ALGEBRAIC SUBGROUPS OF THE PLANE CREMONA GROUP OVER A PERFECT FIELD

JULIA SCHNEIDER AND SUSANNA ZIMMERMANN

Contents

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1. Correction of main theorems

Let $\mathbf{k}$ be a perfect field. For a conic fibration $\pi\colon X \longrightarrow \mathbb{P}^1$ we denote by $\mathrm{Aut}(X,\pi) \subset \mathrm{Aut}(X)$ the subgroup preserving the conic fibration, by $\mathrm{Aut}(X/\pi) \subset \mathrm{Aut}(X,\pi)$ its subgroup inducing the identity on $\mathbb{P}^1$, and by $\mathrm{Aut}_{\mathbf{k}}(X,\pi)$ and $\mathrm{Aut}_{\mathbf{k}}(X/\pi)$ their $\mathbf{k}$-points. For a $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-invariant collection $p_1,\dots,p_r \in X(\overline{\mathbf{k}})$ of points, we denote by $\mathrm{Aut}_{\mathbf{k}}(X,p_1,\dots,p_r)$, resp. $\mathrm{Aut}_{\mathbf{k}}(X,\{p_1,\dots,p_r\})$, the subgroup of $\mathrm{Aut}_{\mathbf{k}}(X)$ fixing each p_i, resp. preserving the set $\{p_1,\dots,p_r\}$. A *splitting field* of $\{p_1,\dots,p_r\}$ is a finite normal extension $L/\mathbf{k}$ of smallest degree such that $p_1,\dots,p_r \in X(L)$ and such that $\{p_1,\dots,p_r\}$ is a union of $\mathrm{Gal}(L/\mathbf{k})$-orbits.

Suppose that $\mathbf{k}$ has a quadratic extension $L/\mathbf{k}$ and let g be the generator of $\mathrm{Gal}(L/\mathbf{k}) \simeq \mathbb{Z}/2$. By $\mathcal{Q}^L$ we denote the $\mathbf{k}$-structure on $\mathbb{P}^1_L\times\mathbb{P}^1_L$ given by $(x,y)^g=(y^g,x^g)$. By $\mathcal{S}^{L,L'}$ we denote a surface obtained by blowing up $\mathcal{Q}^L$ in a point p of degree 2, where $L'/\mathbf{k}$ is the splitting field of p, whose geometric components are not on the same ruling of $\mathbb{P}^1_L\times\mathbb{P}^1_L$. We will show in Lemma **??** that its isomorphism class depends only on the isomorphism classes of L, L'. In Theorem 1.1(6b), we denote by $E\subset\mathcal{S}^{L,L'}$ its exceptional divisor.

Below in Theorem 1.1 and Theorem 1.2 we correct case (5a) of Theorem 1.1 and Theorem 1.4 in [2].

Theorem 1.1 ([2, Theorem 1.1]). *Let $\mathbf{k}$ be a perfect field and G an infinite algebraic subgroup of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$. Then there is a $\mathbf{k}$-birational map $\mathbb{P}^2 \dashrightarrow X$ that conjugates G to a subgroup of $\mathrm{Aut}(X)$, with X one of the following surfaces, where no indication of the $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-action means the canonical action.*

(1) $X=\mathbb{P}^2$ and $\mathrm{Aut}(\mathbb{P}^2)\simeq \mathrm{PGL}_3$
(2) $X=\mathbb{F}_0$ and $\mathrm{Aut}(\mathbb{F}_0)\simeq \mathrm{Aut}(\mathbb{P}^1)^2\rtimes\mathbb{Z}/2\simeq\mathrm{PGL}_2^2\rtimes\mathbb{Z}/2$
(3) $X=\mathcal{Q}^L$ and $\mathrm{Aut}(\mathcal{Q}^L)$ is the $\mathbf{k}$-structure on $\mathrm{Aut}(\mathbb{P}^1_L)^2\rtimes\mathbb{Z}/2$ given by the $\mathrm{Gal}(L/\mathbf{k})$-action $(A,B,\tau)^g=(B^g,A^g,\tau)$, where $L/\mathbf{k}$ is a quadratic extension.

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(4) *$X = \mathbb{F}_n$, $n \geqslant 2$, and the action of* $\mathrm{Aut}(\mathbb{F}_n)$ *on* $\mathbb{P}^1$ *induces a split exact sequence*

$$1 \longrightarrow V_{n+1} \longrightarrow \mathrm{Aut}(\mathbb{F}_n) \longrightarrow \mathrm{GL}_2/\mu_n \longrightarrow 1$$

where $\mu_n = \{a\,\mathrm{id} \mid a^n = 1\}$ *and* V_{n+1} *is a vector space of dimension* $n+1$.

(5) *X is a del Pezzo surface of degree 6 with* $\mathrm{NS}(X_{\overline{\mathbf{k}}})^{\mathrm{Aut}_{\overline{\mathbf{k}}}(X)} = 1$. *The action of* $\mathrm{Aut}_{\overline{\mathbf{k}}}(X)$ *on* $\mathrm{NS}(X_{\overline{\mathbf{k}}})$ *induces the split exact sequence*

$$1 \to (\overline{\mathbf{k}}^*)^2 \longrightarrow \mathrm{Aut}_{\overline{\mathbf{k}}}(X) \longrightarrow \mathrm{Sym}_3 \times \mathbb{Z}/2 \to 1.$$

Moreover, we are in one of the following cases.

