemma 4.4. *Let $f\in\mathcal{J}_*\cup\mathcal{J}_\circ$ be non-linear and Λ be a real linear system of degree $\deg(\Lambda)=D$. Suppose that*

$$\deg(f(\Lambda))\leq D\quad (\textit{resp. } \deg(f(\Lambda))<D).$$

(1) If $f\in\mathcal{J}_*$, there exist two real or a pair of non-real conjugate base-points q_1,q_2 of f such that

$$m_\Lambda([1:0:0])+m_\Lambda(q_1)+m_\Lambda(q_2)\geq D\quad (\textit{resp. } >D)$$

(2) Suppose that $f\in\mathcal{J}_\circ$. Then there exists a base-point $q\notin\{p_1,\bar{p}_1,p_2,\bar{p}_2\}$ of f of multiplicity 2 such that

$$(2.1)\qquad m_\Lambda(p_1)+m_\Lambda(p_2)+m_\Lambda(q)\geq D\quad (\textit{resp. } >D)$$

or f has a simple base-point r and there exists $i\in\{1,2\}$ such that

$$(2.2)\qquad 2m_\Lambda(p_i)+m_\Lambda(r)\geq D\quad (\textit{resp. } >).$$

Moreover, if $\deg(f)>2$ and inequality (2.1) does not hold, then the inequality in (2.2) is strict.

Proof. Define $d:=\deg(f)$ to be the degree of f.

(1) Suppose that $f\in\mathcal{J}_*$. Its characteristic is $(d;d-1,1^{2d-2})$ because it preserves the pencil of lines through $[1:0:0]$. Let $r_1,\dots,r_{2d-2}$ be its simple base-points. Since non-real base-points come in pairs, f has an even number N of real base-points. Call $m_i:=m_\Lambda(r_i)$ the multiplicity of Λ in r_i and $m_0=m_\Lambda([1:0:0])$ the one in $[1:0:0]$. We order the base-points such that either r_{2i-1},r_{2i} are real or $r_{2i}=\bar{r}_{2i-1}$ for $i=1,\dots,d-1$. Then

$$\begin{aligned}D\geq\deg(f(\Lambda))=&dD-(d-1)m_0-\sum_{i=1}^{d-1}(m_{2i-1}+m_{2i})\\=&D+\sum_{i=1}^{d-1}(D-m_0-m_{2i-1}-m_{2i})\end{aligned}$$

Hence there exists i_0 such that $D\leq m_0-m_{2i_0-1}-m_{2i_0}$. The claim for ">" follows analogously.

(2) Suppose that $f\in\mathcal{J}_\circ$. By Lemma 3.1, its characteristic is $(d;\frac{d-1}{2}^4,2^{\frac{d-1}{2}})$ or $(d;\frac{d}{2}^2,\frac{d-2}{2}^2,2^{\frac{d-2}{2}},1)$.

Assume that f has no simple base-point, i.e. no base-point of multiplicity 1. Call $r_1,\dots,r_{(d-1)/2}$ its base-points of multiplicity 2. Let $m_i:=m_\Lambda(p_i)$ be the multiplicity of Λ in p_i, $i=1,2$ and $a_i:=m_\Lambda(r_i)$ the one in r_i. Then

$$\begin{aligned}D\geq\deg(f(\Lambda))&=dD-2m_1\cdot\frac{d-1}{2}-2m_2\cdot\frac{d-1}{2}-2\sum_{i=1}^{(d-1)/2}a_i\\&=D+2\sum_{i=1}^{(d-1)/2}(D-m_1-m_2-a_i)\end{aligned}$$

which implies that there exists i_0 such that $0 \geq D - m_1 - m_2 - a_{i_0}$. The claim for ">" follows analogously.

Assume that f has a simple base-point r. Let $r_1, \ldots, r_{(d-2)/2}$ be its base-points of multiplicity 2, $a_i := m_\Lambda(r_i)$ the multiplicity of Λ in r_i, and $m_i := m_\Lambda(p_i)$ the one in p_i. Then

$$\begin{aligned} D \geq \deg(f(\Lambda)) &= dD - 2m_j \cdot \frac{d}{2} - 2m_k \cdot \frac{d-2}{2} - (2\sum_{i=1}^{(d-2)/2} a_i) - m_\Lambda(r) \\ &= D + (D - 2m_j - m_\Lambda(r)) + 2\sum_{i=1}^{(d-2)/2}(D - m_j - m_k - a_i) \end{aligned}$$

where $\{j,k\} = \{1,2\}$. The inequality implies there exist i_0 such that $0 \geq D - m_j - m_k - a_{i_0}$ or that $0 \geq D - 2m_j - m_\Lambda(r)$. The claim for ">" follows analougously.

Suppose that $0 < D - m_j - m_k - a_i$ for all $i = 1, \ldots, \frac{d-2}{2}$, i.e. , $1 \leq D - m_j - m_k - a_i$ for all $i = 1, \ldots, \frac{d-2}{2}$. We obtain from the calculations above that

$$\begin{aligned} 0 \geq\ & (D - 2m_j - m_\Lambda(r)) + 2\sum_{i=1}^{(d-2)/2}(D - m_j - m_k - a_i) \\ \geq\ & (D - 2m_j - m_\Lambda(r)) + (d-2) \end{aligned}$$

Assume that $d > 2$, i.e. since d is even here, $d \geq 4$. The inequality above implies

$$-2 \geq -(d-2) \geq D - 2m_j - m_\Lambda(r)$$

and so

$$2m_j + m_\Lambda(r) \geq D + 2 > D.$$

□

Notation 4.5. For a pair of non-real points $q, \bar{q} \in \mathbb{P}^2$ or infinitely near, we denote by $\mathfrak{q}$ the set $\{q, \bar{q}\}$.

In the following diagrams, the points in the brackets are the base-points of the corresponding birational map (arrow). A dashed arrow indicates a birational map, and a drawn out arrow a linear tranformation.

Let $f_1, \ldots, f_n \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) \cup \mathcal{J}_* \cup \mathcal{J}_\circ$ such that $f_n \cdots f_1 = \mathrm{Id}$. If $w(f_n) \cdots w(f_1) = 1$, we say that the diagram

$$\begin{tikzcd} \mathbb{P}^2 \arrow[r, dashed, "f_2"] & \mathbb{P}^2 \arrow[r, dotted] & \mathbb{P}^2 \arrow[r, dashed, "f_{n-1}"] & \mathbb{P}^2 \arrow[dlll, dashed, "f_n"] \\ \mathbb{P}^2 \arrow[u, dashed, "f_1"] & & & \end{tikzcd}$$

corresponds to a relation in $\mathcal{G}$ or is *generated by relations* in $\mathcal{G}$. In the sequel, we replace $\mathbb{P}^2$ by a linear system Λ of curves in $\mathbb{P}^2$ and its images by $f_1, \ldots, f_{n-1}$.

Let $X_n \overset{\pi_n}{\to} X_{n-1} \cdots X_1 \to \mathbb{P}^n$ be a sequence of blow-ups of points $q_0 \in \mathbb{P}^2, q_1 \in X_1, \ldots, p_{n-1} \in X_{n-1}$. A point in the *first neighbourhood* of q_i is a point in $\pi_{i+1}^{-1}(q_i)$. A point of X_n is *proximate* to q_i if it is contained in the strict transform of $\pi_{i+1}^{-1}(q_i)$.

Lemma 4.6. *Let $f, h \in \mathcal{J}_\circ$ be standard or special quintic transformations, $g \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ and Λ be a real linear system of degree D. Suppose that*

$$\deg(h^{-1}(\Lambda)) \leq D \quad \textit{and} \quad \deg(fg(\Lambda)) < D.$$

Then there exists $\theta_1 \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, $\theta_2, \ldots, \theta_n \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) \cup \mathcal{J}_\circ$ such that

(1) $w(f)w(g)w(h) = w(\theta_n) \cdots w(\theta_1)$ holds in $\mathcal{G}$, i.e. the following diagram corresponds to a relation in $\mathcal{G}$:

$$\begin{tikzcd} & \Lambda \arrow[r, "g"] & g(\Lambda) \arrow[dr, dashed, "f"] & \\ h^{-1}(\Lambda) \arrow[ur, dashed, "h"] \arrow[r, dashed, "\theta_1"] & {} \arrow[r, dotted] & {} \arrow[r, dashed, "\theta_n"] & fg(\Lambda) \end{tikzcd}$$

(2) $\deg(\theta_i\cdots\theta_1 h^{-1}(\Lambda))<D$ *for* $i=2,\dots,n$.

Proof. The maps h^{-1} and f have base-points $p_1,\bar{p}_1,p_2,\bar{p}_2,p_3,\bar{p}_3$ and $p_1,\bar{p}_1,p_2,\bar{p}_2,p_4,\bar{p}_4$ respectively, for some non-real points p_3,p_4 that are in $\mathbb{P}^2$ or infinitely near one of $p_1,\dots,\bar{p}_2$. Denote by $m(q):=m_\Lambda(q)$ the multiplicity of Λ at q. According to Lemma 4.4 we have

$$(\mathrm{Ineq}^0)\qquad m(p_1)+m(p_2)+m(p_3)\geq D,\quad m_{g(\Lambda)}(p_1)+m_{g(\Lambda)}(p_2)+m_{g(\Lambda)}(p_4)>D$$

We choose $\mathfrak{r}_1,\mathfrak{r}_2,\mathfrak{r}_3$ with $\{\mathfrak{r}_1,\mathfrak{r}_2,\mathfrak{r}_3\}=\{\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3\}$ such that $m(r_1)\geq m(r_2)\geq m(r_3)$ and such that if r_i is infinitely near r_j, then $j<i$. Similarly, we choose $\mathfrak{r}_4,\mathfrak{r}_5,\mathfrak{r}_6$ with $\{\mathfrak{r}_4,\mathfrak{r}_5,\mathfrak{r}_6\}=g^{-1}(\{\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_4\})$. In particular, r_1,r_4 are proper points of $\mathbb{P}^2$.

The two inequalities (Ineq^0) translate to

$$(\mathrm{Ineq}^1)\qquad m(r_1)+m(r_2)+m(r_3)\geq D,\quad m(r_4)+m(r_5)+m(r_6)>D$$

We now look at four cases, depending of the number of common base-points of fg and h^{-1}.

<u>Case 0:</u> If h^{-1} and fg have six common base-points, then $\alpha:=fgh$ is linear and $w(g)w(h)w(\alpha^{-1})=w(f^{-1})$ by Definition 2.7 **(rel. 1)** and Remark 4.2 (**rel. 4**).

<u>Case 1</u>: *Suppose that* h^{-1} *and* fg *have exactly four common base-points.* There exists $\alpha_1\in\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ such that α_1 sends the common base-points onto $p_1,\dots,\bar{p}_2$ if all the common points are proper points of $\mathbb{P}^2$, and onto $p_i,\bar{p}_i,p_3,\bar{p}_3$ if $p_3,\bar{p}_3$ are infinitely near $p_i,\bar{p}_i$. There exist $\alpha_2,\alpha_3\in\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ such that $\tilde{f}:=\alpha_3 fg(\alpha_1)^{-1}\in\mathcal{J}_\circ$ and $\tilde{h}:=\alpha_1 h\alpha_2\in\mathcal{J}_\circ$ (see Lemma 2.5). The commutative diagram

$$\begin{tikzcd}
 & & \Lambda \arrow[d, "\alpha_1"] \arrow[drr, dashed, "fg"] & & \\
h^{-1}(\Lambda) \arrow[urr, dashed, "h"] \arrow[r, "\alpha_2"'] & \alpha_2 h^{-1}(\Lambda) \arrow[r, dashed, "\tilde{h}"'] \arrow[rr, dashed, bend right, "\theta_2"'] & \alpha_1(\Lambda) \arrow[r, dashed, "\tilde{f}"'] & \alpha_3 fg(\Lambda) & fg(\Lambda) \arrow[l, "\alpha_3"]
\end{tikzcd}$$

is generated by relations in $\mathcal{G}$: more specifically, each triangle corresponds to relation Definition 2.7 **(rel. 1)** (see Remark 4.3 (1)) or Remark 4.2 (**rel. 4**) and the round triangle on the bottom corresponds to a relation in $\mathcal{J}_\circ$. Write $\theta_2:=\tilde{f}\tilde{h}\in\mathcal{J}_\circ$. The claim now follows with $\theta_1:=\alpha_1,\theta_2,\ \theta_3:=(\alpha_3)^{-1}$.

<u>Case 2:</u> *Suppose that the set* $\mathfrak{r}_1\cup\mathfrak{r}_2\cup\mathfrak{r}_4\cup\mathfrak{r}_5$ *consists of* 6 *points* $r_{i_1},\bar{r}_{i_1},\dots,r_{i_3},\bar{r}_{i_3}$.

If at least four of them are proper points of $\mathbb{P}^2$, inequality (Ineq^1) yields

$$2m(r_{i_1})+2m(r_{i_2})+2m(r_{i_3})>D,$$

which implies that the six points $r_{i_1},\bar{r}_{i_1},\dots,r_{i_3},\bar{r}_{i_3}$ are not contained in one conic. By this and by the chosen ordering of the points, there exists a standard or special quintic transformation $\theta\in\mathcal{J}_\circ$, $\alpha\in\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ such that those six points are the base-points of $\theta\alpha$. By construction, we have

$$\deg(\theta\alpha(\Lambda))=5D-4m(r_{i_1})-4m(r_{i_2})-4m(r_{i_3})<D,$$

and $h^{-1},\theta\alpha$ and $\theta\alpha,fg$ each have four common base-points. We apply Case 1 to h,α,θ and to $\theta^{-1},g\alpha^{-1},f$.

If only two of the six points are proper points of $\mathbb{P}^2$, then the chosen ordering yields $\mathfrak{q}=\mathfrak{r}_1=\mathfrak{r}_4$ and the points in $\mathfrak{r}_2\cup\mathfrak{r}_5$ are infinitely near points. Since h,f are standard or special quintic transformations, it follows that r_3,r_6 are both proper points of $\mathbb{P}^2$. We choose $i\in\{3,6\}$, $j\in\{2,5\}$ with $m(r_i)=\max\{m(r_3),m(r_6)\}$ and $m(r_j)=\max\{m(r_2),m(r_5)\}$. We have

$$2m(r_1)+2m(r_j)+2m(r_i)\geq 2m(r_4)+2m(r_5)+2m(r_6)>D.$$

Thus the six points in $\mathfrak{r}_1\cup\mathfrak{r}_i\cup\mathfrak{r}_j$ are not contained in one conic and there exists a standard or special quintic transformation $\theta\in\mathcal{J}_\circ$, $\alpha\in\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ such that the base-points of $\theta\alpha$ are $\mathfrak{r}_1\cup\mathfrak{r}_i\cup\mathfrak{r}_j$. Again, the maps $h^{-1},\theta\alpha$ and $\theta\alpha,fg$ have four common base-points, $\deg(\theta\alpha(\Lambda))<D$ and we apply Case 1 to h,α,θ and to $\theta^{-1},g\alpha^{-1},f$.

