THE CREMONA GROUP OF THE PLANE IS COMPACTLY PRESENTED

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ABSTRACT. This article shows that the Cremona group of the plane is compactly presented. To do this we prove that it is a generalised amalgamated product of three of its algebraic subgroups (automorphisms of the plane and Hirzebruch surfaces) divided by one relation.

CONTENTS

1. INTRODUCTION

Let k be a field. The Cremona group $\mathrm{Bir}(\mathbb{P}^n)$ is the group of birational transformations of the projective space $\mathbb{P}^n_k = \mathbb{P}^n$. It corresponds to a very intensively studied topic in algebraic geometry (see [Ser08, Des12, Can13] and references therein.)

A birational transformation of $\mathbb{P}^n$ is simply a birational change of coordinates, so $\mathrm{Bir}(\mathbb{P}^n)$ is a natural generalisation of $\mathrm{Aut}(\mathbb{P}^n) = \mathrm{PGL}_{n+1}(k)$ and in many aspects the Cremona group behaves like semi-simple groups, but also in many aspects it does not. Some analogies between the Cremona groups and semi-simple groups have been presented by J.-P. Serre in the 1000th Bourbaki seminar [Ser08], and by S. Cantat in [Can11].

Endowed with the Euclidean topology, constructed in [BlaFur13], the Cremona group becomes a Hausdorff topological group (for k a local field). For $k = \mathbb{C}$ and $k = \mathbb{R}$ the restriction to its subgroup $\mathrm{PGL}_{n+1}(k)$ of linear coordinate changes of $\mathbb{P}^n$ is the Euclidean topology. This not only opens the path to study the geometric properties of the Cremona group coming from the Euclidean topology but also presents the opportunity to study the Cremona group from the point of view of geometric group theory and rises the question of analogies to Lie groups.

In this article we will present one of these analogies, namely the property of being compactly presented (see Definition 6.1).

Lets take a closer look at the Cremona group endowed with the Euclidean topology:

The group $\mathrm{Bir}(\mathbb{P}^1_\mathbb{C}) = \mathrm{PGL}_2(\mathbb{C})$ is compactly presented by any neighbourhood of 1 because it is a connected complex algebraic group (see for example [Abe72, Satz 3.1]).

For $n \geq 2$ and k any local field, the group $\mathrm{Bir}(\mathbb{P}^n_k)$ is not locally compact [BlaFur13, Lemma 5.15], though the topology is the inductive topology given by the family of closed sets $\mathrm{Bir}(\mathbb{P}^n)_{\leq d} = \{f \in \mathrm{Bir}(\mathbb{P}^n) \mid \deg(f) \leq d\}$, which are locally compact [BlaFur13, Proposition 2.10, Lemma 5.4]. Furthermore, any compact subset of $\mathrm{Bir}(\mathbb{P}^n)$ is of bounded degree.

For $n \geq 3$, the group $\mathrm{Bir}(\mathbb{P}^n_\mathbb{C})$ is not compactly generated [BlaFur13, Lemma 5.17], hence not compactly presented.

The group $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ is generated by $\mathrm{Aut}(\mathbb{P}^2_\mathbb{C}) = \mathrm{PGL}_2(\mathbb{C})$ and the standard quadratic transformation $\sigma : [x : y : z] \dashrightarrow [yz : xz : xy]$ [Cas01]. Its subgroup $\mathrm{Aut}(\mathbb{P}^2_\mathbb{C})$, being a connected complex algebraic group, is compactly presented by any neighourhood of 1. Hence $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ is compactly generated by any compact neighbourhood of 1 in $\mathrm{Aut}(\mathbb{P}^2_\mathbb{C})$ and σ.

The aim of this article is to show that, even though it is neither an algebraic group nor locally compact, $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ is moreover compactly presented:

Theorem A ((Corollary 6.8))**.** *Endowed with the Euclidean topology the Cremona group* $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ *is compactly presented by* $\{\sigma\} \cup K$ *where K is any compact neighbourhood of 1 in* $\mathrm{Aut}(\mathbb{P}^2_\mathbb{C})$ *and* $\sigma : [x : y : z] \dashrightarrow [yz : xz : xy]$ *is the standard involution of* $\mathbb{P}^2_\mathbb{C}$.

For algebraically closed fields, the generating sets and generating relations of $\mathrm{Bir}(\mathbb{P}^2)$ have been studied throughoutly: The famous Noether-Castelnuovo theorem [Cas01] states that if k is algebraically closed then $\mathrm{Bir}(\mathbb{P}^2)$ is generated by $\mathrm{Aut}(\mathbb{P}^2)$ and the standard quadratic involution $\sigma : [x : y : z] \dashrightarrow [yz : xz : xy]$, i.e. the generating set is the union of two complex linear algebraic groups.

A presentation was given in [Giz82] where the generating set consists of all quadratic transformations of $\mathbb{P}^2$ and the generating relations are of the form $q_1 q_2 q_3 = 1$ where q_i are quadratic transformations. Another presentation was given in [Isk85] where it is shown that $\mathrm{Bir}(\mathbb{P}^1 \times \mathbb{P}^1)$ (isomorphic to $\mathrm{Bir}(\mathbb{P}^2)$) is the amalgamated product of $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ and the de Jonquières group of birational maps of $\mathbb{P}^1 \times \mathbb{P}^1$ preserving the first projection along their intersection modulo one relation. In [Bla12] a similar result is presented; the group $\mathrm{Bir}(\mathbb{P}^2)$ is the amalgamated product of $\mathrm{Aut}(\mathbb{P}^2)$ and the de Jonquières group $J_{[1:0:0]}$ of birational maps of $\mathbb{P}^2$ preserving the pencil of lines through $[1 : 0 : 0]$ along their intersection modulo one relation.

Since neither $\mathrm{Aut}(\mathbb{P}^2) = \mathrm{PGL}_3(k)$ nor the set of quadratic transformations nor the de Jonquières group are compact in the Euclidean topology, these presentations yield no compact presentation but at least the latter two yield a bounded presentation (the length of the generating relations are universally bounded).

In [Wri92], using [Isk85], a presentation of $\mathrm{Bir}(\mathbb{P}^2)$ is given by the generalised amalgamated product of $\mathrm{Aut}(\mathbb{P}^2)$, $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ and $J_{[0:1:0]}$ (as subgroups of $\mathrm{Bir}(\mathbb{P}^2)$) along their pairwise intersection, where $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ is viewed as a subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ via a birational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ given by the pencils of lines through $[0 : 1 : 0]$ and $[1 : 0 : 0]$. Again, since $J_{[0:1:0]}$ is not compact, this does not yield

a compact but only a bounded presentation but it gives rise to the following idea, which is the key step in the proof of Theorem A:

Theorem B ((Theorem 5.5)). *Let k be algebraically closed. Then the Cremona group* $\mathrm{Bir}(\mathbb{P}^2)$ *is isomorphic to the amalgamated product of* $\mathrm{Aut}(\mathbb{P}^2)$, $\mathrm{Aut}(\mathbb{F}_2)$, $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ *(as subgroups of* $\mathrm{Bir}(\mathbb{P}^2)$*) along their pairwise intersection in* $\mathrm{Bir}(\mathbb{P}^2)$ *modulo the relation* $\tau_{13}\sigma\tau_{13}\sigma$, *where* $\tau_{13} \in \mathrm{Aut}(\mathbb{P}^2)$ *is given by* $\tau_{13} \colon [x : y : z] \mapsto [z : y : x]$.

Here the inclusion of $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ into $\mathrm{Bir}(\mathbb{P}^2)$ is the same as before, $\mathbb{F}_2$ is the second Hirzebruch surface and the inclusion of $\mathrm{Aut}(\mathbb{F}_2)$ into $\mathrm{Bir}(\mathbb{P}^2)$ is given by a birational map $\mathbb{P}^2 \dashrightarrow \mathbb{F}_2$ given by the system of lines through $[1 : 0 : 0]$ and the point infinitely near corresponding to the tangent direction $\{y = 0\}$.

The method used to prove Theorem B is, like in [Bla12] and [Isk85], to study linear systems and their base-points. The difference here is that our maps have bounded degree which rigidifies the situation and changes the possibilities for simplifications. The proof of Theorem B does not use [Wri92].

For $k = \mathbb{C}$, the three groups $\mathrm{Aut}(\mathbb{P}^2)$, $\mathrm{Aut}(\mathbb{F}_2)$, $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ are locally compact algebraic groups. Using this and Theorem B we prove Theorem A.

The plan of the article is as follows:

In Section 2 and Section 3 we give basic definitions and results on $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ and $\mathrm{Aut}(\mathbb{F}_2)$. Section 4 is devoted to relations in the generalised amalgamated product of $\mathrm{Aut}(\mathbb{P}^2)$, $\mathrm{Aut}(\mathbb{F}_0)$, $\mathrm{Aut}(\mathbb{F}_2)$ modulo the relation $\tau_{13}\sigma\tau_{13}\sigma$. These are the backbone of the proof of Theorem B, which will be given in Section 5. In Section 6 we visit some facts about compactly presented groups and then finally prove Theorem A.

In Section 2 to 5 we work over any algebraically closed field k and Section 6 restricts to $k = \mathbb{C}$.

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2. DESCRIPTION OF $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ AND $\mathrm{Aut}(\mathbb{F}_2)$ INSIDE THE CREMONA GROUP

This section is devoted to the description of the subgroups $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ and $\mathrm{Aut}(\mathbb{F}_2)$ of $\mathrm{Bir}(\mathbb{P}^2)$.

Remember that the n-th Hirzebruch surface $\mathbb{F}_n$, $n \in \mathbb{N}$, is given by

$$\mathbb{F}_n = \{([x : y : z], [u : v]) \in \mathbb{P}^2 \times \mathbb{P}^1 \mid yv^n = zu^n\}.$$

Observe that $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and that $\mathbb{F}_1$ is isomorphic to the blow-up of one point in $\mathbb{P}^2$.

Consider the birational maps $\varphi_0 : \mathbb{P}^2 \dashrightarrow \mathbb{F}_0$ and $\varphi_2 : \mathbb{P}^2 \dashrightarrow \mathbb{F}_2$ given as follows: The map φ_0 is given by the blow-up of the points $[1 : 0 : 0]$ and $[0 : 1 : 0]$ followed by the contraction of the line passing through them. The map φ_2 is given by the blow up of $[1 : 0 : 0]$ and the point infinitely near $[1 : 0 : 0]$ lying on the strict transform of $\{y = 0\}$ followed by the contraction of the strict transform of $\{y = 0\}$. The birational maps φ_0 and φ_2 are only defined up to automorphism of $\mathbb{F}_0$ and $\mathbb{F}_2$.

They induce homomorphisms of groups

$$\mathrm{Aut}(\mathbb{F}_0) \to \mathrm{Bir}(\mathbb{P}^2), \quad \psi \mapsto \varphi_0^{-1}\psi\varphi_0$$

$$\mathrm{Aut}(\mathbb{F}_2) \to \mathrm{Bir}(\mathbb{P}^2), \quad \psi \mapsto \varphi_2^{-1}\psi\varphi_2$$

whose image is uniquely determined by the choice of points blown-up in $\mathbb{P}^2$. We will denote the image of $\mathrm{Aut}(\mathbb{F}_i)$ also by $\mathrm{Aut}(\mathbb{F}_i)$ for $i = 0, 2$ since no confusion occurs.

Remark 2.1 (and Notation).   (1) We can check that

$$\mathbb{P}^2 \dashrightarrow \mathbb{F}_0, \quad [x : y : z] \mapsto ([x : z], [y : z])$$

with inverse $([u_0 : u_1], [v_0 : v_1]) \mapsto [u_0v_1 : v_0u_1 : u_1v_1]$, and

$$\mathbb{P}^2 \dashrightarrow \mathbb{F}_2, \quad [x : y : z] \mapsto ([xy : y^2 : z^2], [y : z])$$

with inverse $([u : v : w], [a : b]) \mapsto [ua : va : vb]$ are examples for φ_0 and φ_2.

(2) For $i = 0, 2$, the map $(\varphi_i)^{-1}$ has exactly one base-point, which we denote by p_i.

(3) The image of the linear system of lines of $\mathbb{P}^2$ by φ_i has a unique base-point, namely p_i.

(4) We denote by C_1 the curve of self intersection 0 in $\mathbb{F}_0$ which is contracted by $(\varphi_0)^{-1}$ onto $[1 : 0 : 0]$ and by C_2 the curve of self intersection 0 which is contracted onto $[0 : 1 : 0]$. Remark that $p_0 = \varphi_0(\{z = 0\})$ and that $\{p_0\} = C_1 \cap C_2$.

(5) We denote by E the exceptional curve of self intersection -2 in $\mathbb{F}_2$. It is contracted onto $[1 : 0 : 0]$ by $(\varphi_2)^{-1}$. Denote by C the curve of self intersection 0 in $\mathbb{F}_2$ which is contracted by $(\varphi_2)^{-1}$ onto the point infinitely near $[1 : 0 : 0]$ corresponding to the tangent $\{y = 0\}$. Remark that $p_2 = \varphi_2(\{y = 0\})$ and $p_2 \in C \setminus E$.

(6) Let $L \subset \mathbb{P}^2$ be a general line. Then $C_j \cdot \varphi_0(L) = 1$, $j = 1, 2$, and $C \cdot \varphi_2(L) = 1$, $E \cdot \varphi_2(L) = 0$.