(a) $\mathrm{rk}\,\mathrm{NS}(X) = 1$ *and there is a quadratic extension* $L/\mathbf{k}$ *and a birational morphism* $\pi\colon X_L \longrightarrow \mathbb{P}^2_L$ *blowing up a point* $p = \{p_1, p_2, p_3\}$ *of degree 3 with splitting field F over* $\mathbf{k}$ *containing L, and one of the following cases holds:*

(i) $\mathrm{Gal}(F/L) \simeq \mathbb{Z}/3$ *and* $\mathrm{Gal}(F/\mathbf{k}) \simeq \mathbb{Z}/6$ *and the action of* $\mathrm{Aut}_{\mathbf{k}}(X)$ *on* $\mathrm{NS}(X)$ *induces the split exact sequence*

$$1 \to \mathrm{Aut}_L(\mathbb{P}^2, p_1, p_2, p_3)^{\pi\,\mathrm{Gal}(L/\mathbf{k})\pi^{-1}} \longrightarrow \mathrm{Aut}_{\mathbf{k}}(X) \longrightarrow \mathbb{Z}/6 \to 1$$

(ii) $\mathrm{Gal}(F/L) \simeq \mathrm{Sym}_3$ *and* $\mathrm{Gal}(F/\mathbf{k}) \simeq \mathrm{Sym}_3 \times \mathbb{Z}/2$ *and the action of* $\mathrm{Aut}_{\mathbf{k}}(X)$ *on* $\mathrm{NS}(X)$ *induces the split exact sequence*

$$1 \to \mathrm{Aut}_L(\mathbb{P}^2, p_1, p_2, p_3)^{\pi\,\mathrm{Gal}(L/\mathbf{k})\pi^{-1}} \longrightarrow \mathrm{Aut}_{\mathbf{k}}(X) \longrightarrow \mathbb{Z}/2 \to 1,$$

(iii) $\mathrm{Gal}(F/L) \simeq \mathbb{Z}/3$, *and* $\mathrm{Gal}(F/\mathbf{k}) \simeq \mathrm{Sym}_3$ *acts on the hexagon of X by a rotation of order 3 and a reflection at an axis through two vertices. The action of* $\mathrm{Aut}_{\mathbf{k}}(X)$ *on* $\mathrm{NS}(X)$ *induces the split exact sequence*

$$1 \to \mathrm{Aut}_L(\mathbb{P}^2, p_1, p_2, p_3)^{\pi\,\mathrm{Gal}(L/\mathbf{k})\pi^{-1}} \longrightarrow \mathrm{Aut}_{\mathbf{k}}(X) \longrightarrow \mathbb{Z}/2 \to 1$$

(b) $\mathrm{rk}\,\mathrm{NS}(X) \geqslant 2$, $\mathrm{rk}\,\mathrm{NS}(X)^{\mathrm{Aut}_{\mathbf{k}}(X)} = 1$ *and X is one of the following:*

(i) *X is the blow-up of* $\mathbb{P}^2$ *in the coordinate points, and the action of* $\mathrm{Aut}_{\mathbf{k}}(X)$ *on* $\mathrm{NS}(X)$ *induces the split exact sequence*

$$1 \to (\mathbf{k}^*)^2 \longrightarrow \mathrm{Aut}_{\mathbf{k}}(X) \longrightarrow \mathrm{Sym}_3 \times \mathbb{Z}/2 \to 1.$$

(ii) *X is the blow-up of* $\mathbb{F}_0$ *in a point* $p = \{(p_1, p_1), (p_2, p_2)\}$ *of degree 2. The action of* $\mathrm{Aut}_{\mathbf{k}}(X)$ *on* $\mathrm{NS}(X)$ *induces the exact sequence,*

$$1 \to \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^1, p_1, p_2)^2 \longrightarrow \mathrm{Aut}_{\mathbf{k}}(X) \longrightarrow \mathrm{Sym}_3 \times \mathbb{Z}/2 \to 1$$

which is split if $\mathrm{char}(\mathbf{k}) \neq 2$.

(iii) *X is the blow-up of* $\mathbb{P}^2$ *in a point* $p = \{p_1, p_2, p_3\}$ *of degree 3 with splitting field L such that* $\mathrm{Gal}(L/\mathbf{k}) \simeq \mathbb{Z}/3$. *The action of* $\mathrm{Aut}_{\mathbf{k}}(X)$ *on* $\mathrm{NS}(X)$ *induces the split exact sequence*

$$1 \to \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2, p_1, p_2, p_3) \longrightarrow \mathrm{Aut}_{\mathbf{k}}(X) \longrightarrow \mathbb{Z}/6 \to 1$$

(iv) *X is the blow-up of* $\mathbb{P}^2$ *in a point* $p = \{p_1, p_2, p_3\}$ *of degree 3 with splitting field L such that* $\mathrm{Gal}(L/\mathbf{k}) \simeq \mathrm{Sym}_3$. *The action of* $\mathrm{Aut}_{\mathbf{k}}(X)$ *on* $\mathrm{NS}(X)$ *induces the split exact sequence*

$$1 \to \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2, p_1, p_2, p_3) \longrightarrow \mathrm{Aut}_{\mathbf{k}}(X) \longrightarrow \mathbb{Z}/2 \to 1$$

where $\mathbb{Z}/2$ *is generated by a rotation.*

(c) $\mathrm{rk}\,\mathrm{NS}(X)^{\mathrm{Aut}_{\mathbf{k}}(X)} = 2$ *and there is a quadratic extension* $L/\mathbf{k}$ *and a birational morphism* $\nu\colon X \longrightarrow \mathcal{Q}^L$ *contracting two curves onto rational points* p_1, p_2 *or one curve onto a point* $\{p_1, p_2\}$ *of degree 2 with splitting field* $L'/\mathbf{k}$. *The action* *of* $\mathrm{Aut}_{\mathbf{k}}(X)$ *on* $\mathrm{NS}(X)$ *induces the split exact sequence*

$$1 \to T^{L,L'}(\mathbf{k}) \longrightarrow \mathrm{Aut}_{\mathbf{k}}(X) \longrightarrow \mathbb{Z}/2 \times \mathbb{Z}/2 \to 1$$

where $\nu \,\mathrm{Aut}_{\mathbf{k}}(X)\nu^{-1} = \mathrm{Aut}_{\mathbf{k}}(\mathcal{Q}^L, \{p_1, p_2\})$ *and* $T^{L,L'}$ *is the subgroup of* $\mathrm{Aut}_{\mathbf{k}}(\mathcal{Q}^L, p_1, p_2)$ *preserving the rulings of* $\mathcal{Q}^L_L$.