Case 3: *Suppose that* $\mathfrak{r}_1 \cup \mathfrak{r}_2 \cup \mathfrak{r}_4 \cup \mathfrak{r}_5$ *consists of eight points.* Then $\mathfrak{r}_1 \cup \mathfrak{r}_2 \cup \mathfrak{r}_4$ and $\mathfrak{r}_1 \cup \mathfrak{r}_4 \cup \mathfrak{r}_5$ each consist of six points. We have by inequality Ineq[1] and by the chosen ordering that

$$2m(r_1) + 2m(r_2) + 2m(r_4) > 2D, \qquad 2m(r_1) + 2m(r_4) + 2m(r_5) > 2D,$$

so the points in each set $\mathfrak{r}_1 \cup \mathfrak{r}_2 \cup \mathfrak{r}_4$ and $\mathfrak{r}_1 \cup \mathfrak{r}_4 \cup \mathfrak{r}_5$ are not on one conic. Moreover, at least four points in each set are proper points of $\mathbb{P}^2$ ($r_1, r_4 \in \mathbb{P}^2$). Therefore, there exist standard or special quintic transformations $\theta_1, \theta_2 \in \mathcal{J}_\circ$, $\alpha_1, \alpha_2 \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ such that $\theta_1\alpha_1$ (resp. $\theta_2\alpha_2$) has base-points $\mathfrak{r}_1 \cup \mathfrak{r}_2 \cup \mathfrak{r}_4$ (resp. $\mathfrak{r}_1 \cup \mathfrak{r}_4 \cup \mathfrak{r}_5$). Then $\deg(\theta_i\alpha_i(\Lambda)) < D$ and we can apply Case 1 to $h, \alpha_1^{-1}, \theta_1$ and to $(\theta_1)^{-1}, \alpha_2(\alpha_1)^{-1}, \theta_2$ and to $(\theta_2)^{-1}, g(\alpha_2)^{-1}, f$. □

Remark 4.7. Let $f \in \mathcal{J}_*$, and q_1, q_2 two simple base-points of f. Then the points $[1:0:0], q_1, q_2$ are not collinear. (This means that they do not belong, as proper points of $\mathbb{P}^2$ or infinitely near points, to the same line.)

Lemma 4.8. *Let $f, h \in \mathcal{J}_*$, $g \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ and Λ be a real linear system of degree D. Suppose that*

$$\deg(h^{-1}(\Lambda)) \leq D, \quad \deg(fg(\Lambda)) < D$$

Then there exist $\theta_1, \ldots, \theta_n \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) \cup \mathcal{J}_* \cup \mathcal{J}_\circ$ such that

(1) $w(f)w(g)w(h) = w(\theta_n)\cdots w(\theta_1)$ holds in $\mathcal{G}$, i.e. the following commutative diagram corresponds to a relation in $\mathcal{G}$:

$$\begin{tikzcd} & \Lambda \arrow[r, "g"] & g(\Lambda) \arrow[dr, dashed, "f"] & \\ h^{-1}(\Lambda) \arrow[ur, dashed, "h"] \arrow[r, dashed, "\theta_1"] & \cdots \arrow[r, dotted] & \cdots \arrow[r, dashed, "\theta_n"] & fg(\Lambda) \end{tikzcd}$$

(2) $\theta_1 \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, $\deg(\theta_i \cdots \theta_1 h^{-1}(\Lambda)) < D$ for $i = 2, \ldots, n$
(3) or $\theta_1 \in \mathcal{J}_*$, $\theta_2 \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, $\deg(\theta_1) = \deg(h) - 1$ and

$$\deg(\theta_1(\Lambda)) = \deg(\theta_2\theta_1(\Lambda)) \leq D$$
$$deg(\theta_i \cdots \theta_1 h^{-1}(\Lambda)) < D, \quad i = 3, \ldots, n.$$

Proof. If $g \in \mathcal{J}_*$ then $w(f)w(g)w(h) = w(fgh)$ in $\mathcal{J}_*$. So, lets assume that $g \notin \mathcal{J}_*$. Let $p := [1:0:0], q := g^{-1}([1:0:0])$. Let $m(q)$ be the multiplicity of Λ in q. By Lemma 4.4 there exists r_1, r_2 base-points of h^{-1} and s_1, s_2 base-points of fg such that

$$(\bigstar) \qquad m(p) + m(r_1) + m(r_2) \geq D, \quad m(q) + m(s_1) + m(s_2) > D$$

and either r_1, r_2 (resp. s_1, s_2) are both real or a pair of non-real conjugate points. We can assume that $m(r_1) \geq m(r_2)$, $m(s_1) \geq m(s_2)$ and that r_1 (resp. s_1) is a proper point of $\mathbb{P}^2$ or in the first neighbourhood of p (resp. q) and that r_2 (resp. s_2) is a proper point of $\mathbb{P}^2$ or in the first neighbourhood of p (resp. q) or r_1 (resp. s_1).

Note that if $\deg(h^{-1}(\Lambda)) < D$, then by Lemma 4.4 "$>$" holds in all inequalities. We split the remain of the proof into three Situations, depending on whether or not there exist $\tau_1, \tau_2 \in \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ with base-point p, r_1, r_2 and $p = g(q), g(s_1), g(s_2)$ respectively.

- Situation 1 - *Assume that there exist $\tau_1, \tau_2 \in \mathcal{J}_*$ of degree 2 with base-points p, r_1, r_2 and $p = g(q), g(s_1), g(s_2)$ respectively, and that $\tau_1, \tau_2 g$ have common base-points.*

Observe that $\tau_1 h,\ f(\tau_2)^{-1} \in \mathcal{J}_*$ and

$$\deg(\tau_1 h) = \deg(h) - 1,$$

and by inequality ($\bigstar$) that

$$\deg(\tau_1(\Lambda)) = 2D - m(p) - m(r_1) - m(r_2) \leq D,$$
$$deg(\tau_2 g(\Lambda)) = 2D - m(q) - m(s_1) - m(s_2) < D$$

We are going to look at three cases, depending on the common base-points of τ_1, τ_2.

• If τ_1 and $\tau_2 g$ have three common base-points, the map $\tau_2 g(\tau_1)^{-1}$ is linear. The commutative diagram

$$\begin{tikzcd}
& \Lambda \arrow[r, "g"] \arrow[d, dashed, "\tau_1"] & g(\Lambda) \arrow[d, dashed, "\tau_2"] \arrow[dr, dashed, "f"] & \\
h^{-1}(\Lambda) \arrow[ur, dashed, "h"] \arrow[r, dashed] & \tau_1(\Lambda) \arrow[r] & \tau_2 g(\Lambda) \arrow[r, dashed] & fg(\Lambda)
\end{tikzcd}$$

is generated by relations in $\mathcal{G}$: more specifically, the triangles correspond to relations in $\mathcal{J}_*$ and the square corresponds to relation Definition 2.7 **(rel. 2)**. The claim follows with $\theta_1 := \mathrm{Id},\ \theta_2; = \tau_1 h,\ \theta_3 := \tau_2 g(\tau_1)^{-1},\ \theta_4 := f(\tau_2)^{-1}$.

• If τ_1 and $\tau_2 g$ have exactly two common base-points, the map $\tau_2 g(\tau_1)^{-1}$ is of degree 2 and there exists $\alpha_1, \alpha_2 \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, $\tau_3 \in \mathcal{J}_*$ of degree 2 such that $\tau_2 g(\tau_1)^{-1} = \alpha_2 \tau_3 \alpha_1$.

The situation is summarised in the following commutative diagram

$$\begin{tikzcd}
& \Lambda \arrow[rr, "g"] \arrow[d, dashed, "{[p,r_1,r_2]}"', "\tau_1"] & & g(\Lambda) \arrow[dr, dashed, "{[g(q),g(s_1),g(s_2)]}"', "\tau_2"] \arrow[drr, "f"] & & \\
h^{-1}(\Lambda) \arrow[ur, dashed, bend left, "h"] \arrow[r, dashed, "\deg(h)-1"'] & \tau_1(\Lambda) \arrow[r, "\alpha_1"] & \alpha_1\tau_1(\Lambda) \arrow[r, dashed, "\tau_3"] & \tau_3\alpha_1\tau_1(\Lambda) \arrow[r, "\alpha_2"] & \tau_2(\Lambda) \arrow[r, dashed] & fg(\Lambda)
\end{tikzcd}$$

It is generated by relations in $\mathcal{G}$: more specifically, the left and right triangle correspond to relations in $\mathcal{J}_*$ and the middle part corresponds to relation Definition 2.7 **(rel. 3)**. Moreover, observe that

$$\deg(\alpha_1\tau_1(\Lambda)) = \deg(\tau_1(\Lambda)) \leq D, \quad \deg(\tau_3\alpha_1\tau_1(\Lambda)) = \deg(\tau_2(\Lambda)) < D.$$

The claim follows with $\theta_1 := \tau_1 h$, $\theta_2 := \alpha_1$, $\theta_3 := \tau_3$, $\theta_4 := \alpha_2$, $\theta_5 := f(\tau_2)^{-1}$.

• If τ_1 and $\tau_2 g$ have exactly one common base-point, then $\tau_2 g(\tau_1)^{-1}$ is of degree 3 and there exists $\alpha_1, \alpha_2 \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, $\tau_3 \in \mathcal{J}_*$ of degree 3 such that $\tau_2 g(\tau_1)^{-1} = \alpha_2\tau_3\alpha_1$, which corresponds to relation Definition 2.7 **(rel. 3)**. The situation can be visualised with the diagram of the previous case, with $\deg(\tau_3) = 3$ instead of $\deg(\tau_3) = 2$, and here too, $\deg(\alpha_1\tau_1(\Lambda)) = \deg(\tau_1(\Lambda)) \leq D$, $\deg(\tau_3\alpha_1\tau_1(\Lambda)) = \deg(\tau_2(\Lambda)) < D$. The claim follows, as above, with $\theta_1 := \tau_1 h$, $\theta_2 := \alpha_1$, $\theta_3 := \tau_3$, $\theta_4 := \alpha_2$, $\theta_5 := f(\tau_2)^{-1}$.

- Situation 2 - *As in Situation 1, we assume that there exist $\tau_1, \tau_2 \in \mathcal{J}_*$ of degree 2 with base-points p, r_1, r_2 and $p = g(q), g(s_1), g(s_2)$ respectively. Opposed to Situation 1, we now assume that $\tau_1, \tau_2 g$ have no common base-points.*

We put $\theta_1 := \tau_1 h \in \mathcal{J}_*$, $\theta_n := f(\tau_2)^{-1} \in \mathcal{J}_*$ and note that

$$\deg(\tau_1 h) = \deg(h) - 1.$$

Now, we construct $\theta_2, \ldots, \theta_{n-1}$ as follows in the below three cases, which depend on the r_i's and s_i's being real point or non-real points:

• If r_1, r_2, s_1, s_2 are real points, let $\{a_1, a_2, a_3\} = \{p, r_1, r_2\}$ and $\{b_1, b_2, b_3\} = \{g(q), g(s_1), g(s_2)\}$ such that $m(a_i) \leq m(a_{i+1})$ and $m(b_i) \leq m(b_{i+1})$, $i = 1, 2, 3$, and if a_i (resp. b_i) is infinitely near a_j (resp. b_j) then $j > i$. From inequalities ($\bigstar$), we obtain

$$m(a_1) + m(a_2) + m(b_1) > D, \quad m(a_1) + m(b_1) + m(b_2) > D.$$

By them and the chosen ordering, there exists $\tau_3, \tau_4 \in \mathcal{J}_*$ of degree 2, $\alpha_1, \alpha_2 \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ such that $\tau_3\alpha_1$, $\tau_4\alpha_2$ have base-points a_1, a_2, b_1 and a_1, b_1, b_2 respectively. The situation is summarised in the following commutative diagram

$$\begin{tikzcd}
& & \Lambda \arrow[r] \arrow[dl, dashed, "{[a_1,a_2,a_3]}"', "\tau_1"] \arrow[d, dashed, "{[a_1,a_2,b_1]}", "\tau_3\alpha_1"'] & g(\Lambda) \arrow[d, dashed, "{[a_1,b_1,b_2]}"', "\tau_4\alpha_2"] \arrow[dr, dashed, "{[b_1,b_2,b_3]}", "\tau_2"'] \arrow[drr, dashed, bend left, "f"] & & \\
h^{-1}(\Lambda) \arrow[urr, dashed, bend left, "h"] \arrow[r, dashed, "\deg(h)-1"'] & \tau_1(\Lambda) & \tau_3\alpha_1(\Lambda) & \tau_4\alpha_2 g(\Lambda) & \tau_2 g(\Lambda) \arrow[r, dashed] & fg(\Lambda)
\end{tikzcd}$$

By construction of τ_3, τ_4, we have

$$\deg(\tau_3\alpha_1(\Lambda)) = 2D - m(a_1) - m(a_2) - m(b_1) < D,$$
$$\deg(\tau_4\alpha_2 g(\Lambda)) = 2D - m(a_1) - m(b_1) - m(b_2) < D$$

The maps $\tau_1, \tau_3\alpha_1$, the maps $\tau_3\alpha_1, \tau_4\alpha_2$ and the maps $\tau_4\alpha_2, \tau_2$ each have two common base-points, and we proceed with each pair as in Situation 1 to obtain $\theta_2, \ldots, \theta_{n-1}$.

• Assume that $r_2 = \bar{r}_1$ and s_1, s_2 are real points. If $m(q) \geq m(p)$, then

$$m(q) + 2m(r_1) > D$$

hence $q, r_1, \bar{r}_1$ are not collinear and there exists $\tau_3 \in \mathcal{J}_*$ of degree 2 with base-points $g(q), g(r_1), g(r_2)$.

If $m(q) < m(p)$, then

$$m(p) + m(q) + m(s_1) > m(q) + m(s_1) + m(s_2) > D$$

hence there exists $\tau_4 \in \mathcal{J}_*$ of degree 2 with base-points p, q, s_1. Note that $\tau_2(\tau_3)^{-1}, \tau_4(\tau_1)^{-1} \in \mathcal{J}_*$. The situation is summarised in the following commutative diagrams.

$$\begin{tikzcd}
\Lambda \arrow[rr, "g"] \arrow[d, dashed, "\tau_1"] & & g(\Lambda) \arrow[d, dashed, "\tau_3", "{[g(q),g(r_1),g(r_2)]}"'] \arrow[dr, dashed, "\tau_2", "{[g(q),g(s_1),g(s_2)]}"'] & \\
\tau_1(\Lambda) & & \tau_3(\Lambda) \arrow[r, dashed] & \tau_2 g(\Lambda),
\end{tikzcd}
\qquad
\begin{tikzcd}
& \Lambda \arrow[r, "g"] \arrow[dl, dashed, "\tau_1", "{[p,r_1,r_2]}"'] \arrow[d, dashed, "\tau_4", "{[p,q,s_1]}"'] & g(\Lambda) \arrow[d, dashed, "\tau_2"] \\
\tau_1(\Lambda) \arrow[r, dashed] & \tau_4(\Lambda) & \tau_2 g(\Lambda)
\end{tikzcd}$$

By construction of τ_3, τ_4, we have

$$\deg(\tau_3 g(\Lambda)) < D, \quad \deg(\tau_4(\Lambda)) < D$$

The maps $\tau_1, \tau_3 g$, the maps $\tau_4, \tau_2 g$ are of degree 2 with one common base-point and we obtain $\theta_2, \ldots, \theta_{n-1}$ as in Situation 1.