The following picture illustrates for $i = 0, 2$ the transformation $(\varphi_i)^{-1}\psi_i\varphi_i$ where ψ_i is some automorphism of $\mathbb{F}_i$. At the same time it shows the blow-up diagram of $(\varphi_i)^{-1}\psi_i\varphi_i$.
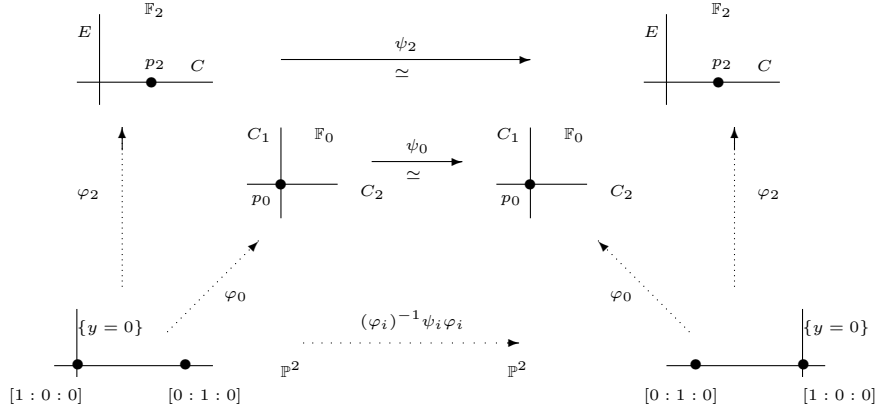

FIGURE 1. The transformation $(\varphi_i)^{-1}\psi_i\varphi_i$ for $i = 0, 2$

Consider the birational transformations of $\mathbb{P}^2$ given by

$$\sigma_1 : [x : y : z] \dashrightarrow [-xy + z^2 : y^2 : yz]$$
$$\sigma_2 : [x : y : z] \dashrightarrow [xy : z^2 : yz]$$
$$\sigma_3 : [x : y : z] \dashrightarrow [yz : xz : xy]$$

They are three quadratic involutions of $\mathbb{P}^2$ with respectively exactly one, two and three proper base-points in $\mathbb{P}^2$. The map σ_3 is usually referred to as standard quadratic involution of $\mathbb{P}^2$.

The map σ_3 has base-points $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]$, the map σ_2 has base-points $[1 : 0 : 0], [0 : 1 : 0]$ and the point p infinitely near $[1 : 0 : 0]$ corresponding to the direction $\{y = 0\}$, and the map σ_1 has base-points $[1 : 0 : 0], p, q$ where q is a point infinitely near p not contained in the intersection of the strict transform of the exceptional divisor of $[1 : 0 : 0]$.

Remark 2.2. For any quadratic map $\tau \in \mathrm{Bir}(\mathbb{P}^2)$ we can find $i = 1, 2, 3$ and $\alpha, \beta \in \mathrm{Aut}(\mathbb{P}^2)$ such that α sends the base-points of τ onto the base-points of σ_i and β sends the base-points of τ^{-1} onto the base-points of σ_i. We can then write $\tau = \beta^{-1} \sigma_i \alpha$. [Alb02, §2.1 and §2.8]

It follows that the linear system of τ is the image of the linear system of σ_i by α^{-1}, and that τ and σ_i have the same amount of proper base-points in $\mathbb{P}^2$. Since $\sigma_1, \sigma_2, \sigma_3$ have respectively one, two and three proper base-points in $\mathbb{P}^2$ the amount of proper base-points of τ determines i.

The following is the description of the groups $\mathrm{Aut}(\mathbb{F}_0)$ and $\mathrm{Aut}(\mathbb{F}_2)$ as subgroups of $\mathrm{Bir}(\mathbb{P}^2)$ given by the above inclusions:

Lemma 2.3. (1) *For $i = 0, 2$ the group $\mathcal{A}_i := \mathrm{Aut}(\mathbb{F}_i) \cap \mathrm{Aut}(\mathbb{P}^2)$ is the group of automorphisms of $\mathbb{P}^2$ fixing the set of base-points of φ_i, i.e. the set $\{[1 : 0 : 0], [0 : 1 : 0]\}$ if $i = 0$, and the point $[1 : 0 : 0]$ and the line $\{y = 0\}$ if $i = 2$.*

For each $i \in \{0, 2\}$, $\mathcal{A}_i$ corresponds via φ_i to the set of automorphisms of $\mathbb{F}_i$ that fix p_i.

(2) *The set $\mathrm{Aut}(\mathbb{F}_0) \setminus \mathrm{Aut}(\mathbb{P}^2)$ consists of all elements of the form $\beta \sigma_i \alpha$, where $i = 2, 3$ and $\alpha, \beta \in \mathrm{Aut}(\mathbb{F}_0) \cap \mathrm{Aut}(\mathbb{P}^2)$.*

(3) *The set $\mathcal{A}_0 \cup \mathcal{A}_0 \sigma_2 \mathcal{A}_0$ corresponds via φ_0 to the set of automorphisms of $\mathbb{F}_0$ sending p_0 into $C_1 \cup C_2$.*

(4) *The set $\mathrm{Aut}(\mathbb{F}_2) \setminus \mathrm{Aut}(\mathbb{P}^2)$ consists of all elements of the form $\beta \sigma_i \alpha$, where $i = 1, 2$ and $\alpha, \beta \in \mathrm{Aut}(\mathbb{F}_2) \cap \mathrm{Aut}(\mathbb{P}^2)$.*

(5) *The set $\mathcal{A}_2 \cup \mathcal{A}_2 \sigma_1 \mathcal{A}_2$ corresponds via φ_2 to the set of automorphisms of $\mathbb{F}_2$ that send p_2 into C.*

Proof. For $i = 0, 2$ let ψ_i be an automorphism of $\mathbb{F}_i$ and consider the following commutative diagram

$$\mathbb{F}_i \xrightarrow{\psi_i} \mathbb{F}_i$$
$$\varphi_i \nearrow \qquad \searrow (\varphi_i)^{-1}$$
$$\mathbb{P}^2 \dashrightarrow_{(\varphi_i)^{-1} \psi_i \varphi_i} \mathbb{P}^2.$$

1: The map $(\varphi_i)^{-1}\psi_i\varphi_i$ is an automorphism if and only if it does not have any base-points, which is equivalent to ψ_i preserving the union of the curves contracted by $(\varphi_i)^{-1}$ and fixing the point p_i blown-up by $(\varphi_i)^{-1}$. This is equivalent to $(\varphi_i)^{-1}\psi_i\varphi_i$ being an automorphism preserving the set of base-points of φ_i.

2 to 5: Let Δ be the linear system of lines in $\mathbb{P}^2$. We will determine the linear system $(\varphi_i)^{-1}\psi_i\varphi_i(\Delta)$. Note that 1 shows that $(\varphi_i)^{-1}\psi_i\varphi_i(\Delta) = \Delta$ if and only if ψ_i preserves p_i and the union of lines contracted by $(\varphi_i)^{-1}$, which is equivalent to ψ_i fixing p_i.

Assume that $\psi_i(p_i) \neq p_i$ holds. From this it follows that $(\varphi_i)^{-1}\psi_i\varphi_i$ has at least one and at most three base-points, hence is a quadratic map. In particular, $(\varphi_i)^{-1}\psi_i\varphi_i(\Delta)$ is a linear system of conics.

2 and 3: If $i = 0$, we can check that σ_2, σ_3 are elements of $\mathrm{Aut}(\mathbb{F}_0)$. In fact, if we take φ_0 as in Remark 2.1 1 they are given by the automorphisms $([u_0 : u_1], [v_0 : v_1]) \dashmapsto [u_0v_1 : v_0u_1 : u_1v_1]$ and $([u_1 : v_1], [u_2 : v_2]) \dashmapsto ([v_1 : u_1], [v_2 : u_2])$ respectively. It follows that the set $\mathcal{A}_0\sigma_2\mathcal{A}_0 \cup \mathcal{A}_0\sigma_3\mathcal{A}_0$ is contained in $\mathrm{Aut}(\mathbb{F}_0)$.

A general element of $\psi_0\varphi_0(\Delta)$ intersects each C_j in exactly one point different from p_0 (Remark 2.1 3,6), which means that $[1 : 0 : 0], [0 : 1 : 0]$ are base-points of the linear system of conics $(\varphi_0)^{-1}\psi_0\varphi_0(\Delta)$. The third base-point corresponds via φ_0 to the point $\psi_0(p_0)$. In particular, it is infinitely near to $[1 : 0 : 0]$ (resp. $[0 : 1 : 0]$) if and only if $\psi_0(p_0) \in C_1$ (resp. $\psi_0(p_0) \in C_2$). By Remark 2.2 we can write $(\varphi_0)^{-1}\psi_0\varphi_0 = \beta\sigma_j\alpha$ for some $j = 2, 3$ and $\alpha, \beta \in \mathrm{Aut}(\mathbb{P}^2)$, where α, β^{-1} respectively send the base-points of $(\varphi_0)^{-1}\psi_0\varphi_0$, $(\varphi_0)^{-1}(\psi_0)^{-1}\varphi_0$ onto the base-points of σ_j. If $j = 2$, it follows that α, β fix the set $\{[1 : 0 : 0], [0 : 1 : 0]\}$. If $j = 3$, we have $(\beta\theta)\sigma_3(\theta\alpha) = \beta\sigma_3\alpha$ for any permutation θ of coordinates x, y, z, hence we can assume that α fixes the set $\{[1 : 0 : 0], [0 : 1 : 0]\}$ and it follows that $\alpha \in \mathcal{A}_0$. Since $\sigma_2, \sigma_3 \in \mathrm{Aut}(\mathbb{F}_0)$, it follows that $\beta \in \mathcal{A}_0$.

Note that $(\varphi_0)^{-1}\psi_0\varphi_0$ has an infinitely near base-point if and only if $\psi_0(p_0) \in (C_1 \cup C_2) \setminus \{p_0\}$.

4 and 5: If $i = 2$, we can check that $\sigma_1, \sigma_2 \in \mathrm{Aut}(\mathbb{F}_2)$. In fact, if we take φ_2 as in Remark 2.1 1 then they are given by the automorphisms $([u : v : w], [a : b]) \mapsto ([-u + w : v : w], [a : b])$ and $([u : v : w], [a : b]) \mapsto ([u : w : v], [b : a])$ respectively. It follows that $\mathcal{A}_2\sigma_1\mathcal{A}_2 \cup \mathcal{A}_2\sigma_2\mathcal{A}_2 \subset \mathrm{Aut}(\mathbb{F}_2)$.

A general element of $\psi_2\varphi_2(\Delta)$ does not intersect E and intersects C in exactly one point different from p_2 (Remark 2.1 3,6). Therefore, $[1 : 0 : 0]$, the point p infinitely near to it corresponding to the tangent direction $\{y = 0\}$ are base-points of the linear system $(\varphi_2)^{-1}\psi_2\varphi_2(\Delta)$. The third base-point correspond via φ_2 to the point $\psi_2(p_2)$. In particular, it is infinitely near to p if and only if $\psi_2(p_2) \in C$ and it is a proper point of $\mathbb{P}^2$ otherwise. It follows from Remark 2.2 that we can write $(\varphi_2)^{-1}\psi_2\varphi_2 = \beta\sigma_j\alpha$ for $j = 1, 2$ and $\alpha, \beta \in \mathrm{Aut}(\mathbb{P}^2)$ where α, β^{-1} respectively send the linear system of $(\varphi_2)^{-1}\psi_2\varphi_2$, $(\varphi_2)^{-1}(\psi_2)^{-1}\varphi_2$ onto the linear system of σ_j. It follows that α, β fix $[1 : 0 : 0], p$ and hence $\alpha, \beta \in \mathcal{A}_2$.

Note that $(\varphi_2)^{-1}\psi_2\varphi_2$ has exactly one proper base-points in $\mathbb{P}^2$ if and only if ψ_2 sends p_2 into $C \setminus \{p_2\}$. $\square$

Lemma 2.3 allows us to present the following classical results and also describe a Zariski-open set in each $\mathrm{Aut}(\mathbb{F}_i)$ which will be useful in Section 6 when we prove that $\mathrm{Bir}(\mathbb{P}^2)$ is compactly presented.

Lemma 2.4. *For $i = 0, 2$, let $\mathcal{A}_i := \mathrm{Aut}(\mathbb{P}^2) \cap \mathrm{Aut}(\mathbb{F}_i)$.*

(1) *The groups* $\mathrm{Aut}(\mathbb{F}_0)$ *and* $\mathrm{Aut}(\mathbb{F}_2)$ *are linear algebraic subgroups of* $\mathrm{Bir}(\mathbb{P}^2)$.
(2) *The group* $\mathrm{Aut}(\mathbb{F}_0)$ *has two irreducible components, namely the component* $\mathrm{Aut}(\mathbb{F}_0)^0$ *containing* 1 *and* $\tau_{12}\,\mathrm{Aut}(\mathbb{F}_0)^0$, *where* $\tau_{12} \in \mathrm{Aut}(\mathbb{F}_0)$ *is given by* $\tau_{12}\colon [x:y:z] \mapsto [y:x:z]$.
(3) *The group* $\mathrm{Aut}(\mathbb{F}_2)$ *is irreducible.*
(4) *The set* $\mathcal{A}_0\sigma_3\mathcal{A}_0$ *is a Zariski-open set of* $\mathrm{Aut}(\mathbb{F}_0)$.
(5) *The set* $\mathcal{A}_2\sigma_2\mathcal{A}_2$ *is a Zariski-open set of* $\mathrm{Aut}(\mathbb{F}_2)$.

Proof. 1, 2 and 3 are classical results, which for example can be found in [Bla09, Proposition 2.2.6, Théorème 2].

4 By Lemma 2.3 the set $\mathrm{Aut}(\mathbb{F}_0) \setminus (\mathcal{A}_0\sigma_3\mathcal{A}_0)$ is the set of elements of $\mathrm{Aut}(\mathbb{F}_0)$ that send the point p_0 into the curve $C_1 \cup C_2$ and is therefore closed.

