THE REAL PLANE CREMONA GROUP IS AN AMALGAMATED PRODUCT

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Abstract. We show that the real Cremona group of the plane is a non-trivial amalgam of two groups amalgamated along their intersection and give an alternative proof of its abelianisation.

1. Intoduction

The plane Cremona group is the group $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2)$ of birational transformations of $\mathbb{P}^2$ defined over a field k. For algebraically closed fields k, the Noether-Castelnuovo theorem [5] shows that $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2)$ is generated by $\mathrm{Aut}_{\mathrm{k}}(\mathbb{P}^2)$ and the subgroup preserving the pencil of lines through $[1:0:0]$. It implies that the normal subgroup generated by $\mathrm{Aut}_{\mathrm{k}}(\mathbb{P}^2)$ is equal to $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2)$. Furthermore, [4, Appendix by Cornulier] shows that $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2)$ is not isomorphic to a non-trivial amalgam of two groups. However, it is isomorphic to a non-trivial amalgam modulo one simple relation [1, 10, 7], and it is isomorphic to a generalised amalgamated product of three groups, amalgamated along all pairwise intersections [16]. For $\mathrm{k}=\mathbb{R}$, the group $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ and the two subgroups

$$\mathcal{J}_* = \{ f\in \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2) \mid f \text{ preserves the pencil of lines through } [1:0:0]\ \}$$

$$\mathcal{J}_\circ = \{ f\in \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2) \mid f \text{ preserves the pencil of conics through } p_1,\bar{p}_1,p_2,\bar{p}_2\ \}$$

where $p_1,p_2\in\mathbb{P}^2$ are two fixed non-real points such that $p_1,\bar{p}_1,p_2,\bar{p}_2$ are not collinear [3, Theorem 1.1]. Over $\mathbb{C}$, the analogon of $\mathcal{J}_\circ$ is conjguate to $\mathcal{J}_*$ since a pencil of conics through four points in $\mathbb{P}^2$ in general position can be sent onto a pencil of lines through point.

We define $\mathcal{G}_\circ\subset\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ to be the subgroup generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ and $\mathcal{J}_\circ$, and by $\mathcal{G}_*\subset\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ the subgroup generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ and $\mathcal{J}_*$. Then $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is generated by $\mathcal{G}_*\cup\mathcal{G}_\circ$, and the intersection $\mathcal{G}_*\cap\mathcal{G}_\circ$ contains the subgroup $\mathcal{H}$ generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ and the involution $[x:y:z]\dashrightarrow[xz:yz:x^2+y^2]$, which is contained $\mathcal{J}_\circ\cap\mathcal{J}_*$.

Theorem 1.1. *We have $\mathcal{G}_*\cap\mathcal{G}_\circ=\mathcal{H}$, it is a proper subgroup of $\mathcal{G}_\circ$ and $\mathcal{G}_*$ and*

$$\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)\simeq\mathcal{G}_\circ \mathop{\ast}_{\mathcal{H}} \mathcal{G}_*.$$

Moreover, both $\mathcal{G}_*$ and $\mathcal{G}_\circ$ have uncountable index in $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$.

The action of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ on the Bass-Serre tree associated to the amalgamated product yields the following:

2010 *Mathematics Subject Classification.* 14E07; 20F05; 14P99.

During this work, the author was supported by the Swiss National Science Foundation, by Projet PEPS 2018 JC/JC and by ANR Project FIBALGA ANR-18-CE40-0003-01.

Corollary 1.2. *The group* $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ *acts on a tree, and all its algebraic subgroups are conjugate to a subgroup of* $\mathcal{G}_*$ *or of* $\mathcal{G}_\circ$.

For the finite subgroups of odd order Corollary 1.2 can also be verified by checking their classification in [17].

An earlier version of this article used an explicit presentation of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ and of $\mathcal{G}_\circ$ in terms of generators and generating relations, the first of which is proven in [18, Theorem 4.4], and the second was proven analogously in the earlier version of this article. The present version does not use either presentation. Instead, we look at the groupoid $\mathrm{Sar}_{\mathbb{R}}(\mathbb{P}^2)$ of birational maps between rational real Mori fibre spaces of dimension 2. It contains $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ as subgroupoid, is generated by Sarkisov links and and the elementary relations are a set of generating relations [11, Theorem 3.1], see also Theorem 2.5. This information is encoded in a square complex on which $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ acts [11, §2–3], see also § 2. The groups $\mathcal{G}_\circ$, $\mathcal{G}_*$ and $\mathcal{H}$ will turn out to be stabilisers of the union of elementary discs containing vertices marked with elements of $\mathcal{G}_\circ$, $\mathcal{G}_*$ and $\mathcal{G}_\circ \cap \mathcal{G}_*$.

This allows us to provide a new proof of the abelianisation of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ given in [18]. Using the presentation of $\mathrm{Sar}_{\mathbb{R}}(\mathbb{P}^2)$ we can construct a surjective homomorphism of groupoids $\mathrm{Sar}_{\mathbb{R}}(\mathbb{P}^2) \longrightarrow \bigoplus_{(0,1]} \mathbb{Z}/2\mathbb{Z}$, whose restriction onto $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is surjective as well and coincides with its abelianisation by construction. This will give an alternative proof of the following statement.

Theorem 1.3. [18, Theorem 1.1(1)&(3)] *There is a surjective homomorphism of groups*

$$\Phi\colon\ \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2) \longrightarrow \bigoplus_{(0,1]} \mathbb{Z}/2\mathbb{Z}$$

such that its restriction to $\mathcal{J}_\circ$ *is surjective and* $\mathcal{G}_* \subset \ker(\Phi) = [\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2), \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)]$, *where the right hand side is also equal to the normal subgroup generated by* $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$.

Acknowledgement: I would like thank Anne Lonjou for asking me whether the real plane Cremona group is isomorphic to a generalised amalgamated product of several groups, and the interesting discussions that followed. I would also like to thank Stéphane Lamy for discussions on the square complex, and Jérémy Blanc, Yves de Cornulier for helpful remarks, questions and discussions.

2. A square complex associated to the Cremona group

In this section we recall the square complex constructed in [11], whose vertices are marked rational surfaces, its edges of length two are Sarkisov links between them and discs relations between Sarkisov links.

By a surface S, we mean a smooth projective surface defined over $\mathbb{R}$, and by $S_{\mathbb{C}}$ the same surface but defined over $\mathbb{C}$. We define the Néron-Severi space $N^1(S_{\mathbb{C}})$ as the space of $\mathbb{R}$-divisors $N^1(S_{\mathbb{C}}) := \mathrm{Div}(S_{\mathbb{C}}) \otimes \mathbb{R}/\equiv$. The action of the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on $N^1(S_{\mathbb{C}})$ factors through a finite group, and we denote by $N^1(S)$ the subspace of the $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-invariant classes. Since we only consider $\mathbb{R}$-rational surfaces and $\mathbb{C}[S_{\mathbb{C}}]^* = \mathbb{C}^*$, $N^1(S)$ is also the space of classes of divisors defined over $\mathbb{R}$ (see for instance [14, Lemma 6.3(iii)]). The dimension of $N^1(S_{\mathbb{C}})$ is the Picard rank of S and is denoted by $\rho(S)$. If not stated otherwise, all morphisms are defined over $\mathbb{R}$.

2.1. **Rank r fibrations.**

Definition 2.1. Let S be a smooth rational surface, and $r \geq 1$ an integer. We say that S is a *rank r fibration* if there exists a morphism $\pi\colon S \to B$, where B is a point or $B = \mathbb{P}^1$, with relative Picard number $\rho(S/B) = r$, and such that the anticanonical divisor $-K_S$ is π-ample.

The last condition means that for any curve C contracted to a point by π, we have $K_S \cdot C < 0$. Observe that the condition on the Picard number is that $\rho(S) = r$ if B is a point, and $\rho(S) = r + 1$ if $B = \mathbb{P}^1$. If S is a rank r fibration, we will write S/B if we want to emphasize the basis of the fibration, and S^r when we want to emphasize the rank.

An isomorphism between two fibrations S/B and S'/B' (necessarily of the same rank r) is an isomorphism $S \overset{\simeq}{\to} S'$ such that there exists an isomorphism on the bases (necessarily uniquely defined) that makes the following diagram commute:

$$\begin{tikzcd} S \arrow[r, "\simeq"] \arrow[d, "\pi"'] & S' \arrow[d, "\pi'"] \\ B \arrow[r, "\simeq"] & B' \end{tikzcd}$$

Observe that the definition of a rank r fibration puts together several notions. If B is a point, then S is just a del Pezzo surface of Picard rank r (over the base field $\mathbb{R}$). If B is a curve, then S is just a conic bundle of relative Picard rank r: a general fiber is isomorphic to a smooth plane conic over $\mathbb{R}$, and over $\mathbb{C}$ any singular fiber is the union of two (-1)-curves secant at one point. Remark also that being a rank 1 fibrations is equivalent to being a (smooth) rational Mori fibre space of dimension 2.

As the following examples make it clear, there are sometimes several choices for a structure of rank r fibration on a given surface, that may even correspond to distinct ranks.

Example 2.2. (1) $\mathbb{P}^2$ with the morphism $\mathbb{P}^2 \to \mathrm{pt}$, or the Hirzebruch surface $\mathbb{F}_n$ with the morphism $\mathbb{F}_n \to \mathbb{P}^1$, are rank 1 fibrations.