• If $r_2 = \bar{r}_1$ and $s_2 = \bar{s}_1$, then $r_1, \bar{r}_2, s_1, \bar{s}_1$ are proper points of $\mathbb{P}^2$. Moreover, no three collinear: Else, all four would be on one line and so $2m(r_1) + 2m(s_1) \leq D$. But then the inequality (obtained from inequalities ($\bigstar$))

$$(m(p) + 2m(r_1)) + (m(q) + 2m(s_1)) > 2D \tag{Ineq^2}$$

would imply $m(p) + m(q) > D$, which is impossible by Bézout. Since no three are collinear, there exists $\alpha, \beta, \gamma \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ such that $\alpha(r_1) = p_1, \alpha(s_1) = p_2$ and $\tilde{\tau}_1 := \beta\tau_1\alpha^{-1} \in \mathcal{J}_\circ$, $\tilde{\tau}_2 := \gamma\tau_2 g\alpha^{-1} \in \mathcal{J}_\circ$ (see Remark 4.3). These correspond to relation Definition 2.7 **(rel. 2)** and relations in $\mathrm{Aut}(\mathbb{P}^2)$.

$$\begin{tikzcd}
& \Lambda \arrow[r, "g"] \arrow[dl, "\tau_1"] \arrow[d, "\alpha"] & g(\Lambda) \arrow[d, dashed, "\tau_2"] \\
\tau_1(\Lambda) \arrow[dr, "\beta"] & \alpha(\Lambda) \arrow[d, dashed, "\tilde{\tau}_1"] \arrow[dr, dashed, "\tilde{\tau}_2"] & \tau_2(\Lambda) \arrow[d, "\gamma"] \\
& \beta\tau_1(\Lambda) \arrow[r, dashed] & \gamma\tau_2 g(\Lambda)
\end{tikzcd}$$

Note that $\tilde{\tau}_2(\tilde{\tau}_1)^{-1} \in \mathcal{J}_\circ$ and we get from the inequalities at the very beginning of the proof that

$$\deg(\beta_1\tau_1(\Lambda)) = \deg(\tau_1(\Lambda)) \leq D, \quad \deg(\gamma\tau_2 g(\Lambda)) = \deg(\tau_2 g(\Lambda)) < D.$$

The claim follows with $\theta_2 := \beta, \theta_3 := \tilde{\tau}_2(\tilde{\tau}_1)^{-1}, \theta_3 = \theta_{n-1} = \gamma^{-1}$.

- Situation 3 - *Assume that there exists no $\tau_1 \in \mathcal{J}_*$ or no $\tau_2 \in \mathcal{J}_*$ of degree 2 with base-points p, r_1, r_2 and $p = g(q), g(s_1), g(s_2)$ respectively.*

We essentially look at two cases, depending on who of τ_1, τ_2 exists:

• Assume that neither τ_1 nor τ_2 exists. Since p, r_1, r_2 (resp. q, s_1, s_2) are not collinear by Lemma 4.7, it follows that r_1, r_2 are both proximate to p and s_1, s_2 are both proximate to q [AC2002, §2]. Then $m(p) \geq m(r_1) + m(r_2)$, and from Inequalities ($\bigstar$) we obtain $2m(p) \geq m(p) + m(r_1) + m(r_2) \geq D$. Similarly we get $2m(q) > D$. But then $m(p) \geq \frac{D}{2}$ and $m(q) > \frac{D}{2}$, which is impossible by Bézout's intersection theorem. So, this case does not appear.

• Assume that τ_1 exists, but τ_2 does not. As above, it follows that s_1, s_2 are both proximate to q and hence $m(q) > \frac{D}{2}$. In particular, by Bézout's intersection theorem,

$$m(q) > m(s_1), m(s_2), m(p), m(r_1), m(r_2).$$

Furthermore, $\tau_1 h \in \mathcal{J}_*$ and (from Inequalities ($\bigstar$))

$$\deg(\tau_1 h) = \deg(h) - 1, \quad \deg(\tau_1(\Lambda)) = 2D - m(p) - m(r_1) - m(r_2) \leq D.$$

We define $\theta_1 := \tau_1 h$ and construct $\theta_2, \ldots, \theta_n$.

If r_1, r_2 are real, let $\{t_1, t_2, t_3\} = \{p, r_1, r_2\}$ such that $m(t_i) \geq m(t_{i+1})$ and such that if t_i is infinitely near t_j then $i > j$. By the chosen ordering, we have

$$m(t_1) + m(t_2) + m(q) \geq \frac{2D}{3} + \frac{D}{2} > D.$$

Moreover, t_1, t_2 are proper points of $\mathbb{P}^2$ or t_2 is in the first neighbourhood of t_1, hence there exist $\tau_3 \in \mathcal{J}_*$ with base-points $[1:0:0] = g(q), g(t_1), g(t_2)$.

If $r_2 = \bar{r}_1$, then $r_1, \bar{r}_2$ are proper points of $\mathbb{P}^2$ (they are base-points of τ_1). We have from inequalities ($\bigstar$) and $m(q) > m(p)$ and that

$$m(q) + 2m(r_1) > m(p) + 2m(r_1) \geq D.$$

Thus there exists $\tau_4 \in \mathcal{J}_*$ with base-points $[1:0:0] = g(q), g(r_1), g(\bar{r}_2)$.

$$\begin{tikzcd}
 & \Lambda \arrow[rr, "g"] \arrow[d, dashed, "{[p,r_1,r_2]}", "\tau_1"'] & & g(\Lambda) \arrow[d, dashed, "{[g(q),g(t_1),g(t_2)]/}"', "{\tau_3/}"] \arrow[d, dashed, "{[g(q),g(r_1),g(\bar{r}_2)]}"', "\tau_4"] \arrow[dr, dashed, "f"] & \\
h^{-1}(\Lambda) \arrow[ur, dashed, "h"] \arrow[r, dashed, "\theta_1", "\deg(h)-1"'] & \tau_1(\Lambda) & & \tau_3 g(\Lambda)/\tau_4 g(\Lambda) \arrow[r, dashed, "\theta_7"] & fg(\Lambda)
\end{tikzcd}$$

The maps $f(\tau_3)^{-1}$ and $f(\tau_4)^{-1}$ are contained in $\mathcal{J}_*$ and

$$\deg(\tau_3 g(\Lambda)) = 2D - m(q) - m(t_1) - m(t_2) < D,$$
$$\deg(\tau_4 g(\Lambda)) = 2D - m(q) - 2m(r_1) < D$$

Define $\theta_7 := f(\tau_3)^{-1}$ (resp. $= f(\tau_4)^{-1}$). We obtain $\theta_2, \ldots, \theta_6$ by applying Situation 1 to $\tau_1, \tau_3 g$ (resp. $\tau_1, \tau_4 g$).

• The case where τ_1 does not exist and τ_2 exists is treated similarly; although the diagram is not symmetric, we can proceed analogously to above. In fact, all linear systems obtained during the process will be strictly smaller than D and we will obtain statement (2) of the lemma. $\square$

Lemma 4.9. *Let $g \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ and either $f \in \mathcal{J}_\circ$ be a standard or special quintic transformation and $h \in \mathcal{J}_*$ or $f \in \mathcal{J}_*$ and $h \in \mathcal{J}_\circ$ be a standard or special quintic transformation. Let Λ be a real linear system of degree D. Suppose that*

$$\deg(h^{-1}(\Lambda)) \leq D, \quad \deg(fg(\Lambda)) < D$$

Then there exist $\theta_1, \ldots, \theta_n \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) \cup \mathcal{J}_* \cup \mathcal{J}_\circ$ such that

(1) $w(f)w(g)w(h) = w(\theta_n) \cdots w(\theta_1)$ holds in $\mathcal{G}$, i.e. the following commutative diagram corresponds to a relation in $\mathcal{G}$:

$$\begin{tikzcd}
 & \Lambda \arrow[r, "g"] & g(\Lambda) \arrow[dr, dashed, "f"] & \\
h^{-1}(\Lambda) \arrow[ur, dashed, "h"] \arrow[r, dashed, "\theta_1"] & {} \arrow[r, dotted] & {} \arrow[r, dashed, "\theta_n"] & fg(\Lambda)
\end{tikzcd}$$

(2) $\theta_1 \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, $\deg(\theta_i \cdots \theta_1 h^{-1}(\Lambda)) < D$ for $i = 2, \ldots, n$ or $\theta_1 \in \mathcal{J}_*$, $\theta_2 \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, $\deg(\theta_1) = \deg(h) - 1$ and

$$\deg(\theta_1(\Lambda)) = \deg(\theta_2\theta_1(\Lambda)) \leq D,$$
$$\deg(\theta_i \cdots \theta_1 h^{-1}(\Lambda)) < D, \quad i = 3, \ldots, n.$$

(3) If $h \in \mathcal{J}_\circ$ is a standard or special quintic transformation and $f \in \mathcal{J}_*$, the same statements holds with

$$\theta_1 \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2),\ \deg(\theta_i \cdots \theta_1 h^{-1}(\Lambda)) < D,\ i = 2, \ldots, n$$

If $\deg(h^{-1}(\Lambda)) < D$, then "$<$" holds everywhere.

Proof. We only look at the situation, where $f \in \mathcal{J}_\circ$, $h \in \mathcal{J}_*$, since for $f \in \mathcal{J}_*$, $h \in \mathcal{J}_\circ$ the proof works similarly.

Let $p := [1:0:0]$, and define $m(q) := m_\Lambda(q)$ to be the multiplicity of Λ at q. Call $p_1, \dots, \bar{p}_2, p_3, \bar{p}_3$ the base-points of f. By Lemma 4.4 we have

$$m(p_1) + m(p_2) + m(p_3) > D \tag{Ineq^3}$$

By Lemma 4.4 there exist two real or two non-real conjugate base-points r_1, r_2 of h, such that

$$m(p) + m(r_1) + m(r_2) \geq D \tag{Ineq^4}$$

Note that if $\deg(h^{-1}(\Lambda)) < D$, then "$>$" holds everywhere (Lemma 4.4) and we will have "$<$" everywhere.

We order r_1, r_2 such that $m(r_1) \geq m(r_2)$ and such that r_1 is a proper point of $\mathbb{P}^2$ or infinitely near p and r_2 is a proper base-point of $\mathbb{P}^2$ or infinitely near p or r_1. Let $\mathfrak{s}_1 \cup \mathfrak{s}_2 \cup \mathfrak{s}_3 = g^{-1}(\mathfrak{p}_1 \cup \mathfrak{p}_2 \cup \mathfrak{p}_3)$ such that $m(s_1) \geq m(s_2) \geq m(s_3)$ and if s_i is infinitely near s_j, then $i > j$. In particular, s_1 is a proper point of $\mathbb{P}^2$. We now look at two cases, depending on whether r_1, r_2 are real or not. Inequality (Ineq^3) translates to

$$m(s_1) + m(s_2) + m(s_3) > D \tag{Ineq^5}$$

We look at two cases, depending on whether r_1, r_2 are real or not.

Case 1: Suppose that r_1, r_2 are real points. Let $t \in \{p, r_1, r_2\} \cap \mathbb{P}^2$ such that $m(t) = \max\{m(p), m(r_1), m(r_2)\}$. Then

$$m(t) + 2m(s_1) > D$$

There exists $\tau \in \mathcal{J}_*$ of degree 2, $\alpha \in \operatorname{Aut}_\mathbb{R}(\mathbb{P}^2)$ such that $\theta\alpha$ has base-points $t, s_1, \bar{s}_1$. There exists $\beta_1, \beta_2, \beta_3 \in \operatorname{Aut}_\mathbb{R}(\mathbb{P}^2)$ such that $\tilde{\tau} := \beta_1 g(\tau\alpha)^{-1}\beta_1$, $\tilde{f} := \beta_3 f(\beta_2)^{-1} \in \mathcal{J}_\circ$ (see Remark 4.3). The situation is summarised in the following commutative diagram:

$$\begin{tikzcd}
 & \Lambda \arrow[r, "g"] \arrow[d, dashed, "{[q,s_1,\bar{s}_1]}", "\tau\alpha"'] & g(\Lambda) \arrow[dr, "\beta_2"] \arrow[rrrd, dashed, bend left, "f"] & & & \\
h^{-1}(\Lambda) \arrow[ur, dashed, "h"] & \tau\alpha(\Lambda) & (\beta_1)^{-1}\tau\alpha(\Lambda) \arrow[l, "\beta_1"] \arrow[r, dashed, "\tilde{\tau}"] \arrow[rr, dashed, bend right] & {} \arrow[r, dashed, "\tilde{f}"] & \beta_3 fg(\Lambda) & fg(\Lambda) \arrow[l, "\beta_3"']
\end{tikzcd}$$

It is generated by relations in $\mathcal{G}$: more specifically, the middle part corresponds to relation Definition 2.7 **(rel. 2)** and the right part corresponds to Definition 2.7 **(rel. 1)** or Remark 4.2 (**rel. 4**). Moreover,

$$\deg((\beta_1)^{-1}\tau\alpha(\Lambda)) = \deg(\tau\alpha(\Lambda)) = 2D - m(q) - 2m(s_1) < D$$

The claim now follows from applying Lemma 4.8 to h, α, τ.

Case 2: Assume that $r_2 = \bar{r}_1$. If $r_1, \bar{r}_1 \in \mathfrak{s}_1 \cup \mathfrak{s}_2 \cup \mathfrak{s}_2$, then in particular, $r_1, \bar{r}_1$ are proper points of $\mathbb{P}^2$, and by Remark 4.7 the points $p, r_1, \bar{r}_1$ are not collinear. So, there exists $\tau \in \mathcal{J}_*$ of degree 2 with base-points $p, r_1, \bar{r}_1$. Let $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{J}_\circ$ such that $\tilde{\tau} := \alpha_2\tau\alpha_1 \in \mathcal{J}_\circ$, $\tilde{f} := \alpha_3 fg\alpha_1 \in \mathcal{J}_\circ$. The situation is summarised in the following commutative diagram:

$$\begin{tikzcd}
 & h(\Lambda) \arrow[rr, "g"] \arrow[dd, dashed, "\tau"] & & g(\Lambda) \arrow[dr, dashed, "f"] & \\
h^{-1}(\Lambda) \arrow[ur, dashed, "h"] \arrow[dr, dashed, "\deg(h)-1"'] & & (\alpha_1)^{-1}(\Lambda) \arrow[ul, "\alpha_1"'] \arrow[d, dashed, "\tilde{\tau}"] \arrow[dr, dashed, "\tilde{f}"] & & fg(\Lambda) \arrow[dl, "\alpha_3"] \\
 & \tau(\Lambda) \arrow[r, "\alpha_2"] & \tilde{\tau}(\alpha_1)^{-1}(\Lambda) \arrow[r, dashed] & \alpha_3 fg(\Lambda) &
\end{tikzcd}$$

It is generated by relations in $\mathcal{G}$: more specifically, the most left part is corresponds to a relation in $\mathcal{J}_*$, the upper right part to relation Definition 2.7 **(rel. 1)** or Remark 4.2 (**rel. 4**) with a relation in

$\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, the middle romboid correspond to relations Definition 2.7 **(rel. 2)** and the right triangle to a relation in $\mathcal{J}_\circ$. Note that $\deg(\tau h) = \deg(h) - 1$ and

$$\deg(\tilde{\tau}(\alpha_1)^{-1}(\Lambda)) = \deg(\tau(\Lambda)) \leq D, \quad \deg(\alpha_3 fgh(\Lambda)) = \deg(fg(\Lambda)) < D$$

The claim follows with $\theta_1 := \tau h, \theta_2 := \alpha_2, \theta_3 := \tilde{f}\tilde{\tau}, \theta_4 := (\alpha_3)^{-1}$.

So, lets assume that $r_1, \bar{r}_1 \notin \mathfrak{s}_1 \cup \mathfrak{s}_2 \cup \mathfrak{s}_2$.