5 By Lemma 2.3 the set $\mathrm{Aut}(\mathbb{F}_2) \setminus (\mathcal{A}_2\sigma_2\mathcal{A}_2)$ is the set of element of $\mathrm{Aut}(\mathbb{F}_2)$ that fix the curve C and is therefore closed. $\qquad\square$

Remark 2.5. The Noether-Castelnuvo Theorem states that $\mathrm{Bir}(\mathbb{P}^2)$ is generated by $\mathrm{Aut}(\mathbb{P}^2)$ and σ_3 [Cas01] (see also [Alb02, §8]). Furthermore, we can write $\sigma_3 = \tau_{12}\sigma_2\tau_{12}\sigma_2$ where $\tau_{12}([x:y:z]) = ([y:x:z])$, hence the group $\mathrm{Bir}(\mathbb{P}^2)$ is also generated by $\mathrm{Aut}(\mathbb{P}^2)$ and σ_2. Therefore, for any $i = 0, 2$, the group $\mathrm{Bir}(\mathbb{P}^2)$ is generated by its subgroups $\mathrm{Aut}(\mathbb{P}^2)$ and $\mathrm{Aut}(\mathbb{F}_i)$ and thus also generated by all three subgroups $\mathrm{Aut}(\mathbb{P}^2)$, $\mathrm{Aut}(\mathbb{F}_0)$ and $\mathrm{Aut}(\mathbb{F}_2)$.

Note that Lemma 2.3 in particular implies that all elements of $\mathrm{Aut}(\mathbb{F}_0)\cup\mathrm{Aut}(\mathbb{F}_2)$ are linear or quadratic.

Definition 2.6. For a set S let F_S be the free group generated by S.
For the set $S := \mathrm{Aut}(\mathbb{P}^2) \cup \mathrm{Aut}(\mathbb{F}_0) \cup \mathrm{Aut}(\mathbb{F}_2) \subset \mathrm{Bir}(\mathbb{P}^2)$, define

$$\mathfrak{G} := F_S \Big/ \left\langle \begin{array}{ll} fgh^{-1}, & \text{if } fg = h \text{ in } \mathrm{Aut}(\mathbb{P}^2) \\ fgh^{-1}, & \text{if } fg = h \text{ in } \mathrm{Aut}(\mathbb{F}_0) \\ fgh^{-1}, & \text{if } fg = h \text{ in } \mathrm{Aut}(\mathbb{F}_2) \\ \tau_{13}\sigma_3\tau_{13}\sigma_3 \end{array} \right\rangle$$

where $\tau_{13} \in \mathrm{Aut}(\mathbb{P}^2)$ is given by $\tau_{13}\colon [x:y:z] \mapsto [z:y:x]$.

Remark 2.7. The group $\mathfrak{G}$ is isomorphic to the free product of the three groups $\mathrm{Aut}(\mathbb{P}^2), \mathrm{Aut}(\mathbb{F}_2), \mathrm{Aut}(\mathbb{F}_2)$ amalgamated along all the pairwise intersections (generalised amalgamated product of the three groups) modulo the relation $\tau_{13}\sigma_3 = \sigma_3\tau_{13}$.

A geometric approach to generalised amalgamated products can be found in [Hae90], [Ser77] and [Sta90]. The generalised amalgamated product

$$F_S \Big/ \left\langle \begin{array}{ll} fgh^{-1}, & \text{if } fg = h \text{ in } \mathrm{Aut}(\mathbb{P}^2) \\ fgh^{-1}, & \text{if } fg = h \text{ in } \mathrm{Aut}(\mathbb{F}_0) \\ fgh^{-1}, & \text{if } fg = h \text{ in } \mathrm{Aut}(\mathbb{F}_2) \end{array} \right\rangle$$

is in [Sta90, §1.3] the colimit of the diagram

$$\mathrm{Aut}(\mathbb{P}^2)$$

$$\mathcal{A}_0 \text{———} \mathcal{A}_0 \cap \mathcal{A}_2 \text{———} \mathcal{A}_2$$

$$\mathrm{Aut}(\mathbb{F}_0) \text{————} \mathrm{Aut}(\mathbb{F}_0) \cap \mathrm{Aut}(\mathbb{F}_2) \text{————} \mathrm{Aut}(\mathbb{F}_2)$$

Equivalently, it is the fundamental group of a 2-complex of groups. The vertices are $\mathrm{Aut}(\mathbb{P}^2)$, $\mathrm{Aut}(\mathbb{F}_0)$, $\mathrm{Aut}(\mathbb{F}_2)$, the edges are their pairwise intersection and the 2-simplex is the group $\mathrm{Aut}(\mathbb{P}^2) \cap \mathrm{Aut}(\mathbb{F}_0) \cap \mathrm{Aut}(\mathbb{F}_2)$ [Hae90, §2.1, §3.4], [Ser77, 4.4].

Remark 2.8. By Remark 2.5 there exists a canonical surjective homomorphism of groups

$$\pi : \mathfrak{G} \to \mathrm{Bir}(\mathbb{P}^2)$$

and by definition of $\mathfrak{G}$ a natural map

$$w \colon \mathrm{Aut}(\mathbb{P}^2) \cup \mathrm{Aut}(\mathbb{F}_0) \cup \mathrm{Aut}(\mathbb{F}_2) \to \mathfrak{G}$$

which sends an element to its corresponding word. Note that $\pi \circ w$ is the identity map.

3. Base-points, Multiplicities, de Jonquières

The methods we use mainly consist of studying linear systems of $\mathbb{P}^2$ and their base-points. In this section we recall some definitions, notions and formulae which will be used almost constantly in Section 4 and 5, which have the aim to prove Theorem B (Theorem 5.5).

Definition 3.1. A *point over* $\mathbb{P}^2$ is a point $p \in S$, where $S := S_{n+1} \overset{\nu_n}{\to} S_{n-1} \overset{\nu_{n-1}}{\to} \cdots \overset{\nu_1}{\to} S_0 := \mathbb{P}^2$ is a sequence of blow-ups, and where we identify $p \in S$ with $p_i \in S_i$ if $\nu_{i+1} \cdots \nu_n \colon S \to S_i$ is a local isomorphism around p sending p to p_i.

A point $p \in S$ over $\mathbb{P}^2$ is *proper* if it is equivalent to a point $p' \in \mathbb{P}^2$, and *infinitely near* otherwise.

Definition 3.2. Let $f \in \mathrm{Bir}(\mathbb{P}^2)$ be a quadratic birational transformation and call p_1, p_2, p_3 its base-points and q_1, q_2, q_3 the base-points of f^{-1}. We say that *the base-points of f are ordered consistently* if the following holds: The base-points of f and of f^{-1} are ordered such that

(1) If p_1, p_2, p_3 are proper points of $\mathbb{P}^2$ then all the lines through p_1 (respectively p_2, p_3) are sent onto lines through q_1 (respectively q_2, q_3).

(2) If p_1, p_2 are proper points of $\mathbb{P}^2$ and p_3 is infinitely near to p_1, then all the lines through p_1 (respectively p_2) are sent onto lines through q_1 (respectively q_2).

(3) If p_1 is a proper point of $\mathbb{P}^2$, p_2 infinitely near p_1 and p_3 infinitely near p_2 then the lines through p_1 are sent onto lines through q_1 and the exceptional curve associated to p_3 is sent onto the tangent associated to q_2.

Remark 3.3. Writing down the blow-up diagram of the three quadratic involutions $\sigma_1, \sigma_2, \sigma_3$, we see that we can always order their base-points consistenly (for example for σ_3 the ordering $p_1 = q_1 = [1 : 0 : 0], p_2 = q_2 = [0 : 1 : 0], p_3 = q_3 = [0 : 0 : 1]$

is consistent). Since any quadratic birational transformation of $\mathbb{P}^2$ can be written $\beta\sigma_i\alpha$ for some suitable $i \in \{1,2,3\}$, $\alpha, \beta \in \mathrm{Aut}(\mathbb{P}^2)$ (Remark 2.2), it is always possible to order its base-points consistently.

Throughout the article, we will always assume that the base-points of a quadratic transformation of $\mathbb{P}^2$ are ordered consistently.

Let us remind of the following formula: Let Δ be a linear system and $f \in \mathrm{Bir}(\mathbb{P}^2)$ a quadratic transformation with base-points p_1, p_2, p_3, and q_1, q_2, q_3 the base-points of f^{-1}. Let a_i be the multiplicity of Δ in p_i and b_i the multiplicity of $f(\Delta)$ in q_i. If the base-points of f are ordered consistently then

$$\deg(f(\Delta)) = 2\deg(\Delta) - \varepsilon, \qquad b_i = \deg(\Delta) - \varepsilon + a_i$$

for $i = 1, 2, 3$ and $\varepsilon = a_1 + a_2 + a_3$ [Alb02, §4.2].

Definition 3.4. We define

$$J := \{f \in \mathrm{Bir}(\mathbb{P}^2) : f \text{ preserves the pencil of lines through } [1:0:0]\}.$$

The elements of J are called *de Jonquières transformations*.

A linear system Δ of $\mathbb{P}^2$ of degree $\deg(\Delta) = d$ and with base-points $p_1, \ldots, p_n$ of multiplicity $a_1, \ldots, a_n$ is called *de Jonquières linear system* if it has multiplicity $d-1$ at $[1:0:0]$ and satisfies the conditions $d^2 - 1 = \sum_{i=1}^n a_i^2$ and $3(d-1) = \sum_{i=1}^n a_i$.

We call a base-point of f a *simple base-point* if it is different from $[1:0:0]$ and denote the set of simple base-points by $\mathrm{sBp}(f)$.

Remark 3.5.
(1) We have the following inclusions: $\mathrm{Aut}(\mathbb{F}_2) \subset J$ and $\mathrm{Aut}(\mathbb{F}_0)^0 \subset J$, where $\mathrm{Aut}(\mathbb{F}_0)^0$ is the connected component of $\mathrm{Aut}(\mathbb{F}_0)$ containing Id and which is equal to $(\mathcal{A}\sigma_2\mathcal{A}) \cup (\mathcal{A}\tau_{12}\sigma_2\tau_{12}\mathcal{A}) \cup (\mathcal{A}\sigma_3\mathcal{A}) \cup \mathcal{A}$, where $\mathcal{A} = \{\alpha \in \mathrm{Aut}(\mathbb{P}^2) \cap \mathrm{Aut}(\mathbb{F}_2) \mid \alpha([1:0:0]) = [1:0:0]\}$ and $\tau_{12} \in \mathrm{Aut}(\mathbb{P}^2)$ is given by $\tau_{12}\colon [x:y:z] \mapsto [y:x:z]$ (Lemma 2.4).

(2) Any element of $f \in J \setminus \mathrm{Aut}(\mathbb{P}^2)$ of degree d has $2d-1$ base-points: the base-point $[1:0:0]$ of multiplicity $d-1$ and $2d-2$ other base-points of multiplicity one (this follows from the conditions on the degree and multiplicities). Thus the definition of simple base-point of f is quite natural. If $f \in J$ is of degree 2, it has exactly three base-points, all of multiplicity one. Its simple base-points are just the ones different from $[1:0:0]$.

(3) A de Jonquières linear system of $\mathbb{P}^2$ of degree d has $2d-1$ base-points and the multiplicity at any base-point different from $[1:0:0]$ is one. Such a point is called a *simple base-point* of Δ. Observe that for $f \in J$ and Δ a de Jonquières linear system, $f(\Delta)$ is a de Jonquières linear system, and the linear system of f is a de Jonquières linear system.

Lemma 3.6. *For any quadratic de Jonquières transformation $f \in \mathrm{Bir}(\mathbb{P}^2)$ there exist $\alpha_1, \alpha_2 \in \mathrm{Aut}(\mathbb{P}^2) \cap J$, $\tau \in \{\sigma_1, \sigma_2, \tau_{12}\sigma_2\tau_{12}, \sigma_3\} \subset \mathrm{Aut}(\mathbb{F}_0) \cup \mathrm{Aut}(\mathbb{F}_2)$, where $\tau_{12} \in \mathrm{Aut}(\mathbb{P}^2)$ is given by $\tau_{12}\colon [x:y:z] \mapsto [y:x:z]$, such that $f = \alpha_2\tau\alpha_1$.*

In particular, $(\mathrm{Aut}(\mathbb{P}^2) \cup \mathrm{Aut}(\mathbb{F}_0) \cup \mathrm{Aut}(\mathbb{F}_2)) \cap \mathcal{J}$ generates $\mathcal{J}$.

Proof. By Remark 2.1, we can write $f = \alpha_2\sigma_i\alpha_1$ for some $\alpha_1, \alpha_2 \in \mathrm{Bir}(\mathbb{P}^2)$ and i is determined by the amount of proper base-points of f. Since f is de Jonquières the point $[1:0:0]$ is a base-point of f.

If f has only one proper base-point in $\mathbb{P}^2$, it has to be fixed by α_1 and α_2, which belong thus to J. This gives the result.

Suppose that f has exactly two proper base-points, namely $[1:0:0]$ and p. This implies that $\sigma_i = \sigma_2$, which has base-points $[1:0:0], [0:1:0]$ and a third one, infinitely near $[1:0:0]$. The base-point of f which is not a proper point of $\mathbb{P}^2$ is either infinitely near $[1:0:0]$ or p. If it is infinitely near $[1:0:0]$, then α_1, α_2 fix $[1:0:0]$ and are therefore de Jonquières. If it is infinitely near p then α_1 sends p onto $[1:0:0]$ and $[1:0:0]$ onto $[0:1:0]$. We write $\alpha_1 = \tau_{12}\beta_1$, $\alpha_2 = \beta_2\tau_{12}$, for some $\beta_1, \beta_2 \in \mathrm{Aut}(\mathbb{P}^2)$, which means that β_1 fixes $[1:0:0]$, i.e. $\beta_1 \in J \cap \mathrm{Aut}(\mathbb{P}^2)$ and $f = \beta_2(\tau_{12}\sigma_2\tau_{12})\beta_1$. Since $f, \beta_1, \tau_{12}\sigma_2\tau_{12} \in J$, we have $\beta_2 \in J$.