(6) $\pi\colon X \longrightarrow \mathbb{P}^1$ *is one of the following conic fibrations with*

$$\mathrm{rk}\,\mathrm{NS}(X_{\overline{\mathbf{k}}}/\mathbb{P}^1)^{\mathrm{Aut}_{\overline{\mathbf{k}}}(X,\pi)} = \mathrm{rk}\,\mathrm{NS}(X/\mathbb{P}^1)^{\mathrm{Aut}_{\mathbf{k}}(X,\pi)} = 1 :$$

(a) $X/\mathbb{P}^1$ *is the blow-up of points* $p_1, \ldots, p_r \in \mathbb{F}_n$, $n \geqslant 2$, *contained in a section* $S_n \subset \mathbb{F}_n$ *with* $S_n^2 = n$. *The geometric components of the* p_i *are on pairwise distinct geometric fibres and* $\sum_{i=1}^r \deg(p_i) = 2n$. *There are split exact sequences*

$$\begin{tikzcd}
 & (T_1/\mu_n) \rtimes \mathbb{Z}/2 \arrow[d, phantom, "\wr"] & \mathrm{Aut}(X) \arrow[d, equal] & & \\
1 \arrow[r] & \mathrm{Aut}(X/\pi_X) \arrow[r] & \mathrm{Aut}(X,\pi_X) \arrow[r] & \mathrm{Aut}(\mathbb{P}^1,\Delta) \arrow[r] & 1 \\
1 \arrow[r] & \mathrm{Aut}_{\mathbf{k}}(X/\pi_X) \arrow[r] \arrow[d, phantom, "\wr"] & \mathrm{Aut}_{\mathbf{k}}(X,\pi_X) \arrow[r] \arrow[d, equal] & \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^1,\Delta) \arrow[r] & 1 \\
 & (\mathbf{k}^*/\mu_n(\mathbf{k})) \rtimes \mathbb{Z}/2 & \mathrm{Aut}_{\mathbf{k}}(X) & &
\end{tikzcd}$$

where $\Delta = \pi(\{p_1, \ldots, p_r\}) \subset \mathbb{P}^1$, T_1 *is the split one-dimensional torus and* μ_n *its subgroup of* n^{th} *roots of unity.*

(b) There exist quadratic extensions L *and* L' *of* $\mathbf{k}$ *such that* $X/\mathbb{P}^1$ *is the blow-up of* $\mathcal{S}^{L,L'}$ *in points* $p_1, \ldots, p_r \in E$, $r \geqslant 1$. *The* p_i *are all of even degree, their geometric components are on pairwise distinct geometric components of smooth fibres and each geometric component of* E *contains half of the geometric components of each* p_i. *There are exact sequences*

$$\begin{tikzcd}
 & \mathrm{SO}^{L,L'} \rtimes \mathbb{Z}/2 \arrow[d, phantom, "\wr"] & \mathrm{Aut}(X) \arrow[d, equal] & & \\
1 \arrow[r] & \mathrm{Aut}(X/\pi_X) \arrow[r] & \mathrm{Aut}(X,\pi_X) \arrow[r] & \mathrm{Aut}(\mathbb{P}^1,\Delta) \arrow[r] & 1 \\
1 \arrow[r] & \mathrm{Aut}_{\mathbf{k}}(X/\pi_X) \arrow[r] \arrow[d, phantom, "\wr"] & \mathrm{Aut}_{\mathbf{k}}(X,\pi_X) \arrow[r] \arrow[d, equal] & (D_{\mathbf{k}}^{L,L'} \rtimes \mathbb{Z}/2) \cap \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^1,\Delta) \arrow[r] & 1 \\
 & \mathrm{SO}^{L,L'}(\mathbf{k}) \rtimes \mathbb{Z}/2 & \mathrm{Aut}_{\mathbf{k}}(X) & &
\end{tikzcd}$$

with $\Delta = \pi(\{p_1, \ldots, p_r\}) \subset \mathbb{P}^1$ *and* $\mathrm{SO}^{L,L'} = \{(a,b) \in T^L \mid ab = 1\}$, *and*

- *if* L, L' *are* $\mathbf{k}$*-isomorphic, then* $\mathrm{SO}^{L,L'}(\mathbf{k}) \simeq \{a \in L^* \mid aa^g = 1\}$ *and* $D_{\mathbf{k}}^{L,L'} \simeq \{\alpha \in k^* \mid \alpha = \lambda\lambda^g, \lambda \in L\}$, *where* g *is the generator of* $\mathrm{Gal}(L/\mathbf{k})$,
- *if* L, L' *are not* $\mathbf{k}$*-isomorphic, then* $\mathrm{SO}^{L,L'}(\mathbf{k}) \simeq \mathbf{k}^*$ *and* $D_{\mathbf{k}}^{L,L'} \simeq \{\lambda\lambda^{gg'} \in F \mid \lambda \in K, \lambda\lambda^{g'} = 1\}$, *where* $\mathbf{k} \subset F \subset LL'$ *is the intermediate extension such that* $\mathrm{Gal}(F/\mathbf{k}) \simeq \langle gg' \rangle \subset \mathrm{Gal}(L/\mathbf{k}) \times \mathrm{Gal}(L'/\mathbf{k})$, *where* g, g' *are the generators of* $\mathrm{Gal}(L/\mathbf{k}), \mathrm{Gal}(L'/\mathbf{k})$, *respectively.*

The statement of [2, Theorem 1.2 and Corollary 1.3] remain correct, and we complete their proofs further below.