• If $m(p) < m(r_1)$, then in particular $r_1, \bar{r}_1$ are proper points of $\mathbb{P}^2$ and there exists $\tau \in \mathcal{J}_*$ with base-points $p, r_1, \bar{r}_1$. Remark that

$$\deg(\tau(\Lambda)) \leq D, \quad \deg(\tau h) = \deg(h) - 1.$$

Furthermore, from inequality (Ineq[5]) and the order of the s_i's we derive the inequality $2m(r_1) + 2m(s_1) + 2m(s_2) > 2D$. Since moreover r_1, s_1 are proper points of $\mathbb{P}^2$, there exists a standard or special quintic transformation $\theta \in \mathcal{J}_\circ$, $\alpha \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ such that $\theta\alpha$ has base-points $g(\mathfrak{r}_1 \cup \mathfrak{s}_1 \cup \mathfrak{s}_2)$. Consider the following diagram

$$\begin{tikzcd}
 & \Lambda \arrow[r, "g"] \arrow[d, dashed, "\tau"] & g(\Lambda) \arrow[d, dashed, "\theta\alpha"] \arrow[dr, dashed, "f"] & \\
h^{-1}(\Lambda) \arrow[ur, dashed, "h"] \arrow[r, dashed, "\deg(h)-1"'] & \tau(\Lambda) & \theta\alpha g(\Lambda) & fg(\Lambda)
\end{tikzcd}$$

Note that by construction of θ, we have

$$\deg(\theta\alpha g(\Lambda)) = 5D - 4m(r_1) - 4m(s_1) - 4m(s_2) < D$$

The maps $\tau, \alpha g, \theta$ are in the situation of the Case 1, and θ, α, f satisfy the assumptions of Lemma 4.6, and the claim follows from them.

• If $m(p) \geq m(r_1)$, then $m(p) + 2m(s_1) > D$ and so there exists $\tau \in \mathcal{J}_*$ with base-points $p, s_1, \bar{s}_1$. We proceed as in Case 1 (where r_1, r_2 are real but the map we construct is of the same kind). □

Proof of Theorem 2.9. We prove the following: If $f_1, \ldots, f_m \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) \cup \mathcal{J}_* \cup \mathcal{J}_\circ$ such that

$$f_m \cdots f_1 = \mathrm{Id} \quad \text{in } \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2),$$

then

$$w(f_m) \cdots w(f_1) = 1 \quad \text{in } \mathcal{G}.$$

It then follows that the natural surjective homomorphism $\mathcal{G} \to \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is an isomorphism.

Let Λ_0 be the linear system of lines in $\mathbb{P}^2$, and define

$$\Lambda_i := (f_i \cdots f_1)(\Lambda_0)$$

It is the linear system of the map $(f_i \cdots f_1)^{-1}$ and of degree $d_i := \deg(f_i \cdots f_1)$. Define

$$D := \max\{d_i \mid i = 1, \ldots, m\}, \ n := \max\{i \mid d_i = D\}, \ k := \sum_{i=1}^{n}(\deg(f_i) - 1)$$

We use induction on the lexicographically ordered pair (D, k).

If $D = 1$, then $f_1, \ldots, f_m$ are linear maps, and thus $w(f_m) \cdots w(f_1) = 1$ holds in $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ (and hence in $\mathcal{G}$). So, lets assume that $D > 1$. Note that by construction $\deg(f_{n+1}) \geq 2$. We may assume that f_n is a linear map - else we can insert Id after f_n, i.e. $w(f_m) \cdots w(f_1) = w(f_m) \cdots w(f_{n+1}) w(\mathrm{Id}) w(f_n) \cdots w(f_1)$, which does not change (D, k).

We now construct maps $\theta_1, \ldots, \theta_N \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) \cup \mathcal{J}_* \cup \mathcal{J}_\circ$ such that

$$w(f_{n+1}) w(f_n) w(f_{n-1}) = w(\theta_N) \cdots w(\theta_1)$$

and such that the pair $(\tilde{D}, \tilde{k})$ associated to $f_m \cdots f_{n+1} \theta_N \cdots \theta_1 f_{n-2} \cdots f_1$ is strictly smaller than (D, k).

If $f_{n-1}, f_{n+1} \in \mathcal{J}_*$, we apply Lemma 4.8 to f_{n-1}, f_n, f_{n+1}. We obtain a new pair (D', k') where $D' < D$ or $D' = D$ and in that case, we have

$$\text{Lemma 4.8 (2)}: \quad n' = n-1 \text{ and hence } k' = \sum_{i=1}^{n-2}(\deg(f_i) - 1) < k$$

$$\text{or Lemma 4.8 (3)}: \quad k' = \sum_{i=1}^{n-2}(\deg(f_i) - 1) + (\deg(\theta_1) - 1) + (\deg(\theta_2) - 1) = k - 1.$$

In particular, the pair (D, k) decreases.

If $f_{n-1} \in \mathcal{J}_\circ$ or $f_{n+1} \in \mathcal{J}_\circ$, we have to look at three cases, depending on to which group they belong to. We will only do one case as the other two are done similarly.

Suppose that $f_{n-1} \in \mathcal{J}_\circ$ and $f_{n+1} \in \mathcal{J}_*$. By Lemma 4.4, there exists a base-point q of $(f_{n-1})^{-1}$ of multiplicity 2 such that $m(p_1)+m(p_2)+m(q) \geq D$, or there exists $i \in \{1, 2\}$ such that $2m(p_i)+m(r) \geq D$, where r is the simple base-point of $(f_{n-1})^{-1}$. We can assume that q is either a proper point of $\mathbb{P}^2$ or in the first neighbourhood of one of $p_1, \bar{p}_1, p_2, \bar{p}_2$.

• If $m(p_1) + m(p_2) + m(q) \geq D$ for some *non-real* base-point q of $(f_{n-1})^{-1}$ of multiplicity 2, then $p_1, \ldots, \bar{p}_2, q, \bar{q}$ are not one one conic (Lemma 3.1). So, there exists a standard or special quintic transformation $\theta \in \mathcal{J}_\circ$ with base-points $p_1, \ldots, \bar{p}_2, q, \bar{q}$. Then $\theta f_{n-1} \in \mathcal{J}_\circ$ and

$$(*) \qquad \deg(\theta f_{n-1}) = \deg(f_{n-1}) - 4 < \deg(f_{n-1}), \quad \deg(\theta(\Lambda_{n-1})) \leq D.$$

Applying Lemma 4.9 to $\theta^{-1}, f_n, f_{n+1}$ decreases (D, k).

• If $m(p_1) + m(p_2) + m(q) \geq D$ for some *real* base-point q of f of multiplicity 2, then q is a proper point of $\mathbb{P}^2$. If $\deg(f_{n-1})$ is odd, then by Bézout, q is not collinear with any two of $p_1, \bar{p}_1, p_2, \bar{p}_2$, and there exists $\theta_1 \in \mathcal{J}_\circ$ of degree 3 with base-points $q, p_1, \ldots, \bar{p}_2$ (Lemma 3.4). If $\deg(f_{n-1})$ is even, let p_i be a base-point of multiplicity $\frac{\deg(f_{n-1})}{2}$. By Bézout, q is not collinear with any two of $\{p_1, \bar{p}_1, p_2, \bar{p}_2\}$ except maybe $p_{3-i}, \bar{p}_{3-i}$. It follows from Lemma 3.3 that there exists $\theta_2 \in \mathcal{J}_\circ$ of degree 2 with base-points $q, p_i, \bar{p}_i$. Note that for $i = 1, 2$, $\theta_i f_{n+1} \in \mathcal{J}_\circ$ and

$$(**) \qquad \deg(\theta_i f_{n-1}) = \deg(f_{n-1}) - 2 < \deg(f_{n-1}), \quad \deg(\theta(\Lambda_{n-1})) \leq D$$

There exist $\tilde{\theta}_i \in \mathcal{J}_*$ and $\alpha_1, \alpha_2 \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ such that $\theta_i = \alpha_2 \tilde{\theta}_i \alpha_1$. By Definition 2.7 **(rel. 2)**, $w(\theta_i) = w(\alpha_2) w(\tilde{\theta}_i) w(\alpha_1)$ and we can apply Lemma 4.8 to $\tilde{\theta}^{-1}$, $f_n(\alpha_1)^{-1}, f_{n+1}$, which decreases (D, k).

• Suppose that there is no base-point q of multiplicity 2 such that $m(p_1)+m(p_2)+m(q) \geq D$, which means by Lemma 4.4 that

(1) d is even,
(2) $m(r) + 2m(p_i) \geq D$, $i \in \{1, 2\}$,
(3) $2m(p_i) + m(r) > D$ if $\deg(f_{n-1}) > 2$.

If $\deg(f_{n-1}) = 2$, there exist $\alpha, \beta \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2), \tau \in \mathcal{J}_*$ such that $f_{n-1} = \beta\tau\alpha \in \mathcal{J}_*$. Aplplying Lemma 4.8 to $\tau, f_n\alpha^{-1}, f_{n+1}$ decreases (D, k).

If $\deg(f_{n-1}) > 2$, the point r may not be a proper point of $\mathbb{P}^2$. We denote by s the proper point of $\mathbb{P}^2$ to which r is infinitely near to, if r is not a proper point of $\mathbb{P}^2$, and $s = r$ if r is a proper point of $\mathbb{P}^2$. The above list still holds if we write s instead of r. In particular, $p_i, \bar{p}_i, s$ are not collinear and so there exists $\tau \in \mathcal{J}_\circ$ of degree 2 with base-points $s, p_i, \bar{p}_i$ (Lemma 3.3). Then $\tau f_{n-1} \in \mathcal{J}_\circ$ and

$$\deg(\tau(\Lambda_{n-1})) = 2D - m(r) - 2m(p_i) < D.$$

The situation is summarised in the following commutative diagram:

$$\begin{tikzcd}
& \Lambda_{n-1} \arrow[r, "f_n"] \arrow[d, dashed, "\tau"] & \Lambda_n \arrow[dr, dashed, "f_{n+1}"] & \\
\Lambda_{n-1} \arrow[ur, dashed, "f_{n-1}"] \arrow[r, dashed] & \tau(\Lambda_{n-1}) & & {}
\end{tikzcd}$$

There exist $\alpha, \beta \in \operatorname{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, $\tilde{\tau} \in \mathcal{J}_*$ of degree 2 such that $\tau = \beta\tilde{\tau}\alpha$. Applying Lemma 4.8 to $\tau, f_n\alpha^{-1}, f_{n+1}$ decreases (D,k). $\square$

5. A quotient of $\operatorname{Bir}_{\mathbb{R}}(\mathbb{P}^2)$

Let $\varphi_0 \colon \mathcal{J}_\circ \to \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$ be the map given in Definition 3.10. By Theorem 2.9, the group $\operatorname{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is isomorphic to $\mathcal{G}$ (see Defintion 2.7), which, according to Remark 2.8, is the quotient of the free product $\operatorname{Aut}_{\mathbb{R}}(\mathbb{P}^2) * \mathcal{J}_* * \mathcal{J}_\circ$ by the normal subgroup generated by all the relations given by the pairwise intersections of $\operatorname{Aut}_{\mathbb{R}}(\mathbb{P}^2), \mathcal{J}_*, \mathcal{J}_\circ$ and the relations **(rel. 1)**, **(rel. 2)**, **(rel. 3)** of Definition 2.7. Define the map

$$\Phi\colon\ \operatorname{Aut}_{\mathbb{R}}(\mathbb{P}^2) * \mathcal{J}_* * \mathcal{J}_\circ \longrightarrow \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}, \quad f \mapsto \begin{cases} \varphi_\circ(f), & f \in \mathcal{J}_\circ \\ 0, & f \in \operatorname{Aut}_{\mathbb{R}}(\mathbb{P}^2) \cup \mathcal{J}_* \end{cases}$$

It is a surjective homomorphism of groups because $\varphi_\circ$ is a surjective homomorphism of groups (Lemma 3.14).

We shall now show that there exists a homomorphism φ such that the diagram

$$\begin{tikzcd} \operatorname{Aut}_{\mathbb{R}}(\mathbb{P}^2) * \mathcal{J}_* * \mathcal{J}_\circ \arrow[r, "\pi"] \arrow[d, "\Phi"] & \mathcal{G} \simeq \operatorname{Bir}_{\mathbb{R}}(\mathbb{P}^2) \arrow[dl, dotted, "\exists\ \varphi"] \\ \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z} & \end{tikzcd}$$

is commutative, where π is the quotient map. For this, it suffices to show that $\ker(\pi) \subset \ker(\Phi)$. We will first show that the relations given by the pairwise intersections of $\operatorname{Aut}_{\mathbb{R}}(\mathbb{P}^2), \mathcal{J}_*, \mathcal{J}_\circ$ are contained in $\ker(\Phi)$ and then it is left to prove that relations **(rel. 1)**, **(rel. 2)**, **(rel. 3)** are contained in $\ker(\Phi)$.

Lemma 5.1.
(1) *Let $f_1 \in \operatorname{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, $f_2 \in \mathcal{J}_\circ$ such that $\pi(f_1) = \pi(f_2)$. Then $\Phi(f_1) = \Phi(f_2) = 0$.* (2) *Let $f_1 \in \mathcal{J}_*$, $f_2 \in \mathcal{J}_\circ$ such that $\pi(f_1) = \pi(f_2)$. Then $\Phi(f_1) = \Phi(f_2) = 0$.*
In particular, Φ induces a homomorphism from the generalised amalgamated product of $\operatorname{Aut}_{\mathbb{R}}(\mathbb{P}^2), \mathcal{J}_*, \mathcal{J}_\circ$ along all pairwise intersections onto $\bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$.

Proof. (1) We have $\pi(f_1) = \pi(f_2) \in \operatorname{Aut}_{\mathbb{R}}(\mathbb{P}^2) \cap \mathcal{J}_\circ \subset \mathcal{J}_\circ$. In particular, $\varphi_\circ(\pi(f_i)) = 0$, $i = 1, 2$ (Remark 3.13, (2)), and so $\Phi(f_1) = \Phi(f_2) = 0$ by definition of Φ.

(2) Lets first figure out what exactly $\mathcal{J}_* \cap \mathcal{J}_\circ$ consists of. First of all, it is not empty because the quadratic involution

$$\tau\colon [x:y:z] \dashrightarrow [y^2+z^2 : xy : xz]$$

is contained in it. Let $f \in \mathcal{J}_* \cap \mathcal{J}_\circ$ be of degree d. By Lemma 3.1, its characteristic is $(d; \frac{d-1}{2}^4, 2^{\frac{d-1}{2}})$ or $(d; \frac{d}{2}^2, \frac{d-2}{2}^2, 2^{\frac{d-2}{2}}, 1)$. Since $f \in \mathcal{J}_*$, it has characteristic $(d; d-1, 1^{2d-2})$. If follows that $d \in \{1, 2, 3\}$.

Linear and quadratic elements of $\mathcal{J}_\circ$ are sent by $\varphi_\circ$ onto 0 (Remark 3.13 (2)). Elements of $\mathcal{J}_\circ$ of degree 3 decompose into quadratic elements of $\mathcal{J}_\circ$ and are hence sent onto zero by $\varphi_\circ$ as well. In particular, $\Phi(f_1) = \Phi(f_2) = 0$.

Since $\Phi(\operatorname{Aut}_{\mathbb{R}}(\mathbb{P}^2)) = \Phi(\mathcal{J}_*) = 0$, (1) and (2) imply that Φ induces a homomorphism from the generalised amalgamated product of $\operatorname{Aut}_{\mathbb{R}}(\mathbb{P}^2), \mathcal{J}_*, \mathcal{J}_\circ$ onto $\bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$. $\square$

Lemma 5.2. *Let $\theta \in \mathcal{J}_\circ$ be a standard quintic with $S(\theta) = \{(q, \bar{q})\}$, $S(\theta^{-1}) = \{(q', \bar{q}')\}$.*

Let $\alpha_q, \alpha_{q'} \in \operatorname{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ that fix p_1 and send q (resp. q') onto p_2. Then $\theta' := \alpha_{q'}\theta(\alpha_q)^{-1} \in \mathcal{J}_\circ$ is a standard quintic transformation and

$$\Phi(\theta) = \Phi(\alpha_{q'}\theta(\alpha_q)^{-1}) = \Phi(\theta')$$

Note that the statement still holds if we write $\bar{p}_2$ instead of p_2.