Suppose that f has three proper base-points. For any $\theta \in \mathrm{Aut}(\mathbb{P}^2)$ that permutes the coordinate x, y, z, we have $\theta\sigma_3\theta = \sigma_3$. Therefore, we can assume that $\alpha_1 \in J$. Since σ_3 is de Jonquières, the map α_2 has to be de Jonquières as well.

Since every element of $\mathcal{J}$ decomposes into quadratic elements of $\mathcal{J}$ [Alb02, Theorem 8.4.3], Lemma 3.6 implies that $\mathcal{J}$ is generated by $(\mathrm{Aut}(\mathbb{P}^2) \cup \mathrm{Aut}(\mathbb{F}_0) \cup \mathrm{Aut}(\mathbb{F}_2)) \cap \mathcal{J}$ generates $\mathcal{J}$. $\qquad\square$

Remark 3.7. Suppose Δ is a de Jonquières linear system of degree d and f a quadratic de Jonquières transformation. We can say the following about the degree of $f(\Delta)$: Let $p_1 = [1:0:0], p_2, p_3$ be the base-points of f and a_i be the multiplicity of Δ in p_i. Then $a_1 = \deg(\Delta) - 1$ and by the formula given above, we have

$$\deg(f(\Delta)) = 2d - (d-1) - a_2 - a_3 = d + 1 - a_2 - a_3$$

Since Δ has one base-point of multiplicity $d-1$ and all the other base-points are of multiplicity one, we know that for $i = 2, 3$, a_i is either zero or one. In fact, $a_i = 0$ if p_i is not a common base-point of f and Δ, and $a_i = 1$ if p_i is a common base-point of f and Δ. Thus the formula implies

$$\deg(f(\Delta)) = \begin{cases} d+1, & \text{if } f \text{ and } \Delta \text{ have no common simple base-points} \\ d, & \text{if } f \text{ and } \Delta \text{ have exactly one common simple base-point} \\ d-1, & \text{if } f \text{ and } \Delta \text{ have exactly two common simple base-points} \end{cases}$$

Furthermore, if p is a simple base-point of Δ that is not base-point of f, then $f^\bullet(p)$ (see definition below) is a simple base-point of $f(\Delta)$ [Alb02, §4.1].

Definition 3.8. Let $f \in \mathrm{Bir}(\mathbb{P}^2)$ and p a point over the domain $\mathbb{P}^2$ that is not a base-point of f. Take a minimal resolution of f

$$
\begin{array}{ccc}
 & S & \\
{}^{\nu_1}\swarrow & & \searrow{}^{\nu_2} \\
\mathbb{P}^2 & \dashrightarrow{\ f\ } & \mathbb{P}^2
\end{array}
$$

where ν_1, ν_2 are sequences of blow-ups. Let $p' \in S$ be a representative of p. We can see p' as a point over the range $\mathbb{P}^2$, and call it $f^\bullet(p)$.

Lets look at an example to understand $f^\bullet(p)$ and $f(p)$:

Example 3.9. Consider the standard quadratic involution $\sigma_3 \in \mathrm{Bir}(\mathbb{P}^2)$ and the point $p = [0:1:1]$, which is on the line $\{x = 0\}$ contracted by σ_3 onto the point $[1:0:0]$, which means that $\sigma_3(p) = [1:0:0]$. The line $L = \{y = z\}$ passing through p and $[1:0:0]$ is sent by σ_3 onto itself. By definition, $(\sigma_3)^\bullet(p)$ is the point in the first neighbourhood of $[1:0:0]$ corresponding to the tangent direction $\{y = z\}$. In conclusion, $\sigma_3(p)$ is a proper point of $\mathbb{P}^2$, whereas $(\sigma_3)^\bullet(p)$ is not. The following picture (Figure 2) illustrates the situation.
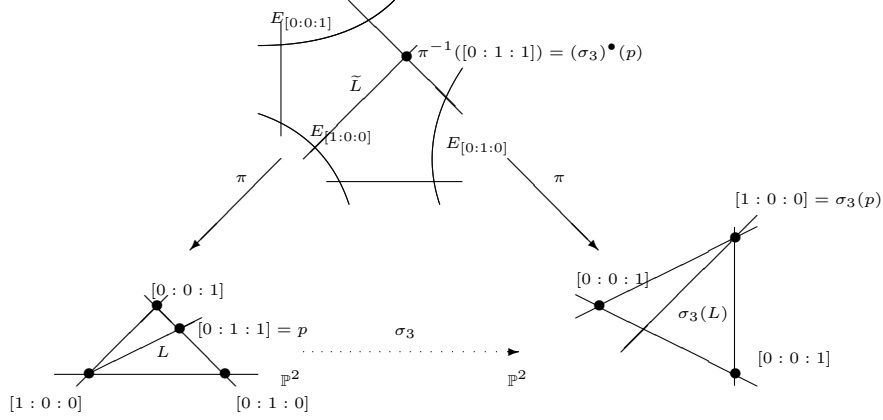
FIGURE 2. The points $(\sigma_3)^\bullet(p)$ and $\sigma_3(p)$

Remark 3.10. Note that $f^\bullet$ is a one-to-one correspondence between the sets

$$(\mathbb{P}^2 \cup \{\text{infinitely near points}\}) \setminus \{\text{base-points of } f\} \quad \text{and}$$
$$(\mathbb{P}^2 \cup \{\text{infinitely near points}\}) \setminus \{\text{base-points of } f^{-1}\}$$

4. BASIC RELATIONS IN $\mathfrak{G}$

In this section, we present basic relations that hold in $\mathfrak{G}$ and which will be the backbone of the proof of Theorem A (Theorem 5.5). We prove relations for words in $\mathfrak{G}$ of length three using properties of the elements of $\mathrm{Aut}(\mathbb{F}_0)$, $\mathrm{Aut}(\mathbb{F}_2)$ and $\mathrm{Aut}(\mathbb{P}^2)$. (Lemma 4.2, Lemma 4.3 and Lemma 4.4). They will then be used in the next section to prove that there exists an injective map $w_J: J \to \mathfrak{G}$ such that $\pi \circ w_J = \mathrm{Id}$ (Lemma 5.1, Corollary 5.2) which will enable us to prove Theorem A (Theorem 5.5) using the result that $\mathrm{Bir}(\mathbb{P}^2)$ is the amalgamated product of $\mathrm{Aut}(\mathbb{P}^2)$ and J modulo one relation [Bla12] (Theorem 5.3).

Lemma 4.1 and 4.2 yield that words of length three in $\mathfrak{G}$ whose image in $\mathrm{Bir}(\mathbb{P}^2)$ is linear or quadratic behave like their images in $\mathrm{Bir}(\mathbb{P}^2)$. Lemma 4.3 and 4.4 yield relations for words of length three whose image in $\mathrm{Bir}(\mathbb{P}^2)$ is de Jonquières and of degree three.

Define $\mathrm{TAut}(\mathbb{P}^2) = D \rtimes S_3$, where $S_3 \subset \mathrm{Aut}(\mathbb{P}^2)$ is the image of the permutation matrices of GL_3 and D is the image of the three dimensional torus. We can check that the group $\mathrm{TAut}(\mathbb{P}^2)$ is normalised by σ_3, and the automorphism of $\mathrm{TAut}(\mathbb{P}^2)$ given by the conjugation of σ_3 will be denoted by ι. Note that $\iota(\alpha) = \alpha$ for $\alpha \in S_3$ and $\iota(\delta) = \delta^{-1}$ for $\delta \in D$.

As subgroup of $\mathrm{Aut}(\mathbb{P}^2)$, we can embed $\mathrm{TAut}(\mathbb{P}^2)$ (as a set) into $\mathfrak{G}$ by the word map w. The next lemma shows that in $\mathfrak{G}$ the image of $\mathrm{TAut}(\mathbb{P}^2)$ is normalised by $w(\sigma_3)$:

Lemma 4.1. *For any* $(\delta, \alpha) \in D \rtimes S_3$ *the relation* $w(\delta\alpha)w(\sigma_3) = w(\sigma_3)w(\iota(\delta\alpha))$ *holds in* $\mathfrak{G}$.

Proof. Let $\tau_{12} : [x : y : z] \mapsto [y : x : z]$. In $\mathrm{Aut}(\mathbb{F}_0)$, the relation $\tau_{12}\sigma_3\tau_{12} = \sigma_3$ holds, hence the relation $w(\tau_{12})w(\sigma_3) = w(\sigma_3)w(\tau_{12})$ holds in $\mathfrak{G}$. By definition,

$w(\tau_{13})w(\sigma_3) = w(\sigma_3)w(\tau_{13})$ is a relation in $\mathfrak{G}$, and τ_{13} and τ_{12} generate S_3. Therefore, the relation $w(\alpha)w(\sigma_3) = w(\sigma_3)w(\alpha) = w(\sigma_3)w(\iota(\alpha))$ holds in $\mathfrak{G}$ for any $\alpha \in S_3$.

Let $\delta \in D$. The relation $\delta\sigma_3\delta = \sigma_3$ holds in $\mathrm{Aut}(\mathbb{F}_0)$, hence $w(\delta)w(\sigma_3) = w(\sigma_3)w(\delta^{-1}) = w(\sigma_3)w(\iota(\delta))$ holds in $\mathfrak{G}$. $\qquad\square$

Using Lemma 4.1, we now show that for $f, g, h \in \mathrm{Aut}(\mathbb{P}^2) \cup \mathrm{Aut}(\mathbb{F}_0) \cup \mathrm{Aut}(\mathbb{F}_2)$ and $\deg(fgh) \leq 2$, the word $w(f)w(g)w(h)$ behaves like the composition fgh:

Lemma 4.2. *Let* $g \in \mathrm{Aut}(\mathbb{P}^2), h, f \in \mathrm{Aut}(\mathbb{F}_0) \cup \mathrm{Aut}(\mathbb{F}_2)$ *such that* $\deg(fgh) \in \{1, 2\}$.

  (1) *If* $\deg(fgh) = 1$*, then* $w(f)w(g)w(h) = w(fgh)$ *in* $\mathfrak{G}$*.*
  (2) *If* $\deg(fgh) = 2$*, there exist* $\alpha, \beta \in \mathrm{Aut}(\mathbb{P}^2)$*,* $\tilde{g} \in \mathrm{Aut}(\mathbb{F}_0) \cup \mathrm{Aut}(\mathbb{F}_2)$ *such that* $w(f)w(g)w(h) = w(\beta)w(\tilde{g})w(\alpha)$ *in* $\mathfrak{G}$*.*
  (3) *If* $\deg(fgh) = 2$ *and* $f, g, h \in J$ *then* $\alpha, \beta, \tilde{g}$ *can be chosen to be in* J*.*

Proof. Suppose that $f \in \mathrm{Aut}(\mathbb{P}^2)$ or $h \in \mathrm{Aut}(\mathbb{P}^2)$. The first claim follows from the definition of $\mathfrak{G}$. The second and third claim follow by putting $\beta := fg$ or $\alpha = gh$ if $\deg(f) = 1$ or $\deg(h) = 1$ respectively.

Assume that $h \in \mathrm{Aut}(\mathbb{F}_i) \setminus \mathrm{Aut}(\mathbb{P}^2)$ and $f \in \mathrm{Aut}(\mathbb{F}_j) \setminus \mathrm{Aut}(\mathbb{P}^2)$. Since $\mathrm{Aut}(\mathbb{F}_0)$ is generated by $\mathrm{Aut}(\mathbb{P}^2) \cap \mathrm{Aut}(\mathbb{F}_0)$ and σ_2, σ_3 and $\mathrm{Aut}(\mathbb{F}_2)$ is generated by $\mathrm{Aut}(\mathbb{P}^2) \cap \mathrm{Aut}(\mathbb{F}_2)$ and σ_1, σ_2 (Lemma 2.3), we can write $f = \beta_2\sigma_k\alpha_2$ and $h = \beta_1\sigma_l\alpha_1$ for some $\alpha_1, \beta_1 \in \mathrm{Aut}(\mathbb{F}_i) \cap \mathrm{Aut}(\mathbb{P}^2)$, $\alpha_2, \beta_2 \in \mathrm{Aut}(\mathbb{F}_j)\,\mathrm{Aut}(\mathbb{P}^2)$ and $k, l \in \{1, 2, 3\}$. By replacing g with $\alpha_2 g \beta_1$ in $\mathrm{Aut}(\mathbb{P}^2)$, we can assume that $\alpha_2 = \beta_1 = \mathrm{Id}$ and hence $f = \beta_2\sigma_k$ and $h = \sigma_l\alpha_1$. It follows from Remark 2.2 that

$(*)$ the base-points of f are exactly the base-points of σ_k,

the base-points of h^{-1} are exactly the base-points of σ_l.

(i) Suppose that $\deg(fgh) = 1$. Then f and $(gh)^{-1}$ have exactly the same base-points, which are respectively the base-points of σ_k and the image of the base-points of σ_l by g. In particular, f, $(gh)^{-1}$ and hence also σ_k, σ_l have the same amount of proper base-points in $\mathbb{P}^2$. Since σ_1, σ_2, σ_3 have exactly one, two and three proper proper base-points, it follows that $\sigma_k = \sigma_l$.

If $k \in \{1, 2\}$ the equation $\sigma_k = \sigma_l$, the fact that f, $(gh)^{-1}$ have the same base-points and $(*)$ imply that $g \in \mathrm{Aut}(\mathbb{F}_2) \cap \mathrm{Aut}(\mathbb{P}^2)$ and so $f, g, h \in \mathrm{Aut}(\mathbb{F}_2)$. The definition of $\mathfrak{G}$ then implies $w(f)w(g)w(h) = w(fgh)$.

