Problem Set 1 Answer Key

1. Fill in the missing prices in the following table of minimum wage levels:

	Nominal Wage ($/hr)	CPI	Real Wage ($$_{1955}$/hr)	Real Wage ($$_{2015}$/hr)
1955	0.75	26.8	0.75	6.63
1985	3.35	107.6	0.83	7.38
2015	7.25	237.0	0.82	7.25

The trick to this problem is thinking about real and nominal prices as a change in units. Each row gives you the quantity in one unit and your job is to convert to the other units. The first row is the most straightforward. We know from the first column that the nominal minimum wage in 1955 was $0.75/hr. To get the real price in $1955, we don't need to do anything. To get the real price in 2015 dollars, we do a standard conversion.

$$\begin{aligned}
\text{Nominal Price}(\$_{1955})\frac{CPI_?}{CPI_?} &= \text{Real Price}(\$_{2015}) \\
\text{Nominal Price}(\$_{1955})\frac{CPI_{2015}}{CPI_{1955}} &= \text{Real Price}(\$_{2015}) \\
\$_{1955}0.75 * \frac{\$_{2015}237.0}{\$_{1955}26.8} &= \$_{2015}6.63
\end{aligned}$$

The second row is perhaps the most confusing, but if we follow the unit conversion problem, we can work it out. Let's consider the first cell of the row first. We know the real wage in 1985 *measured in 2015 dollars* was 7.38. For a nominal price, we always want the dollars we're using to match up with the year we're looking at the price. Thus, we want to put the real wage in 1985 measured in 1985 dollars, so we have to do the reverse of what we did in lecture.

$$\begin{aligned}
\text{Real Price}(\$_{2015})\frac{CPI_?}{CPI_?} &= \text{Nominal Price}(\$_{1985}) \\
\text{Real Price}(\$_{2015})\frac{CPI_{1985}}{CPI_{2015}} &= \text{Nominal Price}(\$_{1985}) \\
\$_{2015}7.38 * \frac{\$_{1985}107.6}{\$_{2015}237.0} &= \$_{1985}3.35
\end{aligned}$$

Once we have the nominal price, we can follow the standard approach to get to the real price in 1955 dollars or we could convert directly as below.

$$\begin{aligned}
\text{Real Price}(\$_{2015})\frac{CPI_?}{CPI_?} &= \text{Real Price}(\$_{1955}) \\
\text{Real Price}(\$_{2015})\frac{CPI_{1955}}{CPI_{2015}} &= \text{Real Price}(\$_{1955}) \\
\$_{2015}7.38 * \frac{\$_{1955}26.8}{\$_{2015}237.0} &= \$_{1955}0.83
\end{aligned}$$

Now, we want to consider the last row. For the first cell, we want to know the nominal wage in 2015 *measured in 2015 dollars*. But that is precisely what we know from the real wage $2015/hr column. For the third cell, we need to convert the wage from 2015 dollars to 1955 dollars.

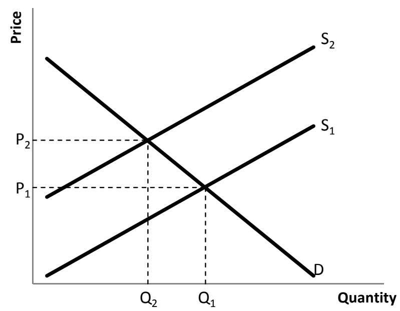
$$\begin{aligned}
\text{Real Price}(\$_{2015})\frac{CPI_?}{CPI_?} &= \text{Real Price}(\$_{1955}) \\
\text{Real Price}(\$_{2015})\frac{CPI_{1955}}{CPI_{2015}} &= \text{Real Price}(\$_{1955})
\end{aligned}$$

$$\$_{2015}7.25 * \frac{\$_{1955}26.8}{\$_{2015}237.0} \quad = \quad \$_{1955}0.82$$

2. Using separate diagrams for each subpart, sketch the impact of the following changes on the market for automobiles. (*Note:* You do not need to find a mathematical answer. Just sketch the qualitative impacts). You should identify the equilibrium both before and after the change and then briefly explain the change.