Proof. Remark that

$$S(\theta') = \{\ (\alpha_q(p_2),\ \alpha_q(\bar{p}_2))\ \}.$$

Hence we need to show that

$$\Phi(\theta') = \varphi_\circ(\theta') = e_{\nu(C_{\alpha_q(p_2)})} = e_{\nu(C_q)} = \varphi_\circ(\theta) = \Phi(\theta).$$

To do this, it suffices to show that $\pi_\circ(C_{\alpha_q(p_2)}) = \lambda\pi_\circ(C_q)$ or $\pi_\circ(C_{\alpha_q(p_2)}) = \lambda\pi_\circ(C_{\bar{q}})$ for some $\lambda \in \mathbb{R}_{>0}$. For this, we need to understand the map α_q. So, we study the non algebraic mapping

$$\psi\colon \mathbb{P}^2(\mathbb{C}) \setminus \{z=0\} \longrightarrow \mathbb{P}^2(\mathbb{C}) \setminus \{z=0\}, \quad q \mapsto \alpha_q(p_2)$$

which we can describe, via the parametrisation

$$\iota\colon \mathbb{R}^2 \to \mathbb{P}^2(\mathbb{C}), \quad (u,v,x,y) \mapsto [u+iv : x+iy : 1],$$

by the real birational involution

$$\hat{\psi}\colon \mathbb{R}^4 \dashrightarrow \mathbb{R}^4, \quad (u,v,x,y) \mapsto\dashrightarrow \left(\frac{ud-vx}{v^2+y^2}, \frac{-v}{v^2+y^2}, \frac{uv+xy}{v^2+y^2}, \frac{y}{v^2+y^2}\right).$$

The domain of $\hat{\psi}$ is $\mathbb{R}^4 \setminus \{v=y=0\} = \iota^{-1}\left(\mathbb{P}^2(\mathbb{C}) \setminus (\{z=0\} \cup \mathbb{P}^2(\mathbb{R}))\right)$. To understand $\psi(C_q \setminus \{z=0\})$, we use the parametrisation

$$\begin{aligned} \mathrm{par}\colon \mathbb{C} &\longrightarrow C_q \setminus \{z=0\}, \\ t &\mapsto \left[\frac{(t-1)(t+1)(\lambda+\mu)}{\lambda t+\mu t+\lambda-\mu} : \frac{i(\lambda t^2+\mu t^2+2\lambda t-2\mu t+\lambda+\mu)}{\lambda t+\mu t+\lambda-\mu} : 1\right], \end{aligned}$$

which is the inverse of the projection of C_q centred at p_1. This yields the commutative diagram

$$\begin{tikzcd} & \iota^{-1}(C_q \setminus \{z=0\}) \arrow[r, "\hat{\psi}"] \arrow[d, "\iota"] & \hat{\psi}(\iota^{-1}(C_q \setminus \{z=0\})) \arrow[d, "\iota"] & \\ \mathbb{C} \arrow[r, "\mathrm{par}"] \arrow[rrr, bend right] & C_q \setminus \{z=0\} \arrow[r, "\psi"] & \psi(C_q \setminus \{z=0\}) \arrow[r, "\pi_\circ"] & \mathbb{P}^1 \end{tikzcd}$$

The map $(\pi_\circ \circ \psi \circ \mathrm{par})$ is given by

$$x+iy \mapsto \left[\frac{-\rho Q_q(x,y)}{4(\nu^2+\rho^2)} + i\frac{-\nu Q_q(x,y)}{4(\rho^2+\nu^2)} : 1\right] = \left[\frac{-Q_q(x,y)}{4(\nu^2+\rho^2)}(\rho+i\nu) : 1\right]$$

where $\rho,\nu \in \mathbb{R}$ are the real coordinates of $\pi_\circ(C_q)$, i.e. $\pi_\circ(C_q) = [\rho+i\nu : 1]$, and

$$Q_q(x,y) = (\nu^2+\rho^2+2\rho+1)(x^2+y^2) + (2x+1)(\nu^2+\rho^2) + 4\nu y - 2\rho - 2x + 1 \in \mathbb{R}[x,y].$$

This shows that

$$\pi_\circ(C_{(\alpha_q)^{-1}(p_2)}) = \pi_\circ((\alpha_q)^{-1}(p_2)) \in \pi_\circ(\psi(C_q \setminus \{z=0\})) = \mathbb{R}_{>0}\pi_\circ(C_q).$$

□

Proposition 5.3. *The homomorphism Φ induces a surjective homomorphism of groups*

$$\varphi\colon \operatorname{Bir}_{\mathbb{R}}(\mathbb{P}^2) \longrightarrow \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$$

which is given as follows:

Let $f \in \operatorname{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ and write $f = f_n \cdots f_1$, where $f_1,\dots,f_n \in \operatorname{Aut}_{\mathbb{R}}(\mathbb{P}^2) \cup \mathcal{J}_* \cup \mathcal{J}_\circ$. Then $\varphi(\operatorname{Aut}_{\mathbb{R}}(\mathbb{P}^2) \cup \mathcal{J}_*) = 0$ and

$$\varphi(f) = \sum_{i=1}^n \Phi(f_i) = \sum_{f_i \in \mathcal{J}_\circ} \varphi_\circ(f_i)$$

Its kernel $\ker(\varphi)$ *contains all elements of degree ≤ 4.*

Proof. Let $\pi\colon \operatorname{Aut}_{\mathbb{R}}(\mathbb{P}^2) * \mathcal{J}_* * \mathcal{J}_\circ \to \mathcal{G} \simeq \operatorname{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ be the quotient map (Remark 2.8). We want to show that there exists a homomorphism $\varphi\colon \operatorname{Bir}_{\mathbb{R}}(\mathbb{P}^2) \to \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$ such that the diagram

$$\begin{tikzcd} \operatorname{Aut}_{\mathbb{R}}(\mathbb{P}^2) * \mathcal{J}_* * \mathcal{J}_\circ \arrow[r, "\pi"] \arrow[d, "\Phi"] & \mathcal{G} \simeq \operatorname{Bir}_{\mathbb{R}}(\mathbb{P}^2) \arrow[dl, dotted, "\exists\ \varphi"] \\ \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z} & \end{tikzcd}$$

is commutative. For this, it suffices to show that $\ker(\pi) \subset \ker(\Phi)$. By Lemma 5.1, Φ induces a homomorphism from the generalised amalgamated product of $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2), \mathcal{J}_*, \mathcal{J}_\circ$ along all intersections onto $\bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$. So, by Remark 2.8 it suffices to show that Φ sends the relations **(rel. 1)**, **(rel. 2)**, **(rel. 3)** in Definition 2.7 onto zero.

Linear, quadratic and cubic transformations in $\mathcal{J}_\circ$ and the group $\mathcal{J}_*$ are sent onto zero by φ (definition of Φ and Remark 3.13 (2)), hence relations **(rel. 2)** and **(rel. 3)** are contained in $\ker(\Phi)$. So, we just have to bother with relation **(rel. 1)**:

Lets $\theta_1, \theta_2 \in \mathcal{J}_\circ$ be standard quintic transformations, $\alpha_1, \alpha_2 \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ such that

$$\theta_2 = \alpha_2 \theta_1 \alpha_1$$

If α_1, α_2 are contained in $\mathcal{J}_\circ$, then $\Phi(\alpha_2\theta_1\alpha_1(\theta_2)^{-1}) = \varphi_\circ(\alpha_2\theta_1\alpha_2(\theta_2)^{-1}) = \varphi_\circ(\mathrm{Id}) = 0$.

So, lets assume that $\alpha_1, \alpha_2 \notin \mathcal{J}_\circ$. Denote $S(\theta_1) = \{(p_3, \bar{p}_3)\}$, $S((\theta_1)^{-1}) = \{(p_4, \bar{p}_4)\}$. There exist $i, j \in \{1, 2, 3\}$ such that

$$(\alpha_1)^{-1}(\{p_i, \bar{p}_i\}) = \{p_1, \bar{p}_1\}, \qquad (\alpha_1)^{-1}(\{p_j, \bar{p}_j\}) = \{p_2, \bar{p}_2\}.$$

Since $\alpha_1 \notin \mathcal{J}_\circ$, we have $3 \in \{i, j\}$. By Remark 3.2 there exist $\beta, \gamma \in \mathcal{J}_\circ \cap \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ such that $(\alpha_1^{-1}\beta^{-1})(p_1) = p_1$ and $(\alpha_2\gamma)(p_1) = p_1$. We obtain that $(\beta\alpha_1)^{-1}(p_3) \in \{p_2, \bar{p}_2\}$ and $(\alpha_2\gamma)(p_4) \in \{p_2, \bar{p}_2\}$. It follows from Lemma 5.2 that

$$\Phi(\theta_2) = \Phi\left((\alpha_2\gamma)(\gamma^{-1}\theta_1\beta^{-1})(\beta\alpha_1)\right) = \Phi(\theta_1),$$

i.e. Φ sends relation **(rel. 1)** onto zero. The surjectivity of φ follows from the surjectivity of φ_0 (Lemma 3.14).

If $f \in \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is of degree 2 or 3 there exists $\alpha, \beta \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ such that $\beta f \alpha \in \mathcal{J}_*$. Hence $\varphi(f) = 0$. If $\deg(f) = 4$, f is a composition of quadratic maps, hence $\varphi(f) = 0$. □

Let X be a real variety. We denote by $X(\mathbb{R})$ its set of real points of and by $\mathrm{Aut}(X(\mathbb{R})) \subset \mathrm{Bir}(X)$ the subgroup of transformations defined at each point of $X(\mathbb{R})$. It is also called the group of *birational diffeomorphisms* of X, and is, in general, strictly larger than the group of automorphisms $\mathrm{Aut}_{\mathbb{R}}(X)$ of X defined over $\mathbb{R}$. The group $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$ is generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ and the standard quintic transformations [RV2005, BM2012]. Until now no similar result has been found for $\mathrm{Aut}(\mathbb{A}^2(\mathbb{R}))$.

Corollary 5.4. *There exist surjective group homomorphisms*

$$\mathrm{Aut}(\mathbb{P}^2(\mathbb{R})) \to \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}, \qquad \mathrm{Aut}(\mathbb{A}^2(\mathbb{R})) \to \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}.$$

Proof. We identify $\mathbb{A}^2(\mathbb{R})$ with $\mathbb{P}^2(\mathbb{R}) \setminus L_{p_1,\bar{p}_1}$. All quintic transformations are contained in $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$ (Lemma 2.3) and preserve $C_3 := L_{p_1,\bar{p}_1} \cup L_{p_2,\bar{p}_2}$. For any standard quintic transformation θ there exists a permutation α of $p_1, \dots, \bar{p}_2$ such that $\alpha\theta$ preserves $L_{p_i,\bar{p}_i}$, $i = 1, 2$, i.e. is contained in $\mathrm{Aut}(\mathbb{A}^2(\mathbb{R}))$. Therefore, the restriction of φ onto $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$ and $\mathrm{Aut}(\mathbb{A}^2(\mathbb{R}))$ is surjective. □

Let $\mathcal{Q}_{3,1} \subset \mathbb{P}^3$ be the variety given by the equation $w^2 = x^2 + y^2 + z^2$. Its real part $\mathcal{Q}_{3,1}(\mathbb{R})$ is the 2-sphere $\mathbb{S}^2$.

Lemma 5.5. *There exists a surjective group homomorphism*

$$\mathrm{Aut}(\mathcal{Q}_{3,1}(\mathbb{R})) \to \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}.$$

Proof. By [KM2009, Theorem 1] (see also [BM2012, Theorem 1.4]), the group $\mathrm{Aut}(\mathcal{Q}_{3,1}(\mathbb{R}))$ is generated by $\mathrm{Aut}_{\mathbb{R}}(\mathcal{Q}_{3,1}) = \mathbb{P}\mathrm{O}(3,1)$ and the family of standard cubic transformations (see [BM2012, Example 5.1] for definition). Consider the stereographic projection

$$\mathrm{p}\colon \mathcal{Q}_{3,1} \dashrightarrow \mathbb{P}^2, \quad [w : x : y : z] \mapsto [w - z : x : y]$$

It is a real birational transformation obtained by first blowing-up the point $[1 : 0 : 0 : 1]$ and then blowing down the singular hyperplane section $w = z$ onto the points $p_2, \bar{p}_2$. The inverse p^{-1} is an isomorphism around $p_1, \bar{p}_1$ and p sends a general hyperplane section onto a general conic passing through $p_2, \bar{p}_2$.

To prove the Lemma, it suffices to show that every standard quintic transformation $\theta \in \mathcal{J}_\circ$ that contracts the conic passing through all its base-points except p_2 onto p_2 is conjugate via p to a standard cubic transformation in $\mathrm{Aut}(\mathcal{Q}_{3,1}(\mathbb{R}))$. The description of p implies that $\mathrm{p}\,\theta\,\mathrm{p}^{-1} \in \mathrm{Aut}(\mathcal{Q}_{3,1}(\mathbb{R}))$. So, by [BM2012, Lemma 5.4 (3)], we have to show that $(\mathrm{p}\,\theta\,\mathrm{p}^{-1})^{-1}$ sends a general hyperplane section onto a cubic section passing through $\mathrm{p}^{-1}(p_1), \mathrm{p}^{-1}(\bar{p}_1)$, $\mathrm{p}^{-1}(p_3), \mathrm{p}^{-1}(\bar{p}_3)$ with multiplicity 2.

Let $p_1, \bar{p}_1$, $p_2, \bar{p}_2$, $p_3, \bar{p}_3$ be the base-points of θ. The map p sends a general hyperplane section onto a general conic C passing through $p_2, \bar{p}_2$. The curve $\theta^{-1}(C)$ is a curve of degree 6 with multiplicity 3 in $p_2, \bar{p}_2$ and multiplicity 2 in $p_1, \bar{p}_1, p_3, \bar{p}_3$. Therefore, $(\mathrm{p}^{-1}\,\theta^{-1})(C) \subset \mathcal{Q}_{3,1}$ is a curve of self-intersection 18 passing through $\mathrm{p}^{-1}(p_1), \mathrm{p}^{-1}(\bar{p}_1)$, $\mathrm{p}^{-1}(p_3), \mathrm{p}^{-1}(\bar{p}_3)$ with multiplicity 2. It follows that $(\mathrm{p}^{-1}\,\theta\,\mathrm{p})^{-1}$ sends a general hyperplane section onto a cubic section having multiplicity 2 at these four points. $\square$

Corollary 5.4 and Lemma 5.5 imply Corollary 1.4.

Corollary 5.6 (Corollary 1.6)**.** *For any real birational map $\psi\colon \mathbb{F}_0 \dashrightarrow \mathbb{P}^2$, the group $\psi\,\mathrm{Aut}(\mathbb{F}_0(\mathbb{R}))\psi^{-1}$ is a subgroup of* $\ker(\varphi)$*.*

Proof. By [BM2012, Theorem 1.4], the group $\mathrm{Aut}(\mathbb{F}_0(\mathbb{R}))$ is generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{F}_0) \simeq \mathrm{PGL}(2,\mathbb{R})^2 \rtimes \mathbb{Z}/\mathbb{Z}$ and the involution

$$\tau\colon ([u_0 : u_1], [v_0 : v_1]) \mapstochar\dashrightarrow ([u_0, u_1], [u_0v_0 + u_1v_1 : u_1v_0 - u_0v_1]).$$

Consider the real birational map

$$\psi\colon \mathbb{P}^2 \dashrightarrow \mathbb{F}_0, \quad [x : y : z] \mapstochar\dashrightarrow ([x : z], [y : z]),$$

with inverse $\psi^{-1}\colon ([u_0 : u_1], [v_0, v_1]) \mapstochar\dashrightarrow [u_0v_1 : u_1v_0 : u_1v_1]$.

