MAXIMAL SUBGROUPS IN THE CREMONA GROUP

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Abstract. We show that for any $n \geq 5$ there exist connected algebraic subgroups in the Cremona group $\mathrm{Bir}(\mathbb{P}^n)$ that are not contained in any maximal connected algebraic subgroup. Our approach exploits the existence of stably rational, non-rational threefolds.

Contents

Introduction

The goal of this work is to elucidate the algebraic structure of higher-dimensional Cremona groups $\mathrm{Bir}(\mathbb{P}^n)$, which are the groups of birational transformations of the n-dimensional projective space.
It is well-known that $\mathrm{Bir}(\mathbb{P}^1) = \mathrm{PGL}_2(\mathbb{C})$ is an algebraic group, while $\mathrm{Bir}(\mathbb{P}^n)$ with $n \geq 2$ has a much more intriguing group-theoretic nature [13, 8, 28, 9] and cannot be endowed with the structure of an algebraic group. Thanks to the seminal work by Blanc and Furter [7], we understand the topological obstruction to equip $\mathrm{Bir}(\mathbb{P}^n)$, $n \geq 2$, even with the structure of an infinite-dimensional (or ind-)algebraic group.
In this context, a natural problem consists in studying algebraic groups lying in $\mathrm{Bir}(\mathbb{P}^n)$, $n \geq 2$, up to conjugation.

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Demazure formalised in [14] the notion of rational action of an algebraic group G on an algebraic variety X, i.e. of *algebraic subgroups of* $\mathrm{Bir}(X)$ (see Definition 1.1). After the work of Matsumura [30], it is known that $\mathrm{Bir}(X)$ is finite, when X is a variety of general type and, from the view-point of the birational classification of algebraic varieties, we expect $\mathbb{P}^n$ to lie as far as possible from varieties of general type. It is then natural to interpret the structure of *connected* algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^n)$ as a measure of complexity for Cremona groups.

Connected algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^2)$ have been classified by Enriques [16]: up to conjugacy, they are all contained in (the connected component of the identity Aut° of) the automorphism group of $\mathbb{P}^2$ or of (minimal) Hirzebruch surfaces. Moreover the Aut° of those rational surfaces are all non-conjugate in $\mathrm{Bir}(\mathbb{P}^2)$. More recently, the classification of maximal finite algebraic subgroups has been completed in [2] (see also [15, 37]).

Maximal connected algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^3)$ have been classified by Umemura, partially in collaboration with Mukai, in a series of papers [39, 40, 41, 32, 42, 43], see [4, 5] for a modern proof using the Minimal Model Program: all connected algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^3)$ are contained, up to conjugacy, in a maximal one and the full classification of those ones involves several discrete and one continuous families. The classification of maximal finite algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^3)$ is not complete, but several results on special classes of finite subgroups of $\mathrm{Bir}(\mathbb{P}^3)$ have been obtained in the last decade [33, 34, 3] and it is now clear how modern results in birational geometry can be exploited in the study of Cremona groups [35, 8] (see also [24]).

Any classification in dimension $n \geq 4$ is currently unreachable, since we lack fundamental ingredients such as the classification of Fano varieties; partial results have been obtained in [6].

In this work we are interested in maximal connected algebraic subgroups of the Cremona groups in higher dimensions. In the seminal work [14], Demazure studied maximal connected algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^n)$ containing a torus of dimension n: his approach originated the study of toric varieties (see also [4, Section 2.5] for more results on conjugacy classes of tori in $\mathrm{Bir}(\mathbb{P}^n)$). An interesting feature of connected algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ and $\mathrm{Bir}(\mathbb{P}^3)$ is the following: they are all contained in a maximal one. Blanc asked 10 years ago the following.

Question. *Is every connected algebraic subgroup of* $\mathrm{Bir}(\mathbb{P}^n)$ *contained in a maximal one?*

In [19, 18], the algebraic subgroups of $\mathrm{Bir}(C \times \mathbb{P}^1)$, where C is a non-rational curve, are classified up to conjugation. Moreover, Fong shows that if X is a surface of Kodaira dimension $-\infty$, then any algebraic subgroup of $\mathrm{Bir}(X)$ is contained in a maximal algebraic subgroup of $\mathrm{Bir}(X)$ if and only if X is rational. Further results for $\mathrm{Bir}(C \times \mathbb{P}^n)$, $n \geq 2$, have been obtained in [20].

The main result of this work provides an answer to Blanc's question.

Main Theorem. *For $n \geq 5$, then there exist connected algebraic subgroups of* $\mathrm{Bir}(\mathbb{P}^n)$ *which are not contained in any maximal one.*

The approach of this work to study this structural question on $\mathrm{Bir}(\mathbb{P}^n)$ is new and does not depend on any classification, but rather on the nature of rationality in higher dimension.

More concretely, we show the following: let X be the smooth stably rational non-rational threefold of [1]. After birational modification, we may assume that X is endowed with a fibration $c\colon X \to \mathbb{P}^2$. Let $n \geq 1$ and consider the projective bundle $\mathcal{P}_n = \mathbb{P}_X(\mathcal{O}_X \oplus c^*\mathcal{O}_{\mathbb{P}^2}(n))$ over X. The total space $\mathcal{P}_n$ has dimension 4 and since $X \times \mathbb{P}^m$ is rational for any $m \geq 2$ [1, 38], the variety $\mathcal{P}_n \times \mathbb{P}^m$ is rational for any $m \geq 1$. Therefore $\mathrm{Aut}^\circ(\mathcal{P}_n \times \mathbb{P}^m) \cong \mathrm{Aut}^\circ(\mathcal{P}_n) \times \mathrm{Aut}^\circ(\mathbb{P}^m)$ is a connected algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^{m+4})$. The $\mathbb{P}^1$-bundle $\mathcal{P}_n$ admits a unique section X_0 fixed pointwise by $\mathrm{Aut}^\circ(\mathcal{P}_n)$, which acts preserving each fibre of π. The key technical step of the proof is to show that *every* $\mathrm{Aut}^\circ(\mathcal{P}_n)$-equivariant birational map $\mathcal{P}_n \dashrightarrow Y$ is *essentially* a birational transformation to another projective bundle obtained in the following way: first blow up a surface $D_0 \subseteq X_0$ and then contract the strict transform of $\pi^{-1}(\pi(D_0))$. Thus Y is again a $\mathbb{P}^1$-bundle over X. A careful study of the automorphisms of $\mathbb{P}^1$-bundles allows us to find another birational transformation as above, with an $\mathrm{Aut}^\circ(Y)$-equivariant birational map $Y \dashrightarrow W$ such that $\mathrm{Aut}^\circ(Y) \subsetneq \mathrm{Aut}^\circ(W)$ (see Proposition 4.1). This essentially proves that $\mathrm{Aut}^\circ(\mathcal{P}_n)$ is not contained in any maximal algebraic subgroup of $\mathrm{Bir}(\mathcal{P}_n)$ (see Theorem 4.2). We then show in §5 that for any $m \geq 1$, the subgroup $\mathrm{Aut}^\circ(\mathcal{P}_n \times \mathbb{P}^m)$ of $\mathrm{Bir}(\mathbb{P}^{m+4})$ is not contained in any maximal connected algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^{m+4})$.

According to the authors' knowledge, it is to date unknown whether $X \times \mathbb{P}^1$ is rational, i.e. whether $\mathcal{P}_n$ is rational, or not. Therefore, we are currently unable to apply our technique to determine whether the Main Theorem holds for $n = 4$ or not. In the work [25] János Kollár developed a new approach which produced new examples and solved the problem in dimension $n = 4$.

Our construction is inspired by the one used in [20], where it is shown that for any $n \geq 2$ and any curve C of genus ≥ 1, the group $\mathrm{Bir}(C \times \mathbb{P}^n)$ contains connected algebraic subgroups that are not contained in a maximal connected algebraic subgroup.

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1. Preliminary results

We work over the field of complex numbers. Varieties are always projective unless stated otherwise. We refer to [27] for the notion of terminality and the basic notions on the minimal model program.

1.1. **Group actions.**

We recall here some fundamental results on algebraic actions on varieties.

Definition 1.1. Let Y be a variety and let G be an algebraic group. We say that G acts *rationally* on Y if there exists a birational map

$$\mu\colon G\times Y \dashrightarrow G\times Y,\ (g,y)\mapsto (g,\mu(g,y))$$

that restricts to an isomorphism $U\to V$ on dense open subsets $U,V\subseteq G\times Y$, whose projections onto G are surjective, and such that $\mu(gh,\cdot)=\mu(g,\cdot)\circ\mu(h,\cdot)$ for any $g,h\in G$. If moreover the kernel of the induced homomorphism $G\to \mathrm{Bir}(Y),\ g\mapsto \mu_g$ is trivial, i.e. if G acts *faithfully* on Y, then G is called an *algebraic subgroup of* $\mathrm{Bir}(X)$.

The algebraic subgroup G of $\mathrm{Bir}(X)$ is called *maximal* if it is maximal with respect to the inclusion among the algebraic subgroups of $\mathrm{Bir}(X)$.

Notice that if W is a rational variety and $\psi\colon W \dashrightarrow Y$ a birational map and $G\subseteq \mathrm{Aut}(W)$ an algebraic group, then $G\times Y\to Y$, $(g,y)\mapsto (g,\psi g\psi^{-1}(y))$ is a rational action of G on Y and G is an algebraic subgroup of $\mathrm{Bir}(Y)$; conjugating G by ψ embeds G into $\mathrm{Bir}(Y)$.

On the other hand, if G is a connected subgroup of $\mathrm{Bir}(Y)$ acting rationally on Y, by the Weil regularisation theorem [44] there is a birational model of Y on which G acts regularly.

Remark 1.2. By [2], any algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ is contained in a maximal algebraic subgroup. Nevertheless, there are infinite increasing sequences of algebraic subgroups, see [20, Remark 2.8].