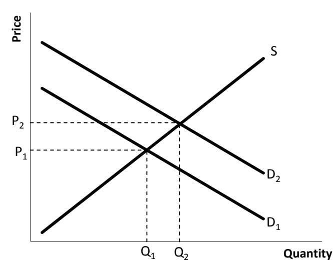
   (a) An increase in the price of steel (a major input for automobile manufacturers)

   This will increase the cost of producing cars and will therefore shift the supply curve in. Demand curve is unchanged so price rises and quantity falls.



   (b) An increase in consumer income

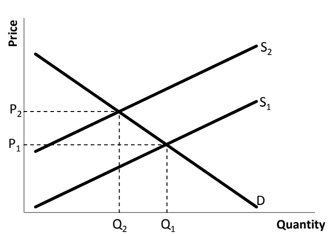
   This will increase consumers' incomes and will therefore shift the demand curve out. The supply curve is unchanged so, the equilibrium price and quantity rise.



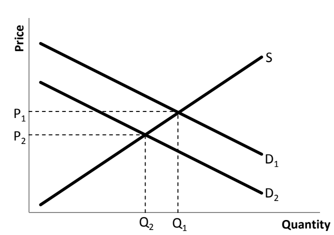
   (c) A decrease in the supply of gasoline (you can assume the market for gasoline is perfectly competitive)

   The decrease in supply of gasoline will cause the price of gasoline to increase. Since gasoline and automobiles are complementary goods, an increase in the price of the complement will cause demand for automobiles to shift in. The supply of automobiles will remain unchanged so the equilibrium price and quantity of automobiles will fall.

Gasoline Market



Automobile Market



3. Suppose the demand for bicycles is given by$Q_D = 10,000P^{-\frac{2}{3}}$ and the supply of bicycles is given by $Q_S = 0.01P^{\frac{4}{3}}$.
   (a) Using calculus and the given supply and demand curves, find mathematical expressions for the elasticities of demand and supply as a function of the price. Use these expressions to demonstrate that both curves have constant elasticity along their entire length.

   We know that $E_d = \frac{dQ}{dP}\frac{P}{Q}$. Using calculus, we can find an expresion for $\frac{dQ}{dP}$.

   $$\frac{dQ}{dP} = -\frac{2}{3}(10,000)\,P^{-\frac{5}{3}}$$

   meaning that

   $$E_d = -\frac{2}{3}(10,000)\,P^{-\frac{5}{3}}\left(\frac{P}{Q}\right)$$

   Now we want to get our expression all in terms of P. We need to substitute in for Q using the demand curve.

   $$\begin{aligned} E_d &= -\frac{2}{3}(10,000)\,P^{-\frac{5}{3}}\frac{P}{10,000P^{-\frac{2}{3}}} \\ &= -\frac{2}{3}P^{-\frac{5}{3}}P^{\frac{5}{3}} \\ &= -\frac{2}{3} \end{aligned}$$

   Since the expression doesn't vary with P, the elasticity of demand is constant.

   We can then do the same thing for the elasticity of supply.

   $$\begin{aligned} E_s &= \frac{dQ}{dP}\frac{P}{Q} \\ &= 0.01\left(\frac{4}{3}\right)P^{\frac{1}{3}}\cdot\frac{P}{0.01P^{\frac{4}{3}}} \\ &= \frac{4}{3} \end{aligned}$$

   Since this expression is also constant and doesn't vary with P, we know that the elasticy of supply is also constant.

   (b) Find the equilibrium price and quantity in this market mathematically.

   In equilibrium, quantity demanded must equal quantity supplied. Set the two expressions equal and solve for the equilibrium P. Then use this value to find the equilibrium Q.
   $10,000P^{-\frac{2}{3}} = 0.01P^{\frac{4}{3}}$
   $1,000,000 = P^2$
   $P = 1000$
   To find the quantity, we substitute back into the demand (or supply) curve.
   $Q_D = 10,000 * 1000^{-\frac{2}{3}}$
   $Q_D = 10,000 * .01$
   $Q_D = 100$
   Although it is not necessary, you can also check your calculations by checking the other curve (in this case supply) as well.
   $Q_S = .01 * 1000^{\frac{4}{3}}$
   $Q_S = .01 * 10,000$
   $Q_S = 100$

4. Suppose you run a factory whose short-run production function is given by

$$f(L) = 180L^2 - 3L^3.$$

(a) Find mathematical expressions for the total product of labor, the average product of labor and the marginal product of labor.