If $k = 3$, the equation $\sigma_k = \sigma_l$, the fact that f, $(gh)^{-1}$ have the same base-points and $(*)$ imply that g permutes the base-points of σ_3. Lemma 4.1 states that $w(g)w(\sigma_3) = w(\sigma_3)w(\iota(g))$. We get

$$\begin{aligned}
w(f)w(g)w(h) &= w(\beta_2\sigma_3)w(g)w(\sigma_3\alpha_1) \\
&= w(\beta_2)w(\sigma_3)w(\sigma_3)w(\iota(g))w(\alpha_1) \\
&= w(\beta_2)w(\iota(g))w(\alpha_1) = w(\beta_2\iota(g)\alpha_1) \\
&= w(\beta_2\sigma_3 g\sigma_3\alpha_1) = w(fgh).
\end{aligned}$$

(ii) Suppose $\deg(fgh) = 2$, i.e. f and $(gh)^{-1}$ have exactly two common base-points s, t, at least one of them being proper. Assume that s is proper.

If t is infinitely near to s, $(*)$ implies that $\{k, l\} \subset \{1, 2\}$, i.e. $f, h \in \mathrm{Aut}(\mathbb{F}_2)$. Then $(*)$ and the fact that t is infinitely near s implies that $s = [1 : 0 : 0]$ and that t lies on the strict transform of $\{y = 0\}$. Then s, t are base-points of both h^{-1} and

$(gh)^{-1}$ and it follows that $g(\{s,t\}) = \{s,t\}$, thus $g \in \mathrm{Aut}(\mathbb{F}_2)$. It follows that in $\mathrm{Aut}(\mathbb{F}_2)$ (hence also in $\mathfrak{G}$)

$$w(f)w(g)w(h) = w(\beta_2\sigma_k)w(g)w(\sigma_l\alpha_1) = w(\beta_2)w(\sigma_k g\sigma_l)w(\alpha_1).$$

Remark that any map contained in $\mathrm{Aut}(\mathbb{F}_2)$ is de Jonquières (Remark 3.5), from which claim (iii) follows for this subcase.

If s and t are both proper, $(*)$ implies that $\{k,l\} \subset \{2,3\}$, i.e. $f,h \in \mathrm{Aut}(\mathbb{F}_0)$. Then $(*)$ yields that $\{s,t\} \subset \{[1:0:0],[0:1:0],[0:0:1]\}$. There exist $\alpha,\beta \in \mathrm{TAut}(\mathbb{P}^2)$ such that

$$\alpha(\{[1:0:0],[0:1:0]\}) = g^{-1}(\{s,t\}), \quad \beta(\{s,t\}) = \{[1:0:0],[0:1:0]\}$$
$$\beta g\alpha([1:0:0]) = [1:0:0], \quad \beta g\alpha([0:1:0]) = [0:1:0].$$

If $k = 2$, we may choose $\beta = \mathrm{Id}$. If $l = 2$, we may choose $\alpha = \mathrm{Id}$. We get

$$w(f)w(g)w(h) = w(\beta_2\sigma_k)w(\beta^{-1})w(\beta)w(g)w(\alpha)w(\alpha^{-1})w(\sigma_l\alpha_1)$$
$$\overset{\mathrm{Lem}4.1}{=} w(\beta_2\iota(\beta^{-1}))w(\sigma_k)w(\beta g\alpha)w(\sigma_l)w(\iota(\alpha^{-1})\alpha_1)$$
$$= w(\beta_2\iota(\beta^{-1}))w(\sigma_k\beta g\alpha\sigma_l)w(\iota(\alpha)\alpha_1)$$

The claim follows with $\alpha = \iota(\alpha^{-1})\alpha_1$, $\tilde{g} = \sigma_k(\tilde{\beta}g\tilde{\alpha})\sigma_l$, $\beta = \beta_1\iota(\beta^{-1})$. It remains to prove claim (iii) for this sub case: If f,g,h are de Jonquières, then $g([1:0:0]) = [1:0:0]$ and $[1:0:0]$ is a common base-point of gf and h^{-1}. Choosing α,β above such that they fix $[1:0:0]$ (i.e. are de Jonquières) it follows that $\tilde{g} = \beta g\alpha$ is de Jonquières. The maps f and h being de Jonquières implies that α_1,β_2 are de Jonquières (Remark 3.5), hence $\iota(\alpha^{-1})\alpha_1$, $\beta_1\iota(\beta^{-1})$ and $\tilde{g}$ are de Jonquières.   □

The next two lemmata yield relations for words of length three whose image in $\mathrm{Bir}(\mathbb{P}^2)$ is of degree three.

Lemma 4.3. *Let $f \in (\mathrm{Aut}(\mathbb{F}_0) \cup \mathrm{Aut}(\mathbb{F}_2)) \cap J$ be a quadratic transformation, $\alpha_1,\ldots,\alpha_4 \in \mathrm{Aut}(\mathbb{P}^2) \cap J$ such that*

    (1) *f is a local isomorphism at the simple base-points q_2,q_3 of $(\alpha_2\sigma_3\alpha_1)^{-1}$,*
    (2) *$\mathrm{sBp}(\alpha_4\sigma_3\alpha_3) = \{f(q_2),f(q_3)\}$.*

Then

    (1) *The map $(\alpha_4\sigma_3\alpha_3)f(\alpha_2\sigma_3\alpha_1)$ is quadratic de Jonquières,*
    (2) *$\mathrm{sBp}((\alpha_4\sigma_3\alpha_3)f(\alpha_2\sigma_3\alpha_1)) = \left((\alpha_2\sigma_3\alpha_1)^{-1}\right)^{\bullet}(\mathrm{sBp}(f))$*
    (3) *there exist $\beta_1,\beta_3 \in \mathrm{Aut}(\mathbb{P}^2) \cap J$ and $\beta_2 \in \{\sigma_2,\sigma_3,\tau_{12}\sigma_2\tau_{12}\}$ such that the following equation holds in $\mathfrak{G}$:*

$$w(\alpha_4)w(\sigma_3)w(\alpha_3)w(f)w(\alpha_2)w(\sigma_3)w(\alpha_1) = w(\beta_3)w(\beta_2)w(\beta_1)$$

i.e. the following diagram corresponds to a relation in $\mathfrak{G}$:

$$
\begin{array}{ccc}
\mathbb{P}^2 & \overset{f}{\dashrightarrow} & \mathbb{P}^2 \\
{\scriptstyle \alpha_2\sigma_3\alpha_1} \Big\uparrow & & \Big\downarrow {\scriptstyle \alpha_4\sigma_3\alpha_3} \\
\mathbb{P}^2 & \underset{\beta_3\beta_2\beta_1}{\dashrightarrow} & \mathbb{P}^2
\end{array}
$$

Proof. Define $\tau_1 := \alpha_2\sigma_3\alpha_1$ and $\tau_2 := \alpha_4\sigma_3\alpha_2$ and denote by $p_1 = [1:0:0], p_2, p_3$ the base-points of f and by $p_1, \bar{p}_2, \bar{p}_3$ the base-points of its inverse (ordered consistently, see Definition 3.2).

Since f is a local isomorphism at q_2, q_3 the map f^{-1} is a local isomorphism at $f(q_2), f(q_3)$. Hence there exist simple base-points $p_i, \bar{p}_i$ of f, f^{-1} respectively, either proper points of $\mathbb{P}^2$ or infinitely near p_1, which do not lie on the lines contracted by $(\tau_1)^{-1}$ and τ_2. Up to order, we can assume that $p_i = p_2$. Therefore, the points $\tilde{p}_2 := (\tau_1^{-1})^\bullet(p_2)$ and $\hat{p}_2 := (\tau_2)^\bullet(\bar{p}_2)$ are proper points of $\mathbb{P}^2$.

Observe that the map $\tau_2 f \tau_1$ is de Jonquières of degree two having base-points $p_1, \tilde{p}_2, \tilde{p}_3 := (\tau_1^{-1})^\bullet(p_3)$ and its inverse having base-points $p_1, \hat{p}_2, \hat{p}_3 := (\tau_2)^\bullet(\bar{p}_3)$. Indeed, the map $f\tau_1$ is of degree three with base-points $p_1, \tilde{p}_2, \tilde{p}_3, \tilde{q}_2, \tilde{q}_3$, where $\tilde{q}_2, \tilde{q}_3$ are the simple base-points of τ_1, and its inverse having base-points $p_1, p_4, p_5, f(q_2), f(q_3)$. Thus $\tau_2 f \tau_1$ is de Jonquières of degree two with base-points $p_1, \tilde{p}_2, \tilde{p}_3$ and its inverse having base-points $p_1, \hat{p}_2, \hat{p}_3$ (by the formula given in Section 3).

Since $\tau_2 f \tau_1$ has at least one simple proper base-point (namely $\tilde{p}_2$), Lemma 3.6 and 2.2 imply that there exist $\beta_1, \beta_2 \in \mathrm{Aut}(\mathbb{P}^2) \cap J$ and $\beta_2 \in \{\sigma_2, \sigma_3, \tau_{12}\sigma_2\tau_{12}\}$ such that $\tau_2 f \tau_1 = \beta_3 \beta_2 \beta_1$.

It is left to prove that $w(\alpha_4)w(\sigma_3)w(\alpha_3)w(f)w(\alpha_2)w(\sigma_3)w(\alpha_1) = w(\beta_3)w(\beta_2)w(\beta_1)$ in $\mathfrak{G}$. We will use Lemma 4.2, and for this we fill the diagram

with triangles corresponding to relations in $\mathfrak{G}$.

The map f is a local isomorphism at q_2, q_3 hence the three points p_1, p_2, q_2 are not collinear. Since moreover p_1, q_2 are both proper points of $\mathbb{P}^2$ there exists a quadratic map $\rho \in \mathrm{Bir}(\mathbb{P}^2) \cap J$ which has base-points p_1, q_2, p_2. The maps $\rho\tau_1$ and ρf^{-1} are quadratic de Jonquières maps with base-points $p_1, \tilde{q}_2, \tilde{p}_2$ and $p_1, \bar{p}_2, f(q_2)$ respectively. It follows that also the map $\rho\tau_1(\beta_3\beta_2\beta_1)^{-1}$ is quadratic. The situation is summarised in the following diagram, where all the arrows are quadratic maps and the points in the brackets are the simple base-points of the corresponding quadratic map:

Writing $\rho = \gamma_3\gamma_2\gamma_1$ for some $\gamma_1, \gamma_2, \gamma_3 \in (\mathrm{Aut}(\mathbb{P}^2) \cup \mathrm{Aut}(\mathbb{F}_0) \cup \mathrm{Aut}(\mathbb{F}_2)) \cap J$, only γ_2 quadratic (possible by Lemma 3.6), Lemma 4.2 implies that each triangle in the above diagram corresponds to a relation in $\mathfrak{G}$, making the whole diagram correspond to a relation in $\mathfrak{G}$. $\qquad\square$

Lemma 4.4. *Let $f, h \in \operatorname{Aut}(\mathbb{F}_0) \cup \operatorname{Aut}(\mathbb{F}_2)$, $g \in \operatorname{Aut}(\mathbb{P}^2)$, $f, g, h \in J$, and let Δ be a de Jonquières linear system. Assume that*

$$\deg(fgh) = 3, \quad \deg(fgh(\Delta)) < \deg(gh(\Delta)), \quad \deg(\Delta) \leq \deg(gh(\Delta))$$

and that $(gh)(\Delta)$ has a proper base-point different from $[1:0:0]$. Then there exist $\alpha_1, \ldots, \alpha_7 \in \left(\operatorname{Aut}(\mathbb{P}^2) \cup \operatorname{Aut}(\mathbb{F}_0) \cup \operatorname{Aut}(\mathbb{F}_2)\right) \cap J$, $\alpha_1, \alpha_3, \alpha_5, \alpha_7 \in \operatorname{Aut}(\mathbb{P}^2)$, such that

(1) *the following equation holds in $\mathfrak{G}$:*

$$w(f)w(g)w(h) = w(\alpha_7) \cdots w(\alpha_1)$$

i.e. the following diagram corresponds to a relation in $\mathfrak{G}$:

(2) *For $i = 2, \ldots, 7$*

$$\deg(\alpha_i \cdots \alpha_1(\Delta)) < \deg((gh)(\Delta)).$$

Proof. The equality $\deg(fgh) = 3$ implies that f and $(gh)^{-1}$ have exactly one common base-point, namely $p_1 = [1:0:0]$. Denote $\operatorname{sBp}((gh)^{-1}) = \{p_2, p_3\}$, $\operatorname{sBp}(f) = \{p_4, p_5\}$ the simple base-points of $(gh)^{-1}$ and f and write $d = \deg(gh(\Delta))$.

By assumption, $gh(\Delta)$ is a de Jonquières linear system which has a proper base-point s different from p_1. For any point r, let $m(r)$ be the multiplicity of $gh(\Delta)$ in r respectively. Then $m(p_1) = d - 1$ and $m(s) = 1$ (Remark 3.5), and Remark 3.7 implies that because $\deg(\Delta) \leq d$ and $\deg(fgh(\Delta)) < d$, we have (up to ordering of p_2, p_3)

$$m(p_2) = 1, \ m(p_3) \leq 1$$
$$\deg(fgh(\Delta)) = d - 1, \quad m(p_4) = m(p_5) = 1$$

We will now construct $\alpha_1, \ldots, \alpha_7$.