Remark 1.3. It is natural to ask if $\mathrm{Bir}(\mathbb{P}^n)$ itself can be endowed with a structure of an (ind-)algebraic group. We know this is not possible, thanks to the work [7] (see also [4, Section 2.5] for the construction of the functor $\mathfrak{Bir}_{\mathbb{P}^n}$).

We also recall the following two classical facts on regular actions. The first follows from [11, Proposition 2, page 8].

Lemma 1.4. *Let G be an algebraic group acting regularly on a projective variety X. Let $n=\max\{\dim(G\cdot x)\mid x\in X\}$ be the maximal dimension of an orbit of G. Then, the set $\{x\in X\mid \dim(G\cdot x)<n\}$ is a closed subset of X. In particular, the union of orbits of dimension n is a dense open G-invariant subset of X.*

The second is the Blanchard's lemma [12, Proposition 4.2.1].

Lemma 1.5. *Let $f\colon X \to Y$ be a proper morphism between varieties such that $f_*(\mathcal{O}_X) = \mathcal{O}_Y$. If a connected algebraic group G acts regularly on X, then there exists a unique regular action of G on Y such that f is G-equivariant.*

By the Blanchard's lemma, there is a homomorphism $\mathrm{Aut}^\circ(X) \to \mathrm{Aut}^\circ(Y)$. We set $\mathrm{Aut}^\circ(X)_Y$ the kernel of this homomorphism.

1.2. **Chow varieties.** We refer to [26] for a presentation of Chow varieties, we introduce here the notation and briefly recall some results.
Let X be a normal projective variety and G a connected group acting regularly on X. Let $\mathrm{Chow}(X)$ be the Chow variety of X. We recall that $\mathrm{Chow}(X)$ has countably many irreducible components (cf. [26, Theorem I.3.21(3)]).

If Z is a subvariety of X we denote by $[Z]$ the corresponding point of $\mathrm{Chow}(X)$. If $\mathcal{W}$ is an irreducible subvariety of $\mathrm{Chow}(X)$, we denote by $\mathcal{U} \subseteq \mathcal{W} \times X$ the universal family defined as

$$\mathcal{U} = \{([Z], x) \in \mathcal{W} \times X |\ x \in Z\}.$$

Denote by $e\colon \mathcal{U} \to X$ and $u\colon \mathcal{U} \to \mathcal{W}$ the natural morphisms. We notice that $u^{-1}[Z] \cong Z$ for every cycle Z in $\mathcal{W}$.
Moreover, if $\mathcal{W}_0 \subseteq \mathcal{W}$, then the universal family over $\mathcal{W}_0$ is $\mathcal{U}_0 = u^{-1}\mathcal{W}_0$: We will refer to it as *the restriction of the universal family over* $\mathcal{W}_0$.

Moreover, G acts on every irreducible component of $\mathrm{Chow}(X)$: indeed, G preserves every irreducible component as those are countable. If $[Z]$ is a subvariety of X and $g \in G$, then the natural action is given by $g\cdot[Z] = [g(Z)]$.

1.3. **Ruled varieties.** This section contains some definitions and facts on ruled varieties. We give first some definitions.

Definition 1.6. Let $\pi\colon X \to B$ be a morphism between normal projective varieties. One says that π is

(1) a $\mathbb{P}^1$-*fibration* if its general fibre is a smooth rational curve;
(2) a *birationally trivial* $\mathbb{P}^1$*-fibration* if its generic fibre is isomorphic to $\mathbb{P}^1_{\mathbb{C}(B)}$.

Let $\pi\colon X \to B$ be a $\mathbb{P}^1$-fibration. Then one says that π is:

(3) a *standard conic bundle* if X and B are smooth and $\rho(X/B) = 1$;
(4) an *embedded conic bundle* if there is a rank 3 vector bundle $\mathcal{E}$ on B and an embedding $X \hookrightarrow \mathbb{P}_B(\mathcal{E})$ such that π is the restriction of the natural morphism $\rho\colon \mathbb{P}_B(\mathcal{E}) \to B$ and X restricted to any fibre of ρ is a conic.

Remark 1.7.

(1) We notice that a birationally-trivial $\mathbb{P}^1$-fibration is a $\mathbb{P}^1$-fibration, and that $\mathbb{P}^1$-fibrations are also called conic bundles.
(2) Moreover, a fibration is birationally trivial if and only if its general fibre is isomorphic to $\mathbb{P}^1$ and it admits a birational section.

(3) By [36, Section 1.5], every standard conic bundle is embedded. If G is an algebraic group acting regularly on a standard conic bundle, then the embedding is equivariant.

We will often consider *projective bundles* of relative dimension 1, i.e. $\mathbb{P}^1$-*bundles*, which are projectivisations of a locally free sheaves. Let V be a projective variety. If $\mathcal{E} \to V$ is a rank r vector bundle, we denote by $\mathbb{P}(\mathcal{E})$ or $\mathbb{P}_V(\mathcal{E})$ the Grothendieck projectivisation in $\mathcal{E}$

$$\mathbb{P}_V(\mathcal{E}) = \mathrm{Proj}(\mathrm{Sym}(\mathcal{E}))$$

together with the natural morphism $\pi\colon \mathbb{P}(\mathcal{E}) \to V$.

Remark 1.8.

(1) In particular, a surjection $\mathcal{E}^\vee \to \mathcal{Q}^\vee$ determines an embedding $\mathbb{P}(\mathcal{Q}) \to \mathbb{P}(\mathcal{E})$ such that $\mathcal{O}_{\mathbb{P}(\mathcal{Q})}(1) \sim \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|_{\mathbb{P}(\mathcal{Q})}$.
(2) By the Noether-Enriques theorem, a smooth $\mathbb{P}^1$-fibration over a curve is a $\mathbb{P}^1$-bundle.
(3) If Y and Z are smooth, then a $\mathbb{P}^1$-bundle $g\colon Y \to Z$ is a standard conic fibration.

We conclude the section with a result that will be useful in the proof of the main theorem.

Proposition 1.9. *Let $g\colon Y \to Z$ be a birationally-trivial $\mathbb{P}^1$-fibration. Then there is a smooth variety $\widetilde{Z}$, a $\mathbb{P}^1$-bundle $\tilde{g}\colon \widetilde{Y} \to \widetilde{Z}$ and a diagram*

$$\begin{tikzcd} Y \arrow[r, dashed] \arrow[d, "g"'] & \widetilde{Y} \arrow[d, "\tilde{g}"] \\ Z \arrow[r, dashed] & \widetilde{Z} \end{tikzcd}$$

such that all the maps are $\mathrm{Aut}^\circ(Y)$*-equivariant and the horizontal arrows are birational.*

Proof. Let $U \subseteq Z$ be the maximal open set such that g admits a section over U. Then U is left invariant by $\mathrm{Aut}^\circ(Y)$. Let $Y_U = g^{-1}(U)$. The section over U gives a relatively very ample line bundle and thus a morphism $U \to \mathrm{Grass}(\mathbb{P}^1, \mathbb{P}^N)$ for some N. Let $\mathcal{U} \subseteq \mathrm{Grass}(\mathbb{P}^1, \mathbb{P}^N) \times \mathbb{P}^N$ be the universal family. Let X be the Zariski closure of the image of U in $\mathrm{Grass}(\mathbb{P}^1, \mathbb{P}^N)$. We have a morphism $Y_U \to \mathcal{U}$ and an isomorphism $Y_U \cong U \times_{\mathrm{Grass}(\mathbb{P}^1,\mathbb{P}^N)} \mathcal{U}$. Let W be the Zariski closure of the image of Y_U in $\mathcal{U}$. The group $\mathrm{Aut}^\circ(Y)$ acts on X and W, making all the morphisms equivariant. We take $\widetilde{Z}$ an $\mathrm{Aut}^\circ(Y)$-equivariant projective smooth resolution of the indeterminacy of $Z \dashrightarrow X$ and $\widetilde{Y} = \widetilde{Z} \times_X W$. Then $\tilde{g}\colon \widetilde{Y} \to \widetilde{Z}$ is the sought $\mathbb{P}^1$-bundle. $\square$

1.4. **Rationally connected and non rational threefolds.** We will also need the following statements on rationally connected irrational threefolds.

Proposition 1.10. *Let X be a rationally connected non-rational threefold. Then* $\mathrm{Aut}^\circ(X)$ *is trivial.*

Proof. Assume by contradiction that $\mathrm{Aut}^\circ(X)$ is nontrivial. Since X is rationally connected, $\mathrm{Aut}^\circ(X)$ is linear and thus contains a 1-parameter subgroup Γ. By [4, Proposition 2.5.1], there is an open set X' of X which is of the form $\Gamma \times U$. Since X is rationally connected, any compactification of U is rationally connected. Since it is a surface, it is also rational. Thus X is birational to $\Gamma \times U$ which is in turn birational to $\mathbb{P}^1 \times \mathbb{P}^2$, a contradiction. □

Remark 1.11. More generally, one can prove that $\mathrm{Bir}(X)$ contains no connected algebraic subgroups, when X is a rationally connected, non-rational threefold [4, Corollary 2.5.9].

Remark 1.12. Let X' be a rationally connected threefold. Assume X' has a conic bundle structure $X' \to S$. Then there is a birational model X of X' with a fibration $c\colon X \to \mathbb{P}^2$ with general fibre $\mathbb{P}^1$ and sitting in a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "c"'] & X' \arrow[d] \\ \mathbb{P}^2 \arrow[r, dashed] & S. \end{tikzcd}$$

Indeed, since X' is rationally connected, the surface S is rational. Let $\mathbb{P}^2 \dashrightarrow S$ be a birational morphism, and let $X \to X' \times \mathbb{P}^2$ be a resolution of the indeterminacies of the induced map $X' \dashrightarrow \mathbb{P}^2$. Then the induced morphism $X \to \mathbb{P}^2$ is the required morphism.

Moreover, if X' is not rational, then the generic fibre of c is not $\mathbb{P}^1_{\mathbb{C}(\mathbb{P}^2)}$.