$TP_L = f(L) = 180L^2 - 3L^3$
$AP_L = \frac{f(L)}{L} = \frac{1580L^2 - 3L^3}{L} = 180L - 3L^2$
$MP_L = \frac{\partial f(L)}{\partial L} = 360L - 9L^2$

(b) On two separate axes, graph TP_L, AP_L, and MP_L. The diagrams do not need to be scale, but be sure they are internally consistent. In other words, make sure that all relevant relationships between the three lines are correct. If you have trouble figuring out what the graph looks like, feel free to use a graphing calculator or an online graphing tool to find the basic shape. Find the quantity of labor associated with each of the following points and label it on your diagram: (i) the maximum value of TP_L, (ii) the maximum value of AP_L, (iii) the maximum value of MP_L.

To distinguish in the expressions below, I will use L_i to refer to the amount of labor that maximizes total product, L_{ii} for the amount that maximizes average product, and L_{iii} for the amount that maximizes marginal product.
The amount of labor that yields the maximum value of TP_L is found by setting its derivative (i.e. MP_L) equal to zero. Using our expressions from part (a), we see that $MP_L = 360L - 9L^2$, so we have

$$\begin{aligned} 360L_i - 9L_i^2 &= 0 \\ 360 - 9L_i &= 0 \\ L_i &= 40 \end{aligned}$$

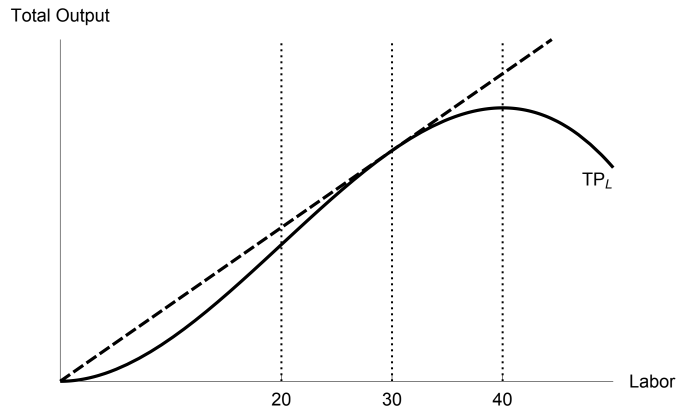
Notice that $L_i = 0$ also solves the first-order condition, but if you graphed the entire function, you would see that $L_i = 0$ is a local minimum not a maximum.
To find the amount of labor that maximizes AP_L, we have two choices. We could maximize AP_L by taking its derivative and setting it equal to zero or we could remember that $MP_L = AP_L$ at the maximum value of AP_L. Using the second approach, we have

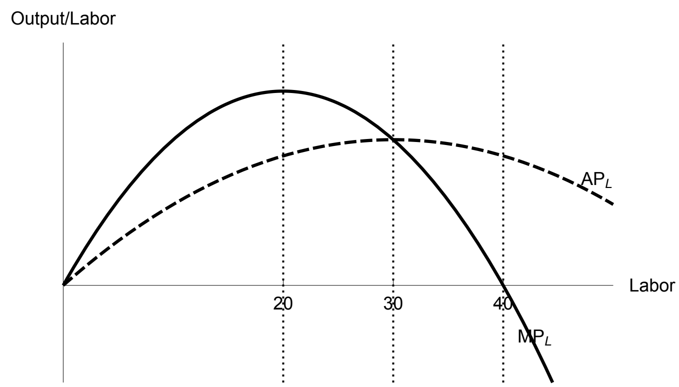
$$\begin{aligned} 360L_{ii} - 9L_{ii}^2 &= 180L_{ii} - 3L_{ii}^2 \\ 180L_{ii} &= 6L_{ii}^2 \\ 30 &= L_{ii} \end{aligned}$$