(2) $\mathbb{F}_1$ with the morphism $\mathbb{F}_1 \to \mathrm{pt}$ is a rank 2 fibration. Idem for $\mathbb{F}_0 \to \mathrm{pt}$. The blow-up $S^2 \to \mathbb{F}_n \to \mathbb{P}^1$ of a Hirzebruch surface in a real point or a pair of non-real conjugate point such that each of them is in a distinct fiber, is a rank 2 fibration over $\mathbb{P}^1$.

(3) The blow-up of two distinct real points on $\mathbb{P}^2$, or of two points of $\mathbb{F}_n$ not lying on the same fiber, gives examples of rank 3 fibrations, with morphisms to the point or to $\mathbb{P}^1$ respectively.

(4) The quadric surface $\mathcal{Q}$ obtained by blowing up $\mathbb{P}^2$ in a pair of non-real conjugate points and then contracting the strict transform of the line passing through them is a del Pezzo surface of degree 8 of Picard rank $\rho(\mathcal{Q}) = 1$, and so $\mathcal{Q} \to \mathrm{pt}$ is a rank 1 fibration.

(5) The blow-up $\mathcal{CB}_6$ of a pair of non-real conjugate points on $\mathcal{Q}$ is a del Pezzo surface of degree 6 of Picard rank $\rho(\mathcal{CB}_6) = 2$. It has a conic bundle structure $\mathcal{CB}_6 \longrightarrow \mathbb{P}^1$, and so $\mathcal{CB}_6/\mathrm{pt}$ is a rank 2 fibration while $\mathcal{CB}_6/\mathbb{P}^1$ is a rank 1 fibration.

(6) The blow-up of a real point in $\mathcal{CB}_6$ yields a rank 2 fibration $\mathcal{CB}_5/\mathbb{P}^1$. Note that $\mathcal{CB}_5$ is the blow-up of $\mathbb{P}^2$ in two pairs of non-real conjugate points and the fibres of $\mathcal{CB}_5/\mathbb{P}^1$ are the strict transforms of conics passing through the two pairs.

We call *marking* on a rank r fibration S/B a choice of a birational map $\varphi\colon S \dashrightarrow \mathbb{P}^2$. We say that two marked fibrations $\varphi\colon S/B \dashrightarrow \mathbb{P}^2$ and $\varphi'\colon S'/B' \dashrightarrow \mathbb{P}^2$ are *equivalent* if $\varphi'^{-1} \circ \varphi\colon S/B \to S'/B'$ is an isomorphism of fibrations. We denote by $(S/B, \varphi)$ an equivalence class under this relation.

If S'/B' and S/B are marked fibrations of respective rank r' and r, we say that S'/B' *factorizes* through S/B if the birational map $S' \to S$ induced by the markings is a morphism, and moreover there exists a (uniquely defined) morphism $B \to B'$ such that the following diagram commutes:

$$\begin{tikzcd} S' \arrow[rr, "\pi'"] \arrow[dr] & & B' \\ & S \arrow[r, "\pi"] & B \arrow[u] \end{tikzcd}$$

In fact if $B' = \mathrm{pt}$ the last condition is empty, and if $B' \simeq \mathbb{P}^1$ it means that $S' \to S$ is a morphism of fibration over a common basis $\mathbb{P}^1$. Note that $r' \geq r$.

2.2. **Square complex.** We define a 2-dimensional complex $\mathcal{X}$ as follows. Vertices are equivalence classes of marked rank r fibrations, with $3 \geq r \geq 1$. We put an oriented edge from $(S'/B', \varphi')$ to $(S/B, \varphi)$ if S'/B' factorizes through S/B. If $r' > r$ are the respective ranks of S'/B' and S/B, we say that the edge has type r', r. For each triplets of pairwise linked vertices $(S''^3/B'', \varphi'')$, $(S'^2/B', \varphi')$, $(S^1/B, \varphi)$, we glue a triangle. In this way we obtain a 2-dimensional simplicial complex $\mathcal{X}$.

Lemma 2.3. [11, Lemma 2.3] *For each edge of type* $3, 1$ *from* S''/B'' *to* S/B*, there exist exactly two triangles that admit this edge as a side.*

In view of the lemma, by gluing all the pairs of triangles along edges of type 3,1, and keeping only edges of types 3,2 and 2,1, we obtain a square complex that we still denote $\mathcal{X}$. When drawing subcomplexes of $\mathcal{X}$ we will often drop part of the information which is clear by context, about the markings, the equivalence classes and/or the fibration. For instance S/B must be understood as $(S/B, \varphi)$ for an implicit marking φ, and $(\mathbb{P}^2, \varphi)$ as $(\mathbb{P}^2/\mathrm{pt}, \varphi)$. The Cremona group $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ acts on $\mathcal{X}$ by

$$f \cdot (S/B, \varphi) := (S/B, f \circ \varphi).$$

2.3. **Sarkisov links and elementary relations.** In this section we show that the complex $\mathcal{X}$ encodes the notion of Sarkisov links (or *links* for short), and of elementary relation between them.

First we rephrase the usual notion of Sarkisov links between 2-dimensional Mori fiber spaces. Let $(S/B, \varphi)$, $(S'/B', \varphi')$ be two marked rank 1 fibrations. We say that the induced birational map $S \dashrightarrow S'$ is a *Sarkisov link* if there exists a marked rank 2 fibration S''/B'' that factorizes through both S/B and S'/B'. Equivalently, the vertices corresponding to S/B and S'/B' are at distance 2 in the complex $\mathcal{X}$, with middle vertex S''/B'':

$$\begin{tikzcd} & S''/B'' \arrow[dl] \arrow[dr] & \\ S/B & & S'/B' \end{tikzcd}$$

This definition is in fact equivalent to the usual definition of a link of type I, II, III or IV from S/B to S'/B' (see [9, Definition 2.14] for the definition in arbitrary dimension). Below we recall these definitions in the context of surfaces, in terms of commutative diagrams where each morphism has relative Picard number 1 (such a diagram corresponds to a "two rays game"), and we give some examples. Remark that these diagrams are not part of the complex $\mathcal{X}$: in each case, the corresponding subcomplex of $\mathcal{X}$ is just a path of two edges, as described above.

- Type I: B is a point, $B' \simeq \mathbb{P}^1$, and $S' \to S$ is the blow-up of a real point or a pair of non-real conjugate points such that we have a diagram

$$\begin{tikzcd} & S' \arrow[dl] \arrow[dr] & \\ S \arrow[dr] & & \mathbb{P}^1 \arrow[dl] \\ & \mathrm{pt} & \end{tikzcd}$$

Then we take $S''/B'' := S'/\mathrm{pt}$. We also refer to the map $S \dashrightarrow S'$ as link of type I.

Examples are given by the blow-up of a real point on $S' = \mathbb{F}_1 \longrightarrow \mathbb{P}^2 = S$ or the blow-up of a pair of non-real conjugate points $S' = \mathcal{CB}_6 \longrightarrow \mathcal{Q} = S$. The fibration $\mathbb{F}_1/\mathbb{P}^1$ corresponds to the lines through the point, and the fibration $\mathcal{CB}_6/\mathbb{P}^1$ corresponds to a pencil of conics in $\mathbb{P}^2$ passing through two pairs of non-real conjugate points.

- Type II: $B = B'$, and there exist two blow-ups $S'' \to S$ and $S'' \to S'$ that fit into a diagram of the form:

$$\begin{tikzcd} & S'' \arrow[dl] \arrow[dr] & \\ S \arrow[dr] & & S' \arrow[dl] \\ & B & \end{tikzcd}$$

Then we take $S''/B'' := S''/B$. We also refer to the map $S \dashrightarrow S'$ as link of type II.

An example is given by $S = \mathbb{P}^2$, S'' the blow-up of a pair of non-real conjugate points and $S' = \mathcal{Q}$. Other examples are the blow-up of a conic bundle in a real point, or in a pair of non-real conjugate points not contained in the same fibre, followed by the contraction of the strict transform of the fibre(s) containing it (them).

- Type III: symmetric situation of a link of type I. We refer to the map $S' \longrightarrow S$ as link of type III.
- Type IV: (S, φ) and (S', φ') are equal as marked surfaces, but the fibrations to B and B' are distinct. In this situation B and B' must be isomorphic to $\mathbb{P}^1$, and we have a diagram

$$\begin{tikzcd} & S \arrow[dl] \arrow[dr] & \\ B \arrow[dr] & & B' \arrow[dl] \\ & \mathrm{pt} & \end{tikzcd}$$

Then we take $S''/B'' := S/\mathrm{pt}$.

For rational surfaces, a type IV link always corresponds to the two rulings on $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$, that is, $S/B = \mathbb{F}_0/\mathbb{P}^1$ is one of the rulings, $S'/B' = \mathbb{F}_0/\mathbb{P}^1$ the other one, and $S''/B'' = \mathbb{F}_0/\mathrm{pt}$. See [8, Theorem 2.6 (iv)] for other examples in the context of non-rational surfaces.

A *path of links* is a finite sequence of marked rank 1 fibrations

$$(S_0/B_0, \varphi_0), \ldots, (S_n/B_n, \varphi_n),$$

such that for all $0 \le i \le n-1$, the induced map $g_i\colon S_i/B_i \dashrightarrow S_{i+1}/B_{i+1}$ is a Sarkisov link.