2. Automorphisms of $\mathbb{P}^1$-bundles

In the first part of this section, we show that invariant sections of projective bundles induce equivariant elementary transformations of $\mathbb{P}^1$-bundles (Lemma 2.1 and Lemma 2.2) and we recall the description of the automorphism group of projective bundles (Lemma 2.3). Recall that for a line bundle $\mathcal{L} \to V$ and a divisor D on V, we write $\mathcal{L}^\vee(-D) := \mathcal{L}^\vee \otimes \mathcal{O}_V(-D)$, and $\mathcal{L}^\vee|_D := \mathcal{L}^\vee \otimes \mathcal{O}_D$. Notice that $\mathcal{L}^\vee(-D) = (\mathcal{L}(D))^\vee$.

Lemma 2.1. *Let V be a smooth variety, $\mathcal{E} \to V$ a rank 2 vector bundle and $\pi\colon \mathbb{P}(\mathcal{E}) \to V$ be the induced $\mathbb{P}^1$-bundle. Let V_0 be a section of π defined by a surjective morphism $\mathcal{E}^\vee \to \mathcal{L}^\vee$. Let D_1 be a smooth effective irreducible divisor in V. Then the following hold:*

1. *The sheaf $\mathcal{E}_1^\vee$ equal to the kernel of the surjection $\mathcal{E}^\vee \to \mathcal{L}^\vee|_{D_1}$ is a rank two vector bundle on V.*
2. *More precisely, if $\mathcal{E}^\vee$ is an extension*
$$0 \to \mathcal{M}^\vee \to \mathcal{E}^\vee \to \mathcal{L}^\vee \to 0$$
then $\mathcal{E}_1^\vee$ is an extension
$$0 \to \mathcal{M}^\vee \to \mathcal{E}_1^\vee \to \mathcal{L}^\vee(-D_1) \to 0.$$

(3) *There is an induced birational map* $\psi\colon \mathbb{P}(\mathcal{E}) \dashrightarrow \mathbb{P}(\mathcal{E}_1)$ *which factors as* $\eta_2 \circ \eta_1^{-1}$, *where* $\eta_1\colon W \to \mathbb{P}(\mathcal{E})$ *is the blow up of the subvariety of* $\mathbb{P}(\mathcal{E})$ *defined by* $\mathcal{E}^\vee \to \mathcal{L}^\vee|_{D_1}$ *and* $\eta_2\colon W \to \mathbb{P}(\mathcal{E}_1)$ *is the contraction of the strict transform of* $\pi^{-1}(D_1)$ *in* W. *In particular,* ψ *is a link*

$$\begin{tikzcd} W \arrow[r, equal] \arrow[d, "\eta_1"'] & W \arrow[d, "\eta_2"] \\ \mathbb{P}(\mathcal{E}) \arrow[r, dashed, "\psi"] \arrow[d, "\pi"'] & \mathbb{P}(\mathcal{E}_1) \arrow[d, "\pi_1"] \\ V \arrow[r, equal] & V \end{tikzcd}$$

Proof. We have a diagram of exact sequences

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & \\ & & \mathcal{E}_1^\vee \arrow[d, "\beta"'] & \mathcal{L}^\vee(-D_1) \arrow[d] & \\ 0 \arrow[r] & \mathcal{M}^\vee \arrow[r, "j"] & \mathcal{E}^\vee \arrow[r, "\alpha"] \arrow[d, "\alpha_1"'] & \mathcal{L}^\vee \arrow[r] \arrow[d, "\alpha_2"] & 0 \\ & & \mathcal{L}^\vee|_{D_1} \arrow[r, equal] & \mathcal{L}^\vee|_{D_1} & \end{tikzcd}$$

Since $\mathcal{M}^\vee = \ker(\alpha)$ and $\mathcal{E}_1^\vee = \ker(\alpha_1 \circ \alpha)$, we have an injection $\mathcal{M}^\vee \hookrightarrow \mathcal{E}_1^\vee$. Moreover, $\alpha\beta(\mathcal{E}_1^\vee)$ is sent to zero by α_2, therefore $\alpha\beta(\mathcal{E}_1^\vee)$ is contained in $\mathcal{L}^\vee(-D_1)$. Via a diagram chase one can prove that the sequence

$$0 \to \mathcal{M}^\vee \to \mathcal{E}_1^\vee \to \mathcal{L}^\vee(-D_1) \to 0.$$

is exact. Since $\mathcal{M}^\vee$ and $\mathcal{L}^\vee$ have constant rank, the rank of $\mathcal{E}_1^\vee$ is constant as well and we have proved (1) and (2).

As for (3), let U be a trivialising set for $\mathcal{E}^\vee$ and $\mathcal{E}_1^\vee$. There is a 2×2 matrix M representing the inclusion $\mathcal{E}_1^\vee|_U \to \mathcal{E}^\vee|_U$. If (e_1, e_2) and (e_1, f_2) are local frames for $\mathcal{E}^\vee$ and $\mathcal{E}_1^\vee$ over U such that e_1 is a local frame for $\mathcal{M}^\vee$, then the matrix has the form

$$M = \begin{pmatrix} a_{1,1} & a_{1,2} \\ 0 & a_{2,2} \end{pmatrix}$$

where $a_{1,1} \in H^0(U, \mathcal{O}_V^*)$, $a_{2,2} \in H^0(U, \mathcal{O}_V(D_1))$, $a_{1,2} \in H^0(U, \mathcal{O}_V)$. The induced map between $\pi^{-1}(U) = U \times \mathbb{P}^1$ and $\pi_1^{-1}(U) = U \times \mathbb{P}^1$ is defined by the action of the transposed of M. Thus we have

$$\psi(z, [x_0 : x_1]) = (z, [a_{1,1}x_0 : a_{1,2}x_0 + a_{2,2}x_1]).$$

Without loss of generality we can assume that $a_{1,1} = 1$ and multiply by $b = a_{2,2}^{-1} \in H^0(U, \mathcal{O}_V(-D_1))$. The section b is a local equation for D_1

because $\mathcal{E}^\vee/\mathcal{E}_1^\vee = \mathcal{L}^\vee|_{D_1}$ is supported on D_1. We can assume that there are local analytic coordinates $z = (z_1, \ldots, z_k)$ in U such that $D_1 \cap U = \{z_1 = 0\}$. Therefore there are a regular function $f(z)$ on U and a constant c such that $\psi(z, [x_0 : x_1]) = (z, [cz_1x_0 : z_1f(z)x_0 + x_1])$. The indeterminacy locus is thus $D_1 \times \{[1 : 0]\}$. We consider the chart $x_0 \neq 0$, set $s = x_1/x_0$ and blow up the ideal (z_1, s). The blow up is

$$W = \{(z_1, \ldots, z_k, s), [u : v]|\ z_1v - su = 0\}.$$

In the chart $u \neq 0$ we have $s = z_1v/u$. Thus on W we extend ψ to a morphism by $\tilde{\psi}((z_1, \ldots, z_k, s), [u : v]) = (z, [cu : f(z)u + v])$. This proves (3). ☐

We give a criterion for the existence of a section fixed by the automorphisms. Recall that for a $\mathbb{P}^1$-bundle $P \to V$, we denote by $\mathrm{Aut}^\circ(P)_V$ the subgroup of $\mathrm{Aut}^\circ(P)$ acting trivially on V.

Lemma 2.2. *Let $\mathcal{E} \to V$ be a rank 2 vector bundle. Assume that $\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2$. If $H^0(V, \mathcal{L}_1 \otimes \mathcal{L}_2^\vee) = \{0\}$, then $\mathrm{Aut}^\circ(\mathbb{P}(\mathcal{E}))_V$ fixes pointwise the section of $\mathbb{P}(\mathcal{E}) \to V$ corresponding to $\mathcal{E}^\vee \to \mathcal{L}_2^\vee$.*

Proof. The projection $\mathcal{E}^\vee \to \mathcal{L}_2^\vee$ induces a section $V_0 \to \mathbb{P}(\mathcal{E})$ such that $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|_{V_0} \sim \mathcal{L}_2^\vee$ (see Remark 1.8). Viceversa, any section $V_0 \to \mathbb{P}(\mathcal{E})$ with $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|_{V_0} \sim \mathcal{L}_2^\vee$ corresponds to a surjective morphism $\mathcal{E}^\vee \to \mathcal{L}_2^\vee$. Now,

$$\begin{aligned}\mathrm{Hom}(\mathcal{E}^\vee, \mathcal{L}_2^\vee) =& H^0(V, \mathcal{L}_2^\vee \otimes \mathcal{E})\\ =& H^0(V, \mathcal{L}_1 \otimes \mathcal{L}_2^\vee) \oplus H^0(V, \mathcal{L}_2 \otimes \mathcal{L}_2^\vee) = \mathbb{C}\end{aligned}$$

where the last equality follows from the hypothesis on $\mathcal{L}_1, \mathcal{L}_2$. It follows that V_0 is unique with the property that the restriction of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ to it is $\mathcal{L}_2^\vee$ and thus is preserved by the automorphism group. ☐

We recall the description of the automorphism group of a projective bundle of relative dimension 1.