Theorem 1.2 ([2, Theorem 1.4])**.** *Let* $\mathbf{k}$ *be a perfect field. The conjugacy classes of the maximal subgroups* $\mathrm{Aut}_{\mathbf{k}}(X)$ *of* $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ *from Theorem 1.1 are parametrised by*

- *(1), (2): one point*
- *(3): one point for each* $\mathbf{k}$*-isomorphism class of quadratic extensions of* $\mathbf{k}$
- *(4): one point for each* $n \geqslant 2$
- *(5(a)i) if* $|\mathbf{k}| \geqslant 3$*, one point for any pair of extensions* $F \supset L \supset \mathbf{k}$*, where* $L/\mathbf{k}$ *is quadratic and* F/L *such that* $\mathrm{Gal}(F/L) \simeq \mathbb{Z}/3$ *and* $\mathrm{Gal}(F/\mathbf{k}) \simeq \mathbb{Z}/6$*, up to the following equivalence class:* $\mathbf{k}$*-isomorphisms* $F \simeq F'$ *that induces an isomorphism* $L \simeq L'$*.*
- *(5(a)ii): one point for any pair* $F \supset L \supset \mathbf{k}$*, where* $L/\mathbf{k}$ *is quadratic and* F/L *such that* $\mathrm{Gal}(F/L) \simeq \mathrm{Sym}_3$ *and* $\mathrm{Gal}(F/\mathbf{k}) \simeq \mathrm{Sym}_3 \times \mathbb{Z}/2$*, up to the following equivalence class:* $\mathbf{k}$*-isomorphisms* $F \simeq F'$ *that induces an isomorphism* $L \simeq L'$*.*
- *(5(a)iii): one point for any pair* $F \supset L \supset \mathbf{k}$*, where* $L/\mathbf{k}$ *is quadratic and* F/L *such that* $\mathrm{Gal}(F/L) \simeq \mathbb{Z}/3$ *and* $\mathrm{Gal}(F/\mathbf{k}) \simeq \mathrm{Sym}_3$*, up to the following equivalence class:* $\mathbf{k}$*-isomorphisms* $F \simeq F'$ *that induces an isomorphism* $L \simeq L'$*.*
- *(5(b)i): one point if* $|\mathbf{k}| \geqslant 3$
- *(5(b)ii): one point for each* $\mathbf{k}$*-isomorphism class of quadratic extensions of* $\mathbf{k}$
- *(5(b)iii): one point for each* $\mathbf{k}$*-isomorphism class of Galois extensions* $F/\mathbf{k}$ *with* $\mathrm{Gal}(F/\mathbf{k}) \simeq \mathbb{Z}/3$*.*
- *(5(b)iv): one point for any* $\mathbf{k}$*-isomorphism class of Galois extensions* $F/\mathbf{k}$ *with* $\mathrm{Gal}(F/\mathbf{k}) \simeq \mathrm{Sym}_3$*.*
- *(6a): for each* $n \geqslant 2$ *the set of points* $\{p_1, \dots, p_r\} \subset \mathbb{P}^1$ *with* $\sum_{i=1}^r \deg(p_i) = 2n$ *up to the action of* $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^1)$
- *(6b): for each* $n \geqslant 1$ *and for each pair of* $\mathbf{k}$*-isomorphism classes of quadratic extensions* (L, L')*, the set of points* $\{p_1, \dots, p_r\} \subset \mathbb{P}^1$ *of even degree with* $\sum_{i=1}^r \deg(p_i) = 2n$ *up to the action of* $D_{\mathbf{k}}^{L,L'}(\mathbf{k}) \rtimes \mathbb{Z}/2$

2. Corrections in remaining article

Throughout the article, $\mathbf{k}$ denotes a perfect field and $\overline{\mathbf{k}}$ an algebraic closure. If not mentioned otherwise, any surface, curve, point and rational map will be defined over the perfect field $\mathbf{k}$. By a geometric component of a curve C (resp. a point $p = \{p_1, \dots, p_d\}$), we mean an irreducible component of $C_{\overline{\mathbf{k}}}$ (resp. one of $p_1, \dots, p_d$).

By the following lemma, whenever we contract a curve onto a point of degree 2 in $\mathcal{Q}^L$ with splitting field L, we can choose the point conveniently.

The proof of the following lemma contained a gap that we now close.

Lemma 2.1. [2, Lemma 3.7] *Let* $p = \{p_1, p_2, p_3\}$ *and* $q = \{q_1, q_2, q_3\}$ *be points in* $\mathcal{Q}^L$ *of degree* 3 *such that for any* $h \in \mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$ *there exists* $\sigma \in \mathrm{Sym}_3$ *such that* $p_i^h = p_{\sigma(i)}$ *and* $q_i^h = q_{\sigma(i)}$*. Suppose that the geometric components of* p *(resp. of* q*) are in general position on* $\mathcal{Q}^L$*. Then there exists* $\alpha \in \mathrm{Aut}_{\mathbf{k}}(\mathcal{Q}^L)$ *such that* $\alpha(p_i) = q_i$ *for* $i = 1, 2, 3$*.*