Proposition 2.4. [11, Proposition 2.6] *Let $(S'/B, \varphi)$ be a marked rank 3 fibration. Then there exist finitely many squares in $\mathcal{X}$ with S' as a corner, and the union of these squares is a subcomplex of $\mathcal{X}$ homeomorphic to a disk with center corresponding to S'.*

In the situation of Proposition 2.4, by going around the boundary of the disc we obtain a path of Sarkisov links whose composition is the identity (or strictly speaking, an automorphism). We say that this path is an *elementary relation* between links, coming from ${S'}^3/B$. More generally, any composition of links that corresponds to a loop in the complex $\mathcal{X}$ is called a *relation* between Sarkisov links.

Theorem 2.5. [11, Proposition 3.14, Proposition 3.15]

(1) Any birational map between rank 1 fibrations is a composition of links, and in particular the complex $\mathcal{X}$ is connected.

(2) Any relation between links is a composition of elementary relations, and in particular $\mathcal{X}$ is simply connected.

The first part of Theorem 2.51 can also be found in [8, Theorem 2.5]. In fact, a relative version can be extracted from the classification of links in [8, Theorem 2.6].

Proposition 2.6. *Let $S/\mathbb{P}^1$ and $S'/\mathbb{P}^1$ be rank 1 fibrations. Any birational map $f\colon S \dashrightarrow S'$ over $\mathbb{P}^1$ is a composition of links of rank 1 fibrations over $\mathbb{P}^1$, that is, a composition of Sarkisov links of type II over $\mathbb{P}^1$. In partiuclar:*

(1) Any element of $\mathcal{J}_*$ is conjugate to a composition of automorphisms of some Hirzebruch surface and links of type II between them.

(2) Any element of $\mathcal{J}_\circ$ is conjugate to a composition automorphisms of $\mathcal{CB}_6$ and links $\mathcal{CB}_6 \dashrightarrow \mathcal{CB}_6$ of type II.

2.4. **Elementary discs.** We call the disc with center a rank 3 fibration from Proposition 2.4 an *elementary disc*. In this section, we classify them and therewith obtain an explicit list of elementary relations among rank 1 fibrations.

Lemma 2.7. *Any edge of $\mathcal{X}$ is contained in a square. In particular, $\mathcal{X}$ is the union of elementary discs.*

Proof. Let e be an edge of $\mathcal{X}$. If it is an edge between a rank 2 fibration and a rank 3 fibration, let e' be an edge from the rank 2 fibration to a rank 1 fibration. By [11, Lemma 2.3], there is a unique square in $\mathcal{X}$ with e and e' among its edges. Suppose that e is the edge between a rank 1 fibration S/B and a rank 2 fibration S'/B'. If $S' = S$ (and so $B = \mathbb{P}^1$, $B' = \mathrm{pt}$), then S is a del Pezzo surface with $\rho(S) = 2$ endowed with a conic bundle structre $S/\mathbb{P}^1$. In particular, $S = \mathbb{F}_n$, $n \in \{0,1\}$ or $S = \mathcal{CB}_6$. Let $S'' \longrightarrow S$ be the blow-up of a general point. Then S'' is also a del Pezzo surface, and so S''/pt is a rank 3 fibration and the square

$$\begin{tikzcd} S/\mathrm{pt} & S''/\mathrm{pt} \arrow[l] \arrow[d] \\ S/\mathbb{P}^1 \arrow[from=u, "e"] & S''/\mathbb{P}^1 \arrow[l] \end{tikzcd}$$

contained in $\mathcal{X}$. Suppose that $S' \longrightarrow S$ is the blow-up of a real point or a pair of non-real conjugate points (and so $B = B'$). Note that if $B = \mathrm{pt}$, the surface S' is a del Pezzo surface of degree ≤ 6. The blow-up $S'' \longrightarrow S'$ of a general point yields a rank 3 fibration S''/B which factorises through S/B. The two blow-ups commute and $\mathcal{X}$ contains a square of the form

$$\begin{tikzcd} S'/B \arrow[d, "e"] & S''/B \arrow[l] \arrow[d] \\ S/B & \tilde{S}/B \arrow[l] \end{tikzcd}$$

It follows that $\mathcal{X}$ is the union of squares. The claim now follows from Proposition 2.4. $\square$

In general, Lemma 2.7 is not true for an arbitrary perfect field k. Indeed, if k has an extension of degree 8, then a Bertini involution whose set of base-points is an orbit of 8 conjugate points is a link of type II

$$\begin{tikzcd} & S \arrow[dl] \arrow[dr] & \\ \mathbb{P}^2 \arrow[rr, dashed] \arrow[dr] & & \mathbb{P}^2 \arrow[dl] \\ & \mathrm{pt} & \end{tikzcd}$$

with S/pt a del Pezzo surface of degre 1, and whose corresponding two edges in $\mathcal{X}$ are not contained in any square [11, Lemma 4.3].

In what follows, X_d is a del Pezzo surface of degree $d = (K_{X_d})^2$. A arrow marked by 1 is the blow-up of a real point and an arrow marked with 2 is the blow-up of a pair of non-real conjugate points.

Proposition 2.8. *Any elementary disc in $\mathcal{X}$ is one of the discs $\mathcal{D}_1, \ldots, \mathcal{D}_6$ in Figures 4 – 1, where the arrows are marked by 1 (or 2) if it is the blow-up of a real point (or a pair of non-real conjugate points).*

Proof. Let $\mathcal{D}$ be an elementary disc and let S/B be the rank 3 fibration that is its center, and let S'/B' be a rank 1 fibration through which the rank 3 fibration S/B factors.

(a) Suppose that $B = \mathrm{pt}$. Then S is a del Pezzo surface with $\rho(S) = 3$, and hence S'/pt is a del Pezzo surface is as well.

If $B' = \mathbb{P}^1$, then $S' \simeq \mathbb{F}_n$, $n \in \{0, 1\}$ or $S' \simeq \mathcal{CB}_6$, and $S \longrightarrow S' \longrightarrow \mathbb{P}^1$ is a rank 2 fibration. Then $S \longrightarrow S'$ is the blow-up of one real point or one pair of non-real conjugate points. If $S' = \mathbb{F}_0$, then $\mathcal{D} = \mathcal{D}_5$. If $S' = \mathbb{F}_1$, then $\mathcal{D} = \mathcal{D}_1$ or $\mathcal{D} = \mathcal{D}_5$. If $S' = \mathcal{CB}_6$ then $\mathcal{D} = \mathcal{D}_3$ or $\mathcal{D} = \mathcal{D}_4$.

If $B' = \mathrm{pt}$, then $S' = \mathbb{P}^2$ or $S' = \mathcal{Q}$, and $\pi\colon S \longrightarrow S'$ is the blow-up of two points (two real points, two pairs of non-real conjugate points or a real point and a pair of non-real points) in general position. If $S' = \mathbb{P}^2$ then $\mathcal{D}$ is one of the discs $\mathcal{D}_1, \mathcal{D}_3, \mathcal{D}_5$. Note that in the discs $\mathcal{D}_1$ and $\mathcal{D}_3$ we cover also the case where $S' = \mathcal{Q}$ and π blows up two points, at least one of which is a real point. If $S' = \mathcal{Q}$ and π blows up two pairs of non-real conjugate points, $\mathcal{D} = \mathcal{D}_4$.

(b) Suppose that $B = \mathbb{P}^1$. Then $B' = \mathbb{P}^1$, $S' = \mathbb{F}_n$ or $S' = \mathcal{CB}_6$ and $\pi\colon S \longrightarrow S'$ is a blow-up in two points (again two real points, two pairs of non-real conjugate points or a real point and a pair of non-real points). Furthermore, any rank 1

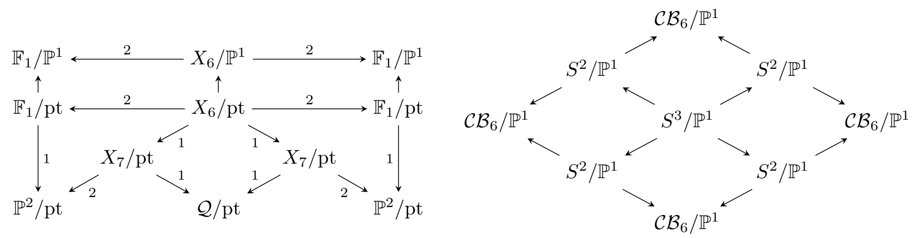

FIGURE 1. Disc $\mathcal{D}_1$ on the left, $\mathcal{D}_2$ on the right.

fibration in the elementary disc is of the form $S''/\mathbb{P}^1$. If $S' = \mathcal{CB}_6$, then S has two singular fibres whose $\mathbb{C}$-components are conjugate, and hence all rank 1 fibrations in the elementary discs are isomorphic to $\mathcal{CB}_6$. The hypothesis that $-K_S$ is π-ample implies that the $\mathbb{C}$-components of the points blown up by π are on pairwise distinct smooth fibres. Thus $\mathcal{D} = \mathcal{D}_2$ or $\mathcal{D} = \mathcal{D}_6$. $\square$

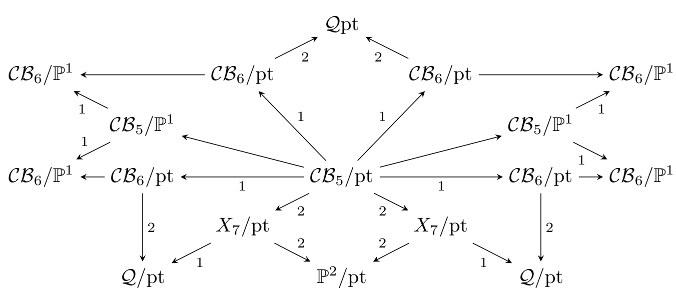

FIGURE 2. Disc $\mathcal{D}_3$.