Lemma 2.3. *Let V be a smooth variety and $\mathcal{E} \to V$ a rank-2 vector bundle and $\pi\colon \mathbb{P}(\mathcal{E}) \to V$ be the induced $\mathbb{P}^1$-fibration. Suppose that $\mathrm{Aut}^\circ(\mathbb{P}(\mathcal{E}))_V$ fixes a section V_0 of π, given by a surjective morphism $\phi^\vee\colon \mathcal{E}^\vee \to \mathcal{L}^\vee$. Let $\Gamma := H^0(V, \det \mathcal{E} \otimes \ker(\phi^\vee)^{\otimes 2})$. Then the following hold:*

1. *If $\mathcal{E}$ is decomposable, then $\mathrm{Aut}^\circ(\mathbb{P}(\mathcal{E}))_V \simeq \Gamma \rtimes \mathbb{G}_m$.*
2. *If $\mathcal{E}$ is indecomposable, then $\mathrm{Aut}^\circ(\mathbb{P}(\mathcal{E}))_V \simeq \Gamma$.*
3. *If $\Gamma \neq 0$, then V_0 is the only $\mathrm{Aut}^\circ(\mathbb{P}(\mathcal{E}))_V$-invariant section.*
4. *If V is rationally connected and irrational, then $\mathrm{Aut}^\circ(\mathbb{P}(\mathcal{E})) \simeq \mathrm{Aut}^\circ(\mathbb{P}(\mathcal{E}))_V$.*
5. *If V is rationally connected and irrational and $\Gamma \neq 0$, then the orbits of the $\mathrm{Aut}^\circ(\mathbb{P}(\mathcal{E}))$-action are included in the fibres of π and are either of the form $\pi^{-1}(v) \cap V_0$ for $v \in V$ or the intersection of a fibre $\pi^{-1}(v)$ of π with the complement of V_0 in $\mathbb{P}(\mathcal{E})$.*

Proof. We essentially follow [29, pp.90–92]. Let $V = \cup V_i$ be a trivialising cover for $\mathbb{P}(\mathcal{E})$. For the morphism $\phi\colon \mathcal{L} \hookrightarrow \mathcal{E}$ induced by the surjection

$\phi^\vee\colon \mathcal{E}^\vee \to \mathcal{L}^\vee$, the image $\mathrm{Im}\phi$ coincides with the annihilator of $\ker(\phi^\vee)$, which is a hyperplane. By hypothesis, $V_0 = \mathbb{P}(\mathrm{Im}\phi)$ is fixed by $\mathrm{Aut}^\circ(\mathbb{P}(\mathcal{E}))_V$. If $(v, [x_0 : x_1])$ are local coordinates above V_i, we can suppose that V_0 is given by $x_0 = 0$. Therefore, an automorphism $\varphi \in \mathrm{Aut}^\circ(\mathbb{P}(\mathcal{E}))_V$ is given by

$$\varphi_i := \varphi|_{V_i} = \begin{pmatrix} \alpha_i & s_i \\ 0 & 1 \end{pmatrix} \in \mathrm{PGL}_2(\mathcal{O}_V(V_i))$$

Moreover, the transition functions $\{g_{ij}\}_{i,j}$ of $\mathbb{P}(\mathcal{E})$ are given by

$$g_{ij} := \begin{pmatrix} a_{ij} & c_{ij} \\ 0 & 1 \end{pmatrix} \in \mathrm{PGL}_2(O_V(V_i \cap V_j))$$

where the $\{a_{ij}\}_{ij} = \frac{b_{ij}}{d_{ij}}$ and b_{ij} are the transition functions of $\ker(\phi^\vee)$. Notice that the φ_i glue to $\varphi \in \mathrm{Aut}^\circ(\mathbb{P}(\mathcal{E}))_V$ if and only if $g_{ij}\varphi_j = \varphi_i g_{ij}$ for all i, j, which is equivalent to

$$\alpha_i a_{ij} = a_{ij}\alpha_j, \quad a_{ij}s_j + c_{ij} = \alpha_j c_{ij} + s_i \quad \text{for all } i, j.$$

The first condition is equivalent to $\alpha_i = \alpha_j =: \alpha \in \Gamma(V, \mathcal{O}_V^*) = \mathbb{G}_m$ for all i, j. The second then becomes $s_j a_{ij} - s_i = c_{ij}(\alpha - 1)$.

Suppose that $\alpha \neq 1$. Then conjugating g_{ij} as follows

$$\begin{pmatrix} 1 & \frac{s_i}{\alpha-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_{ij} & c_{ij} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{s_j}{\alpha-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{ij} & 0 \\ 0 & 1 \end{pmatrix}$$

yields that $\mathbb{P}(\mathcal{E})$ is decomposable, i.e. $\mathbb{P}(\mathcal{E}) \simeq \mathbb{P}(\ker(\phi^\vee) \oplus \mathcal{L}')$ for some subbundle $\mathcal{L}' \subset \mathcal{E}$. We can then assume that $c_{ij} = 0$ and obtain that $s_i = a_{ij}s_j$. Recall that $a_{ij} = \frac{b_{ij}}{d_{ij}}$, where b_{ij} and d_{ij} are respectively the transition functions of $\ker(\phi^\vee)$ and $\mathcal{L}'$. The $\{d_{ij}^{-1}\}$ define the line bundle $\det \mathcal{E} \otimes \ker(\phi^\vee)$, so the s_i glue into a section $s \in \Gamma(V, \det \mathcal{E} \otimes \ker(\phi^\vee)^{\otimes 2})$. This yields (1).

If $\alpha = 1$, then $s_i = a_{ij}s_j$ and again the s_i glue to a section $s \in \Gamma(V, \det \mathcal{E} \otimes \ker(\phi^\vee)^{\otimes 2})$ and we obtain (2).

(3) If Γ is nontrivial, then it is a nontrivial unipotent group it has therefore at most one fixed point on a general fibre of the $\mathbb{P}^1$-bundle π.

(4) If V is rationally connected and irrational, then $\mathrm{Aut}^\circ(V)$ is trivial by Proposition 1.10 and hence $\mathrm{Aut}^\circ(\mathbb{P}(\mathcal{E})) \simeq \mathrm{Aut}^\circ(\mathbb{P}(\mathcal{E}))_V$.

(5) This follows from (3) and (4). □

Remark 2.4. Suppose that V admits a conic fibration $c\colon V \to \mathbb{P}^2$ and that $\mathcal{E} = \mathcal{O}_V \oplus c^*\mathcal{O}_{\mathbb{P}^2}(n)$. Then $\ker(\phi^\vee)$ is trivial and in particular,
$\Gamma = H^0(V, \det \mathcal{E} \otimes \ker(\phi^\vee)^{\otimes 2}) = H^0(V, \det \mathcal{E}) = H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n)) \simeq \mathbb{C}[x, y, z]_n$ is the additive group of homogeneous polynomials of degree n.

We conclude the section with the following proposition.

Proposition 2.5. *Let Y, Z be smooth varieties and let $g\colon Y \to Z$ be a $\mathbb{P}^1$-bundle. If the neutral connected component of* $\mathrm{Aut}^\circ(Y)_Z$ *does not fix any rational section, then $Y = Z \times \mathbb{P}^1$.*

Proof. The $\mathbb{P}^1$-bundle $g\colon Y \to Z$ has generic fibre $\mathbb{P}^1_{\mathbb{C}(Z)}$ by definition. The action of $\mathrm{Aut}^\circ(Y)_Z$ on $\mathbb{P}^1_{\mathbb{C}(Z)}$ induces an injective homomorphism of groups $\mathrm{Aut}^\circ(Y)_Z \hookrightarrow \mathrm{PGL}_2(\mathbb{C}(Y))$. Let G be the neutral connected component of $\mathrm{Aut}^\circ(Y)_Z$. Assume that G is contained in a connected solvable subgroup B of $\mathrm{PGL}_2(\mathbb{C}(Z))$. Since the closure of a solvable subgroup of $\mathrm{PGL}_2(\mathbb{C}(Z))$ is again solvable, we can assume that B is closed. Up to conjugation in $\mathrm{PGL}_2(\mathbb{C}(Z))$, B is contained in the upper or lower triangular group. Then G has a rational fixed point on $\mathbb{P}^1_{\mathbb{C}(Z)}$ and hence fixes a rational section of $g\colon Y \to Z$, against our hypothesis. Assume now that G is not contained in any solvable group, i.e. G is not contained in any conjugate of the upper triangular group. Then there exist a 1-dimensional torus $T \subseteq G$. By the corollary to Proposition 2 of [21], T acts on the rank-2 vector bundle defining Y. The weight decomposition of it yields to T-invariant sub-line bundles and thus two T-invariant sections Z_1 and Z_2. Since T is a reductive group, by [17] its fixed point set is smooth and thus Z_1, Z_2 are disjoint. Since G is not a torus (any torus is contained in a solvable group), there is an element $h' \in G$ that does not preserve Z_1, Z_2 and such that the regular section $h'(Z_1)$ is disjoint from Z_1 and Z_2. A conic bundle with three pairwise disjoint regular sections is trivial (for instance, one can follow the proof of [19, Lemma 2.11]). $\square$

3. A family of projective bundles over a rationally connected non-rational threefold

We introduce the projective bundle $\mathcal{P}_n$, a main player in this article, and prove Proposition 3.4 below that characterises $\mathrm{Aut}^\circ(\mathcal{P}_n)$-equivariant birational maps starting from $\mathcal{P}_n$.

Definition 3.1. Let X be a rationally connected threefold, admitting a fibration $c\colon X \to \mathbb{P}^2$ with general fibre $\mathbb{P}^1$. We define $\mathcal{P}_n = \mathbb{P}_X(\mathcal{O}_X \oplus c^*\mathcal{O}_{\mathbb{P}^2}(n))$ and $\pi\colon \mathcal{P}_n \to X$. Let X_0 be the section defined by the surjective morphism $\mathcal{O}_X \oplus c^*\mathcal{O}_{\mathbb{P}^2}(-n) \to c^*\mathcal{O}_{\mathbb{P}^2}(-n)$.

Remark 3.2. Examples of threefolds X which are stably rational but not rational exist by [1] and come with a conic bundle structure.

Let us recall the properties of $\mathrm{Aut}^\circ(\mathcal{P}_n)$.