A quick calculation shows that the conjugate by ψ of these generators of $\mathrm{Aut}(\mathbb{F}_0(\mathbb{R}))$ are of degree at most 3. Proposition 5.3 implies that they are contained in $\ker(\varphi)$. In particular, $\psi^{-1}\,\mathrm{Aut}(\mathbb{F}_0(\mathbb{R}))\psi \subset \ker(\varphi)$. Since $\ker(\varphi)$ is a normal subgroup of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$, the same statement holds for any other real birational map $\mathbb{P}^2 \dashrightarrow \mathbb{F}_0$. $\square$

Corollary 5.7 (Corollary 1.7)**.** *For any $n \in \mathbb{N}$ there is a normal subgroup of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ of index 2^n containing all elements of degree ≤ 4.*
The same statement holds for $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$, $\mathrm{Aut}(\mathbb{A}^2(\mathbb{R}))$ and $\mathrm{Aut}(\mathcal{Q}_{3,1}(\mathbb{R}))$.

Proof. Let $pr_{\delta_1,\dots,\delta_n}\colon \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z} \to (\mathbb{Z}/2\mathbb{Z})^n$ be the projection onto the $\delta_1,\dots,\delta_n$-th factors. Then $pr_{\delta_1,\dots,\delta_n} \circ \varphi$ has kernel of index 2^n containing $\ker(\varphi)$ and thus all elements of degree ≤ 4. By Corollary 1.4, the same argument works for $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$, $\mathrm{Aut}(\mathbb{A}^2(\mathbb{R}))$ and $\mathrm{Aut}(\mathcal{Q}_{3,1}(\mathbb{R}))$. $\square$

Lemma 2.5 and Theorem 2.4 imply that $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2), \sigma_1, \sigma_0$ and all standard quintic transformations in $\mathcal{J}_\circ$. This generating set is not far from being minimal:

Corollary 5.8 (Theorem 1.2 and Corllary 1.5)**.** *The group $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is not generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ and a countable family of elements.*

The same statement holds for $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$, $\mathrm{Aut}(\mathbb{A}^2(\mathbb{R}))$ and $\mathrm{Aut}(\mathcal{Q}_{3,1}(\mathbb{R}))$, replacing $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ for the latter two by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{A}^2)$, $\mathrm{Aut}_{\mathbb{R}}(\mathcal{Q}_{3,1})$ respectively.

Proof. If $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ was generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ and a countable family $\{f_n\}_{n\in\mathbb{N}}$ of elements of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ then by Proposition 5.3, the countable family would yield a countable generating set of $\oplus_{\mathbb{R}}\mathbb{Z}/2\mathbb{Z}$, which is impossible.

The same argument works for $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$, $\mathrm{Aut}(\mathcal{Q}_{3,1}(\mathbb{R}))$ and $\mathrm{Aut}(\mathbb{A}^2(\mathbb{R}))$ - for the latter two we replace $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ respectively by $\mathbb{P}\mathrm{O}(3,1)$ and by the subgroup of affine automorphisms of $\mathbb{A}^2$, which corresponds to $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) \cap \mathrm{Aut}(\mathbb{A}^2(\mathbb{R})))$. $\square$

Corollary 5.9 (Corollary 1.8)**.** *The normal subgroup of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ generated by any countable set of elements of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is a proper subgroup of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$.*
The same statement holds for $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$, $\mathrm{Aut}(\mathbb{A}^2(\mathbb{R}))$ and $\mathrm{Aut}(\mathcal{Q}_{3,1}(\mathbb{R}))$.

Proof. Let $S \subset \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ be a countable set of elements. Its image $\pi(S) \subset \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$ is a countable set and hence a proper subset of $\bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$. Since π is surjective (Proposition 5.3), the preimage $\pi^{-1}(\pi(S)) \subsetneq \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is a proper subset. The group $\oplus_{\mathbb{R}}\mathbb{Z}/2\mathbb{Z}$ is Abelian, so the set $\pi^{-1}(\pi(S))$ contains the normal subgroup of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ generated by S, which in particular is a proper subgroup of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$. $\square$

Remark 5.10. The group homomorphism $\varphi\colon \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2) \to \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$ does not have any sections: If it had a section, then for any $k \in \mathbb{N}$ the group $(\mathbb{Z}/2\mathbb{Z})^k$ would embed into $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$, which is not possible by [Bea2007].

Remark 5.11. Over $\mathbb{C}$, the group $\mathcal{J}_\circ$ is conjugate to $\mathcal{J}_*$ (f.e. by any quadratic transformation having base-points $p_1, \bar{p}_1, p_2$ and sending $\bar{p}_2$ onto $[1:0:0]$). This is not true over $\mathbb{R}$: By Proposition 5.3, one is contained in $\ker(\varphi)$ and the other is not.

Remark 5.12.

(1) No proper normal subgroup of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ of finite index is closed with respect to the Zariski or the Euclidean topology because $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is connected with respect to either topology [Bla2010].

(2) The group $\mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^2)$ does not contain any proper normal subgroups of countable index: Assume that $\{\mathrm{Id}\} \neq N$ is a normal subgroup of countable index. The image of $\mathrm{PGL}_3(\mathbb{C})$ in the quotient is countable, hence $\mathrm{PGL}_3(\mathbb{C}) \cap N$ is non-trivial. Since $\mathrm{PGL}_3(\mathbb{C})$ is a simple group, we have $\mathrm{PGL}_3(\mathbb{C}) \subset N$. Since the normal subgroup generated by $\mathrm{PGL}_3(\mathbb{C})$ is $\mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^2)$ [Giz1994, Lemma 2], we get that $N = \mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^2)$.

Proposition 5.3 states that $\ker(\varphi)$ contains all elements of degree ≤ 4. For the field of complex numbers, modding out any non-trivial transformation of degree ≤ 4 will result in the trivial quotient, as shown in the following lemma by Gizatullin.

Lemma 5.13 (cf. [Giz1994])**.** *The normal subgroup of $\mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^2)$ generated by any non-trivial element of of degree ≤ 4 is equal to $\mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^2)$.*

Proof. The claim is stated in [Giz1994, Remark on Lemma 2, p. 42] for degree ≤ 7 but only a partial proof is given, which works for all transformations preserving a pencil of lines [Giz1994, Lemma 2].

(deg 1:) For degree 1, it is the fact that the normal subgroup generated by $\mathrm{PGL}_3(\mathbb{C})$ is equal to $\mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^2)$ [Giz1994, Lemma 2, Case 1 of proof].

(deg 2, 3:) Let $f \in \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ be of degree 2 or 3. There exists a proper base-point q (resp. q') of f (resp. f^{-1}) such that f sends the pencil of lines through q onto the pencil of lines through q'. Pick $\alpha \in \mathrm{PGL}_3(\mathbb{C})$ that exchanges q, q' and such that $f\alpha^{-1}f\alpha \neq \mathrm{Id}$. Then $f\alpha^{-1}f\alpha$ is contained in the normal subgroup generated by f and preserves the pencil of lines through q'. Hence the normal subgroup generated by f is $\mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^2)$ [Giz1994, Lemma 2].

(deg 4:) If the transformation has a triple base-point, we prove the claim with a similar argument as above. For transformations without triple points, despite the idea of proof in [Giz1994, Remark on lemma 2], we only succeeded to show the claim with a lot of effort and a rather long case by case study depending on the configuration of the base-points. $\square$

6. The kernel of the quotient

In this section, we prove that the kernel of $\varphi\colon \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2) \to \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$ is the normal subgroup generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, which will turn out to be the commutator subgroup of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$. It implies that the quotient $\varphi\colon \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2) \to \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$ is in fact the Abelianisation of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$.

For this, we will again use the presentation of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ in terms of generators and relations given in Theorem 2.9. We will see that $\ker(\varphi)$ is the normal subgroup generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, $\mathcal{J}_*$ and $\ker(\varphi_0)$, and then it suffices to prove that $\mathcal{J}_*$ and $\ker(\varphi_\circ)$ are contained in the normal subgroup generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$.

The key idea is to show that if two standard quintic transformations θ_1, θ_2 are sent by $\varphi_\circ$ onto the same image, then θ_2 can be obtained by composing θ_1 with a suitable amount of quadratic elements, which will imply that $\theta_1(\theta_2)^{-1}$ is contained the normal subgroup of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$. For this to be useful, we need to be able to put the quintic elements next to each other when decomposing an element of $\mathcal{J}_\circ$ into quadratic and standard quintic elements. To do this we need to sidle off to involve special quintic transformations (see Definition 3.5), which is why they pop up again in this section.

More precisely, Lemma 6.5 shows that if two standard or special quintic transformations in $\mathcal{J}_\circ$ have the same image via $\varphi_\circ$, we can obtain one from the other by composing with a suitable amount of quadratic transformations in $\mathcal{J}_\circ$. We then show that every quadratic transformation in $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is contained in the normal subgroup generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ (Lemma 6.7). Lemma 6.8 shows that $\mathcal{J}_*$ is

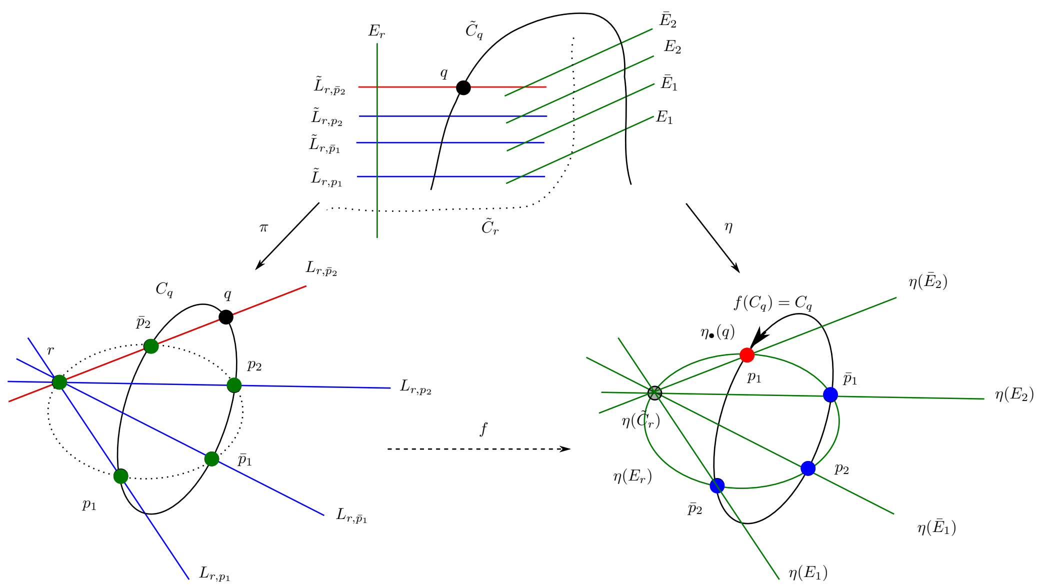

FIGURE 4. The cubic transformation of Lemma 6.3

contained in the normal subgroup generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$. All of this will yield that that the kernel of φ is indeed the normal subgroup generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ (Proposition 6.12)

Definition 6.1. We denote by $\langle\langle \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)\rangle\rangle$ the normal subgroup of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$.

6.1. **Geometry between cubic and quintic transformations.** One idea in the proof that $\ker(\varphi) = \langle\langle \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)\rangle\rangle$ is to see that if two standard quintic transformations are sent onto the same standard vector in $\bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$, then one is obtained from the other by composing from the right and the left with suitable cubic maps, which in turn can be written as composition of quadratic maps. For this, we first have to dig into the geometry of cubic maps.

Remark 6.2. Let $f \in \mathcal{J}_\circ$ of degree 3 and $r \in \mathbb{P}^2(\mathbb{R})$ its double point. The points $p_1, \dots, \bar{p}_2$ are base-points of f of multiplicity 1 (Lemma 3.1). Note that for $i \in \{1,2\}$, the map f contracts the line passing through r, p_i onto one of $p_1, \bar{p}_1, p_2, \bar{p}_2$ and that r is not collinear with any two of $p_1, \bar{p}_1, p_2, \bar{p}_2$.

By Lemma 3.3 there is a quadratic transformation $g \in \mathcal{J}_\circ$ with base-points $r, p_1, \bar{p}_1$. The map fg^{-1} is of degree 2 and thus every element of $\mathcal{J}_\circ$ of degree 3 is the composition of two quadratic elements of $\mathcal{J}_\circ$.

Lemma 6.3. *Let $q \in \mathbb{P}^2(\mathbb{C}) \setminus \{p_1, \bar{p}_1, p_2, \bar{p}_2\}$ be a non-real point such that $C_q = \pi_\circ^{-1}(\pi_\circ(q)) \notin \{C_1, C_2, C_3\}$. Then there exists a real point $r \in \mathbb{P}^2(\mathbb{C})$ and $f \in \mathcal{J}_\circ$ of degree 2 or 3 with r among its base-points such that*

(1) $f(C_q) = C_q$
(2) $f_\bullet(q)$ is infinitely near p_1 corresponding to the tangent direction of $f(C_q)$
(3) either $\deg(f) = 3$ and $C_r \notin \{C_1, C_2, C_3\}$ or $\deg(f) = 2$ and $C_r \in \{C_1, C_2, C_3\}$ is singular.
(4) $q \in L_{r,\bar{p}_2}$.

Proof. Let L be the line passing through $q, \bar{p}_2$. Since C_q is irreducible, q is not collinear with any of $p_1, \bar{p}_1, p_2, \bar{p}_2$. It follows that $L \neq \bar{L}$, and so L and $\bar{L}$ intersect in exactly one point r, which is a real point.

If r is not collinear with any two of $p_1, \bar{p}_1, p_2, \bar{p}_2$, then Lemma 3.4 states that there exists $f \in \mathcal{J}_\circ$ of degree 3 with singular point r. The line L is contracted onto p_i or $\bar{p}_i$, $i \in \{1,2\}$. By composing with elements of $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) \cap \mathcal{J}_\circ$, we can assume that L is contracted onto p_1 and that f preserves the conic $L_{p_1,p_2} \cup L_{\bar{p}_1,\bar{p}_2}$, and thus induces the identity map on $\mathbb{P}^1$ (Lemma 3.9), and therefore preserves C_q. It follows that $f^\bullet(q)$ is infinitely near p_1 and corresponds to the tangent direction of $f(C_q) = C_q$.