Assume that $s \in \{p_2, p_3, p_4, p_5\}$. If $s \in \{p_2, p_3\}$, we choose $r \in \{p_4, p_5\}$. If $s \in \{p_4, p_5\}$, we choose $r \in \{p_i : i = 2, 3 \text{ and } m(p_i) = 1\}$. We choose r to be infinitely near p_1 or a proper point (this is always possible). The points p_1, s, r are not aligned, because $a_1 + m(s) + m(r) > d$, thus there exist $\rho \in \operatorname{Bir}(\mathbb{P}^2)$ quadratic de Jonquières with base-points p_1, r, s. The following commutative diagram, where the points in the brackets are the base-points of the corresponding map, summarises the situation:

Using Remark 3.7, we obtain

$$\deg(\rho gh) = \deg(f\rho^{-1}) = 2,$$
$$\deg(\rho gh(\Delta)) = d - 1 < \deg(gh(\Delta))$$

We write $\rho = \gamma\tilde{\rho}\delta$, $\rho gh = \alpha_3\alpha_2\alpha_1$, $f\rho^{-1} = \alpha_6\alpha_5\alpha_4$ where $\delta, \gamma, \alpha_1, \cdots, \alpha_6 \in \left(\mathrm{Aut}(\mathbb{P}^2) \cup \mathrm{Aut}(\mathbb{F}_0) \cup \mathrm{Aut}(\mathbb{F}_2)\right) \cap J$, only $\tilde{\rho}, \alpha_2, \alpha_5$ quadratic (Lemma 3.6). By Lemma 4.2, the above diagram is generated by relations in $\mathfrak{G}$. Hence $w(f)w(g)w(h) = w(\alpha_6)\cdots w(\alpha_1)$ in $\mathfrak{G}$.

Assume $s \notin \{p_2, p_3, p_4, p_5\}$, we choose $r_1 \in \{p_i : i = 2, 3 \text{ and } m(p_i) = 1\}$, $r_2 \in \{p_4, p_5\}$ such that r_1 (respectively r_2) is either a proper point or infinitely near p_1 (this is always possible). For $i = 1, 2$, the points p_1, s, r_i are not collinear, because $a_1 + m(s) + m(r_i) > d$. Thus there exist $\rho_1, \rho_2 \in \mathrm{Bir}(\mathbb{P}^2)$ quadratic de Jonquières with base-points p_1, s, r_1 and p_1, s, r_2 respectively. The following commutative diagram, where the brackets are the base-points of the corresponding map, summarises the situation:

Using Remark 3.7 we obtain

$$\deg(\rho_1 gh) = \deg(\rho_2\rho_1^{-1}) = \deg(f\rho_2^{-1}) = 2,$$
$$\deg(\rho_1 gh(\Delta)) = d - 1 < \deg(gh(\Delta))$$
$$\deg(\rho_2 gh(\Delta)) = d - 1 < \deg(gh(\Delta))$$

We write $\rho_1 = \gamma_1\tilde{\rho}_1\beta_1$, $\rho_2 = \gamma_2\tilde{\rho}_2\beta_2$, $\rho_1 gh = \alpha_3\alpha_2\alpha_1$, $\rho_2\rho_1^{-1} = \alpha_6\alpha_5\alpha_4$, $f\rho_2^{-1} = \alpha_9\alpha_8\alpha_7$ for $\alpha_1, \ldots, \alpha_9, \beta_1, \beta_2, \gamma_1, \gamma_2, \tilde{\rho}_1, \tilde{\rho}_2 \in (\mathrm{Aut}(\mathbb{P}^2) \cup \mathrm{Aut}(\mathbb{F}_0) \cup \mathrm{Aut}(\mathbb{F}_2)) \cap J$, only $\alpha_2, \alpha_5, \alpha_8, \tilde{\rho}_1, \tilde{\rho}_2$ quadratic (Lemma 3.6). Lemma 4.2 implies that all triangles of the above diagram are generated by relations in $\mathfrak{G}$ and thus $w(f)w(g)w(h) = w(\alpha_9)\cdots w(\alpha_1)$ in $\mathfrak{G}$. We obtain the α_i's in the claim by merging neighbour automorphisms of $\mathbb{P}^2$ in the product $\alpha_9\cdots\alpha_1$. $\qquad\square$

5. The Cremona group is isomorphic to $\mathfrak{G}$

In this section we prove Theorem B (Theorem 5.5). The main tool will be Lemma 5.1 which yields the existence of an injective map $w_J\colon J \to \mathfrak{G}$ such that $\pi \circ w_J = \mathrm{Id}$ (Corollary 5.2) and enables us to use the result (Theorem 5.3) of [Bla12], that $\mathrm{Bir}(\mathbb{P}^2)$ is isomorphic to the amalgamated product of $\mathrm{Aut}(\mathbb{P}^2)$ and J along their intersection modulo one relation, for the proof of Theorem B (Theorem 5.5).

Lemma 5.1. *Let* $f_1, \ldots, f_n \in \left(\mathrm{Aut}(\mathbb{P}^2) \cup \mathrm{Aut}(\mathbb{F}_0) \cup \mathrm{Aut}(\mathbb{F}_2)\right) \cap J$ *such that* $f_n\cdots f_1 = \mathrm{Id}$. *Then* $w(f_n)\cdots w(f_1) = \mathrm{Id}$ *in* $\mathfrak{G}$.

Proof. We can write $w(f_n)\cdots w(f_1) = w(\alpha_{m+1})w(g_m)w(\alpha_m)\cdots w(\alpha_2)w(g_1)w(\alpha_1)$ where $\alpha_i \in \mathrm{Aut}(\mathbb{P}^2) \cap J$ and $g_i \in (\mathrm{Aut}(\mathbb{F}_0) \cup \mathrm{Aut}(\mathbb{F}_2)) \cap J \setminus \mathrm{Aut}(\mathbb{P}^2)$ as follows: We put $g_j := f_i$ if f_i is quadratic, $\alpha_j := f_i$ if f_i is linear. Then we proceed by putting $\alpha_j := \alpha_{i+1}\alpha_i$ ($w(\alpha_j) = w(\alpha_{i+1}\alpha_i)$ by Lemma 4.2). Proceeding like this we will

reach a word where no two consecutive letters both have linear image in $\mathrm{Bir}(\mathbb{P}^2)$. We then insert $\alpha_j = \mathrm{Id}$ between any two consecutive letters whose both image in $\mathrm{Bir}(\mathbb{P}^2)$ is quadratic.

We denote by Δ_0 the linear system of lines in $\mathbb{P}^2$ and define for $i = 1, \ldots, m$

$$\Delta_i := (\alpha_i g_{i-1} \cdots g_1 \alpha_1)(\Delta_0)$$

which is the linear system of the map $(\alpha_i g_{i-1} \cdots g_1 \alpha_1)^{-1}$. We define $d_i := \deg(\Delta_i)$, which is also the degree of the map $(\alpha_i g_{i-1} \cdots g_1 \alpha_1)^{-1}$. Furthermore, we define

$$D := \max\{d_i \mid i = 1, \ldots, m\}, \qquad N := \max\{i \mid d_i = D\}$$

If $D = 1$, it follows that $m = 1$ and $\alpha_1 = \mathrm{Id}$. We can therefore assume that $D > 1$ and prove the result by induction over the lexicographically ordered pair (D, N).

The induction step consists of finding $\tilde{\alpha}_{k+1}, \ldots, \tilde{\alpha}_1 \in \mathrm{Aut}(\mathbb{P}^2) \cap J$ and $\tilde{g}_1, \ldots, \tilde{g}_k \in (\mathrm{Aut}(\mathbb{F}_0) \cup \mathrm{Aut}(\mathbb{F}_2)) \cap J \setminus \mathrm{Aut}(\mathbb{P}^2)$ such that

$$w(g_{N+1})w(\alpha_{N+1})w(g_N) = w(\tilde{\alpha}_{k+1})w(\tilde{g}_k) \cdots w(\tilde{g}_1)w(\tilde{\alpha}_1)$$

and such that the pair $(\tilde{D}, \tilde{N})$ associated to the product

$$\alpha_{m+1}g_m \cdots g_{N+2}(\alpha_{N+2}\tilde{\alpha}_{k+1})\tilde{g}_k \cdots \tilde{g}_1(\tilde{\alpha}_1 \alpha_N)g_{N-1} \cdots g_1 \alpha_1$$

is strictly smaller than (D, N).

We look at three cases, depending on the degree of $g_{N+1}\alpha_{N+1}g_N$, and if the degree is three, we look at two sub cases, the "good case" and the "bad case".

If $\deg(g_{N+1}\alpha_{N+1}g_N) = 1$, define $\mathrm{Aut}(\mathbb{P}^2) \ni \tilde{\alpha} := g_{N+1}\alpha_{N+1}g_N$. It follows from Lemma 4.2 (1) that $w(g_{N+1})w(\alpha_{N+1})w(g_N) = w(\tilde{\alpha})$ in $\mathfrak{G}$. We replace $g_{N+1}\alpha_{N+1}g_N$ by $\tilde{\alpha}$, which decreases (D, N).

If $\deg(g_{N+1}\alpha_{N+1}g_N) = 2$, it follows from Lemma 4.2 (2),(3) that there exists $\tilde{\alpha}, \tilde{\beta} \in \mathrm{Aut}(\mathbb{P}^2) \cap J$ and $\tilde{g} \in (\mathrm{Aut}(\mathbb{F}_0) \cup \mathrm{Aut}(\mathbb{F}_2)) \cap J \setminus \mathrm{Aut}(\mathbb{P}^2)$ such that $w(g_{N+1})w(\alpha_{N+1})w(g_N) = w(\tilde{\beta})w(\tilde{g})w(\tilde{\alpha})$. We replace $g_{N+1}\alpha_{N+1}g_N$ by $\tilde{\beta}\tilde{g}\tilde{\alpha}$, which decreases (D, N).

Finally, suppose that $\deg(g_{N+1}\alpha_{N+1}g_N) = 3$. By definition of N, we have

$$d_{N-1} \leq D, \quad d_N = D, \quad d_{N+1} < D.$$

"Good case": If Δ_N has a proper simple base-point, it follows from Lemma 4.4 (with $\Delta = \Delta_{N-1}$) that there exist $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_4 \in \mathrm{Aut}(\mathbb{P}^2) \cap J$, $\tilde{g}_1, \tilde{g}_2, \tilde{g}_3 \in (\mathrm{Aut}(\mathbb{F}_0) \cup \mathrm{Aut}(\mathbb{F}_2)) \cap J$ such that

$$w(g_{N+1})w(\alpha_{N+1})w(g_N) = w(\tilde{\alpha}_4)w(\tilde{g}_3) \cdots w(\tilde{\alpha}_2)w(\tilde{g}_1)w(\tilde{\alpha}_1) \text{ in } \mathfrak{G}$$

and

$$\deg((\tilde{\alpha}_{i+1}\tilde{g}_i \cdots g_1 \tilde{\alpha}_1)(\Delta_{N-1})) < \deg(\Delta_N) = D$$

for $i = 1, \ldots, 4$. Replacing $g_{N+1}\alpha_{N+1}g_N$ by $\tilde{\alpha}_4\tilde{g}_3 \cdots \tilde{g}_1\tilde{\alpha}_1$ decreases (D, N).

"Bad case": Assume that Δ_N has no simple proper base-points. Without changing the pair (D, N) we will replace the word $w(\alpha_{m+1})w(g_m) \cdots w(g_1)w(\alpha_1)$ in $\mathfrak{G}$ by an equivalent word $w(\hat{\alpha}_{m+1})w(\hat{g}_m) \cdots w(\hat{g})w(\hat{\alpha}_1)$ satisfying the "good case".

Choose two general points p_0, q_0 in $\mathbb{P}^2$ and write $p_1 = (\alpha_2 g_1 \alpha_1)(p_0)$, $q_1 = (\alpha_2 g_1 \alpha_1)(q_0)$ and $p_i = (\alpha_{i+1}g_i)(p_{i-1})$, $q_i = (\alpha_{i+1}g_i)(q_{i-1})$ for $i = 2, \ldots, m$. Note that $p_m = p_0$ and $q_m = q_0$ because $\alpha_{m+1}g_m \cdots g_1 \alpha_1 = \mathrm{Id}$.

For $i = 0, \ldots, m$, we denote by $\beta_i \in \mathrm{Aut}(\mathbb{P}^2)$ an element sending $[1 : 0 : 0]$, $[0 : 1 : 0]$, $[0 : 0 : 1]$ respectively onto $[1 : 0 : 0], p_i, q_i$ (this is possible, because we took p_0, q_0 general), and write $\tau_i := \beta_i \sigma_3(\beta_i)^{-1}$, which is a quadratic de Jonquières

involution having base-points $[1 : 0 : 0], p_i, q_i$. We choose $\beta_m = \beta_0$ and then have $\tau_m = \tau_0$.

By Lemma 4.3 the maps $\tau_1(\alpha_2 g_1 \alpha_1)\tau_0^{-1}$, $\tau_i(g_i \alpha_i)\tau_{i-1}^{-1}$ are quadratic de Jonquières and there exist $\gamma_i, \delta_i \in \mathrm{Aut}(\mathbb{P}^2) \cap J$, $\hat{g}_i \in \{\sigma_2, \sigma_3, \tau_{12}\sigma_2\tau_{12}\}$ such that

$$w(\beta_1)w(\sigma_3)w(\beta_1^{-1})w(\alpha_2)w(g_1)w(\alpha_1)w(\beta_0)w(\sigma_3)w(\beta_0^{-1}) = w(\delta_1)w(\hat{g}_1)w(\gamma_1)$$

$$w(\beta_i)w(\sigma_3)w(\beta_i^{-1})w(\alpha_{i+1})w(g_i)w(\beta_{i-1})w(\sigma_3)w(\beta_{i-1}^{-1}) = w(\delta_i)w(\hat{g}_i)w(\gamma_i)$$

for $i = 1, \ldots, m$. We get the following diagram

where each square in the diagram corresponds to a relation in $\mathfrak{G}$, making the whole diagram correspond to a relation in $\mathfrak{G}$. Therefore, writing $\tilde{\alpha}_i := \delta_i \gamma_{i-1}$ for $i = 2, \ldots, m$, $\tilde{\alpha}_{m+1} := \delta_m$, $\tilde{\alpha}_1 := \gamma_1$, the equality

$$w(\alpha_{m+1})w(g_m)\cdots w(g_1)w(\alpha_1) = w(\hat{\alpha}_{m+1})w(\hat{g}_m)w(\hat{\alpha}_m)\cdots w(\hat{\alpha}_2)w(\hat{g}_1)w(\hat{\alpha}_1)$$

holds in $\mathfrak{G}$. We replace $\alpha_{m+1}g_m \cdots g_1 \alpha_1$ by $\hat{\alpha}_{m+1}\hat{g}_m\hat{\alpha}_m \cdots \hat{\alpha}_2\hat{g}_1\hat{\alpha}_1$.