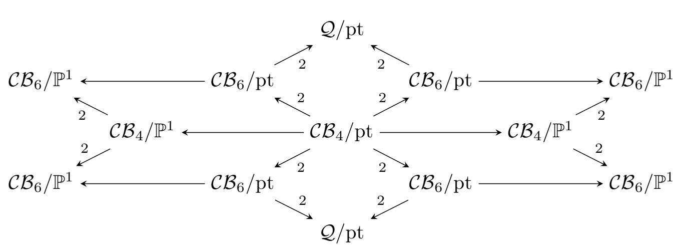

FIGURE 3. Disc $\mathcal{D}_4$.

Lemma 2.9. *Let $i, j \in \{1, \ldots, 6\}$ and $\mathcal{D}_i \neq \mathcal{D}_j$. If they intersect non-trivially, their intersection is a single vertex or is a boundary segment of the form $S/B \leftarrow S''/B'' \rightarrow S'/B'$, where S/B and S'/B' are rank 1 fibrations.*

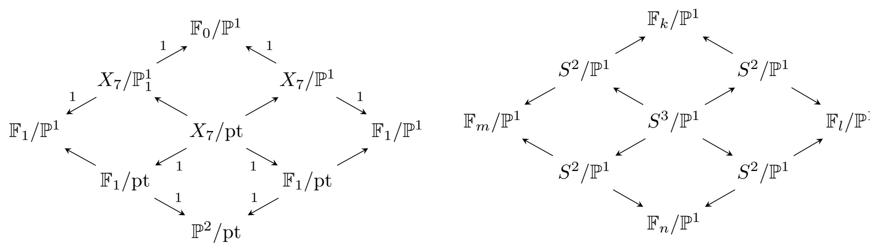

FIGURE 4. Disc $\mathcal{D}_5$ on the left, disc $\mathcal{D}_6$ on the right.

Proof. Suppose that $\mathcal{D}_i \cap \mathcal{D}_j$ is non-empty and does not consist of one vertex. Then it contains at least one edge e. If it is in the relative interior of $\mathcal{D}_i$, the center of $\mathcal{D}_i$ is a vertex of e and hence $\mathcal{D}_i = \mathcal{D}_j$. Suppose that e is on the relative boundary of $\mathcal{D}_i$, i.e. is of the form $S''/B'' \longrightarrow S/B$ with vertices S''/B'' ad S/B a rank 2 and rank 1 fibration, respectively. Following the two-rays game, there is a unique edge $S''/B'' \longrightarrow S'/B'$ to another rank 1 fibration S'/B', which is hence contained in both $\mathcal{D}_i$ and $\mathcal{D}_j$. If $\mathcal{D}_i \cap \mathcal{D}_j$ contains another edge in the relative boundary of $\mathcal{D}_i$, then the list of elementary discs $\mathcal{D}_1, \dots, \mathcal{D}_6$ in Proposition 2.8 implies that $\mathcal{D}_i = \mathcal{D}_j$. □

3. PROOF OF THE MAIN THEOREMS

3.1. **The group $\mathcal{G}_* \cap \mathcal{G}_\circ$.** We denote by $\mathcal{H} \subset \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ the subgroup generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ and the quadratic involution $\sigma\colon [x:y:z] \mapstochar\dashrightarrow [xz:yz:x^2+y^2]$. Equivalently, it is generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ and the quadratic maps in $\mathcal{J}_\circ \cap \mathcal{J}_*$ [18, Lemma 3.2]. In particular, $\mathcal{H} \subseteq \mathcal{G}_\circ \cap \mathcal{G}_*$. We will prove equality in this section.

Remark 3.1. (1) For any quadratic map $f \in \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ with one real base-point and a pair of non-real conjugate base-points there exist $\alpha, \beta \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ such that $\alpha f \beta = \sigma$.
(2) The path along the boundary of an elementary disc $\mathcal{D}_1$ connecting its two vertices $\mathbb{P}^1/\mathrm{pt}$ corresponds to a quadratic map with one real and two non-real conjugate base-points. It is thus contained in $\mathcal{H}$ and hence also in $\mathcal{G}_\circ \cap \mathcal{G}_*$.
(3) We can write $\sigma = \varphi_2 \circ \varphi_1$ where $\varphi_1\colon \mathbb{P}^2 \dashrightarrow \mathcal{Q}$ and $\varphi_2\colon \mathcal{Q} \dashrightarrow \mathbb{P}^2$ are two links of type II. It follows that any element of $\mathcal{H}$ is a composition of links $\mathbb{P}^2 \dashrightarrow \mathcal{Q}$ and $\mathcal{Q} \dashrightarrow \mathbb{P}^2$ of type II.

Definition 3.2.

(1) We call $\mathcal{X}_\circ \subset \mathcal{X}$ the union of the $\mathcal{G}_\circ$-orbits of all discs $\mathcal{D}_1, \dots, \mathcal{D}_4$ containing a vertex $(\mathcal{Q}/\mathrm{pt}, \tau)$ or $(\mathcal{CB}_6/\mathbb{P}^1, \tau \circ \pi)$ with $\tau\colon \mathcal{Q} \dashrightarrow \mathbb{P}^2$ is a link of type II and $\pi\colon \mathcal{CB}_6 \longrightarrow \mathcal{Q}$ a link of type III.

(2) We call $\mathcal{X}_* \subset \mathcal{X}$ the union of $\mathcal{G}_*$-orbits of all discs $\mathcal{D}_1, \mathcal{D}_5, \mathcal{D}_6$ containing a vertex $(\mathbb{F}_n/\mathbb{P}^1, \pi_n)$ for some $n \geq 0$, with $\pi_1\colon \mathbb{F}_1 \longrightarrow \mathbb{P}^2$ a link of type III and $\pi_n = \pi_1 \circ \varphi_n$, $n \neq 1$, where $\varphi_n\colon \mathbb{F}_n \dashrightarrow \mathbb{F}_1$ is a composition of Sarkisov links of type II between Hirzebruch surfaces.

As the type of elementary disc is invariant by the action of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$, any elementary disc contained in $\mathcal{X}_\circ$ is one of type $\mathcal{D}_1,\dots,\mathcal{D}_4$, and any elementary disc contained in $\mathcal{X}_*$ is one of type $\mathcal{D}_1,\mathcal{D}_5$ or $\mathcal{D}_6$.

Lemma 3.3.

(1) The set of vertices $(\mathbb{P}^2,g)$ of $\mathcal{X}_\circ$ is the $\mathcal{G}_\circ$-orbit of $(\mathbb{P}^2,\mathrm{id})$.

(2) The set of vertices $(\mathbb{P}^2,g)$ of $\mathcal{X}_*$ is the $\mathcal{G}_*$-orbit of $(\mathbb{P}^2,\mathrm{id})$.

(3) We have

$$\mathcal{X}=\bigcup_{f\in\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)} f(\mathcal{X}_\circ)\cup\bigcup_{f\in\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)} f(\mathcal{X}_*).$$

(4) The sets $\mathcal{X}_\circ$ and $\mathcal{X}_*$ are connected..

(5) For any $f\in\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$, the set $\mathcal{X}_\circ\cap f(\mathcal{X}_*)$ is the union of all elementary discs $\mathcal{D}_1$ containing a vertex $(\mathbb{P}^2,g)$ with $g\in\mathcal{G}_\circ$ and a vertex $(\mathbb{P}^2,f\circ g')$ with and $g'\in\mathcal{G}_*$.

(6) The set $\mathcal{X}_\circ\cap\mathcal{X}_*$ is the union of all elementary discs $\mathcal{D}_1$ containing a vertex $(\mathbb{P}^2,g)$ with $g\in\mathcal{G}_\circ\cap\mathcal{G}_*$.

Proof. (1) For any $g\in\mathcal{G}_\circ$ we have $(\mathbb{P}^2,g)=g(\mathbb{P}^2,\mathrm{id})$. Let $\tau\colon\mathcal{Q}\dashrightarrow\mathbb{P}^2$ be a link of type II. There is an elementary disc $\mathcal{D}_1$ containing $(\mathbb{P}^2,\mathrm{id})$ and $(\mathcal{Q},\tau)$. Then $g\mathcal{D}_1$ is an elementary disc containing $(\mathbb{P}^2,g)$ and $(\mathcal{Q},g\circ\tau)$, and it follows that $(\mathbb{P}^2,g)\in\mathcal{X}_\circ$. For the converse, if $(\mathbb{P}^2,f)$ is contained in $\mathcal{X}_\circ$, it is the vertex of an elementary disc $\mathcal{D}_i$, $i\in\{1,3\}$ contained $\mathcal{X}_\circ$. There is $g\in\mathcal{G}_\circ$ such that $\mathcal{D}_i=g\mathcal{D}_i'$ is the image by g of another elementary disc of type $\mathcal{D}_i'$ containing a vertex $(\mathcal{Q},\tau)$ where $\tau\colon\mathcal{Q}\dashrightarrow\mathbb{P}^2$ is a link of type II. We can choose the vertex $(\mathcal{Q},\tau)$ inside $\mathcal{D}_i'$ such that there are two consecutive edges connecting it to the vertex $(\mathbb{P}^2,g^{-1}\circ f)\in\mathcal{D}_i'$. The two edges correspond to a link $\tau'\colon\mathcal{Q}\dashrightarrow\mathbb{P}^2$ of type II. In particular, $\tau\circ\tau'=g^{-1}\circ f$. Since $\tau\circ\tau'\in\mathcal{G}_\circ$, it follows that $f\in\mathcal{G}_\circ$.