Proof. The assumption on p and q implies that the residue fields of p and q are $\mathbf{k}$-isomorphic [1, Lemma 3.3]. Since they are points of degree 3, p and q have therefore the same splitting field $F/\mathbf{k}$. Let g be the generator of the Galois group $\mathrm{Gal}(L/\mathbf{k})$ of order 2. We consider the composite field FL. For $i = 1, \dots, 4$, we can write $p_i = (a_i, a_i')$, $q_i = (b_i, b_i')$ with $a_i, a_i', b_i, b_i' \in \mathbb{P}^1_{FL}$. By hypothesis, for any $h \in \mathrm{Gal}(\overline{\mathbf{k}}/L)$ there exists $\sigma \in \mathrm{Sym}_3$ such that one has $a_i^h = a_{\sigma(i)}$; similarly for a_i' and b_i, b_i'. We apply [2, Remark 2.7] to the $\mathrm{Gal}(\overline{FL}/FL)$-invariant sets $\{a_i\}_i$ and $\{a_i'\}$ in $\mathbb{P}^1 \times \mathbb{P}^1$, respectively $\{b_i\}$ and

$\{b_i'\}$, and find a unique $(A,B) \in \mathrm{PGL}_2(FL) \times \mathrm{PGL}_2(FL)$ such that $(A,B)p_i = q_i$ for $i = 1,2,3$. It remains to see that (A,B) gives an automorphism of $\mathcal{Q}^L$, that is, (A,B) commutes with the action of $\mathrm{Gal}(LF/\mathbf{k})$ induced on $\mathbb{P}^1 \times \mathbb{P}^1$. Let $h \in \mathrm{Gal}(LF/\mathbf{k})$ and let $\sigma \in \mathrm{Sym}_3$ be the permutation induced by h. We compute $((A,B)p_i)^h = q_i^h = q_{\sigma(i)}$ and $(A,B)p_i^h = (A,B)p_{\sigma(i)} = q_{\sigma(i)}$ for $i = 1,2,3$. This concludes the proof since matrices in $\mathrm{PGL}_2(\bar{\mathbf{k}})$ are uniquely determined by their action on three points. □

The hexagon of $X_{\bar{\mathbf{k}}}$ is $\mathrm{Gal}(\bar{\mathbf{k}}/\mathbf{k})$-invariant. The Galois group $\mathrm{Gal}(\bar{\mathbf{k}}/\mathbf{k})$ acts on the hexagon by symmetries, so we have homomorphism of groups

$$\mathrm{Gal}(\bar{\mathbf{k}}/\mathbf{k}) \xrightarrow{\rho} \mathrm{Sym}_3 \times \mathbb{Z}/2 \subseteq \mathrm{Aut}(\mathrm{NS}(X_{\bar{\mathbf{k}}})).$$

By *hexagon of* X we mean the hexagon of $X_{\bar{\mathbf{k}}}$ endowed with its canonical $\mathrm{Gal}(\bar{\mathbf{k}}/\mathbf{k})$-action. Since the group $\mathrm{Aut}_{\mathbf{k}}(X)$ acts by symmetries on the hexagon of X, it induces a homomorphism

$$\hat{\rho}\colon\ \mathrm{Aut}_{\mathbf{k}}(X) \longrightarrow \mathrm{Sym}_3 \times \mathbb{Z}/2.$$

In our classification of subgroups of $\mathrm{Sym}_3 \times \mathbb{Z}/2$ there is one missing case, namely Sym_3 acting transitively on the edges of a hexagon as (10) in Figure 1, that is generated by a rotation of degree 3 and a reflection at an axis through two vertices; this is because the dihedral group $\mathrm{Sym}_3 \times \mathbb{Z}/2$ contains two non-conjugate embeddings of Sym_3. (The other one is Figure 1(8).)

In Figure 1 below, we redraw the hexagons of [2, Figure 1] in a slightly different manner, but the numbering remains the same. Here, we choose generators of the respective subgroup of $\mathrm{Sym}_3 \times \mathbb{Z}/2$ as in the proof of Lemma 2.2 and draw the image of each edge under every generator.

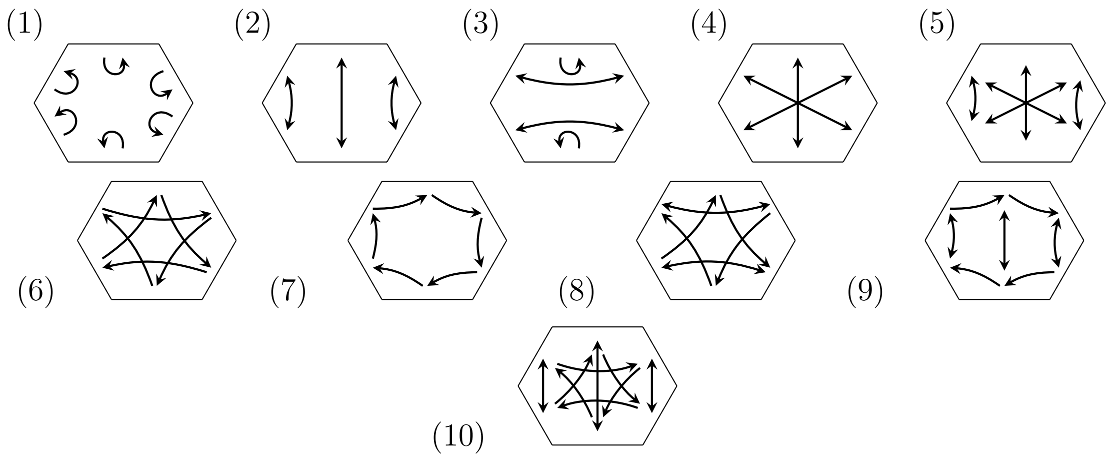

FIGURE 1. The $\mathrm{Gal}(\bar{\mathbf{k}}/\mathbf{k})$-actions on the hexagon of a del Pezzo surface of degree 6.