For $i = 1, \ldots, m$, call $\hat{\Delta}_i := (\hat{\alpha}_i\hat{g}_{i-1} \cdots \hat{g}_1\hat{\alpha}_1)(\Delta_0)$, which is the linear system of the map $(\hat{\alpha}_i\hat{g}_{i-1} \cdots \hat{g}_1\hat{\alpha}_1)^{-1}$, and denote by $\hat{d}_i$ its degree. Using Remark 3.7 we get $\deg(\hat{\alpha}_i\hat{g}_{i-1} \cdots \hat{g}_1\hat{\alpha}_1) = \deg(\alpha_i g_{i-1} \cdots g_1 \alpha_1)$ for each i, thus $\hat{d}_i = d_i$ for $i = 1, \ldots, m$. Therefore, the replacement does not change the pair (D, N), i.e. $(\hat{D}, \hat{N}) = (D, N)$.

It remains to show that $\hat{\alpha}_{n+1}\hat{g}_n \cdots \hat{g}_1\hat{\alpha}_1$ satisfies the "good case", i.e. that $\hat{\Delta}_N$ has a simple proper base-point.

Since $d_{N+1} < D$, it follows from Remark 3.7 that $d_{N+1} = D - 1$ and that all the base-points of g_{N+1} are base-points of Δ_N. Since the base-point of τ_N are general it follows from Remark 3.7 that each point in $(\tau_N)^\bullet(\mathrm{sBp}(g_N))$ is a base-point of $\hat{\Delta}_N$. Lemma 4.3 states that $(\tau_N)^\bullet(\mathrm{sBp}(g_N)) = \mathrm{sBp}(\hat{g}_N)$, and $\hat{g}_N \in \{\sigma_2, \sigma_3, \tau_{12}\sigma_2\tau_{12}\}$ has a simple proper base-point. Hence $\hat{\Delta}_N$ has a simple proper base-point. $\square$

Corollary 5.2. *Let* $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m \in \mathrm{Aut}(\mathbb{P}^2) \cup \mathrm{Aut}(\mathbb{F}_0) \cup \mathrm{Aut}(\mathbb{F}_2)$ *de Jonquières such that* $\alpha_n \cdots \alpha_1 = \beta_m \cdots \beta_1$. *Then*

$$w(\alpha_n) \cdots w(\alpha_1) = w(\beta_m) \cdots w(\beta_1).$$

In particular there exists a homomorphism $w_J : J \to \mathfrak{G}$ *which sends* $\alpha_n \cdots \alpha_1$ *onto* $w(\alpha_n) \cdots w(\alpha_1)$ *and* $\pi \circ w_J = \mathrm{Id}$, *i.e.* w_J *is injective.*

Proof. The claim follows from applying Lemma 5.1 to $\beta_1^{-1} \cdots \beta_m^{-1}\alpha_n \cdots \alpha_1$. $\square$

Proposition 5.3 ([Bla12])**.** *The group* $\mathrm{Bir}(\mathbb{P}^2)$ *is isomorphic to*

$$\left(\mathrm{Aut}(\mathbb{P}^2) *_{\mathrm{Aut}(\mathbb{P}^2) \cap J} J\right) / \langle \tau_{12}\sigma_3\tau_{12}\sigma_3 \rangle,$$

the amalgamated product of $\mathrm{Aut}(\mathbb{P}^2)$ *and* J *along their intersection and divided by the relation* $\tau_{12}\sigma_3 = \sigma_3\tau_{12}$, *where* $\tau_{12}([x : y : z]) = [y : x : z]$.

Remark 5.4. In $\mathrm{Bir}(\mathbb{P}^2)$, the three relations

(1) $\tau_{12}\sigma_3\tau_{12}\sigma_3 = \mathrm{Id}$
(2) $\tau_{13}\sigma_3\tau_{13}\sigma_3 = \mathrm{Id}$
(3) $\tau_{23}\sigma_3\tau_{23}\sigma_3 = \mathrm{Id}$

hold. Choosing two of them, the remaining relation of the three is generated by the chosen two. Relation 3 is a relation holding in J. Thus it suffices to impose relation 1 or 2 in Theorem 5.3. However, since $\tau_{12}, \sigma_3 \in \mathrm{Aut}(\mathbb{F}_0)$, relation 1 holds in $\mathrm{Aut}(\mathbb{F}_0)$, so in particular it holds in the generalised amalgamated product of $\mathrm{Aut}(\mathbb{P}^2), \mathrm{Aut}(\mathbb{F}_0), \mathrm{Aut}(\mathbb{F}_2)$ along all their pairwise intersections.

It is not clear at all whether J embeds into the generalised amalgamated product of $\mathrm{Aut}(\mathbb{P}^2), \mathrm{Aut}(\mathbb{F}_0), \mathrm{Aut}(\mathbb{F}_2)$ along all their pairwise intersections, so it is à priori not clear whether one of the relations 2, 3 holds in there. Thus we need to impose one of them.

Theorem 5.5 ((Theorem B)). *The group $\mathrm{Bir}(\mathbb{P}^2)$ is isomorphic to $\mathfrak{G}$, the generalised amalgamated product of $\mathrm{Aut}(\mathbb{P}^2)$, $\mathrm{Aut}(\mathbb{F}_0)$, $\mathrm{Aut}(\mathbb{F}_2)$ along all the pairwise intersections modulo the relation $\tau_{13}\sigma_3\tau_{13}\sigma_3$ where $\tau_{13}([x:y:z]) = [z:y:x]$.*

Proof. By Corollary 5.2 there exists $w_J : J \to \mathfrak{G}$ such that $\pi \circ w_J = \mathrm{Id}$, and w and w_J coincide on $\mathrm{Aut}(\mathbb{P}^2) \cap J$. Thus the following diagram commutes

$$
\begin{array}{ccc}
\mathfrak{G} & \xleftarrow{\quad w_J \quad} & J \\
{\scriptstyle w}\Big\uparrow & & \Big\uparrow{\scriptstyle \iota_2} \\
\mathrm{Aut}(\mathbb{P}^2) & \xleftarrow{\ \iota_1\ } \mathrm{Aut}(\mathbb{P}^2) \cap J
\end{array}
$$

where ι, ι_1 are the canonical inclusion maps. The universal property of the amalgamated product implies the existence of a unique homomorphism $\varphi : \mathrm{Aut}(\mathbb{P}^2) *_{\mathrm{Aut}(\mathbb{P}^2) \cap J} J \to \mathfrak{G}$ such that the following diagram commutes:

By Proposition 5.3, $\mathrm{Bir}(\mathbb{P}^2)$ is isomorphic to $\mathrm{Aut}(\mathbb{P}^2) *_{\mathrm{Aut}(\mathbb{P}^2) \cap J} J$ modulo the relation $\sigma_3\tau_{12} = \tau_{12}\sigma_3$, where $\tau_{12}([x:y:z]) = [y:x:z]$. Since $\tau_{12}, \sigma_3 \in \mathrm{Aut}(\mathbb{F}_0)$, the relation $\sigma_3\tau_{12} = \tau_{12}\sigma_3$ also holds in $\mathrm{Aut}(\mathbb{F}_0)$ and hence in $\mathfrak{G}$. Thus, the homomorphism φ induces a homomorphism $\bar{\varphi} : (\mathrm{Aut}(\mathbb{P}^2) *_{\mathrm{Aut}(\mathbb{P}^2) \cap J} J)/\langle \sigma_3\tau_{12}\sigma_3\tau_{12}\rangle \longrightarrow \mathfrak{G}$. By construction, $\bar{\varphi}$ and the canonical homomorphism $\pi : \mathfrak{G} \to \mathrm{Bir}(\mathbb{P}^2)$ are inverse to each other. $\square$

6. The Cremona group is compactly presented

In this section, we restrict to case $k = \mathbb{C}$ and show that $\mathrm{Bir}(\mathbb{P}^2)$ is compactly presented using Theorem 5.5 (Theorem B).

Being compactly presented is a notion reserved for Hausdorff topological groups and we consider $\mathrm{Bir}(\mathbb{P}^2)$ endowed with the Euclidean topology as constructed in [BlaFur13, Section 5] which makes $\mathrm{Bir}(\mathbb{P}^2)$ a Hausdorff topological group [BlaFur13, Theorem 3], which is not locally compact [BlaFur13, Lemma 5.15].

Definition 6.1. Let G be a group.

(1) A *presentation* $\langle S \mid R \rangle$ of G is a triple made up of a set S, an epimorphism $\pi : F_S \twoheadrightarrow G$ of the free group on S onto G, a subset R of F_S generating $\ker(\pi)$ as a normal subgroup. The *relations* of the presentation are the elements of $\ker(\pi)$ and the elements of R the *relators* (or *generating relations*) of the presentation.

(2) A *bounded presentation* of G is a presentation $\langle S \mid R \rangle$ of G with R a set of relators of bounded length.

(3) Let G be a Hausdorff topological group. A *compact presentation* of G is a presentation $\langle S \mid R \rangle$ of G with S a compact subset of G and R a set of relators of bounded length. We say that G is *compactly presented by S* if G is given by a compact presentation $\langle S \mid R \rangle$. We also say that G is *compactly presented* if G is compactly presented by some subset.

Lemma 6.2.   (1) *Let G be a group and $S_1, S_2 \subset G$ generating subsets. If $S_1^m \subset S_2^n \subset S_1^{m'}$ for some $m, n, m' \in \mathbb{N}$ then G is boundedly presented by S_1 if and only if it is boundedly presented by S_2.*

(2) *Any connected topological group is generated by any neighbourhood of 1.*

(3) *If G is a locally compact Hausdorff topological group having only finitely many connected components it is compactly presented.*

(4) *If G is a locally compact Hausdorff topological group that is compactly presented then it is compactly presented by all its compact generating subsets.*

(5) *Let G be a locally compact topological group with finitely many connected components $G_0, \ldots, G_n$, where $1 \in G_0$. For each i choose some $g_i \in G$ such that $G_i = g_i G_0$. Then G is generated by any compact neighbourhood of 1 and $g_1, \ldots, g_n$. In particular, it is compactly presented by any compact neighbourhood of 1 and $g_1, \ldots, g_n$.*

Proof. 1 is proved in [CorHar15, Lemma 7.A.9] and [Cor10, Lemma 2.6] and 3 in [Abe72, Satz 3.2] (see also [CorHar15, §8.A]).

2: Let $U \subset G$ be an open neighbourhood of 1. Then the subgroup H of G generated by U is open because $H = \bigcup_{h \in H} hU$. It is also closed because $G \setminus H = \bigcup_{g \in G \setminus H} gH$, which is a open set.

4: If G is compactly generated by a compact set S and $K \subset G$ is a compact set then $K \subset S^n$ for some large n. This follows from the fact that any locally compact topological group is a Baire space and that S is compact. The claim now follows from 1.

5: Let $K \subset G_0$ be a compact neighbourhood of 1. By 2, K generates G_0 and thus the compact set $K \cup \{g_1, \ldots, g_n\}$ generates G. By 3 and 4 the locally compact group G is compactly presented by $K \cup \{g_1, \ldots, g_n\}$. $\qquad\qquad\square$

Remark 6.3. Any irreducible algebraic variety over $\mathbb{C}$ is connected with respect to the Euclidean topology [SGA1, Chapter XII, Proposition 2.4]. Any linear algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ has finitely many irreducible components in the Zariski-topology, which are exactly the cosets of the component containting 1. Thus they are the connected components in the Zariski topology and hence also the connected components in the Euclidean topology.

Furthermore, any linear algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ is a closed subset of $\mathrm{Bir}(\mathbb{P}^2)_{\leq d}$ for some $d \in \mathbb{N}$ [BlaFur13, Lemma 2.19], which is a locally compact set [BlaFur13, Lemma 5.4]. Hence any linear algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ is locally compact and therefore satisfies the conditions of Lemma 6.2 5.

Remark 6.4. Any algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ is a linear algebraic group [Bla09, Théorème 2]. The Euclidean topology on these groups is exactly the topology inherited from the Euclidean topology on $\mathrm{Bir}(\mathbb{P}^2)$ [BlaFur13, Proposition 5.11].

The groups $\mathrm{Aut}(\mathbb{P}^2) = \mathrm{PGL}_3(\mathbb{C})$, $\mathrm{Aut}(\mathbb{F}_0)$ and $\mathrm{Aut}(\mathbb{F}_2)$ are linear algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^2)$ (Lemma 2.4), and thus locally compact by Remark 6.3.