(2) Is proven analogously to (1) with $\mathbb{F}_1$ instead of $\mathcal{Q}$ and links $\pi_1\colon\mathbb{F}_1\longrightarrow\mathbb{P}^2$ of type I and their inverse instead of links $\tau\colon\mathcal{Q}\dashrightarrow\mathbb{P}^2$ of type II.

(3) The complex $\mathcal{X}$ is the union of elementary discs by Lemma 2.7, hence the orbit of $\mathcal{X}_\circ$ and $\mathcal{X}_*$ by $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ covers $\mathcal{X}$.

(4) Let $i\in\{*,\circ\}$. It suffices to check that for any vertex $(\mathbb{P}^2,f)$ of $\mathcal{X}_i$ there is a path in $\mathcal{X}_i$ from $(\mathbb{P}^2,f)$ to $(\mathbb{P}^2,\mathrm{id})$. (1)&(2) imply that $f\in\mathcal{G}_i$. We write $f=\alpha_nf_n\alpha_{n-1}\cdots f_1\alpha_0$, where $\alpha_j\in\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, $f_j\in\mathcal{J}_i$. Proposition 2.6 implies that each f_j is conjugate by $\psi_i\colon\mathbb{P}^2\dashrightarrow S_i$ to a composition of links of type II preserving the associated conic bundle $S_i/\mathbb{P}^1$, where $\psi_*\colon\mathbb{P}^2\dashrightarrow\mathbb{F}_1=S_*$ is the blow-up of $[1:0:0]$ and $\psi_\circ\colon\mathbb{P}^2\dashrightarrow\mathcal{Q}\dashrightarrow\mathcal{CB}_6=S_\circ$ the composition a link of type II and of type I blowing up $[1:i:0],[0:1:i]$ and their conjugates. This yields a decomposition of f into links which corresponds to a path from $(\mathbb{P}^2,\mathrm{id})$ to $(\mathbb{P}^2,f)$ along the boundary of elementary discs contained in $\mathcal{X}_i$.

(5) Lemma 2.9 and Proposition 2.8 imply that the intersection $\mathcal{X}_\circ\cap f(\mathcal{X}_*)$ consists of elementary discs of type $\mathcal{D}_1$. By (1)&(2) such a disc contains a vertex $(\mathbb{P}^2,g)$ with $g\in\mathcal{G}_\circ$ and a vertex $(\mathbb{P}^2,f\circ g')$ with $g'\in\mathcal{G}_*$, and any elementary disc $\mathcal{D}_1$ with this property is contained $\mathcal{X}_*\cap f(\mathcal{X}_\circ)$.

(6) By (5) for $f=\mathrm{id}$, the set $\mathcal{X}_\circ\cap\mathcal{X}_*$ is the union of elementary discs $\mathcal{D}_1$ containing a vertex $(\mathbb{P}^2,g)$ with $g\in\mathcal{G}_\circ$ and a vertex $(\mathbb{P}^2,g')$ with $g'\in\mathcal{G}_*$. Remark 3.1(2) implies that $g,g'\in\mathcal{G}_*\cap\mathcal{G}_\circ$. □

Lemma 3.4. *We have the following stabilisers under the action of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ on $\mathcal{X}$:*

(1) $\mathrm{Stab}(\mathcal{X}_\circ)=\mathcal{G}_\circ$,

(2) $\mathrm{Stab}(\mathcal{X}_*)=\mathcal{G}_*$,

(3) $\mathrm{Stab}(\mathcal{X}_\circ \cap \mathcal{X}_*) = \mathcal{G}_* \cap \mathcal{G}_\circ$.

Proof. (1)&(2) Let $i \in \{\circ, *\}$. As $\mathcal{X}_i$ is the $\mathcal{G}_i$-orbit of certain elementary discs, we have $\mathcal{G}_i \subset \mathrm{Stab}(\mathcal{X}_i)$. Let $f \in \mathrm{Stab}(\mathcal{X}_i)$ and $(\mathbb{P}^2, g) \in \mathcal{X}_i$. Then $(\mathbb{P}^2, f \circ g) \in \mathcal{X}_i$, thus $f \circ g \in \mathcal{G}_i$ by Lemma 3.3(1), and so $f \in \mathcal{G}_i$.

(3) The inclusion $\mathcal{G}_\circ \cap \mathcal{G}_* \subset \mathrm{Stab}(\mathcal{X}_\circ \cap \mathcal{X}_*)$ follows from (1)&(2). Let $f \in \mathrm{Stab}(\mathcal{X}_\circ \cap \mathcal{X}_*)$ and $(\mathbb{P}^2, g) \in \mathcal{X}_\circ \cap \mathcal{X}_*$. Then $(\mathbb{P}^2, g \circ f) \in \mathcal{X}_\circ \cap \mathcal{X}_*$. By Lemma 3.3(6), we have $g, g \circ f \in \mathcal{G}_* \cap \mathcal{G}_\circ$, and so $f \in \mathcal{G}_\circ \cap \mathcal{G}_*$. □

Lemma 3.5. *Let $f \in \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$. Then the following hold:*

(1) *Let $i \in \{\circ, *\}$. If $\mathcal{X}_i$ and $f(\mathcal{X}_i)$ intersect, then they are equal.*
(2) *$\mathcal{X}_* \cap \mathcal{X}_\circ$ is connected.*
(3) *$\mathrm{Stab}(\mathcal{X}_\circ \cap \mathcal{X}_*) = \mathcal{G}_\circ \cap \mathcal{G}_* = \mathcal{H}$.*

Proof. (1) Suppose that $\mathcal{X}_i \cap f(\mathcal{X}_i)$ is non-empty. As the type of elementary disc is invariant by the action of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$, the set $\mathcal{X}_i \cap f(\mathcal{X}_i)$ contains a vertex $(\mathbb{P}^2, h)$ for some $h \in \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$. By Lemma 3.3(1)&(2) we have $h \in \mathcal{G}_i$ and there exists $g \in \mathcal{G}_i$ such that $h = f \circ g$. It follows that $f \in \mathcal{G}_i$. As $\mathcal{G}_i \subset \mathrm{Stab}(\mathcal{X}_i)$, we have $f(\mathcal{X}_i) = \mathcal{X}_i$.

(2)&(3) Let $g \in \mathcal{G}_* \cap \mathcal{G}_\circ$ and write $g = g_1 \circ \cdots \circ g_n = f_1 \circ \cdots \circ f_m$ for some $g_i \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) \cup \mathcal{J}_*$ and $f_i \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) \cup \mathcal{J}_\circ$. Then the relation

$$g_n^{-1} \circ \cdots \circ g_1^{-1} \circ f_1 \circ \cdots \circ f_m = \mathrm{id}$$

induces a loop γ in $\mathcal{X}$ starting at $(\mathbb{P}^2, \mathrm{id})$ and passing through $(\mathbb{P}^2, g_n^{-1})$, $(\mathbb{P}^2, g_n^{-1} \circ g_{n-1}^{-1}), \ldots, (\mathbb{P}^2, g_n^{-1} \circ \cdots \circ g_1^{-1}) = (\mathbb{P}^2, g^{-1}), (\mathbb{P}^2, g^{-1} \circ f_1), \ldots, (\mathbb{P}^2, g^{-1} \circ f_1 \circ \cdots \circ f_{m-1})$, $(\mathbb{P}^2, g^{-1} f) = (\mathbb{P}^2, \mathrm{id})$. We can suppose that $\gamma(\frac{1}{2}) = (\mathbb{P}^2, g^{-1})$. Since $\mathcal{X}_*$ and $\mathcal{X}_\circ$ are connected by Lemma 3.3(4) and $\mathcal{X}$ is simply connected by Proposition 2.5(2), we can assume that γ that $\gamma([0, \frac{1}{2}]) \subset \mathcal{X}_*$ and $\gamma([\frac{1}{2}, 1]) \subset g^{-1}(\mathcal{X}_\circ) = \mathcal{X}_\circ$ (Lemma 3.4(1)). The loop γ is the boundary of a disc D in $\mathcal{X}$, and by Proposition 2.5(2) the disc D is a finite union of elementary discs. In particular, $\gamma([0, \frac{1}{2}]) \subset \mathcal{X}_*$ passes through elementary discs contained in $\mathcal{X}_*$ and $\gamma([\frac{1}{2}, 1]) \subset \mathcal{X}_\circ$ passes through elementary discs contained in $\mathcal{X}_\circ$. The complex $\mathcal{X}$ is covered by its subcomplexes $f(\mathcal{X}_i)$, $i \in \{*, \circ\}$ by Lemma 3.3(3), hence so is the disc D. It follows that D contains a finite connected union U of elementary discs, all contained in the union of sets of the form $f(\mathcal{X}_\circ) \cap f'(\mathcal{X}_*)$, $f, f' \in \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$, and which contains $(\mathbb{P}^2, \mathrm{id})$ and $(\mathbb{P}^2, g^{-1})$. Lemma 3.3(5) implies that U is covered by elementary discs of type $\mathcal{D}_1$. The path from $(\mathbb{P}^2, \mathrm{id})$ to $(\mathbb{P}^2, g^{-1})$ along the boundary of these discs induces a decomposition of g^{-1} into elements of $\mathcal{H}$ (Remark 3.1(2)). This yields $\mathcal{G}_* \cap \mathcal{G}_* \subset \mathcal{H}$. Now (3) follows from Lemma 3.4(3). This and Lemma 3.3(6) imply (2). □

3.2. **The index of the subgroup $\mathcal{G}_\circ$ and the properness of $\mathcal{H}$ in $\mathcal{G}_*$.** We now prove that $\mathcal{G}_*$ is not contained in $\mathcal{G}_\circ$ by showing that quadratic maps having three real base-points in $\mathbb{P}^2$ are in $\mathcal{G}_* \setminus \mathcal{G}_\circ$, and as consequence that the set $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)/\mathcal{G}_\circ$ is uncountable.