Lemma 3.3. *Let X be a rationally connected threefold, admitting a fibration $c\colon X \to \mathbb{P}^2$ with general fibre $\mathbb{P}^1$. Assume that X is not rational. Let $\pi\colon \mathcal{P}_n \to X$ be the $\mathbb{P}^1$-bundle defined in (3.1). The group $\mathrm{Aut}^\circ(\mathcal{P}_n)$ has the following properties:*

1. *the group $\mathrm{Aut}^\circ(\mathcal{P}_n)$ fixes a section V_0 of π;*
2. *there is an equality $\mathrm{Aut}^\circ(\mathcal{P}_n) = \mathrm{Aut}^\circ(\mathcal{P}_n)_X$;*
3. *there is an isomorphism $\mathrm{Aut}^\circ(\mathcal{P}_n) \cong \Gamma \rtimes \mathbb{G}_m$, where Γ is an additive group of dimension $(n+1)(n+2)/2$. In particular, if $n \geq 1$, then $\dim \mathrm{Aut}^\circ(\mathcal{P}_n) \geq 4$;*

(4) *the orbits of the action of* $\mathrm{Aut}^\circ(\mathcal{P}_n)$ *are included in fibres of* π *and are either of the form* $\pi^{-1}(x) \cap X_0$ *for* $x \in X$ *or the intersection of a fibre* $\pi^{-1}(x)$ *of* π *with the complement of* X_0 *in* $\mathcal{P}_n$*.*

Proof. The second statement (1) follows from Lemma 2.2. Statement (2) follows from the short exact sequence induced by the Blanchard's lemma, or, equivalently, by Lemma 2.3(4) and (1). Statement (3) follows from Remark 2.4. We get (4) from Lemma 2.3(5). □

The main goal of this section is to prove the following statement.

Proposition 3.4. *Let* $n \geq 2$ *and let* $\Phi\colon \mathcal{P}_n \dashrightarrow W$ *be a birational* $\mathrm{Aut}^\circ(\mathcal{P}_n)$*-equivariant map. Then the following hold:*

(1) $\mathrm{Aut}^\circ(\mathcal{P}_n)$ *is a normal subgroup of* $\mathrm{Aut}^\circ(W)$*;*
(2) *there are smooth varieties* Y, Z *and a fibration* $Y \to Z$ *with generic fibre* $\mathbb{P}^1_{\mathbb{C}(Z)}$ *and a birational* $\mathrm{Aut}^\circ(W)$*-equivariant map* $\eta\colon W \dashrightarrow Y$ *and a birational map* $\varphi\colon X \dashrightarrow Z$ *fitting into the following commutative diagram*

$$\begin{tikzcd} \mathcal{P}_n \arrow[r, dashed, "\Phi"] \arrow[d] & W \arrow[r, dashed, "\eta"] & Y \arrow[d] \\ X \arrow[rr, dashed, "\varphi"] & & Z \end{tikzcd}$$

To prove Proposition 3.4, let us fix the following notation and construction. For properties and notations regarding $\mathrm{Chow}(X)$ we refer to §1.2.

Construction 3.5. Notation as in Proposition 3.4. Since Φ is $\mathrm{Aut}^\circ(\mathcal{P}_n)$-equivariant, the group $\mathrm{Aut}^\circ(\mathcal{P}_n)$ acts on W. The general orbit has dimension 1, therefore by Lemma 1.4 the orbits of the action of $\mathrm{Aut}^\circ(\mathcal{P}_n)$ on W have dimension 0 or 1. Let $\mathcal{K}_0 \subseteq \mathrm{Chow}(W)$ be the subvariety parametrising the closures of general orbits of $\mathrm{Aut}^\circ(\mathcal{P}_n)$ and $u_0\colon \mathcal{U}_0 \to \mathcal{K}_0$ the restriction of the universal family. Let $\mathcal{K} \supseteq \mathcal{K}_0$ be the smallest $\mathrm{Aut}^\circ(W)$-invariant closed set in $\mathrm{Chow}(W)$ containing $\mathcal{K}_0$ and $u\colon \mathcal{U} \to \mathcal{K}$ the restriction of the universal family.

$$\begin{tikzcd} \mathcal{U}_0 \arrow[r, hook] \arrow[d, "u_0"'] & \mathcal{U} \arrow[r, "e"] \arrow[d, "u"] & W \\ \mathcal{K}_0 \arrow[r, hook] & \mathcal{K} & \end{tikzcd}$$

There are evaluation morphisms $e\colon \mathcal{U} \to W$ and $e_0 = e|_{\mathcal{U}_0}\colon \mathcal{U}_0 \to W$. Those are $\mathrm{Aut}^\circ(W)$-equivariant and $\mathrm{Aut}^\circ(\mathcal{P}_n)$-equivariant respectively. By Lemma 3.3, the conic bundle $\pi\colon \mathcal{P}_n \to X$ has a unique $\mathrm{Aut}^\circ(\mathcal{P}_n)$-invariant section, which we denote by X_0. As recalled in §1.2, this implies that e_0 is birational.

In the following lemmas we follow the notation from Construction 3.5 and Proposition 3.4.

Lemma 3.6. *The group* $\mathrm{Aut}^\circ(\mathcal{P}_n)$ *is a normal subgroup of* $\mathrm{Aut}^\circ(W)$ *if and only if* $\mathcal{K}_0 = \mathcal{K}$.

Proof. The group $\mathrm{Aut}^\circ(\mathcal{P}_n)$ is a normal subgroup of $\mathrm{Aut}^\circ(W)$ if and only if $\mathrm{Aut}^\circ(W)$ permutes the $\mathrm{Aut}^\circ(\mathcal{P}_n)$-orbits in W. This is the case if and only if $\mathrm{Aut}^\circ(W)$ preserves $\mathcal{K}_0$. By minimality of $\mathcal{K}$, this is equivalent to $\mathcal{K}_0 = \mathcal{K}$. □

Lemma 3.7. *If* Φ *does not contract* X_0*, then* $\mathcal{K}_0 = \mathcal{K}$.

Proof. By Lemma 3.3, the only $\mathrm{Aut}^\circ(\mathcal{P}_n)$-invariant proper closed subvarieties of $\mathcal{P}_n$ are union of fibres of π or X_0. If X_0 is not contracted by Φ, then there exists an open nonempty subset $U \subset X$ such that $\Phi|_{\pi^{-1}(U)}$ is an isomorphism.

Suppose that $\mathcal{K}_0 \neq \mathcal{K}$ and let M be the pull-back by $\pi\colon \mathcal{P}_n \to X$ of an ample divisor on X. Let $[\Gamma] \in \mathcal{K}\backslash\mathcal{K}_0$ and $[\Gamma_0] \in \mathcal{K}_0$ be classes of curves Γ, Γ_0 on W such that $\Gamma_0 \subset \Phi(\pi^{-1}(U))$ and such that Γ is not in the exceptional locus of Φ^{-1}.

We denote by $(p,q)\colon \hat{W} \to \mathcal{P}_n \times W$ an $\mathrm{Aut}^\circ(\mathcal{P}_n)$-equivariant resolution of Φ. We denote by $\hat{X}_0$ the strict transform in $\widehat{W}$ of X_0. Let $\widehat{\Gamma}_0$ be the strict transform of Γ_0 in $\widehat{W}$ and let $\widehat{\Gamma}$ be an irreducible curve in $\hat{W}$ such that $q_*(\widehat{\Gamma}) = \Gamma$. Notice that $\widehat{\Gamma}$ is not contracted by $\pi \circ p$, because $[\Gamma] \notin \mathcal{K}_0$.

Claim 3.8.
$$p^*M \cdot \widehat{\Gamma}_0 = p^*M \cdot \widehat{\Gamma} + p^*M \cdot C$$
for some curve C *in* $\hat{W}$.

Assuming the claim, we finish the proof. The left-hand side is zero, because $[\Gamma_0] \in \mathcal{K}_0$, while $p^*M \cdot \widehat{\Gamma} > 0$ and $p^*M \cdot C \geq 0$. This is impossible, so it follows that $\mathcal{K}_0 = \mathcal{K}$.

We are left with the proof of Claim 3.8. Let $(a,b)\colon \widehat{\mathcal{U}} \to \mathcal{U} \times \widehat{W}$ be a resolution of the indeterminacies of $\mathcal{U} \dashrightarrow \widehat{W}$ and let $\hat{u}\colon \widehat{\mathcal{U}} \to \mathcal{K}$ be the induced fibration. Let $\mathcal{C}$ be an irreducible curve in $\mathcal{K}$ such that $[\Gamma], [\Gamma_0] \in \mathcal{C}$. Let S be the component of dimension 2 of $\hat{u}^{-1}(\mathcal{C})$ surjecting onto $\mathcal{C}$. Then $b_*\hat{u}^*[\Gamma_0] = \widehat{\Gamma}_0$ and there is an effective curve C such that $b_*\hat{u}^*[\Gamma] = \widehat{\Gamma} + C$. The claim follows as $b_*\hat{u}^*[\Gamma_0] \equiv b_*\hat{u}^*[\Gamma]$. □

Lemma 3.9. *Suppose that* $\mathrm{Aut}^\circ(\mathcal{P}_n)$ *is a not normal subgroup of* $\mathrm{Aut}^\circ(W)$ *and that every* $\mathrm{Aut}^\circ(W)$*-equivariant desingularisation of* W *extracts* X_0*. Let* $[\Gamma_0] \in \mathcal{K}_0$ *be a general point, let* $G \subset \mathrm{Aut}^\circ(W)$ *be a* 1*-parameter subgroup and* $g \in G$ *a general element. Let* $\widetilde{g\Gamma_0}$ *be the strict transform of* $g\Gamma_0$ *in* $\mathcal{P}_n$*. Then* $\widetilde{g\Gamma_0} \cap X_0$ *is a non-empty finite set.*

Proof. Since $\mathrm{Aut}^\circ(\mathcal{P}_n)$ is not a normal subgroup of $\mathrm{Aut}^\circ(W)$, Lemma 3.6 implies that $\mathcal{K}_0 \subsetneq \mathcal{K}$. Lemma 3.7 implies that X_0 is contracted by Φ. Let $\widehat{W} \to W$ be an $\mathrm{Aut}^\circ(W)$-equivariant desingularisation that extracts X_0. We denote by $\hat{X}_0$ the strict transform of X_0 in $\widehat{W}$. Then the induced birational map $\mathcal{P}_n \dashrightarrow \widehat{W}$ is an isomorphism at the generic point of X_0 and induces a birational map $X_0 \dashrightarrow \widehat{X}_0$. Let $[\Gamma_0] \in \mathcal{K}_0$ be the class of a general curve Γ_0 such that its strict transform $\widehat{\Gamma}_0$ in $\widehat{W}$ meets $\widehat{X}_0$ in a point lying in the open set where $\widehat{W} \dashrightarrow \mathcal{P}_n$ is an isomorphism. Then for general $g \in G$, the curve $g\widehat{\Gamma}_0$ meets $\widehat{X}_0$ in a point lying in the open set where $\widehat{W} \dashrightarrow \mathcal{P}_n$ is an isomorphism. Let $\widetilde{g\Gamma_0}$ be the strict transform of $g\Gamma_0$ in $\mathcal{P}_n$. Then $\widetilde{g\Gamma_0} \cap X_0$ is non-empty. □