Alternatively, the derivative of $AP_L = 180 - 6L$. If we set this equal to zero, we will see that $L_{ii} = 30$ as we found the other way.
Finally, to find the amount of labor that maximizes MP_L, we need to take the derivative of MP_L. We know this for two reasons. First, you may remember from calculus that inflection points occur where the second derivative is zero. But if you forgot this, our goal is simply to maximize MP_L. We can maximize any function by finding where its derivative is zero (and using a little common sense or graphing to make sure we're at a max). Therefore,

$$\begin{aligned} \frac{dMP_L}{dL} &= 360 - 18L_{iii} = 0 \\ L_{iii} &= 20. \end{aligned}$$

In your diagrams, the critical points to label are that the MP_L curve is a parabola that reaches its max at 20, crosses AP_L at 30 and hits zero at 40. For the AP_L curve, it must be below MP_L and rising until 30 where AP_L reaches its max and then falls but remains higher than MP_L. For the TP_L curve, you should see that a line from the origin to the the curve is just tangent to the curve at 30 and that TP_L reaches its max at 40.





(c) For what values of L does the production function exhibit each of the following:

i. increasing marginal labor productivity

Increasing marginal productivity occurs anywhere the MP_L curve is rising, i.e. when $L < 20$.

ii. diminishing marginal labor productivity

Diminishing marginal labor productivity occurs anywhere the marginal productivity curve is falling, i.e. when $L > 20$.

iii. negative marginal labor productivity

This occurs anywhere the MP_L curve is negative, i.e. when $L > 40$.

5. Under what conditions will each of the following production functions exhibit (i) increasing returns to scale, (ii) decreasing returns to scale, and (iii) constant returns to scale? *Notes: your answer should be in the form of inequalities involving a and/or b. For some production functions it may not be possible to exhibit each type of returns to scale.*

The general approach to answering this question is to compute the value of $f(\alpha L, \alpha K)$ and compare it to $\alpha f(L, K)$. If:

- $f(\alpha L, \alpha K) > \alpha f(L, K)$ we have increasing returns to scale (IRTS)
- $f(\alpha L, \alpha K) = \alpha f(L, K)$ we have constant returns to scale (CRTS)

- $f(\alpha L, \alpha K) < \alpha f(L, K)$ we have decreasing returns to scale (IRTS)

(a) $f(L, K) = L^a K^b$

$$\begin{aligned} f(\alpha L, \alpha K) &= (\alpha L)^a (\alpha K)^b \\ &= \alpha^a L^a \alpha^b K^b \\ &= \alpha^{a+b} L^a K^b \\ &= \alpha^{a+b} f(L, K) \end{aligned}$$

So, when $a + b > 1$, we have IRTS; when $a + b = 1$, we have CRTS; and when $a + b < 1$, we have decreasing returns to scale.

(b) $f(L, K) = a\left(K^b + L^b\right)$

$$\begin{aligned} f(\alpha L, \alpha K) &= a\left[(\alpha K)^b + (\alpha L)^b\right] \\ &= a\left[\alpha^b K^b + \alpha^b L^b\right] \\ &= a\left[\alpha^b \left(K^b + L^b\right)\right] \\ &= \alpha^b a\left(K^b + L^b\right) \\ &= \alpha^b f(L, K) \end{aligned}$$

So, when $b > 1$, we have IRTS; when $b = 1$, we have CRTS; and when $b < 1$, we have decreasing returns to scale. The value of a has no influence on returns to scale.

(c) $f(L, K) = (aK + bL)^{\frac{1}{2}}$

$$\begin{aligned} f(\alpha L, \alpha K) &= (a\alpha K + b\alpha L)^2 \\ &= \alpha^{\frac{1}{2}} (aK + bL)^{\frac{1}{2}} \\ &= \alpha^{\frac{1}{2}} f(L, K) \end{aligned}$$

In this case, we have decreasing returns to scale regardless of the values of a and b.

(d) $f(L, K) = aKL^b$

$$\begin{aligned} f(\alpha L, \alpha K) &= a\alpha K (\alpha L)^b \\ &= a\alpha K \alpha^b L^b \\ &= a\alpha^{1+b} K L^b \\ &= \alpha^{1+b} f(L, K) \end{aligned}$$

So, when $b > 0$, we have IRTS; when $b = 0$, we have CRTS; and when $b < 0$, we have decreasing returns to scale. Again, the value of a has no influence on returns to scale.