Recall that any quadratic map $f \in \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ with three real base-points in $\mathbb{P}^2$ is contained in $\mathcal{G}_*$, as there exist $\alpha, \beta \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ such that $\alpha f \beta$ is given by

$$\alpha f \beta \colon [x : y : z] \dashrightarrow [yz : xz : xy]$$

and so $\alpha f \beta \in \mathcal{J}_*$.

Lemma 3.6. *The group $\mathcal{G}_\circ$ does not contain any quadratic map with three real base-points in $\mathbb{P}^2$. In particular, $\mathcal{H} \subsetneq \mathcal{G}_*$.*

Proof. Let $f \in \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ be a quadratic map with three real base-points p_1, p_2, p_3 in $\mathbb{P}^2$ and suppose that $f \in \mathcal{G}_\circ$. Then $f \in \mathcal{G}_\circ \cap \mathcal{G}_* = \mathcal{H}$ by Lemma 3.5(3) and there is a decomposition $f = \psi_r \cdots \psi_1$ into links $\psi_i\colon \mathbb{P}^2 \dashrightarrow \mathcal{Q}$ and $\psi_j\colon \mathcal{Q} \dashrightarrow \mathbb{P}^2$ of type II (Remark 3.1(3)). On the other hand, there is a decomposition of $f = \varphi_4 \cdots \varphi_1$ as follows: The link $\varphi_1\colon \mathbb{P}^2 \dashrightarrow \mathbb{F}_1$ is the blow-up of p_1 of f, $\varphi_2\colon \mathbb{F}_1 \dashrightarrow \mathbb{F}_0$ is the link of type II blowing up of p_2 and contracting the line strict transform of the line through p_1, p_2, and $\varphi_3\colon \mathbb{F}_0 \dashrightarrow \mathbb{F}_1$ is a link of type II blowing up p_3 and contracting the strict transform of the line through p_1, p_3, and $\varphi_4\colon \mathbb{F}_1 \longrightarrow \mathbb{P}^2$ is a link of type III contracting the strict transform of the line through p_2, p_3. This corresponds to a path along the boundary of the two discs $\mathcal{D}_5$ and $\mathcal{D}_5'$ with respective center (X_7, φ) and $(X_7, \varphi' \circ f^{-1})$, where $\varphi\colon X_7 \longrightarrow \mathbb{P}^2$ is the blow-up of the points p_1, p_2 and $\varphi'\colon X_7 \longrightarrow \mathbb{P}^2$ is the blow-up of the two base-points of f^{-1} which are the images of the lines through p_2, p_3 and p_1, p_3. The relation

$$\mathrm{id} = \psi_1^{-1} \cdots \psi_r^{-1} \varphi_4 \cdots \varphi_1$$

corresponds to a loop γ in $\mathcal{X}$ which is the border of a disc $D \subset \mathcal{X}$ that is a finite union of elementary discs (Theorem 2.5). We can assume that $\gamma(\frac{1}{2}) = (\mathbb{P}^2, f^{-1})$ and that $\gamma([0, \frac{1}{2}]) \subset \mathcal{X}_\circ \cap \mathcal{X}_*$. It is therefore covered by elementary discs of the form $\mathcal{D}_1$ (Lemma 3.3(6)). Furthermore, $\gamma([\frac{1}{2}, 1]) \subset \mathcal{D}_5 \cup \mathcal{D}_5' \subset \mathcal{X}_*$ by construction of γ. The discs $\mathcal{D}_5$ and $\mathcal{D}_5'$ intersect only elementary discs of type $\mathcal{D}_1$ (they do so in a vertex $\mathbb{P}^2/\mathrm{pt}$ or in exactly two connected edges corresponding to a link of type I) and elementary discs type $\mathcal{D}_6$, which in turn only intersect other discs of type $\mathcal{D}_6$ or discs of type $\mathcal{D}_5$. It follows that there is a connected union $V \subset D$ of discs of type $\mathcal{D}_1$ that contains $(\mathbb{P}^2, \mathrm{id})$ and $(\mathbb{P}^2, f^{-1})$ such that the subset of $\tilde{D}$ bounded by V and $\gamma([\frac{1}{2}, 1])$ is only covered by discs of type $\mathcal{D}_1$, $\mathcal{D}_5$ and $\mathcal{D}_6$. We can then assume that $\gamma([0, \frac{1}{2}]) \subset V$ and that the set $\tilde{D}$ is homeomorphic to a disc. As discs of type $\mathcal{D}_6$ only intersect other discs $\mathcal{D}_6$ or discs of type $\mathcal{D}_5$, it follows that there is a connected union $\tilde{V} \subset \tilde{D}$ of discs $\mathcal{D}_5$ containing $(\mathbb{P}^2, \mathrm{id})$ and $(\mathbb{P}^2, f^{-1})$ intersecting the discs of type $\mathcal{D}_1$ in V along their boundary. We obtain a path along boundaries of discs of type $\mathcal{D}_1$ of links of type I and type III from $(\mathbb{P}^2, \mathrm{id})$ to $(\mathbb{P}^2, f^{-1})$. This is impossible as f^{-1} has not such decomposition. It follows that $f \notin \mathcal{G}_\circ$. The rest of the claim follows from the fact that $\mathcal{H} = \mathcal{G}_* \cap \mathcal{G}_\circ$ (Lemma 3.5(3)). □

Lemma 3.7. *The group $\mathcal{G}_\circ$ has uncountable index in $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$.*

Proof. Denote by $\sigma\colon [x : y : z] \mapstochar\dashrightarrow [yz : xz : xy]$ the standard quadratic involution of $\mathbb{P}^2$, and define the group

$$A := \{[x : y : z] \mapsto [x + az : y + bz : z] \mid a, b \in \mathbb{R}\} \subset \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$$

Consider the map between sets

$$\psi\colon A \longrightarrow \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)/\mathcal{G}_\circ, \quad \alpha \mapsto (\alpha \circ \sigma)\mathcal{G}_\circ$$

We now prove that it is injective, which will yield the claim. First, note that $\sigma \circ \alpha \circ \sigma$ is of degree ≤ 2 for all $\alpha \in A$ since $\sigma \circ \alpha$ and σ have two common base-points, namely $[1 : 0 : 0]$ and $[0 : 0 : 1]$. Then $\sigma \circ \alpha \circ \sigma \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ if and only if σ and $\sigma \circ \alpha$ have three common base-points. This is the case if and only if $\alpha = \mathrm{Id}$, as $\alpha \in A$. Moreover, if $\sigma \circ \alpha \circ \sigma$ is of degree 2, it has three real base-points.

Now, let $\beta, \gamma \in A$ such that $(\beta \circ \sigma)\mathcal{G}_\circ = (\gamma \circ \sigma)\mathcal{G}_\circ$. Then $\sigma \circ (\beta^{-1} \circ \gamma) \circ \sigma \in \mathcal{G}_\circ$. It follows from Lemma 3.6 and the above that $\sigma \circ \beta^{-1} \circ \gamma \circ \sigma \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, which is the case if and only if $\beta^{-1} \circ \gamma = \mathrm{id}$. Thus the map φ is injective. □

3.3. **The index of the subgroup $\mathcal{G}_\circ$ and a quotient of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$.** We recall the construction of a surjective homomorphism

$$\psi\colon \mathcal{J}_\circ \longrightarrow \bigoplus_{(0,1]} \mathbb{Z}/2\mathbb{Z}$$

given in [18, §3.2.1, §3.2.2].