If r is collinear with two of $p_1, \bar{p}_1, p_2, \bar{p}_2$, it is collinear with $p_1, \bar{p}_1$ and not collinear with any other two. Lemma 3.3 implies that there exists $f \in \mathcal{J}_\circ$ of degree 2 with base-points $r, p_2, \bar{p}_2$, and we can choose

f such that the line L (through $q, \bar{p}_2, r$) is contracted onto p_1 (then $f(\{p_1, \bar{p}_1\}) = \{p_2, \bar{p}_2\}$) and such that $f(p_1) = p_2$. Then $f_\bullet(q)$ is infinitely near p_1. We claim that $f(C_q) = C_q$: Call $\hat{f}$ the automorphism of $\mathbb{P}^1$ induced by f. We calculate $\hat{f}^{-1}$ (cf. proof of Lemma 3.9). Since $f(L_{p_1,p_2}) = L_{p_1,p_2}$, we see that $\hat{f}^{-1}\colon [u:v] \mapsto [(r_1^2 + (r_0+r_2)^2)u : (r_1^2 + (r_0-r_2)^2)v]$, where $r = [r_0 : r_1 : r_2]$. Since $r \in L_{p_1,\bar{p}_1}$, we have $r_2 = 0$ and so $\hat{f}^{-1} = \mathrm{Id}$. In particular, $f(C_q) = C_q$. □

Lemma 6.4. *Let $\theta_1, \theta_2 \in \mathcal{J}_\circ$ be special quintic transformations with $S(\theta_i) = \{(q_i, \bar{q}_i)\}$. If $C_{q_1} = C_{q_2}$ or $C_{q_1} = C_{\bar{q}_2}$, then there exist $\alpha_1, \alpha_2 \in \mathcal{J}_\circ \cap \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ such that $\theta_2 = \alpha_2\theta_1\alpha_1$.*

Proof. We can assume that q_1 is infinitely near p_1 as the proof is the same if it is infinitely near $\bar{p}_1, p_2$ or $\bar{p}_2$.

If q_2 or $\bar{q}_2$ are infinitely near p_1 as well, then $\{q_1, \bar{q}_1\} = \{q_2, \bar{q}_2\}$. Therefore $\theta_2\theta_1^{-1} \in \mathcal{J}_\circ \cap \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$.

Suppose that q_2 or $\bar{q}_2$ is infinitely near p_2. The automorphism $\alpha\colon [x:y:z] \mapsto [z:-y:x]$ is contained in $\mathcal{J}_\circ$ and exchanges p_1 and p_2, while inducing the identity map on $\mathbb{P}^1$. Then $\theta_2\alpha$ and θ_1 are in the case above. □

Lemma 6.5. *Let $\theta_1, \theta_2 \in \mathcal{J}_\circ$ be standard quintic transformations with $S(\theta_i) = \{(q_i, \bar{q}_i)\}$, $i = 1, 2$. Assume that $C_{q_1} = C_{q_2}$ or $C_{q_1} = C_{\bar{q}_2}$.*

Then there exist $\tau_1, \ldots, \tau_8 \in \mathcal{J}_\circ$ of degree ≤ 2 such that $\theta_2 = \tau_8 \cdots \tau_5\theta_1\tau_4 \cdots \tau_1$.

Proof. By exchanging the names of $q_2, \bar{q}_2$, we can assume that $C_{q_1} = C_{q_2}$. It suffices to show that there exist $g_1, \ldots, g_4 \in \mathcal{J}_\circ$ of degree ≤ 3 such that $\theta_2 = g_4g_3\theta_1g_2g_2$, since every element of $\mathcal{J}_\circ$ of degree 3 can be written as composition of two qudratic elements of $\mathcal{J}_\circ$ (Remark 6.2). We give an explicit construction of the g_i's.

According to Lemma 6.3 there exist for $i = 1, 2$ a real point r_i and $f_i \in \mathcal{J}_\circ$ of degree $d_i \in \{2, 3\}$ with base-point r_i such that f_i preserves C_{q_i} and $t_i := (f_i)_\bullet(q_i)$ is infinitely near p_1 corresponding to the tangent direction of C_{q_i} and that $q_i \in L_{r_i,\bar{p}_2} =: L$. Since C_{r_i} is real, r_i is not on a conic contracted by θ_i, and so $s_i := (\theta_i)_\bullet(r_i) = \theta_i(r_i)$ is a proper point of $\mathbb{P}^2$.

If C_{r_i} is irreducible (and hence $d_i = 3$), then r_i is not collinear with any two of $p_1, \ldots, \bar{p}_2$, and so s_i is not collinear with any two of $p_1, \ldots, \bar{p}_2$ either. Therefore, there exists $h_i \in \mathcal{J}_\circ$ of degree 3 with singular base-point s_i (Lemma 3.4). If C_{r_i} is singular (and hence $d_i = 2$), then $r_i \in L_{p_1,\bar{p}_1}$, and so $s_i \in \theta_i(L_{p_1,\bar{p}_1}) = L_{p_j,\bar{p}_j}$ for some $j \in \{1, 2\}$. Therefore, there exists $h_i \in \mathcal{J}_\circ$ of degree 2 with base-points $s_i, p_{3-j}, \bar{p}_{3-j}$ (Lemma 3.3).

By composing h_i with elements in $\mathcal{J}_\circ \cap \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, we can assume that h_i sends the line $\theta_i(L_{r_i,\bar{p}_2})$ onto p_1 (Remark 6.2). Then $h_i\theta_i(f_i)^{-1} \in \mathcal{J}_\circ$ is of degree 5. Its base-points are $p_1, \bar{p}_1, p_2, \bar{p}_2, (f_i)_\bullet(q_i), (f_i)_\bullet(\bar{q}_i)$, where the latter ones are infinitely near $p_1, \bar{p}_1$ corresponding to the tangent direction of $C_{q_i}, C_{\bar{q}_i}$. By Lemma 6.4, $h_1\theta_1(f_1)^{-1}$ and $h_2\theta_2(f_2)^{-1}$ have exactly the same base-points, hence $h_1\theta_1(f_1)^{-1} = \beta h_2\theta_2(f_2)^{-1}\alpha$ for some $\alpha, \beta \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) \cap \mathcal{J}_\circ$. In particular, $\theta_2 = (h_2)^{-1}\beta^{-1}h_1\theta_1(f_1)^{-1}\alpha^{-1}f_2$. The claim follows with $g_1 = \alpha^{-1}f_2,\ g_2 = (f_1)^{-1},\ g_3 = \beta^{-1}h_1,\ g_4 = (h_2)^{-1}$. □

Lemma 6.6. *Let $\theta_1, \theta_2 \in \mathcal{J}_\circ$ be a standard and a special quintic transformation respectively with $S(\theta_i) = \{(q_i, \bar{q}_i)\}$. Assume that $C_{q_1} = C_{q_2}$ or $C_{q_1} = C_{\bar{q}_2}$.*

Then there exists $\tau_1, \ldots, \tau_4 \in \mathcal{J}_\circ$ of degree ≤ 2 such that $\theta_2 = \tau_4\tau_3\theta_1\tau_2\tau_1$.

Proof. By exchanging the names of $q_1, \bar{q}_1, q_2, \bar{q}_2$, we can assume that $C_{q_1} = C_{q_2}$ and that q_2 is infinitely near p_i, $i \in \{1, 2\}$. By Lemma 6.3 there exists $f \in \mathcal{J}_\circ$ of degree $d \in \{2, 3\}$ such that $f(C_{q_1}) = C_{q_1} = C_{q_2}$ and $f_\bullet(q_1)$ is infinitely near p_i. Let r be the real base-point of f. Since r is real, it is not on a conic contracted by θ_1, and so $(\theta_1)_\bullet(r) = \theta_1(r)$ is a proper point of $\mathbb{P}^2$.

If C_r is irreducible (i.e. $d = 3$), the conic $\theta_1(C_r) = C_{\theta(r)}$ is irreducible as well. By Lemma 3.4 there exists $g \in \mathcal{J}_\circ$ of degree 3 with double point $\theta_1(r)$. If C_r is singular (i.e. $d = 2$), the conic $\theta_1(C_r) = C_{\theta(r)}$ is singular as well. By Lemma 3.3 there exists $g \in \mathcal{J}_\circ$ of degree 2 with $\theta(r)$ among its base-points.

The map $g\theta_1f^{-1}$ is of degree 5 with base-points $p_1, \bar{p}_1, p_2, \bar{p}_2, f_\bullet(q_1), f_\bullet(q_1)$, where the latter two are infinitely near $p_i, \bar{p}_i$ corresponding to the tangent directions $C_{q_1} = C_{q_2}, C_{\bar{q}_2}$. By Lemma 6.4 there exists $\alpha \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) \cap \mathcal{J}_\circ$ such that $\alpha g\theta_1f^{-1} = \theta_2$. The claim follows from the fact that we can write f, g as composition of at most two quadratic transformations in $\mathcal{J}_\circ$. □

6.2. **The normal subgroup generated by** $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$**.** Lemma 6.5 implies that if two standard or special quintic transformations θ_1, θ_2 contract the same conics through $p_1, \bar{p}_1, p_2, \bar{p}_2$, then θ_2 is obtained from θ_1 by composing with suitable quadratic transformations. So, one step of proving that $\ker(\varphi) = \langle\langle \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)\rangle\rangle$ is to see that all quadratic transformations are contained in $\langle\langle \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)\rangle\rangle$.

Lemma 6.7. *Any quadratic map in* $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ *is contained in* $\langle\langle \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)\rangle\rangle$.

Proof. Let $\tau \in \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ be of degree 2. Pick two base-points q_1, q_2 of τ that are either a pair of non-real conjugate points or two real base-points, such that either both are proper points of $\mathbb{P}^2$ or q_1 is a proper point of $\mathbb{P}^2$ and q_2 is in the first neighbourhood of q_1. Let t_1, t_2 be base-points of τ^{-1} such that τ sends the pencil of conics through q_1, q_2 onto the pencil of conics through t_1, t_2. Pick a general point $r \in \mathbb{P}^2$ and let $s := \tau(r)$. There exists $\alpha \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ that sends q_1, q_2 onto t_1, t_2 and exchanges r, s. The map $\tilde{\tau} := \tau\alpha$ is of degree 2, fixes s, and t_1, t_2 are base-points of $\tilde{\tau}$ and $\tilde{\tau}^{-1}$.

Since r is general, also s is general, and there exists $\theta \in \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ of degree 2 with base-points t_1, t_2, s. Observe that the map $\theta\tilde{\tau}\theta^{-1}$ is linear. In particular, τ is contained in $\langle\langle \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)\rangle\rangle$. □

Recall that $\mathcal{J}_*$ is contained in $\ker(\varphi)$. Using Lemma 6.7, we now prove that $\mathcal{J}_*$ is contained in $\langle\langle \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)\rangle\rangle$. The following lemma is classical over algebraically closed fields (see for instance [AC2002, §8.4]). Not having found a reference over the real numbers, we have included a proof.

Lemma 6.8. *The group* $\mathcal{J}_*$ *is generated by its quadratic and linear elements. In particular,* $\mathcal{J}_* \subset \langle\langle \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)\rangle\rangle$.

Proof. Let $f \in \mathcal{J}_*$. We do induction on the degree $d = \deg(f)$ of f. If f is linear or quadratic, there is nothing to do. So, we can assume that $d \geq 3$.

Case 1: Assume that there exist two simple base-points q_1, q_2 of f that are proper points of $\mathbb{P}^2$ and either non-real conjugate points or both real points. The points $[1:0:0], q_1, q_2$ are not collinear by Bézout, hence there exists a quadratic map $\tau \in \mathcal{J}_*$ with base-points $[1:0:0], q_1, q_2$. The map $f\tau^{-1} \in \mathcal{J}_*$ is of degree $d-1$.

Case 2: Assume that f has exactly one simple (real) base-point q that is a proper point of $\mathbb{P}^2$. Let r be a general real point in $\mathbb{P}^2$. There exists $\tau_1 \in \mathcal{J}_*$ of degree 2 with base-points $[1:0:0], q, r$ and the map $f(\tau_1)^{-1} \in \mathcal{J}_*$ is of degree d. If t is a base-point of f in the first neighbourhood of $[1:0:0]$ or q, then $(\tau_1)^\bullet(t)$ is a base-point of $f(\tau_1)^{-1}$ that is a proper point of $\mathbb{P}^2$. Thus $f(\tau_1)^{-1}$ has at least two simple base-points that are proper points of $\mathbb{P}^2$ and either non-real conjuagte points (if t is non-real) or both real (if t is real). We proceed as above.

Case 3: Assume that f has no simple proper base-points at all, i.e. any simple base-point is infinitely near $[1:0:0]$.

• If there are at least two base-points q_1, q_2 in the first neighbourhood of $[1:0:0]$, let $r, s \in \mathbb{P}^2$ be general points. There exists $\tau_1 \in \mathcal{J}_*$ of degree 2 with base-points $[1:0:0], r, s$. Call $[1:0:0], r', s'$ the base-points of $(\tau_1)^{-1}$. The map $f(\tau_1)^{-1}$ is of degree $d+1$. We may assume that q_1, q_2 are both real or a pair of non-real conjugate points. Then $(\tau_1)_\bullet(q_1), (\tau_1)_\bullet(q_2)$ are proper points of $\mathbb{P}^2$ and base-points of $f(\tau_1)^{-1}$. Since $[1:0:0], \tau_1(q_1), \tau_1(q_2)$ are not collinear, there exists $\tau_2 \in \mathcal{J}_*$ of degree 2 with base-points $[1:0:0], \tau_1(q_1), \tau_1(q_2)$. The map $f(\tau_1)^{-1}(\tau_2)^{-1}$ is of degree d. We claim that the image by $(\tau_2)_\bullet(\tau_1)^\bullet$ of all base-points of f different from q_1, q_2 in the first neighbourhood of $[1:0:0]$ or of q_1, q_2 are base-points of $f(\tau_1)_{-1}(\tau_2)^{-1}$ that are proper points of $\mathbb{P}^2$ or are in the 1^{st} neighbourhood of $[1:0:0]$: Indeed, let t be a base-point of f in the 1^{st} neighbourhood of $[1:0:0]$ or q_1. Then $(\tau_1)_\bullet(t)$ is either a proper point of $\mathbb{P}^2$ on the line $L_{r',s'}$ or is infinitely near $\tau_1(q_1)$. By Bézout, $[1:0:0], \tau_1(q_1), (\tau_1)_\bullet(t)$ are not collinear. It follows that $(\tau_2)_\bullet((\tau_1)_\bullet(t))$ is either in the 1^{st} neighbourhood of $[1:0:0]$ (if t is in the 1^{st} neighbourhood of $[1:0:0]$ or q_1 and proximate to $[1:0:0]$) or a proper point of $\mathbb{P}^2$ (if t is in the 1^{st} neighbourhood of q_1 but not proximate to $[1:0:0]$). The situation is visualised in the following picture:

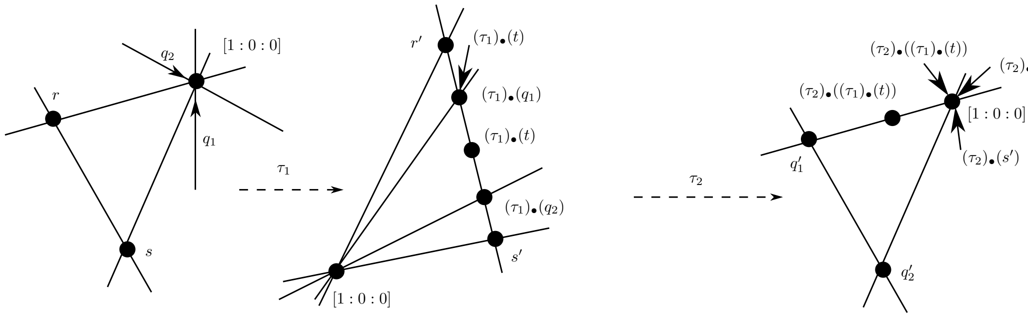

FIGURE: The quadratic maps τ_1, τ_2, and the possibilities for the point $(\tau_2)_\bullet((\tau_1)_\bullet(t))$.

Since not all base-points of f are proximate to $[1:0:0]$, we can repeat all of this until we obtain an element of $\mathcal{J}_*$ of degree d with simple proper base-points. We continue as in Case 1 or Case 2.