Lemma 3.10. *Suppose that* $\mathrm{Aut}^\circ(\mathcal{P}_n)$ *is a not normal subgroup of* $\mathrm{Aut}^\circ(W)$*. Then there is an* $\mathrm{Aut}^\circ(W)$*-equivariant desingularisation of* W *which does not extract* X_0*.*

Proof. We prove the statement by contradiction. Suppose that all $\mathrm{Aut}^\circ(W)$-equivariant desingularisation of W extract X_0.
Since $\mathrm{Aut}^\circ(\mathcal{P}_n)$ is not normal in $\mathrm{Aut}^\circ(W)$, by Lemma 3.6 we have $\mathcal{K}_0 \subsetneq \mathcal{K}$. Then there is a 1-parameter subgroup $G \subset \mathrm{Aut}^\circ(W)$ with $G \not\subseteq \mathrm{Aut}^\circ(\mathcal{P}_n)$ and such that for a general $g \in G$, for a general $[\Gamma_0] \in \mathcal{K}_0$, we have $[g\Gamma_0] \notin \mathcal{K}_0$. Let C be the strict transform of $g\Gamma_0$ in $\mathcal{P}_n$. Lemma 3.9 implies that $C \cap X_0$ is non-empty. Let $H \subset \mathrm{Aut}^\circ(\mathcal{P}_n)$ be an additive 1-parameter subgroup, set $C_t := tC$, $t \in H$, and consider the pencil $\{C_t\}_{t\in H}$. The pencil defines a morphism $\mu\colon \mathbb{A}^1 \times \mathbb{P}^1 \to \mathcal{P}_n$. Let F be the normalisation of the Zariski-closure of $\mu(\mathbb{A}^1 \times \mathbb{P}^1)$ and $n\colon F \to \mathcal{P}_n$ the induced morphism. The image $\pi(C)$ is a curve because $[g\Gamma_0] \notin \mathcal{K}_0$. Let $n\colon D \to \pi(C)$ be the normalisation of $\pi(C)$. By abuse of notation, the strict transform of C_t (resp. C) on F is denoted by C_t (resp. C) as well, as no confusion will arise.

Notice that F is smooth and that it is a $\mathbb{P}^1$-bundle above D. Let $\theta\colon S \to F$ be a minimal resolution of the base-locus of the pencil $\{C_t\}_{t\in H}$. Then there is a conic fibration $u\colon S \to \mathbb{P}^1$, whose fibres are the strict transforms $\overline{C_t}$ of C_t, such that the following diagram commutes.

$$\begin{tikzcd}
\mathbb{A}^1 \times \mathbb{P}^1 \arrow[r, hook] \arrow[d, "u"] \arrow[rrr, bend left, "\mu"] & S \arrow[r, "\theta"] \arrow[d, "u"] & F \arrow[r, "n"] \arrow[d, "\pi"] & \mathcal{P}_n \arrow[d, "\pi"] \\
\mathbb{A}^1 \arrow[r, hook] & \mathbb{P}^1 & D \arrow[r, "n"] & X
\end{tikzcd}$$

Notice that since Γ_0 is rational, so is $C = g\Gamma_0$ and thus $D \simeq \mathbb{P}^1$. By abuse of notation, the strict transform of X_0 on F will be denoted by X_0 as well, as no confusion will arise. Notice that H fixes X_0 pointwise as $H \subset \mathrm{Aut}^\circ(\mathcal{P}_n)$, so all the points in $C \cap X_0$ are base-points of the pencil $\{C_t\}_{t\in H}$.

The H-action on F lifts to S and permutes non-trivially the fibres of u, since any fibre of π intersects C in only finitely many points. By the Blanchard's lemma ([10], [12, Proposition 4.2.1]), H acts on $\mathbb{P}^1$ (base of the fibration u) and, since it is additive, it fixes exactly one point $[\infty] \in \mathbb{P}^1$. Moreover, a curve E contained in the exceptional locus of θ is either contained in a fibre of u or it is a section of u, and the latter is the case if and only if E is the exceptional divisor of a point that is blown up last, or, equivalently, a (-1)-curve.

Claim. C_t is a section of $\pi\colon F \to D$ if $t \in H$ is general.

Proof. Let f be a fibre of π, which is disjoint from the exception locus of θ and such that H acts non-trivially on f. The fibre being disjoint from the exceptional locus implies that the pullback $\theta^* f$ and the strict transform $\bar{f}$ of f in S coincide. The action of H being non-trivial implies that $\bar{f}$ is a section of u. Indeed, the restriction $u\colon \bar{f} \to \mathbb{P}^1$ is surjective and H-equivariant, therefore the ramification and branch locus are preserved by H. But those are both supported on at most one point and by Hurwitz formula this is possible only if u has degree 1. Let t be such that $\overline{C}_t$ is irreducible. Thus we get

$$f \cdot C_t = \theta^* f \cdot \overline{C}_t = \bar{f} \cdot \overline{C}_t = 1$$

where the first equality is the projection formula, the second is because the pullback $\theta^* f$ coincides with the strict transform $\bar{f}$, and the third because $\bar{f}$ is a section of u. This finishes the proof that C_t is a section of $\pi\colon F \to D$ if $t \in H$ is general.

In particular, $C = C_0$ is a section of π. It follows that every fibre of π meets the base-locus in F in at most one point. Since $C \cap X_0$ is non empty, we pick a point $y_0 \in C \cap X_0$, we set $0 = \pi(y_0) \in D$ and $f_0 = \pi^{-1}(0)$. Let $f_0 = f_{t_0}, f_{t_1}, \ldots, f_{t_l}$ be the fibres of π meeting the base-locus of the pencil $\{C_t\}_{t \in H}$, and let $\overline{f_{t_i}}$ (resp. $\overline{f_0}$) be the strict transform of f_{t_i} (resp. f_0) in S. Since C_t is a section of π, the f_{t_i} are not contained in the union $\cup_{t \in H} C_t = \theta(\mathbb{A}^1 \times \mathbb{P}^1)$, and more precisely the intersection of f_{t_i} with $\cup_{t \in H} C_t$ coincides with one point. Therefore each f_{t_i} is contained in $\theta(u^{-1}[\infty])$.

For each $i \geq 0$, denote by E_{ij} the irreducible components of $\theta^{-1}(y_i)$ that are not sections of u and by E_i^{sec} the unique irreducible component that is a section of u. Since $\mathrm{Exc}(\theta)$ is preserved by H, we have $\theta(E_{ij}) = [\infty]$ for every i, j. Then

$$\theta^* C_0 = \overline{C}_0 + \sum a_{ij} E_{ij} + \sum_{i=0}^{l} a_i E_i^{sec}$$

for some integers $a_{ij}, a_i \geq 0$. Denote by $\overline{C}_\infty$ the fibre of u above $[\infty]$. Since all E_{ij} are contained in $\overline{C}_\infty$ and the E_i^{sec} are sections of u, we have

$$\overline{C}_\infty \cdot \theta^* C_0 = C_t \cdot \theta^* C_0 = \sum_{i=0}^{l} a_i, \quad \text{for any } t \in H.$$

We also have

$$\overline{C}_\infty = \theta^*(\alpha X_0 + \sum_{i=0}^{l} \beta_i f_{t_i}) - \sum_{i=0}^{l} b_i E_i^{sec} + C'$$

for some integers $\alpha, \beta_i, b_i \geq 0$ and some effective divisor C' having no common component with $\sum E_i^{sec}$. In fact, $b_i \geq \beta_i \geq 1$ for all $i \geq 1$, because $E_i^{sec} \subseteq \operatorname{supp}(\theta^* f_{t_i})$ and $\bar{f}_{t_i}$ is contained in $\overline{C_\infty}$ for $i \geq 1$. Furthermore, $b_0 \geq \beta_0 + \alpha \geq 2$, because $E_0^{sec} \subset \operatorname{supp}(\theta^* f_0) \cap \operatorname{supp}(\theta^* X_0)$ and $\bar{f}_0$ is contained in $\overline{C_\infty}$. We compute

$$\begin{aligned}
\sum_{i=0}^{l} a_i = \bar{C}_\infty \cdot \theta^* C_0 = \bar{C}_\infty \cdot \left(\sum_{i=0}^{l} a_i E_i^{sec} \right) \\
&= \left(\theta^*(\alpha X_0 + \sum_{i=0}^{l} \beta_i f_{t_i}) - \sum_{i=0}^{l} b_i E_i^{sec} + C' \right) \left(\sum_{i=0}^{l} a_i E_i^{sec} \right) \\
&\overset{(E_i^{sec})^2=-1}{=} \sum_{i=1}^{l} b_i a_i + C' \cdot \left(\sum_{i=1}^{l} a_i E_i^{sec} \right) \geq \sum_{i=0}^{l} b_i a_i \geq a_0 + \sum_{i=0}^{l} a_i.
\end{aligned}$$

where the last inequality holds because $b_0 \geq 2$ and $b_i \geq 1$ for $i \geq 1$. It follows that $a_0 = 0$, which contradicts f_0 containing a base-point of $\{C_t\}_{t \in H}$. This proves that there is an $\operatorname{Aut}^\circ(W)$-equivariant desingularisation of W which does not extract X_0. $\square$

Lemma 3.11. *Suppose that* $\operatorname{Aut}^\circ(\mathcal{P}_n)$ *is a not normal subgroup of* $\operatorname{Aut}^\circ(W)$. *Then* Φ *does not contract* X_0.