Lemma 2.2. *The action of* $\rho(\mathrm{Gal}(\bar{\mathbf{k}}/\mathbf{k}))$ *on the hexagon of a del Pezzo surface of degree* 6 *is as in Figure 1.*

Proof. The dihedral group $D_6 = \langle r, s \mid r^6 = s^2 = \mathrm{id}, srs = r^{-1}\rangle$ equals $\mathrm{Sym}_3 \times \mathbb{Z}/2 = \langle r^2, s\rangle \times \langle r^3\rangle$, where r is a rotation of order 6, as in Figure 1(7), and s is a reflection as in Figure 1(2). Writing $t = r^3 s$ for a reflection as in Figure 1(3), the action of $\rho(\mathrm{Gal}(\bar{\mathbf{k}}/\mathbf{k}))$ is one of the following, up to conjugation. (For convenience, we give references to the lemmas that deal with the respective cases.)

(1) The trivial subgroup as in Figure 1(1), (see [2, Lemma 4.1])
(2) $\langle s\rangle = \mathbb{Z}/2$ as in Figure 1(2), (see [2, Lemma 4.11])
(3) $\langle t\rangle = \mathbb{Z}/2$ as in Figure 1(3), (see [2, Lemma 4.9])
(4) $\langle r^3\rangle = \mathbb{Z}/2$ as in Figure 1(4), (see [2, Lemma 4.10])
(5) $\langle r^3, s\rangle = \mathbb{Z}/2\times\mathbb{Z}/2$ as in Figure 1(5), (see [2, Lemma 4.12])
(6) $\langle r^2\rangle = \mathbb{Z}/3$ as in Figure 1(6), (see [2, Lemma 4.2])
(7) $\langle r\rangle = \mathbb{Z}/6$ as in Figure 1(7), (see [2, Lemma 4.6] and Remark 2.3)
(8) $\langle r^2, t\rangle = \mathrm{Sym}_3$ as in Figure 1(8), (see [2, Lemma 4.3])
(9) $\langle r, s\rangle = \mathrm{Sym}_3\times\mathbb{Z}/2$ as in Figure 1(9), (see [2, Lemma 4.7] and Remark 2.3),
(10) $\langle r^2, s\rangle = \mathrm{Sym}_3$ as in Figure 1(10), (see Lemma 2.4 below).

□

We now discuss the cases where $\rho(\mathrm{Gal}(\bar{\mathbf{k}}/\mathbf{k}))$ acts transitively on the six edges of the hexagon, that is, (7), (9) and the missing case (10) in Figure 1.

Remark 2.3. In [2, Lemma 4.6 and Lemma 4.7], which describe the situation of (7) and (9) in Figure 1, the correct statement is that there is a quadratic extension $L/\mathbf{k}$ and a Galois extension F/L with respectively

(1) $\mathrm{Gal}(F/L)\simeq\mathbb{Z}/3$ and $\mathrm{Gal}(F/\mathbf{k})\simeq\mathbb{Z}/6$ for [2, Lemma 4.6], and
(2) $\mathrm{Gal}(F/L)\simeq\mathrm{Sym}_3$ and $\mathrm{Gal}(F/\mathbf{k})\simeq\mathrm{Sym}_3\times\mathbb{Z}/2$ for [2, Lemma 4.7].

This comes out of the analysis of the del Pezzo surfaces in Figure 1(7) and (9) in [2, Lemma 4.6 and Lemma 4.7] and is stated incorrectly in the statements of the two lemmas.

Similarly, the correct assumptions in [2, Example 4.8] should be F/L as above. In this case it holds that $\mathrm{Gal}(FL/L)\simeq\mathrm{Gal}(F/\mathbf{k})$, as claimed in said example. In fact, the missing case is $\mathrm{Gal}(F/L)\simeq\mathbb{Z}/3$ and $\mathrm{Gal}(F/\mathbf{k})=\mathrm{Sym}_3$.

Lemma 2.4. *Let X be a del Pezzo surface of degree 6 such that $\rho(\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k}))\simeq\mathrm{Sym}_3$ as indicated in Figure 1(10). Then $X\longrightarrow *$ is a Mori fibre space and the following hold:*

(1) There exists a quadratic field extension $L/\mathbf{k}$ and a point $p=\{p_1,p_2,p_3\}$ in $\mathbb{P}^2_L$ of degree 3 such that X_L is isomorphic to the blow-up of $\mathbb{P}^2_L$ in p. Moreover there is a cyclic extension F/L of degree 3 such that each (-1)-curve in the hexagon of X is defined over F.
(2) Two such surfaces X and X' are isomorphic if and only if there exists an isomorphism $F\longrightarrow F'$ over $\mathbf{k}$ that sends L onto L'.
(3) If X_L is L-rational, then the action of $\mathrm{Aut}_{\mathbf{k}}(X)$ on the hexagon of X induces a split exact sequence

$$1\to(\pi^{-1}\mathrm{Aut}_L(\mathbb{P}^2_L,p_1,p_2,p_3)\pi)^g\to\mathrm{Aut}_{\mathbf{k}}(X)\xrightarrow{\hat{\rho}}\mathbb{Z}/2\to 1$$

where $\mathbb{Z}/2$ is generated by a rotation and g is the generator of $\mathrm{Gal}(L/\mathbf{k})$.

Proof. Every (-1)-curve in the hexagon of X is contained in the same $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-orbit, hence X is a Mori fibre space.