Corollary 6.5.   (1) *The group $\mathrm{Aut}(\mathbb{P}^2)$ is compactly presented by any compact neighbourhood of 1.*
   (2) *The group $\mathrm{Aut}(\mathbb{F}_0)$ is compactly presented by the union of the linear map $\tau_{12}\colon [x:y:z] \to [y:x:z]$ and any compact neighbourhood of 1.*
   (3) *The group $\mathrm{Aut}(\mathbb{F}_2)$ is compactly presented by any compact neighbourhood of 1.*

Proof. The groups $\mathrm{Aut}(\mathbb{P}^2), \mathrm{Aut}(\mathbb{F}_0), \mathrm{Aut}(\mathbb{F}_2)$ are linear algebraic groups and locally compact by Remark 6.4.

The group $\mathrm{Aut}(\mathbb{P}^2) = \mathrm{PGL}_3(k)$ is irreducible, hence connected (Remark 6.3), and the group $\mathrm{Aut}(\mathbb{F}_2)$ is connected by Lemma 2.4 3. By Lemma 2.4 2, the group $\mathrm{Aut}(\mathbb{F}_0)$ has two connected components, namely $\mathrm{Aut}(\mathbb{F}_0)^0$ containing the identity element and $\tau_{12} \mathrm{Aut}(\mathbb{F}_0)^0$. The claim now follows from Remark 6.3 and Proposition 6.2 5.   □

Using Corollary 6.5 and the fact that $\mathrm{Bir}(\mathbb{P}^2)$ is isomorphic to the generalised amalgamated product of $\mathrm{Aut}(\mathbb{P}^2)$, $\mathrm{Aut}(\mathbb{F}_0)$, $\mathrm{Aut}(\mathbb{F}_2)$ along their pairwise intersection divided by one relation (Theorem 5.5) we prove that $\mathrm{Bir}(\mathbb{P}^2)$ is compactly presentable.

Lemma 6.6. *Let G be a group, $n \geqslant 2$ be an integer, and $G_1, \ldots, G_n \subset G$, be subgroups of G such that the following hold:*

   (1) *The group G admits the presentation $G = \left\langle \bigcup\limits_{i=1}^{n} G_i \mid R_G \right\rangle$, where R_G is the set of all relators of the form $ab = c$, where $a, b, c \in G_i$ for some $i \in \{1, \ldots, n\}$.*
   (2) *For $i = 1, \ldots, n$, there exists a presentation $\langle K_i \mid R_i \rangle$ of G_i such that for any subset $I \subset \{1, \ldots, n\}$ the set $\bigcap\limits_{i \in I} K_i$ generates $\bigcap\limits_{i \in I} G_i$.*

Then, G admits the presentation $G = \left\langle \bigcup\limits_{i=1}^{n} K_i \mid \bigcup\limits_{i=1}^{n} R_i \right\rangle$.

Proof. Denote by F_G the free group generated by $\bigcup\limits_{i=1}^{n} G_i$ and by F_K the free group generated by $K = \bigcup\limits_{i=1}^{n} K_i$; we view F_K as a subgroup of F_G.

The natural group homomorphism $\pi\colon F_K \to G$ is surjective, because G is generated by $\bigcup\limits_{i=1}^{n} G_i$ and each G_i is generated by K_i. Moreover, each set of relators R_i corresponds to a subset of $\ker(\pi)$. It remains to see that $\ker \pi$ is contained in the normal subgroup generated by $\bigcup\limits_{i=1}^{n} R_i$.

We take an element in $\ker(\pi)$, which in F_K is a word

$$w = s_1 s_2 \ldots s_m$$

such that each s_i belongs to K and $s_1 \cdots s_m = 1$ in G. Because G admits the presentation $G = \left\langle \bigcup_{i=1}^{n} G_i \mid R_G \right\rangle$, we can write w in F_G as a product

$$w = a_1 r_1 a_1^{-1} a_2 r_2 a_2^{-1} \ldots a_l r_l a_l^{-1}$$

where all the a_i, r_i are elements of F_G and $r_i \in R_G$, which means by definition of R_G that $r_i = a_i b_i c_i$ for $a_i, b_i, c_i \in G_{j(i)}$, i.e. each r_i is a word in elements of $G_{j(i)}$.

The word w is equal in F_G to a reduced word, whose letters are elements of K because $w \in F_K$. Hence, each $g = \bigcup_{i=1}^{n} G_i \setminus K$ which appears in the word $a_1 r_1 a_1^{-1} \ldots a_l r_l a_l^{-1}$ disappears after the reduction. We can thus replace each occurrence of g with any chosen element of F_G and do not change the value of the word. We do this in the following way: if $g \in G_i \setminus K_i$, we replace then g with a word with letters in K_i, which belongs to $\pi^{-1}(\pi(g))$ (this is possible since K_i generates G_i). If g belongs to more than one of the G_i we can moreover assume that the letters of the word also belong to these K_i, because of the second hypothesis.

After this replacement, we obtain an equality in F_K

$$s_1 \ldots s_m = b_1 t_1 b_1^{-1} b_2 t_2 b_2^{-1} \ldots b_l t_l b_l^{-1},$$

where each t_i is a word with letters in $K_{j(i)}$, such that $\pi(t_i) = 1$. For each $i = 1, \ldots, n$ denote by F_{K_i} the free group generated by K_i and by $\pi_i : F_{K_i} \to G_i$ the natural group homomorphism onto G_i whose kernel is generated by R_i. We consider F_{K_i} as subgroup of F_K, which means that $\pi_i = \pi|_{F_{K_i}}$ and hence $\ker(\pi_i) = \ker(\pi) \cap F_{K_i}$. Therefore $\pi_i(t_i) = 1$ and thus t_i is a product of conjugates of $R_{j(i)}$. This yields the result. $\square$

Corollary 6.7. *Let $K \subset \mathrm{Aut}(\mathbb{P}^2)$, $K_0 \subset \mathrm{Aut}(\mathbb{F}_0)$, $K_2 \subset \mathrm{Aut}(\mathbb{F}_2)$ be compact neighbourhoods of 1 in the respective groups. Then $\mathrm{Bir}(\mathbb{P}^2)$ is compactly presented by $K \cup K_0 \cup K_2 \cup \{\tau_{12}\}$.*

Proof. Lemma 6.6 yields that the union of any compact generating sets of $\mathrm{Aut}(\mathbb{P}^2)$, $\mathrm{Aut}(\mathbb{F}_0)$, $\mathrm{Aut}(\mathbb{F}_2)$ giving a compact presentation of the respective groups yields a compact presentation of $\mathfrak{G}$, the generalised amalgamated product of $\mathrm{Aut}(\mathbb{P}^2)$, $\mathrm{Aut}(\mathbb{F}_0)$, $\mathrm{Aut}(\mathbb{F}_2)$ along their pairwise intersection divided by one relation. Such compact generating sets are given by Corollary 6.5: Any compact neighbourhood of 1 of the groups $\mathrm{Aut}(\mathbb{P}^2)$ and $\mathrm{Aut}(\mathbb{F}_2)$ respectively, and the union of τ_{12} and any compact neighbourhood of 1 in $\mathrm{Aut}(\mathbb{F}_0)$. Since $\mathrm{Bir}(\mathbb{P}^2)$ and $\mathfrak{G}$ are isomorphic (Theorem 5.5), the claim follows. $\square$

Corollary 6.8. *Let $K \subset \mathrm{Aut}(\mathbb{P}^2)$, $K_0 \subset \mathrm{Aut}(\mathbb{F}_0)$, $K_2 \subset \mathrm{Aut}(\mathbb{F}_2)$ be compact neighbourhoods of 1 in the respective groups. Then $\mathrm{Bir}(\mathbb{P}^2)$ is compactly presented by $K \cup K_0 \cup K_2$.*

Proof. We define $S_1 := K \cup K_0 \cup K_2$ and $S_2 := K \cup K_0 \cup K_2 \cup \{\tau_{12}\}$. The set S_2 generates $\mathrm{Bir}(\mathbb{P}^2)$ by Corollary 6.7. The set K generates $\mathrm{Aut}(\mathbb{P}^2)$ (Corollary 6.5), hence there exists $n \in \mathbb{N}$ such that $\tau_{12} \in K^n$. It follows that also S_2 generates $\mathrm{Bir}(\mathbb{P}^2)$ and moreover that $S_1 \subset S_2 \subset (S_1)^n$. The claim now follows from Lemma 6.2 1 and Corollary 6.7. $\square$

Lemma 6.2 1 and Corollary 6.8 imply that to prove Theorem A (Corollary 6.10) we only need to check that for any compact neighbourhood $K \subset \mathrm{Aut}(\mathbb{P}^2)$ of 1

there exist $K_i \subset \mathrm{Aut}(\mathbb{F}_i)$, $i = 0, 2$, compact neighbourhoods of 1 and integers $m, m', n \in \mathbb{N}$ such that $(K \cup \{\sigma_3\})^m \subset (K \cup K_0 \cup K_2)^n \subset (K \cup \{\sigma_3\})^{m'}$.

Lemma 6.9. *Let $K \subset \mathrm{Aut}(\mathbb{P}^2)$ be a compact neighbourhood of 1. Then there exists $N \in \mathbb{N}$ such that $(K \cup \{\sigma_3\})^N$ contains compact neighbourhoods of 1 in $\mathrm{Aut}(\mathbb{F}_i)$, for $i = 0, 2$.*

Proof. Let $\mathcal{A}_0 = \mathrm{Aut}(\mathbb{F}_0)^0 \cap \mathrm{Aut}(\mathbb{P}^2)$ and $\mathcal{A}_2 = \mathrm{Aut}(\mathbb{F}_2) \cap \mathrm{Aut}(\mathbb{P}^2)$, which are connected algebraic subgroups of $\mathrm{Aut}(\mathbb{F}_0)$, $\mathrm{Aut}(\mathbb{F}_2)$ respectively (Lemma 2.4). For $i = 0, 2$, the set $K_i = K \cap \mathcal{A}_i$ is a compact neighbourhood of 1 in $\mathcal{A}_i$. Corollary 6.5 implies that $\mathcal{A}_i = \bigcup_{n \in \mathbb{N}} (K_i)^n$, for $i = 0, 2$. It follows that

$$\mathcal{A}_0 \sigma_3 \mathcal{A}_0 = \bigcup_{n \in \mathbb{N}} (K_0)^n \sigma_3 (K_0)^n, \qquad \mathcal{A}_2 \sigma_2 \mathcal{A}_2 = \bigcup_{n \in \mathbb{N}} (K_2)^n \sigma_2 (K_2)^n.$$

The sets $\mathcal{A}_0 \sigma_3 \mathcal{A}_0$ and $\mathcal{A}_2 \sigma_2 \mathcal{A}_2$ are Zariski-open subsets of $\mathrm{Aut}(\mathbb{F}_0)$, $\mathrm{Aut}(\mathbb{F}_2)$ respectively (Lemma 2.4), and are thus locally compact and hence Baire spaces. There exists then some $m \in \mathbb{N}$ such that that $(K_0)^m \sigma_3 (K_0)^m$ and $(K_2)^m \sigma_2 (K_2)^m$ have non-empty interior in $\mathcal{A}_0 \sigma_3 \mathcal{A}_0$ and $\mathcal{A}_2 \sigma_2 \mathcal{A}_2$, and thus in $\mathrm{Aut}(\mathbb{F}_0)$ and $\mathrm{Aut}(\mathbb{F}_2)$ respectively.

Since $(K_i)^m \sigma_j (K_i)^m \subset (K_i \cup \{\sigma_j\})^{2m+1}$, the sets $(K_0 \cup \{\sigma_3\})^{2m+1}$ and $(K_2 \cup \{\sigma_2\})^{2m+1}$ also have non-empty interior in $\mathrm{Aut}(\mathbb{F}_0)$, $\mathrm{Aut}(\mathbb{F}_2)$ respectively. Since $(K_i)^{-1} \subset K_i^{m_i}$ for some big m_i and $(\sigma_j)^{-1} = \sigma_j$, the sets $(K_0 \cup \{\sigma_3\})^{m'}$ and $(K_2 \cup \{\sigma_2\})^{m'}$ are neighbourhoods of 1 in the corresponding groups for some m' big enough. Since $\mathrm{Bir}(\mathbb{P}^2)$ is generated by $K \cup \{\sigma_3\}$ (by the Noether-Castelnuovo theorem), we find m'' such that $\sigma_2 \in (K \cup \{\sigma_3\})^{m''}$. A suitable power of $K \cup \{\sigma_3\}$ contains thus $(K_0 \cup \{\sigma_3\})^{m'}$ and $(K_2 \cup \{\sigma_2\})^{m'}$. $\qquad \square$

Corollary 6.10 ((Theorem A)). *Let $K \subset \mathrm{Aut}(\mathbb{P}^2)$ be a compact neighbourhood of 1. Then $\mathrm{Bir}(\mathbb{P}^2)$ is compactly presented by $K \cup \{\sigma_3\}$.*

Proof. According to Lemma 6.9, there exists $N \in \mathbb{N}$ and compact neighbourhoods K_0, K_2 of 1 in $\mathrm{Aut}(\mathbb{F}_0)$ and $\mathrm{Aut}(\mathbb{F}_2)$ respectively, such that $(K \cup \{\sigma_3\})^N$ contains $K_0 \cup K_2$.

We define $S_1 := K \cup \{\sigma_3\}$ and $S_2 := K \cup K_0 \cup K_2$. Because $\sigma_3 \in \mathrm{Aut}(\mathbb{F}_0)^0$ and $\mathrm{Aut}(\mathbb{F}_0)^0$ is compactly generated by K_0 (Lemma 6.2 2) there exists $M \in \mathbb{N}$ such that $\sigma_3 \in (K_0)^M$. It follows that $S_1 \subset (S_2)^M$. Since $S_2 \subset (S_1)^N$, we have $S_1 \subset (S_2)^M \subset (S_1)^{MN}$. The group $\mathrm{Bir}(\mathbb{P}^2)$ being compactly presented by S_2 (Corollary 6.8), it is also compactly presented by S_1 (Lemma 6.2 1). $\qquad \square$

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