Proof. Suppose that Φ contracts X_0. By Lemma 3.10, there exists $\mu \colon \widetilde{W} \to W$ an $\operatorname{Aut}^\circ(W)$-equivariant desingularisation of W which does not extract X_0.

We denote by $(p, q) \colon \widehat{W} \to \mathcal{P}_n \times W$ an $\operatorname{Aut}^\circ(\mathcal{P}_n)$-equivariant resolution of the indeterminacy of Φ. By Lemma 3.3(3), the conic bundle $\pi \colon \mathcal{P}_n \to X$ has a unique $\operatorname{Aut}^\circ(\mathcal{P}_n)$-invariant section $X_0 \subset \mathcal{P}_n$, and by $\widehat{X}_0$ we denote its strict transform in $\widehat{W}$.

Let $(\bar{p}, \bar{q}) \colon \overline{W} \to \widehat{W} \times \widetilde{W}$ be a resolution of the indeterminacies of $\mu^{-1} q \colon \widehat{W} \dashrightarrow \widetilde{W}$, such that $\bar{q}$ is a composition of blow-ups of smooth centres.

$$\begin{tikzcd}
 & & \overline{W} \arrow[dl, "\bar{p}"'] \arrow[dr, "\bar{q}"] & \\
 & \widehat{W} \arrow[dl, "p"'] \arrow[dr, "q"] & & \widetilde{W} \arrow[dl, "\mu"'] \\
\mathcal{P}_n \arrow[rr, dashed, "\Phi"] & & W &
\end{tikzcd}$$

Then the strict transform $\overline{X}_0$ in $\overline{W}$ is among the exceptional divisors of those blow-ups. It follows that X_0 is birational to $\mathbb{P}^k \times Z$ with $k \in \{1, 2, 3\}$ and $\dim Z = 3 - k$. Since $\dim Z \leq 2$ and Z is rationally connected, Z is rational and so is X_0. This contradicts the hypothesis that X is not rational. $\square$

Proof of Proposition 3.4. We first prove that $\mathcal{K}_0 = \mathcal{K}$. If not, by Lemma 3.6, the group $\mathrm{Aut}^\circ(\mathcal{P}_n)$ is not normal in $\mathrm{Aut}^\circ(W)$. Then, by Lemma 3.11, the map Φ does not contract X_0. This is a contradiction with Lemma 3.7.

Therefore, $\mathcal{K}_0 = \mathcal{K}$, and hence $\mathrm{Aut}^\circ(\mathcal{P}_n)$ is normal in $\mathrm{Aut}^\circ(W)$ by Lemma 3.6. Let $\overline{\mathcal{U}}$ and $\overline{\mathcal{K}}$ be $\mathrm{Aut}^\circ(W)$-equivariant compactifications of $\mathcal{U}$ and $\mathcal{K}$ such that there is a morphism $u\colon \overline{\mathcal{U}} \to \overline{\mathcal{K}}$ extending u. The map $e\colon \overline{\mathcal{U}} \dashrightarrow W$ is birational and $\mathrm{Aut}^\circ(W)$-equivariant, see Construction 3.5. We set $Y = \overline{\mathcal{U}}$ and $Z = \overline{\mathcal{K}}$.

Moreover, Z is birational to X because $\mathcal{K}_0 = \mathcal{K}$ parametrises the 1-dimensional orbits of $\mathrm{Aut}^\circ(\mathcal{P}_n)$. □

4. $\mathrm{Aut}^\circ(\mathcal{P}_n)$ is not contained in a maximal subgroup of $\mathrm{Bir}(\mathcal{P}_n)$

The aim of this section is to show in Theorem 4.2 that $\mathrm{Aut}^\circ(\mathcal{P}_n)$ is not contained in a maximal connected algebraic subgroup of $\mathrm{Bir}(\mathcal{P}_n)$ if $n \geq 2$.

Proposition 4.1. *Let V be a smooth variety of dimension at least 3. Let $\mathcal{E} \to V$ be a rank 2 vector bundle. Assume that $\mathrm{Aut}^\circ(\mathbb{P}(\mathcal{E}))_V$ contains a non-trivial additive group (that is, a copy of $(\mathbb{G}_a)^d$ for some $d \geq 1$) and fixes a section of $\pi\colon \mathbb{P}(\mathcal{E}) \to V$. Then there is a rank 2 vector bundle $\mathcal{E}_1 \to V$ and an $\mathrm{Aut}^\circ(\mathbb{P}(\mathcal{E}))_V$-equivariant birational map $\mathbb{P}(\mathcal{E}) \dashrightarrow \mathbb{P}(\mathcal{E}_1)$ over V such that $\mathrm{Aut}^\circ(\mathbb{P}(\mathcal{E}))_V \subsetneq \mathrm{Aut}^\circ(\mathbb{P}(\mathcal{E}_1))_V$.*

Proof. From the two hypotheses on $\mathrm{Aut}^\circ(\mathbb{P}(\mathcal{E}))_V$, it follows that there is a unique section V_0 fixed by $\mathrm{Aut}^\circ(\mathbb{P}(\mathcal{E}))_V$ (uniqueness follows from Lemma 2.3). This section corresponds to the data of a line bundle $\mathcal{L}$ on V and a surjective morphism $\mathcal{E}^\vee \to \mathcal{L}^\vee$ (see Remark 1.8). Let $\mathcal{M}^\vee$ be the kernel of $\mathcal{E}^\vee \to \mathcal{L}^\vee$. Then $\mathcal{M}^\vee$ is a rank 1 torsion-free sheaf on V and it is locally free by [23, Proposition 1.9].

We have $\mathrm{Aut}^\circ(\mathbb{P}(\mathcal{E}))_V \simeq \Gamma \rtimes G$, for some additive group Γ which is non-trivial by our hypothesis on $\mathrm{Aut}^\circ(\mathbb{P}(\mathcal{E}))_V$ and $G = \mathbb{G}_m$ or $G = \{1\}$, see Lemma 2.3. Let D be a very ample divisor on V such that

- there is a smooth element $D_1 \in |D|$;
- the line bundle $\mathcal{M}^\vee \otimes \mathcal{L}(D)$ is very ample, so that $\mathrm{Ext}^1_V(\mathcal{L}^\vee(-D), \mathcal{M}^\vee) = H^1(\mathcal{M}^\vee \otimes \mathcal{L}(D)) = 0$; and
- $\dim \Gamma < \dim H^0(V, \mathcal{M}^\vee \otimes \mathcal{L}(D))$.

Consider now the kernel $\mathcal{E}_1^\vee$ of the surjection $\mathcal{E}^\vee \to \mathcal{L}^\vee|_{D_1}$. By Lemma 2.1(2), the sheaf $\mathcal{E}_1^\vee$ is an extension of $\mathcal{L}^\vee(-D)$ and $\mathcal{M}^\vee$ and since $\mathrm{Ext}^1_V(\mathcal{L}^\vee(-D), \mathcal{M}^\vee) = 0$ we have $\mathcal{E}_1^\vee \cong \mathcal{M}^\vee \oplus \mathcal{L}^\vee(-D)$. Therefore $\mathcal{E}_1$ is decomposable and $\mathcal{E}_1 \cong \mathcal{M} \oplus \mathcal{L}(D)$. By Lemma 2.2, the group $\mathrm{Aut}^\circ(\mathbb{P}(\mathcal{E}_1))_V$ fixes the section corresponding to $\phi^\vee\colon \mathcal{E}_1^\vee \to \mathcal{L}^\vee(-D)$. We have $\ker(\phi^\vee) = \mathcal{M}^\vee$ and $\det \mathcal{E}_1 = \mathcal{M} \otimes \mathcal{L}(D)$. Then by Lemma 2.3(1) $\mathrm{Aut}^\circ(\mathbb{P}(\mathcal{E}_1))_V \simeq H^0(V, \mathcal{M}^\vee \otimes \mathcal{L}(D)) \rtimes \mathbb{G}_m$.

Moreover, by the invariance of V_0, the link or birational map $\psi\colon \mathbb{P}(\mathcal{E}) \dashrightarrow \mathbb{P}(\mathcal{E}_1)$ obtained by Lemma 2.1(3) is $\mathrm{Aut}^\circ(\mathbb{P}(\mathcal{E}))_V$-equivariant. Since $\Gamma \subsetneq H^0(V, \mathcal{M}^\vee \otimes \mathcal{L}(D))$ by assumption on D, we have $\mathrm{Aut}^\circ(\mathbb{P}(\mathcal{E}))_V \subsetneq \mathrm{Aut}^\circ(\mathbb{P}(\mathcal{E}_1))_V$. This proves the claim. □

We are now ready to prove the main theorem of this section.

Theorem 4.2. *Let $n \geq 2$ be a positive integer. Let X be a non-rational and rationally connected variety carrying a non-trivial conic bundle structure and admitting a $\mathbb{P}^1$-fibration $c\colon X \to \mathbb{P}^2$. Set $\mathcal{P}_n = \mathbb{P}_X(\mathcal{O}_X \oplus c^*\mathcal{O}_{\mathbb{P}^2}(n))$. The group $\mathrm{Aut}^\circ(\mathcal{P}_n)$ is not contained in a maximal group of $\mathrm{Bir}(\mathcal{P}_n)$. More precisely, for every variety W, for every $\mathrm{Aut}^\circ(\mathcal{P}_n)$-equivariant birational map $\mathcal{P}_n \dashrightarrow W$, there is a variety Y and an $\mathrm{Aut}^\circ(\mathcal{P}_n)$-equivariant birational map $W \dashrightarrow Y$ with $\mathrm{Aut}^\circ(W) \subsetneq \mathrm{Aut}^\circ(Y)$.*

Proof. Assume that $\mathrm{Aut}^\circ(\mathcal{P}_n)$ is contained in a connected algebraic subgroup H of $\mathrm{Bir}(\mathcal{P}_n)$ acting rationally on $\mathcal{P}_n$. We will prove that there is a connected algebraic subgroup G of $\mathrm{Bir}(\mathcal{P}_n)$ acting rationally on $\mathcal{P}_n$ such that $H \subsetneq G$.