Definition–Lemma 3.8 ([18, §3.2.1, §3.2.2])**.** Any element $f \in \mathcal{J}_\circ$ is conjugate to a birational map $\hat{f}\colon \mathcal{CB}_6/\mathbb{P}^1 \dashrightarrow \mathcal{CB}_6/\mathbb{P}^1$ preserving the conic bundle structure. Such a map $\hat{f}$ has a decomposition $\hat{f} = \varphi_s \cdots \varphi_1$ into links $\mathcal{CB}_6 \dashrightarrow \mathcal{CB}_6$ of type II (Proposition 2.6), and such a link does not blow up any point on a singular fibre of $\mathcal{CB}_6/\mathbb{P}^1$. For $j = 1, \dots, s$, let C_j be a (real or non-real) fibre of $\mathcal{CB}_6/\mathbb{P}^1$ contracted by φ_j and $[a_j + ib_j : 1] \in \mathbb{P}^1$ its image by the projection onto $\mathbb{P}^1$. We define $v_j = 1 - \frac{|a_j|}{a_j^2+b_j^2} \in (0,1]$ if $a_j + b_j \notin \mathbb{R}$, and $v_j = 0$ otherwise. Then

$$\psi\colon \mathcal{J}_\circ \longrightarrow \bigoplus_{(0,1]} \mathbb{Z}/2\mathbb{Z}, \quad f \mapsto \sum_{j=1}^{s} e_{v_j}$$

is a surjective homomorphism of groups whose kernel contains all elements of $\mathcal{J}_\circ$ of degree ≤ 4.

Remark 3.9. A standard quintic transformation $f \in \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is a transformation of degree 5 having three pairs of non-real conjugate base-points in $\mathbb{P}^2$, and no three of the six base-points are collinear (see [3, Example] or [13, §1] for equivalent definitions). In particular, the six points are not on one conic, and f sends the pencil of conics through two of the pairs of base-points of f onto a pencil of conics through two pairs of base-points of f^{-1}. As two non-collinear pairs of non-real conjugate points can be sent by an automorphism of $\mathbb{P}^2$ onto any other two non-collinear pairs of non-real conjugate points, there are $\alpha, \beta \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ such that $g := \alpha \circ f \circ \beta \in \mathcal{J}_\circ$. In particular, $f \in \mathcal{G}_\circ$. Moreover, it is conjugate to a link of type II of the conic bundle $\mathcal{CB}_6/\mathbb{P}^1$. Moreover, the image $\psi(g)$ of g under the homomorphism $\psi\colon \mathcal{J}_\circ \longrightarrow \bigoplus_{(0,1]} \mathbb{Z}/2\mathbb{Z}$ from Definition-Lemma 3.8 is a generator of a $\mathbb{Z}/2\mathbb{Z}$.

We denote by $\mathrm{Sar}_{\mathbb{R}}(\mathbb{P}^2)$ the set of birational transformations between rank 1 fibrations. It is a groupoid and contains $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ as subgroupoid.

Remark 3.10. By Theorem 2.5, the groupoid $\mathrm{Sar}_{\mathbb{R}}(\mathbb{P}^2)$ is generated by links and any relation in $\mathrm{Sar}_{\mathbb{R}}(\mathbb{P}^2)$ is a product of conjugates of elementary relations.

Proposition 3.11 (cf. [18, Proposition 5.3])**.** *The homomorphism $\psi\colon \mathcal{J}_\circ \to \bigoplus_{(0,1]} \mathbb{Z}/2\mathbb{Z}$ lift to a surjective homomorphism*

$$\Psi\colon\ \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2) \longrightarrow \bigoplus_{(0,1]} \mathbb{Z}/2\mathbb{Z}$$

whose kernel contains $\mathcal{G}_*$.

Proof. It suffices to construct a homomorphism of groupoids

$$\Psi\colon\ \mathrm{Sar}_{\mathbb{R}}(\mathbb{P}^2) \longrightarrow \bigoplus_{(0,1]} \mathbb{Z}/2\mathbb{Z}$$

whose restriction to its subgroup $\mathcal{J}_\circ$ is exactly $\psi\colon \mathcal{J}_\circ \longrightarrow \bigoplus_{(0,1]} \mathbb{Z}/2\mathbb{Z}$ and whose kernel contains $\mathcal{G}_*$.

Let $\varphi\colon \mathcal{CB}_6/\mathbb{P}^1 \dashrightarrow \mathcal{CB}_6/\mathbb{P}^1$ be a link of type II blowing up a pair of non-real conjugate points. Let $\tau_1\colon \mathbb{P}^2 \dashrightarrow \mathcal{Q}$ be a link of type II blowing up $[1:i:0],[1:-i:0]$, and $\tau_2\colon \mathcal{Q} \dashrightarrow \mathcal{CB}_6$ the link of type I blowing up $\tau_1([0:1:i]),\tau_1([0:1:-i])$. Then $f := \tau_1^{-1}\circ\tau_2^{-1}\circ\varphi\circ\tau_2\circ\tau_1\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is a standard quintic transformation contained in $\mathcal{J}_\circ$ and we define we define $\Psi(\varphi) := \psi(f)$. For any other link φ we define $\Psi(\varphi) := 0$. Note that by Definition–Lemma 3.8 and Remark 3.9, we have $\Psi(f) = \psi(f)$ for any $f \in \mathcal{J}_\circ$. By Remark 3.10 it remains to check that any relation in $\mathrm{Sar}_{\mathbb{R}}(\mathbb{P}^2)$ is sent onto zero by Ψ. Any relation in $\mathrm{Sar}_{\mathbb{R}}(\mathbb{P}^2)$ is the composition of conjugates of elementary relations. Let $\varphi_r\circ\cdots\circ\varphi_1$ be an elementary relation. We can assume that one of the φ_i is a link of type II of $\mathcal{CB}_6/\mathbb{P}^1$, as otherwise the relation is sent onto zero by definition of Ψ. The elementary relation corresponds to an elementary disc $\mathcal{D}$ in $\mathcal{X}$ (Theorem 2.5(2)) and φ_i corresponds to a segment on the boundary of the elementary disc. The list of elementary discs in Proposition 2.8 yields that $\mathcal{D}$ is one of $\mathcal{D}_2$, $\mathcal{D}_3$ or $\mathcal{D}_4$. The boundary of each such disc is conjugate via τ_1 or $\tau_1\tau_2$ to a relation inside the group $\mathcal{J}_\circ$, so its image by ψ is zero (as ψ is a homomorphism of groups). Thus its image by Ψ is zero. Note that $\mathcal{G}_*$ is contained in $\ker(\Psi)$ by definition of Ψ. □

The proof of Proposition 3.11 uses the principal idea of the construction of the quotients constructed in [2]; instead of working with a presentation of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ in terms of generators and generating relations, we pass to such a presentation of the groupoid of birational transformations between rational rank 1 fibrations (rational Mori fibre spaces) and construct a homomorphism from this groupoid to $\bigoplus_{(0,1]} \mathbb{Z}/2\mathbb{Z}$. In [2] one has then to prove in a second step that it is surjective. Here, this is not necessary, as we know that by construction its restriction to $\mathcal{J}_\circ$ is surjective. The proof of Proposition 3.11 given in [18, Proposition 5.2] uses a presentation of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ in terms of generators and relations. The proof given here is independent from it.

For a perfect field with an extension of degree 8 (for instance $\mathrm{k} = \mathbb{Q}$), there is a surjective homomorphism $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2) \to \ast_I\,\mathbb{Z}/2\mathbb{Z}$ with $|I| \geq |\mathrm{k}|$ [11]. It is induced by the free product structure $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2) \simeq G \ast (\ast_I\,\mathbb{Z}/2\mathbb{Z})$, where the factors $\mathbb{Z}/2\mathbb{Z}$ are generated by Bertini involutions whose set of base-points consist of one orbit of eight conjugate points and G is generated by the other generators given in [6] and $\mathrm{Aut}_{\mathrm{k}}(\mathbb{P}^2)$. The free product structure is proven by constructing a quotient of $\mathcal{X}$ on which $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2)$ acts and by showing that it is its Bass-Serre tree. So at its heart, also the construction of the homomorphism from $\mathrm{Bir}_{\mathrm{k}}(\mathbb{P}^2) \to \ast_I\,\mathbb{Z}/2\mathbb{Z}$ uses the presentation of $\mathrm{Sar}_{\mathrm{k}}(\mathbb{P}^2)$ in terms of generators and generating relations.

Corollary 3.12. *The group $\mathcal{G}_*$ does not contain any standard quintic transformations (in particular, $\mathcal{H} \subsetneq \mathcal{G}_\circ$) and the index of $\mathcal{G}_*$ in $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is uncountable.*

Proof. Let $\Psi\colon \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2) \longrightarrow \bigoplus_{(0,1]} \mathbb{Z}/2\mathbb{Z}$ be the surjective homomorphism from Proposition 3.11 and $f \in \mathcal{G}_\circ$ a standard quintic transformation. It is conjugate to a standard quintic transformation in $\mathcal{J}_\circ$ (Remark 3.9), so $\Psi(f) = \psi(f) \neq 0$ by Proposition 3.11 and Lemma 3.8. Therefore $f \notin \ker\Psi$, while $\mathcal{G}_* \subset \ker\Psi$. It follows that $\mathcal{G}_* \cap \mathcal{G}_\circ$ is a proper subgroups of $\mathcal{G}_\circ$. Moreover, Ψ induces a surjective between map $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)/\mathcal{G}_* \longrightarrow \bigoplus_{(0,1]} \mathbb{Z}/2\mathbb{Z}$ and hence the quotient $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)/\mathcal{G}_*$ is uncountable. □

3.4. **Proof of the main results.** We finally assemble the proofs of Theorem 1.1, Corollary 1.2 and Theorem 1.3.