6. Indira runs a bakery using labor (L, measured in hours of her time) and capital (K, measured in ovens). Indira's production function (measured in loaves of bread) is given by

$$f(L, K) = 4L^{\frac{1}{3}} K^{\frac{2}{3}}.$$

(a) Indira currently has 8 ovens.

i. Find an expression for her (short-run) average and marginal product of labor.

First, we need to substitute in the fixed value of capital. This gives us

$$f\left(L,K\right)=4L^{\frac{1}{3}}8^{\frac{2}{3}}=16L^{\frac{1}{3}}.$$

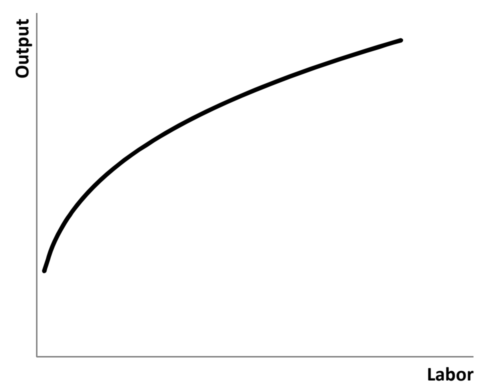
We can then follow our standard approach to computing average and marginal products.

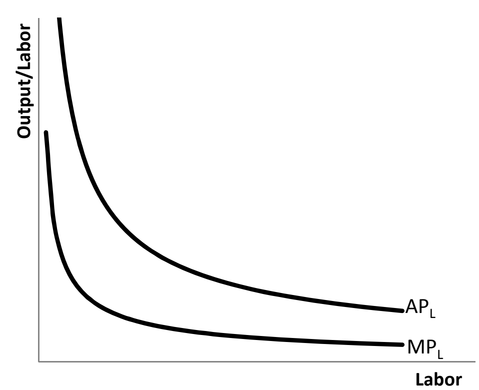
$$AP_L=\frac{16L^{\frac{1}{3}}}{L}=16L^{-\frac{2}{3}}$$

and

$$MP_L=\frac{16}{3}L^{-\frac{2}{3}}$$

ii. In two separate diagrams, graph Indira's total, average, and marginal product of labor. Do these curves follow all the rules we discussed about the relationships between curves? Does Indira's production technology satisfy the law of diminishing marginal returns?





The curves do follow all of our rules, but several of them do not apply. We see that the total product is increasing at a decreasing rate for all values of L. This implies that the marginal product of labor is always positive, but also always declining as we see in the bottom graph. Moreover, a line from the origin to a point on our total product curve gets progressively flatter as L increases. This implies that the AP_L curve is always declining. If AP_L is falling, we know that MP_L is less than AP_L which is indeed what we see in the diagram. (In fact, our mathematical expressions tell us that MP_L is always exactly $\frac{1}{3}$ of AP_L.

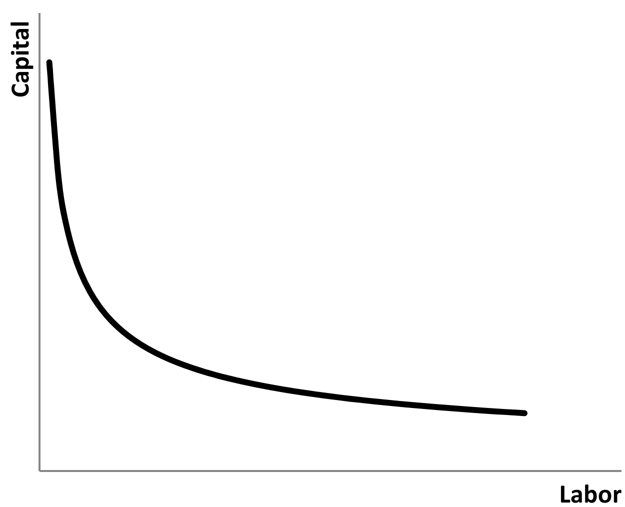
As we see from the diagram and the math, Indira's production technology does exhibit diminishing marginal returns to labor. While she never experiences negative marginal returns to labor, her marginal product is always declining.