By the Weil regularisation theorem [44], there is a variety W birational to $\mathcal{P}_n$ such that $H \subseteq \mathrm{Aut}^\circ(W)$. By Proposition 3.4, there are smooth varieties Y, Z and a fibration $g\colon Y \to Z$ with generic fibre $\mathbb{P}^1_{\mathbb{C}(Z)}$, a birational $\mathrm{Aut}^\circ(W)$-equivariant map $W \dashrightarrow Y$ and a birational map $X \dashrightarrow Z$ fitting into a commutative diagram

$$\begin{tikzcd} \mathcal{P}_n \arrow[r, dashed] \arrow[d] & W \arrow[r, dashed] & Y \arrow[d, "g"] \\ X \arrow[rr, dashed] & & Z \end{tikzcd}$$

By Proposition 1.9, we can assume that Z is smooth and g is a $\mathbb{P}^1$-bundle.

We claim the following.

Claim 4.3. *The $\mathbb{P}^1$-bundle g has an $\mathrm{Aut}^\circ(Y)$-equivariant rational section.*

Assuming the claim, we finish the proof. Since Z is birational to X, by Proposition 1.10 we have $\mathrm{Aut}^\circ(Y) = \mathrm{Aut}^\circ(Y)_Z$. Since $\mathrm{Aut}^\circ(\mathcal{P}_n) \subseteq \mathrm{Aut}^\circ(Y)$, there is a non-trivial additive subgroup of $\mathrm{Aut}^\circ(Y) = \mathrm{Aut}^\circ(Y)_Z$. By the Claim 4.3, the group $\mathrm{Aut}^\circ(Y)$ fixes a rational section Z_0 of g. Since $\mathrm{Aut}^\circ(Y)$ acts trivially on Z, the base change $Y_0 = Z_0 \times_Z Y$ yields an $\mathrm{Aut}^\circ(Y)$-equivariant birational morphism $Y_0 \to Y$. Thus $\mathrm{Aut}^\circ(Y)_Z = \mathrm{Aut}^\circ(Y) = \mathrm{Aut}^\circ(Y_0) = \mathrm{Aut}^\circ(Y_0)_{Z_0}$. Moreover, Y_0 admits a section fixed pointwise by $\mathrm{Aut}^\circ(Y_0)$. Therefore, by Proposition 4.1, there is a $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{E}_1) \to Z_0$ such that $\mathrm{Aut}^\circ(Y_0) = \mathrm{Aut}^\circ(Y_0)_{Z_0} \subsetneq \mathrm{Aut}^\circ(\mathbb{P}(\mathcal{E}_1))_{Z_0} = \mathrm{Aut}^\circ(\mathbb{P}(\mathcal{E}_1))$. We may set $G = \mathrm{Aut}^\circ(\mathbb{P}(\mathcal{E}_1))$.

We are left with the proof of the Claim 4.3: by Proposition 2.5, if there is no $\mathrm{Aut}^\circ(Y)$-equivariant rational section then $Y = Z \times \mathbb{P}^1$. Then we would have $\mathrm{Aut}^\circ(Y) \cong \mathrm{PGL}_2(\mathbb{C})$. But this contradicts the fact that $\mathrm{Aut}^\circ(Y)$ contains $\mathrm{Aut}^\circ(\mathcal{P}_n)$, which has dimension at least 4 by Lemma 3.3(3). □

5. Proof of Main Theorem

We start with some preliminary lemmas on birational map from products with special properties.

Lemma 5.1. *Let X_1 and X_2 be normal projective varieties such that $h^1(X_i, \mathcal{O}_{X_i}) = 0$ for $i = 1, 2$ and let $p_i \colon X_1 \times X_2 \to X_i$ be the projection onto X_i. Then*

$$\mathrm{Nef}(X_1 \times X_2) = p_1^*\mathrm{Nef}(X_1) \oplus p_2^*\mathrm{Nef}(X_2).$$

Proof. By [22, Exercise III 12.6(b)] and since $h^1(X_i, \mathcal{O}_{X_i}) = 0$ for $i = 1, 2$, we have $\mathrm{Pic}(X_1 \times X_2) = p_1^*\mathrm{Pic}(X_1) \oplus p_2^*\mathrm{Pic}(X_2)$. Let $L \subset \mathrm{Nef}(X_1 \times X_2)$ and write $L = p_1^*D_1 + p_2^*D_2$ for some $D_i \in \mathrm{Pic}(X_i)$, $i = 1, 2$. Let $C_1 \subset X_1$ be a curve and consider the curve $\hat{C} = C_1 \times \{x_2\} \subset X_1 \times X_2$. Then $0 \le L \cdot \hat{C} = D_1 \cdot C$ and hence D_1 is nef. The same argument shows that D_2 is nef. □

Proposition 5.2. *Let P be a smooth projective variety and Y a homogeneous variety with $\rho(Y) = 1$ and let $\varphi \colon P \times Y \dashrightarrow Q$ be an $\mathrm{Aut}^\circ(P \times Y)$-equivariant birational map. Then $Q \simeq P' \times Y$, where P' is projective and $\varphi = (\varphi_1, \varphi_2)$ with $\varphi_1 \colon P \dashrightarrow P'$ birational and $\varphi_2 \colon Y \to Y$ an isomorphism.*

Proof. Let $(p, q) \colon W \to P \times Y \times Q$ be a functorial resolution of the indeterminacies of φ such that p is a composition of blow-ups of smooth centres. Since p is $\mathrm{Aut}^\circ(P \times Y)$-equivariant and Y is homogeneous, the morphism p blows up centres that are products of the form $C_i \times Y$. It follows that $W \simeq \hat{P} \times Y$ and that $p = (p_{\hat{P}}, p_Y)$, with $p_{\hat{P}}$ birational and p_Y an isomorphism.

Since q is a birational morphism, it is induced by a Cartier divisor D on $\hat{P} \times Y$ that is big and nef. By Lemma 5.1, we can write $D = p_{\hat{P}}^*D_1 + p_Y^*D_2$ and $D_1 \in \mathrm{Nef}(\hat{P})$ and $D_2 \in \mathrm{Nef}(Y)$. Then q is of the form $q = (f_1, f_2) \colon \hat{P} \times Y \to P' \times Y'$, where $f_1 \colon \hat{P} \to P'$ is defined by D_1 and $f_2 \colon Y \to Y_1$ is defined by D_2, and both f_1, f_2 are birational since q is birational (so D_1, D_2 are nef and big). Since $\rho(Y) = 1$ and D_2 is nef and big, it follows that D_2 is ample and hence that f_2 is an isomorphism. □

We are now ready for the proof of our main result.

Theorem 5.3. *Let $n \ge 2$ and $m \ge 0$ be a positive integers. Let X be a non-rational and rationally connected variety carrying a non-trivial conic bundle structure and admitting a fibration $c \colon X \to \mathbb{P}^2$. Set $\mathcal{P}_n = \mathbb{P}_X(\mathcal{O}_X \oplus c^*\mathcal{O}_{\mathbb{P}^2}(n))$. Then the group $\mathrm{Aut}^\circ(\mathcal{P}_n \times \mathbb{P}^m)$ is not contained in a maximal connected algebraic group of $\mathrm{Bir}(\mathcal{P}_n \times \mathbb{P}^m)$.*

Proof. If $m = 0$, the statement is Theorem 4.2, so let us assume that $m \ge 1$. Notice that $\mathrm{Aut}^\circ(\mathcal{P}_n \times \mathbb{P}^m) \simeq \mathrm{Aut}^\circ(\mathcal{P}_n) \times \mathrm{Aut}(\mathbb{P}^m)$. Assume that $\mathrm{Aut}^\circ(\mathcal{P}_n \times \mathbb{P}^m)$ is contained in a connected algebraic subgroup H of $\mathrm{Bir}(\mathcal{P}_n \times \mathbb{P}^m)$ acting rationally on $\mathcal{P}_n \times \mathbb{P}^m$. By Weil regularisation theorem [44], there is a birational map $\varphi \colon \mathcal{P}_n \times \mathbb{P}^m \dashrightarrow V$ to a variety V such that $H \subseteq \mathrm{Aut}^\circ(V)$. By Proposition 5.2, we have $V \simeq W \times \mathbb{P}^m$ and $\varphi = (\varphi_1, \varphi_2)$, where $\varphi_1 \colon \mathcal{P}_n \dashrightarrow W$ is birational and φ_2 is an isomorphism. Notice that $\mathrm{Aut}^\circ(V) \simeq \mathrm{Aut}^\circ(W) \times \mathrm{Aut}(\mathbb{P}^m)$. By Theorem 4.2, there exists a variety Y and an $\mathrm{Aut}^\circ(W)$-equivariant birational map $W \dashrightarrow Y$ such that $\mathrm{Aut}^\circ(W) \subsetneq \mathrm{Aut}^\circ(Y) =: G$. Then $H \subsetneq G \times \mathrm{Aut}(\mathbb{P}^m)$. □

Proof of Main Theorem. The threefold X from [31, Example 2-6] is irrational and has a fibration $X \to \mathbb{P}^2$ and $X \times \mathbb{P}^3$ is rational [1]. In [38] it is shown that already $X \times \mathbb{P}^2$ is rational. In particular, $\mathcal{P}_n \times \mathbb{P}^m$ is rational for $n \geq 0$ and $m \geq 1$. The claim now follows from Theorem 5.3 applied to the rational variety $Y := \mathcal{P}_n \times \mathbb{P}^m$ for $n \geq 2$ and $m \geq 1$, which is of dimension $\dim Y = 4 + m \geq 5$. $\square$

Remark 5.4. We notice that if X is any stably rational and non-rational variety of dimension 3, and k_0 is the smallest positive integer such that $X \times \mathbb{P}^{k_0}$ is rational, then by Theorem 5.3, the group $\mathrm{Aut}^\circ(\mathcal{P}_n \times \mathbb{P}^m)$ is not contained in a maximal connected algebraic group of $\mathrm{Bir}(\mathbb{P}^{m+4})$ for any $m \in \mathbb{N}$ such that $m + 4 \geq k_0 + 3$.

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