• If there is exactly one base-point q of f in the first neighbourhood of $[1:0:0]$, then in particular, q is a real point. Let $r \in \mathbb{P}^2$ be a general real point. There exists $\tau \in \mathcal{J}_*$ of degree 2 with base-points $[1:0:0], q, r$. The map $f\tau^{-1} \in \mathcal{J}_*$ is of degree d and the image by $\tau_\bullet$ of any base-point in the first neighbourhood of q is a base-points of $f\tau^{-1}$ in the first neighbourhood of $[1:0:0]$. We repeat this step until we reach one of the above cases or until we obtain a linear map. □

6.3. **The kernel is equal to $\langle\langle \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)\rangle\rangle$.** It now remains to actually prove that $\ker(\varphi) = \langle\langle \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)\rangle\rangle$. Take an element of $\ker(\varphi)$. It is the composition of linear, quadratic and standard and special quintic elements (Lemma 3.7). The next three lemmata show that we can choose the order of the linear, quadratic and standard and special quintic elements so that the ones belonging to the same coset are just one after another. These lemmata will be the remaining ingredients to prove that $\ker(\varphi) = \langle\langle \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)\rangle\rangle$.

Lemma 6.9. *Let $\tau, \theta \in \mathcal{J}_\circ$ be a quadratic and a standard (or special) quintic transformation respectively. Then there exist $\tilde{\tau}_1, \tilde{\tau}_2 \in \mathcal{J}_\circ$ of degree 2 and $\tilde{\theta}_1, \tilde{\theta}_2 \in \mathcal{J}_\circ$ standard (or special quintic) transformations such that $\tau\theta = \tilde{\theta}_1\tilde{\tau}_1$ and $\theta\tau = \tilde{\tau}_2\tilde{\theta}_2$, i.e. we can "permute" τ, θ.*

Proof. The map τ^{-1} has base-points $p_i, \bar{p}_i, r$, for some $r \in \mathbb{P}^2(\mathbb{R})$, $i \in \{1,2\}$. Since r is not on a conic contracted by θ, the point $\theta_\bullet(r) = \theta(r)$ is a proper point of $\mathbb{P}^2$ that is a base-point of $(\theta\tau)^{-1}$. Let p_{j_i} be the image by θ of the contracted conic not passing through p_i. The map $\theta\tau$ is of degree 6 and $p_{j_i}, \bar{p}_{j_i}$ are base-points of $(\theta\tau)^{-1}$ of multiplicity 3. By Lemma 3.3 there exists $\tilde{\tau} \in \mathcal{J}_\circ$ of degree 2 with base-points $\theta(r), p_{j_i}, \bar{p}_{j_i}$. The map $\tilde{\theta} := \tilde{\tau}\theta\tau \in \mathcal{J}_\circ$ is a standard (or special) quintic transformation. We put $\tilde{\tau}_2 := \tilde{\tau}^{-1}$, $\tilde{\theta}_2 := \tilde{\theta}$. A similar construction yields $\tilde{\theta}_1, \tilde{\tau}_1$. □

Lemma 6.10. *Let $\theta_1, \theta_2 \in \mathcal{J}_\circ$ be standard or special quintic transformations (both can be either) such that $\varphi_0(\theta_1) \neq \varphi_0(\theta_2)$. Then there exist $\theta_3, \theta_4 \in \mathcal{J}_\circ$ standard or special quintic transformations, such that*

$$\theta_2\theta_1 = \theta_4\theta_3, \quad \varphi_0(\theta_1) = \varphi_0(\theta_4), \quad \varphi_0(\theta_2) = \varphi_0(\theta_3)$$

i.e. we can "permute" θ_1, θ_2.

Proof. Let $S(\theta_1) = \{(p_3, \bar{p}_3)\}$ and $S(\theta_2) = \{(p_4, \bar{p}_4)\}$. By definition of φ_0 the assumption $\varphi_0(\theta_1) \neq \varphi_0(\theta_2)$ implies $p_4 \notin C_{p_3} \cup C_{\bar{p}_3}$.

The point $p_5 := ((\theta_1)^{-1})_\bullet(p_4)$ is either a proper point of $\mathbb{P}^2$ or in the first neighbourhood of one of $p_1, \bar{p}_1, p_2, \bar{p}_2$. Because $p_4, \bar{p}_4, p_1, \bar{p}_1, p_2, \bar{p}_2$ are not on one conic, the points $p_5, \bar{p}_5, p_1, \ldots, \bar{p}_2$ are not on one conic. So, there exists a standard or special quintic transformation $\theta_3 \in \mathcal{J}_\circ$ with base-points $p_1, \ldots, \bar{p}_2, p_5, \bar{p}_5$. The map $\theta_4 := \theta_2\theta_1(\theta_3)^{-1} \in \mathcal{J}_\circ$ is a standard or special quintic transformation. In fact, its inverse has base-points $p_1, \ldots, \bar{p}_2, (\theta_2)_\bullet(p_3), (\theta_2)_\bullet(\bar{p}_3)$. We have by construction $\theta_2\theta_1 = \theta_4\theta_3$. The equalities $\varphi_\circ(\theta_1) = \varphi_\circ(\theta_4)$ and $\varphi_\circ(\theta_2) = \varphi_\circ(\theta_3)$ follow from the construction and Remark 3.13 (7). □

Lemma 6.11. *Let $\theta_1, \theta_2 \in \mathcal{J}_\circ$ be standard or special quintic transformations (both can be either) such that $\varphi_0(\theta_1) = \varphi_0(\theta_2)$. Then $\theta_1(\theta_2)^{-1} \in \langle\langle \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)\rangle\rangle$.*

Proof. Let $S(\theta_1) = \{(p_3, \bar{p}_3)\}$ and $S(\theta_2) = \{(p_4, \bar{p}_4)\}$. The assumption $\varphi_0(\theta_1) = \varphi_0(\theta_2)$ implies that there exists some $\lambda \in \mathbb{R}_{>0}$ such that $\pi_\circ(C_{p_3}) = \lambda\pi_\circ(C_{p_4})$ or $\pi_\circ(C_{p_3}) = \lambda\pi_\circ(C_{\bar{p}_4})$ in $\mathbb{P}^1$. By Lemma 3.9 there exist $\tau_1 \in \mathcal{J}_\circ$ of degree 2 such that $\pi_\circ(\tau_1(C_{p_3})) = \pi_\circ(C_{p_4})$ (resp. $\pi_\circ(C_{\bar{p}_4})$), i.e. $\tau_1(C_{p_3}) = C_{p_4}$ (resp. $C_{\bar{p}_4}$). Let r be the real base-points of τ. Since C_r is a real conic, it is not contracted by θ_1 and hence $(\theta_1)_\bullet(r) = \theta_1(r)$ is a proper point of $\mathbb{P}^2$ and a base-point of $(\theta_1\tau_1)^{-1}$. Let p_{j_i} be the image by θ_1 of the contracted conic not passing through p_i. The map $\theta_1\tau_1$ is of degree 6 and $p_{j_i}, \bar{p}_{j_i}$ are base-points of $(\theta_1\tau_1)^{-1}$ of multiplicity 3. By Lemma 3.3 there exists $\tau_2 \in \mathcal{J}_\circ$ of degree 2 with base-points $\theta(r), p_{j_i}, \bar{p}_{j_i}$. The map $\tau_2\theta_1\tau_1 \in \mathcal{J}_\circ$ is a standard or special quintic transformation contracting the conics $C_{p_4}, C_{\bar{p}_4}$. Hence, by Lemma 6.5, Lemma 6.4 and Lemma 6.6, there exist $\nu_1, \dots, \nu_{2m} \in \mathcal{J}_\circ$ of degree ≤ 2 such that $\theta_2 = \nu_{2m}\cdots\nu_{m+1}(\tau_2\theta_1\tau_n^{-1})\nu_m\cdots\nu_1$. Then

$$\theta_1(\theta_2)^{-1} = \big(\theta_1(\nu_m\cdots\nu_1)^{-1}(\tau_1)\theta_1^{-1}\big)\big(\tau_2)^{-1}(\nu_{2m}\cdots\nu_{m+1})^{-1}.$$

By Lemma 6.7, all quadratic elements of $\mathcal{J}_\circ$ belong to $\langle\langle \mathrm{Aut}_\mathbb{R}(\mathbb{P}^2)\rangle\rangle$, so $\theta_1(\theta_2)^{-1}$ is contained in $\langle\langle \mathrm{Aut}_\mathbb{R}(\mathbb{P}^2)\rangle\rangle$. □

Proposition 6.12. *Let $\varphi\colon \mathrm{Bir}_\mathbb{R}(\mathbb{P}^2) \to \bigoplus_\mathbb{R} \mathbb{Z}/2\mathbb{Z}$ be the surjective group homomorphism defined in Theorem 5.3. Then*

$$\ker(\varphi) = \langle\langle \mathrm{Aut}_\mathbb{R}(\mathbb{P}^2)\rangle\rangle$$

Proof. By definition of φ (see Proposition 5.3), $\mathrm{Aut}_\mathbb{R}(\mathbb{P}^2)$ is contained in $\ker(\varphi)$, hence $\langle\langle \mathrm{Aut}_\mathbb{R}(\mathbb{P}^2)\rangle\rangle \subset \ker(\varphi)$. Lets prove the other inclusion. Consider the commutative diagram from Proposition 5.3:

$$\begin{tikzcd} \mathrm{Aut}_\mathbb{R}(\mathbb{P}^2) * \mathcal{J}_* * \mathcal{J}_\circ \arrow[r, "\pi"] \arrow[d, "\Phi"] & \mathcal{G} \simeq \mathrm{Bir}_\mathbb{R}(\mathbb{P}^2) \arrow[dl, "\varphi"] \\ \bigoplus_\mathbb{R} \mathbb{Z}/2\mathbb{Z} & \end{tikzcd}$$

It follows that $\ker(\varphi) = \pi(\ker(\Phi))$, which is the normal subgroup generated by $\mathrm{Aut}_\mathbb{R}(\mathbb{P}^2), \mathcal{J}_\circ$ and $\ker(\varphi_\circ)$. Moreover, $\mathrm{Aut}_\mathbb{R}(\mathbb{P}^2)$ and $\mathcal{J}_*$ are contained in $\langle\langle \mathrm{Aut}_\mathbb{R}(\mathbb{P}^2)\rangle\rangle$ (Lemma 6.8), thus it suffices to prove that $\ker(\varphi_0)$ is contained in $\langle\langle \mathrm{Aut}_\mathbb{R}(\mathbb{P}^2)\rangle\rangle$.

By Lemma 3.7, every $f \in \ker(\varphi_\circ)$ is the composition of linear, quadratic and standard quintic elements of $\mathcal{J}_\circ$. Note that a quadratic or quintic element composed with a linear element is still a quadratic or standard quintic element respectively, so we can assume that f decomposes into quadratic and standard quintic elements. For every $\delta \in \mathbb{R}$ the number of standard quintic elements in the decomposition of f with image e_δ is even. According to Lemma 6.9 and Lemma 6.10, we can write f as a composition of quadratic, and standard and special quintic transformations, such that for each $\delta \in \mathbb{R}$, all the standard and special quintic transformations with image e_δ are next to each other. In particular, for any δ the number of standard and special quintic transformations next to each other that are sent onto e_δ is even. It follows from Lemma 6.11, Lemma 6.7 and $\varphi_0(\theta) = \varphi_0(\theta^{-1})$ (Remark 3.13 (6)) that $f \in \langle\langle \mathrm{Aut}_\mathbb{R}(\mathbb{P}^2)\rangle\rangle$. □

Corollary 6.13. *We have*

$$\langle\langle \mathrm{Aut}_\mathbb{R}(\mathbb{P}^2)\rangle\rangle = \ker(\varphi) = \big[\mathrm{Bir}_\mathbb{R}(\mathbb{P}^2), \mathrm{Bir}_\mathbb{R}(\mathbb{P}^2)\big]$$

Proof. The first equality is Proposition 6.12. The normal subgroup $\big[\mathrm{Bir}_\mathbb{R}(\mathbb{P}^2), \mathrm{Bir}_\mathbb{R}(\mathbb{P}^2)\big]$ contains non-trivial linear elements, and since $\mathrm{Aut}_\mathbb{R}(\mathbb{P}^2) \simeq \mathrm{PGL}_3(\mathbb{R}) = \mathrm{PSL}_3(\mathbb{R})$ is a simple group (see for instance [Gri1999, Theorem 3.6.7]), $\big[\mathrm{Bir}_\mathbb{R}(\mathbb{P}^2), \mathrm{Bir}_\mathbb{R}(\mathbb{P}^2)\big]$ contains $\mathrm{Aut}_\mathbb{R}(\mathbb{P}^2)$ and therefore also $\langle\langle \mathrm{Aut}_\mathbb{R}(\mathbb{P}^2)\rangle\rangle$. Thus, the Abelianisation homomorphism factors through φ. As φ is a homomorphism onto an Abelian group it implies that φ *is* the Abelianisation homomorphism. □

Corollary 6.14 (Corollary 1.3)**.** *The sequence of iterated commutated subgroups of* $\mathrm{Bir}_\mathbb{R}(\mathbb{P}^2)$ *is stationary. More specifically: Let $H := [\mathrm{Bir}_\mathbb{R}(\mathbb{P}^2), \mathrm{Bir}_\mathbb{R}(\mathbb{P}^2)]$. Then $[H, H] = H$.*

Proof. Since $\mathrm{Aut}_\mathbb{R}(\mathbb{P}^2) \subset H$, the group $[H, H]$ contains non-trivial elements of $\mathrm{Aut}_\mathbb{R}(\mathbb{P}^2)$. But $\mathrm{Aut}_\mathbb{R}(\mathbb{P}^2)$ is simple, hence $\mathrm{Aut}_\mathbb{R}(\mathbb{P}^2) \subset [H, H]$. By Corollary 6.13, we have

$$H = \langle\langle \mathrm{Aut}_\mathbb{R}(\mathbb{P}^2)\rangle\rangle \subset [H, H].$$

□

Theorem 6.15 (Theorem 1.1)**.** *The group* $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ *is not perfect: its Abelianisation is isomorphic to*

$$\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)/[\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2),\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)] \simeq \bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}.$$

Moreover, the commutator subgroup of $[\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2),\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)]$ *is the normal subgroup generated by* $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)=\mathrm{PGL}_3(\mathbb{R})$*, and contains all elements of* $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ *of degree* ≤ 4*.*

Proof. Follows from Proposition 5.3, Proposition 6.12 and Corollary 6.13. □

Remark 6.16. The kernel of φ is the normal subgroup N generated by all squares in $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$: On one hand, for any group G, its commutator subgroup $[G,G]$ is contained in the normal subgroup of G generated by all squares. On the other hand, since $\bigoplus_{\mathbb{R}}\mathbb{Z}/2\mathbb{Z}$ is Abelian and all its elements are of order 2, the normal subgroup of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ generated by the squares is contained in $\ker(\varphi)$. The claim now follows from $\ker(\varphi)=[\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2),\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)]$ (Corollary 6.13).

Remark 6.17. Endowed with the Zariski topology or the Euclidean topology (see [BF2013]), the group $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ does not contain any non-trivial proper closed normal subgroups and $\langle\langle \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)\rangle\rangle$ is dense in $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ [BZ2015]. In particular, the quotient topology on $\bigoplus_{\mathbb{R}}\mathbb{Z}/2\mathbb{Z}$ is the trivial topology.

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Susanna Zimmermann, Mathematisches Institut, Universität Basel, Spiegelgasse 1, 4051 Basel, Switzerland

E-mail address: Susanna.Zimmermann@unibas.ch