(1)&(3) The group $\mathrm{Aut}_{\mathbf{k}}(X)$ acts by symmetries on the hexagon of X. The only element of $\mathrm{Sym}_3\times\mathbb{Z}/2$ that commutes with action of $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$ is the rotation of order 2, so $\hat{\rho}(\mathrm{Aut}_{\mathbf{k}}(X))\subset\mathbb{Z}/2$. Let us show that $\hat{\rho}(\mathrm{Aut}_{\mathbf{k}}(X))=\mathbb{Z}/2$ and that $\hat{\rho}$ has a section. Let $F/\mathbf{k}$ be the splitting field of a (-1)-curve in the hexagon of X, i.e. the smallest normal field extension of $\mathbf{k}$ over which the curve is defined. Then $\mathrm{Gal}(F/\mathbf{k})\simeq\mathrm{Sym}_3$ with action on the hexagon as in Figure 1(10). Let $r\in\mathrm{Gal}(F/\mathbf{k})$ be the rotation of order 3 indicated in Figure 1(10) and set $L:=F^r\subset F$ the subfield of F fixed by r. Then

$\mathrm{Gal}(F/L) = \langle r\rangle \simeq \mathbb{Z}/3$ and $\mathrm{Gal}(L/\mathbf{k}) \simeq \mathrm{Gal}(F/\mathbf{k})/\mathrm{Gal}(F/L) \simeq \mathbb{Z}/2$ is generated by the reflection in Figure 1(10). The action of $\mathrm{Gal}(\overline{\mathbf{k}}/L)$ on the hexagon of X_L factors through $\mathrm{Gal}(F/L)$, forming two $\mathrm{Gal}(\overline{\mathbf{k}}/L)$-orbits of pairwise disjoint (-1)-curves of size 3. This gives (1).

If X_L is L-rational, the hexagon of X_L is as in [2, Lemma 4.2] and the contraction of one of the $\mathrm{Gal}(\overline{\mathbf{k}}/L)$-orbits in the hexagon of X_L onto a point p of degree 3 is a birational morphism $\pi\colon X_L \longrightarrow \mathbb{P}^2_L$. The kernel of $\widehat{\rho}$ is isomorphic to $(\pi^{-1}\mathrm{Aut}_L(\mathbb{P}^2_L, p_1, p_2, p_3)\pi)^g$, where $p_1, p_2, p_3 \in \mathbb{P}^2(L)$ are the geometric components of p and g is the generator of $\mathrm{Gal}(L/\mathbf{k}) \simeq \mathbb{Z}/2$. The only non-trivial element of $\mathrm{Sym}_3 \times \mathbb{Z}/2$ commuting with $\rho(\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k}))$ is the rotation of order 2, hence $\widehat{\rho}(\mathrm{Aut}_{\mathbf{k}}(X)) \subset \mathbb{Z}/2$. There exists a quadratic involution $\varphi_p \in \mathrm{Bir}_L(\mathbb{P}^2)$ with base-point p that induces a rotation of order 2 on the hexagon of X_L. Since X_L is L-rational, it has an L-rational point q. It is not contained in the hexagon of X_L and we assume moreover that φ_p fixes $\pi(q)$. It remains to check that $\psi := \pi^{-1}\varphi_p\pi \in \mathrm{Aut}(X_L)$ is defined over $\mathbf{k}$. The automorphism $\psi^{-1}g\psi g$ of X_L is conjugate by π to an automorphism of $\mathbb{P}^2_L$ fixing $\pi(q)$ and each geometric component of p, and is hence the identity map. Thus the involution φ_p lifts to a $\mathbf{k}$-automorphism of X. It acts like a rotation of order 2 on the hexagon of X and thus $\hat{\rho}(\mathrm{Aut}_{\mathbf{k}}(X)) = \mathbb{Z}/2$ and the sequence splits.

(2) If there is an automorphism $\tau\colon F \longrightarrow F'$ that sends L onto L' and fixes $\mathbf{k}$, we can identify L and L'. Then the surfaces X_L and $X'_{L'}$ are L-isomorphic [2, Lemma 4.2]. Since τ fixes $\mathbf{k}$, it induces a $\mathbf{k}$-isomorphism of X and X' by the above construction. On the other hand, suppose that X and X' are $\mathbf{k}$-isomorphic. Then the smallest normal field extensions F and F' over which all (-1)-curves of $X_{\overline{\mathbf{k}}}$ and $X'_{\overline{\mathbf{k}}}$ are defined are $\mathbf{k}$-isomorphic. This isomorphism sends the fixed field $L = F^r$ onto the fixed field $L' = F'^r$. □

Example 2.5 (Construction of rational del Pezzo surfaces as in Figure 1(7),(9),(10))**.** (See also [2, Example 4.8].) Let $q = \{q_1, q_2\}$ in $\mathbb{P}^2$ be a point of degree 2, with splitting field $L/\mathbf{k}$ being a quadratic extension, and let $p = \{p_1, p_2, p_3\}$ in $\mathbb{P}^2$ be a point of degree 3, with splitting field $F/\mathbf{k}$ (which is an extension of degree 3 or 6). Assume that the components of p and q are in general position, that is, no three of the five geometric components are collinear. Denote by $D \subset \mathbb{P}^2$ the conic passing through the five geometric components.