Proof of Theorem 1.1. The group $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is generated by $\mathcal{G}_*$ and $\mathcal{G}_\circ$. Let $f_n\circ\cdots\circ f_1=\mathrm{id}_{\mathbb{P}^2}$ be a relation in $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ with $f_i\in\mathcal{G}_*\cup\mathcal{G}_\circ$. By Theorem 2.5(1) for each $i=1,\dots,r$ there are links $\varphi_{i1},\dots,\varphi_{ir_i}$ such that $f_i=\varphi_{ir_i}\circ\cdots\circ\varphi_{i1}$. Then

$$\varphi_{nr_n}\circ\cdots\circ\varphi_{n1}\circ\cdots\circ\varphi_{1r_1}\circ\cdots\circ\varphi_{11}=\mathrm{id}_{\mathbb{P}^2}$$

is a relation inside the groupoid $\mathrm{Sar}_{\mathbb{R}}(\mathbb{P}^2)$ and is thus a composition of conjugates of elementary relations (Remark 3.10), and each elementary relation corresponds to a loop around the boundary of an elementary disc (Theorem 2.5). Elementary discs are classified in Proposition 2.8. The boundary of discs $\mathcal{D}_1,\mathcal{D}_3$ and $\mathcal{D}_5$ respectively corresponds to a relation in $\mathcal{G}_\circ\cap\mathcal{G}_*$, in $\mathcal{G}_\circ$ and in $\mathcal{G}_*$, respectively. We attach to the boundary of a disc $\mathcal{D}_2$ a segment e of the form $\mathcal{Q}/\mathrm{pt}\leftarrow X_7/\mathrm{pt}\rightarrow\mathbb{P}^2/\mathrm{pt}$. To the boundary of a disc $\mathcal{D}_4$ we attach first the segment $\mathcal{CB}_6/\mathbb{P}^1\leftarrow\mathcal{CB}_6/\mathrm{pt}\rightarrow\mathcal{Q}/\mathrm{pt}$ and at that one the segment e. This yields that the elementary relation associate to the disc is conjugate to a relation in $\mathcal{G}_\circ$. To the boundary of a disc $\mathcal{D}_6$ we attach the segment corresponding to a composition of links of type II (or an isomorphism if $n=1$) and a link of type III $\mathbb{F}_n \dashrightarrow \mathbb{F}_1\longrightarrow\mathbb{P}^2$. Then the elementary relation associated to $\mathcal{D}_6$ is conjugate to a relation in $\mathcal{G}_*$.

It follows that the relation $f_n\circ\cdots\circ f_1=\mathrm{id}$ is conjugate to a composition of conjugates of relations in $\mathcal{G}_*$ or $\mathcal{G}_\circ$ or $\mathcal{G}_\circ\cap\mathcal{G}_*$. This implies that

$$\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)=\mathcal{G}_*\mathop{\ast}_{\mathcal{G}_*\cap\mathcal{G}_\circ}\mathcal{G}_\circ.$$

Moreover, we have $\mathcal{G}_*\cap\mathcal{G}_\circ=\mathcal{H}$ by Lemma 3.5(3), and it is a proper subgroup of $\mathcal{G}_*$ and $\mathcal{G}_\circ$ by Lemma 3.6 and Corollary 3.12. So the amamlgamated structure on $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is nontrivial. Finally, the index of $\mathcal{G}_*$ is uncountable by Lemma 3.7 and the index of $\mathcal{G}_\circ$ is uncountable by Corollary 3.12. □

Theorem 1.3. The homomorphism $\Psi\colon\ \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)\to\bigoplus_{(0,1]}\mathbb{Z}/2\mathbb{Z}$ from Proposition 3.11 coincides with the one given in [18, Proposition 5.3] since its restriction to $\mathcal{J}_\circ$ is the surjective homomorphism $\psi\colon\mathcal{J}_\circ\to\bigoplus_{(0,1]}\mathbb{Z}/2\mathbb{Z}$ and its kernel contains $\mathcal{G}_*$ by construction, hence it also contains $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ and $\mathcal{J}_*$. The kernel of Ψ is computed in [18, §6] by using [18, §2–3] and is equal to $[\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2),\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)]$ and to the normal subgroup generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$. □

Proof of Corollary 1.2. By Theorem 1.1, the group $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ acts on the Bass-Serre tree T of the amalgamated product $\mathcal{G}_*\ast_{\mathcal{G}_*\cap\mathcal{G}_\circ}\mathcal{G}_\circ$. Then every element of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ of finite order has a fixed point on T. It follows that very finite subgroup of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ has a fixed point on T [15, §I.6.5, Corollary 3], and is in particular conjugate to a subgroup of $\mathcal{G}_*$ or of $\mathcal{G}_\circ$. For infinite algebraic subgroups of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$, it suffices to check the claim for the maximal algebraic subgroups of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$. By [12, Theorem 1.1], the infinite maximal algebraic subgroups of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ are conjugate to the group $\mathrm{Aut}_{\mathbb{R}}(X)$ of real automorphisms of a real rational surface X from the following list:

(1) $X=\mathbb{P}^2$,
(2) $X=\mathcal{Q}$,
(3) $X=\mathbb{F}_n$, $n=0$, $n\geq 2$,
(4) X is a del Pezzo surface of degree 6 with a birational morphism $X\longrightarrow\mathbb{F}_0$ blowing-up a pair of non-real conjugate points.

(5) X is a del Pezzo surface of degree 6 with a birational morphism $X \longrightarrow \mathbb{F}_0$ blowing-up two real points on $\mathbb{F}_0$,

(6) There is a birational morphism $X/\mathbb{P}^1 \longrightarrow \mathcal{CB}_6/\mathbb{P}^1$ of conic bundles blowing up $n \geq 1$ pairs of non-real conjugate points on non-real fibres on the pair of non-real conjugate disjoint (-1)-curves of $\mathcal{CB}_6$ (the exceptional divisors of the link $\mathcal{CB}_6 \longrightarrow \mathcal{Q}$ of type III),

(7) There is a birational morphism $X \longrightarrow \mathbb{F}_n$ of conic bundles blowing up $2n \geq 4$ points on the zero section of self-intersection n.

(1)&(2)&(3) We have $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) \subset \mathcal{H} = \mathcal{G}_* \cap \mathcal{G}_\circ$. The group $\mathrm{Aut}_{\mathbb{R}}(\mathcal{Q})$ is conjugate to a subgroup of $\mathcal{G}_\circ$ via a link $f\colon \mathcal{Q} \dashrightarrow \mathbb{P}^2$ of type II, and $\mathrm{Aut}_{\mathbb{R}}(\mathbb{F}_n)$ is conjugate to a subgroup of $\mathcal{G}_*$ via a (possibly empty) composition of links of type II and one link of type III $\mathbb{F}_n \dashrightarrow \mathbb{F}_1 \longrightarrow \mathbb{P}^2$.

(4) The surface X contains exactly three pairs of non-real conjugate disjoint (-1)-curves. The group $\mathrm{Aut}_{\mathbb{R}}(X)$ is generated by the lift of a subgroup of $\mathrm{Aut}_{\mathbb{R}}(\mathbb{F}_0)$ and two elements, each of which descends via the contraction $X \longrightarrow \mathbb{F}_0$ of one pair of (-1)-curves to birational maps of $\mathbb{F}_0$ preserving one of the two fibrations $\mathbb{F}_0/\mathbb{P}^1$ [12, Proposition 3.5(2)&(3)]. It follows that $\mathrm{Aut}_{\mathbb{R}}(X)$ is conjugate to a subgroup of $\mathcal{G}_*$.

(5) The surface X contains exactly six real (-1)-curves. Via the blow-down $\eta\colon X \to \mathbb{P}^2$ of three disjoint ones, the group $\mathrm{Aut}_{\mathbb{R}}(X)$ is conjugate to a subgroup of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ generated by the of $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ preserving the images of the (-1)-curves and by $[x:y:z] \mapstochar\dashrightarrow [yz:xz:xy]$ [12, Proposition 3.6(2)&(3)]. So, $\eta\,\mathrm{Aut}_{\mathbb{R}}(X)\eta^{-1} \subset \mathcal{G}_*$.

(6) The group $\mathrm{Aut}_{\mathbb{R}}(X)$ is generated by the lift of a subgroup of $\mathrm{Aut}_{\mathbb{R}}(\mathcal{Q})$ by $X \longrightarrow \mathcal{CB}_6 \longrightarrow \mathcal{Q}$ and by elements descending via $X \longrightarrow \mathcal{CB}_6$ to birational maps of $\mathcal{CB}_6$ preserving the conic bundle structure [12, Propositio 4.5(1)&(2)]. It follows that $\mathrm{Aut}_{\mathbb{R}}(X)$ is conjugate to a subgroup of $\mathcal{G}_\circ$.

(7) The group $\mathrm{Aut}_{\mathbb{R}}(X)$ is generated by the lift of a subgroups of $\mathrm{Aut}_{\mathbb{R}}(\mathbb{F}_n)$ by $X \longrightarrow \mathbb{F}_n$ and by elements descending via $X \longrightarrow \mathbb{F}_n$ to birational maps of $\mathbb{F}_n$ preserving the conic bundle structure [12, Proposition 4.8(1)&(2)]. It follows that $\mathrm{Aut}_{\mathbb{R}}(X)$ is conjugate to a subgroup of $\mathcal{G}_*$. □

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