(b) Suppose Indira is trying to decide whether to buy more ovens or sell some of the ones she has and wants to understand the tradeoffs between more ovens and her time. If Indira wants to produce 100 loaves a week, help her find out how many ovens she'll need for various amounts of her time. That is, find an expression for the number of ovens she'll need to produce 100 loaves as a function of the number of hours she works. (Hint: if you're stuck, trying picking a couple specific numbers of hours she might work and see how many ovens she'd need to get 100 loaves).

If Indira wants to produce 100 loaves, she must have $4L^{\frac{1}{3}}K^{\frac{2}{3}} = 100$. We want to turn this into an expression for the necessary capital as a function of labor. To do this, we have

$$\begin{aligned} L^{\frac{1}{3}}K^{\frac{2}{3}} &= 25 \\ LK^2 &= 25^3 \\ K^2 &= \frac{25^3}{L} \\ K &= \frac{5^3}{L^{\frac{1}{2}}} \\ K &= 125L^{-\frac{1}{2}} \end{aligned}$$

(c) Plot your equation from part (b). What does this line represent?



This line represents the isoquant associated with production of 100 loaves of bread.

(d) Find an expression for Indira's marginal rate of technical substitution between ovens and labor. What is the connection between your answer to this question and the picture you drew in part (c).

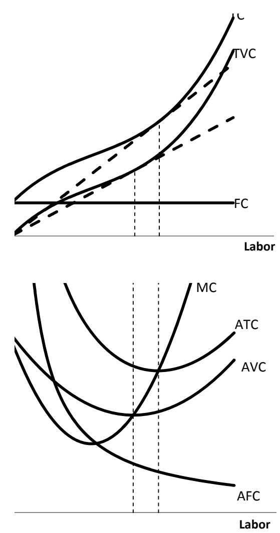
Indira's marginal rate of technical subsitution is given by

$$MRTS = \frac{MP_L}{MP_K} = \frac{\frac{4}{3}K^{\frac{2}{3}}L^{-\frac{2}{3}}}{\frac{8}{3}L^{\frac{1}{3}}K^{\frac{-1}{3}}} = \frac{1}{2}\frac{K}{L}$$

This expression tells us the magnitude of the slope of the isoquant. At first glance, this may seem odd. The slope of the line we found in part (b) is $\frac{dK}{dL} = -\frac{125}{2}L^{-\frac{3}{2}}$. Since the answer to part (c) was that the line found in part (b) **is** the isoquant, shouldn't the slope be equal to the negative MRTS?

With a little math, we can verify that there is no inconsistency. Along the isoquant for 100 loaves, we know that $K = 125L^{-\frac{1}{2}}$. If we substitue this into the epxression for the MRTS, we get $MRTS = \frac{1}{2}\frac{K}{L} = \frac{1}{2}\left(\frac{125L^{-\frac{1}{2}}}{L}\right)$. This is equivalent to $\frac{125}{2}L^{-\frac{3}{2}}$, i.e. the magnitude of the slope of the line we found in part (b).

7. XYZ corporation produces widgets. Its short-run marginal cost curve is given by $MC(q) = 10 - 5q + q^2$ (this is a parabola whose minimum occurs at $q = 2.5$). XYZ's fixed costs are 10. In two separate diagrams, graph the following cost curves: (a) total cost, (b) total variable cost, (c) total fixed cost, (d) marginal cost, (e) average variable cost, and (f) average total cost. Your diagrams do not need to be scale, but must be internally consistent (i.e. the relationships between different curves must be correct). You ***do not*** need to find mathematical expressions for the other cost curves – you only need to sketch lines that are consistent with the shape of the marginal cost curve.



If marginal cost is falling and then rising, total variable cost and total cost must increase at a decreasing rate while marginal cost is falling and then increase at an increasing rate while marginal cost is rising. This implies that both AVC and ATC must also initally fall and then rise. Moreover, their minima must occur as they cross the marginal cost curve, which is where a line from the origin to the TC and TVC curves are just tangent.
Fixed cost is horizontal and average fixed cost (which you were not asked to graph) decreases at a decreasing rate, approaching but never reaching zero.