Blowing up q and contracting the line passing through q gives a k-birational map $\mathbb{P}^2 \dashrightarrow \mathcal{Q}^L$, where $\mathcal{Q}^L$ is a del Pezzo surface of degree 8 as in [2, Definition 3.1]. The image of p in $\mathcal{Q}^L$, again denoted by p, is in general position, and the strict transform of D is the diagonal passing through p. Blowing up p and then contracting the strict transform of D gives a k-birational map $\mathcal{Q}^L \dashrightarrow X$, where X is a del Pezzo surface of degree 6, and the action on the hexagon of X is one of the following:

(1) If $\mathrm{Gal}(F/\mathbf{k}) \simeq \mathbb{Z}/3\mathbb{Z}$ and so $\mathrm{Gal}(LF/\mathbf{k}) \simeq \mathbb{Z}/6\mathbb{Z}$, then the hexagon of X is as in Figure 1(7). (For example, take $\mathbf{k} = \mathbb{F}_q$, $L = \mathbb{F}_{q^2}$, $F = \mathbb{F}_{q^3}$, or $\mathbf{k} = \mathbb{Q}(\zeta)$ where ζ is a primitive third root of 1, and $L = \mathbf{k}[\sqrt[2]{3}]$ and $F = \mathbf{k}[\sqrt[3]{2}]$.)
(2) If $\mathrm{Gal}(F/\mathbf{k}) \simeq \mathrm{Sym}_3$ and $L \subset F$, hence $\mathrm{Gal}(LF/\mathbf{k}) \simeq \mathrm{Sym}_3$, then the hexagon of X is as in Figure 1(10). (For example, take $\mathbf{k} = \mathbb{Q}$, $L = \mathbb{Q}(\zeta)$, and $F = \mathbb{Q}[\sqrt[3]{2}, \zeta]$.)
(3) If $\mathrm{Gal}(F/\mathbf{k}) \simeq \mathrm{Sym}_3$ and $\mathrm{Gal}(LF/\mathbf{k}) \simeq \mathrm{Sym}_3 \times \mathbb{Z}/2\mathbb{Z}$, then the hexagon of X is as in Figure 1(9). (For example, take $\mathbf{k} = \mathbb{Q}$, $L = \mathbb{Q}[i]$, $F = \mathbb{Q}[\sqrt[3]{2}, \zeta]$.)

Proof of Theorem 1.1. The proof remains the same as in [2, Theorem 1.1], but we have to add the missing case (5(a)iii) from the previous section: if the $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-action is as in Figure 1(10), Lemma 2.4 implies that $(X, \mathrm{Aut}(X))$ is as in Theorem 1.1(5(a)iii). □

We now check whether the missing del Pezzo surface case is $\mathrm{Aut}_{\mathbf{k}}(X)$-birationally rigid or superrigid or neither of them. Let X be as in Figure 1(10). Then the field $\mathbf{k}$ is infinite because any finite field extension of a finite field is cyclic. Suppose that X is $\mathbf{k}$-rational and pick $q \in X(\mathbf{k})$. Let $\pi\colon X_L \longrightarrow \mathbb{P}^2_L$ be the contraction from Lemma 2.4. As in [2, Lemma 7.6], one shows that the map

$$H := (\pi^{-1}\operatorname{Aut}_L(\mathbb{P}^2_L, p_1, p_2, p_3)\pi)^g \longrightarrow X(\mathbf{k}), \quad \alpha \mapsto \pi^{-1}\alpha\pi(q)$$

is a bijection. Since $\mathbf{k}$ is infinite, the group H is infinite, and it acts faithfully on the $\overline{\mathbf{k}}$-points of X outside the hexagon.

Lemma 2.6. *Let X be a del Pezzo surface of degree 6 as in Figure 1(10). Then there are no $\mathrm{Aut}_{\mathbf{k}}(X)$-orbits with $\leqslant 5$ geometric components. In particular, there are no $\mathrm{Aut}_{\mathbf{k}}(X)$-equivariant Sarkisov links starting from X.*

Proof. The group $\mathrm{Aut}_{\mathbf{k}}(X)$ acts transitively on the (-1)-curves of the hexagon of X. By the above remark, H has no orbits with $\leqslant 5$ geometric components outside the hexagon of X. Therefore, there are no $\mathrm{Aut}_{\mathbf{k}}(X)$-equivariant Sarkisov links starting from X. □

While the statement of [2, Theorem 1.2] remains correct, we need to complete its proof with the additional case in the classification of del Pezzo surfaces of degree 6.

Completion of proof of [2, Theorem 1.2]. The proof remains the same, however we have to add the case of the del Pezzo surface X of degree 6 in Figure 1. By the above Lemma 2.6, X is $\mathrm{Aut}_{\mathbf{k}}(X)$-birationally superrigid. □

The proof of [2, Corollary 1.3] remains correct, because the statement of [2, Theorem 1.2] is correct.

Completion of proof of [2, Corollary 1.4]. The correction for the parameter in class (5(a)i) and (5(a)ii) follows from Remark 2.3. The parameter for the additional class (5(a)iii) follows from Lemma 2.4. □

Finally, the new class (5(a)iii) of del Pezzo surfaces X of degree 6 are Mori fibre spaces by Lemma 2.4. As [2, Remark 8.3] explains, this means that the homomorphism

$$\Psi\colon \mathrm{BirMori}(\mathbb{P}^2) \longrightarrow \Big(\bigoplus_{\chi\in M(\mathbb{F}_1)} \mathbb{Z}/2\Big) \mathop{\ast}_{C\in J_5} \Big(\bigoplus_{\chi\in M(C)} \mathbb{Z}/2\Big) \ast \Big(\mathop{\ast}_{C\in J_6} \bigoplus_{\chi\in M(C)} \mathbb{Z}/2\Big)$$

sends the conjugacy class of $\mathrm{Aut}_{\mathbf{k}}(X)$ onto zero. Therefore, the statements and proofs of [2, Theorem 1.6 and Proposition 1.7] remain correct.

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Institut de mathématiques d'Orsay, Université Paris-Saclay, Orsay Ville, France
Email address: susanna.zimmermann@universite-paris-saclay.fr

École Polytechnique Fédérale de Lausanne, Chair of Algebraic Geometry (Bâtiment MA), Station 8, CH-1015 Lausanne, Switzerland
Email address: julia.schneider@epfl.ch