8. Zoya runs a cement pouring business. The rental rate for cement trucks is \$80 day. Zoya currently has 9 trucks in her fleet. Her drivers are paid \$18 an hour. The total number of cement driveways she can pave in a day is given by $f(L, K) = L^{\frac{1}{2}}K^{\frac{1}{2}}$ where K measures the number of trucks and L measures the total hours worked by all her drivers (Note that the production function is unrealistic because it assumes that 1 driver working 10 hours is the same as 10 drivers working 1 hour each). Throughout this question, you can think of Zoya renting fractional trucks if necessary.
   (a) Find Zoya's ***short-run*** total cost, average total cost, average variable cost and marginal cost as a function of the number of cement driveways she produces. Assume that labor

is the variable input and capital is fixed in the short run (i.e. Zoya has long term leases on the cement trucks).

Since capital is fixed, Zoya's short-run production function is $f(L) = 3L^{\frac{1}{2}}$. This means that $q = 3L^{\frac{1}{2}}$ and that $L = \frac{q^2}{9}$. Zoya's short-run total cost is $TC(q) = wL + r\bar{K} = 18L(q) + 80(9) = 720 + 18\frac{q^2}{9} = 720 + 2q^2$. Her average total cost is $ATC(q) = \frac{TC(q)}{q} = \frac{720}{q} + 2q$. Her average variable cost is just $AVC = 2q$ and her marginal cost is $MC(q) = 4q$.

(b) Zoya currently has 1 driver for each of her 9 trucks and each driver works 4 hours per day for a total of 36 hours of labor. Is Zoya's current production regime minimizing her long-run cost of production? If yes, explain how you know. If not, explain a small change that Zoya could make to produce the same number of driveways at a lower cost. In either case, your answer should focus on the intuition and should not rely on either the method of Lagrange multipliers or an isoquant/isocost diagram.

We need to explain to Zoya that she could keep her production constant and lower her costs if she changed her labor and capital mix. Before we can do this, we need to compute her marginal rate of technical substitution by computing her marginal products of labor and capital so we can compare these to the ratio of input prices and figure out how to shift production.
Given her production function, $MP_L = \frac{1}{2}L^{-\frac{1}{2}}K^{\frac{1}{2}}$. Since she has 9 drivers working 4 hours per day, $L = 36$. Substituting in we have $MP_L = \frac{1}{2}(36)^{-\frac{1}{2}}(9)^{\frac{1}{2}} = \frac{3}{2*6} = \frac{1}{4}$. Similarly, $MP_K = \frac{1}{2}L^{\frac{1}{2}}K^{-\frac{1}{2}} = \frac{1}{2}(36)^{\frac{1}{2}}(9)^{-\frac{1}{2}} = \frac{6}{2*3} = 1$. This means her $MRTS = \frac{1/4}{1} = \frac{1}{4}$, while the ratio of input prices is $\frac{w}{r} = \frac{18}{80} = \frac{9}{40}$. So, in this case, we have $MRTS > \frac{w}{r}$ which means Zoya is not cost-minimizing. Equivalently, $\frac{w}{MP_L} = \frac{18}{1/4} = 72$ and $\frac{r}{MP_K} = \frac{80}{1} = 80$. This implies that $\frac{w}{MP_L} < \frac{r}{MP_K}$ or that the marginal cost of production using capital is higher than the marginal cost of production using labor.
To explain this to Zoya , we want to show her that she can produce the same amount for less money if she shifts to a more labor intensive regime. To pave 1 more driveway with her current capital input, Zoya would need 4 hours of labor ($\frac{1}{MP_L}$), which would cost her $4(18)$ or $72. But then she could cut back capital by 1 truck ($\frac{1}{MP_L}$), which would save her $1(80) = 80$. This would allow her to produce the same number of driveways for $8 less. This means that she was not cost-minimizing before.

9. Join our Slack workspace (see Moodle or email from me for the link) and answer the survey regarding how you meet the pre-requisites for this class.