Robust Trajectory Planning for a Multirotor Using Funnel Library

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***Abstract*— This paper is about funnel library that can be used to generate robust trajectories for multirotors.**

I. INTRODUCTION

Throughout the paper, 0_{ij} stands for the zero matrix in $\mathbb{R}^{i\times j}$, and I_i denotes the identity matrix in $\mathbb{R}^{i\times i}$. For a matrix, $\|\cdot\|$ represents the induced 2 norm, and $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ indicate the maximum and minimum eigenvalues. Also, $|\cdot|$ is the Euclidean norm for a vector. For two vectors $\alpha, \beta \in \mathbb{R}^{3\times 1}$, we denote the inner and cross products as $\langle \alpha, \beta\rangle = \alpha^\top \beta$ and $\mathbf{S}(\alpha)\beta = \alpha \times \beta$. For two quaternions $\mathbf{q}_1$ and $\mathbf{q}_2$, the quaternion multiplication expressed as $\mathbf{q}_1 \otimes \mathbf{q}_2$. Also, $\mathbf{P}(\cdot)$ is the quaternion representation of a vector $\omega \in$ so(3) such as $\mathbf{P} = [0\ \omega^\top]^\top$. The rotation of a vector is indicated as $\mathbf{q}_1 \odot \omega = \mathbf{q}_1 \otimes \mathbf{P}(\omega) \otimes \mathbf{q}_1^{-1}$. Furthremore, c and s are shorthands of cos and sin, respectively.

II. QUADROTOR DYNAMICS AND CONTROL

A. Quadrotor dynamics

To describe the dynamic model of a multirotor, we define the inertial $O_I\{x_I, y_I, z_I\}$ and the multirotor body-fixed $O_b\{x_b, y_b, z_b\}$ frames. The body-fixed frame is located at the center of the multirotor. The translational and angular dynamics of a multirotor is described as follows:

$$\ddot{p} = -g z_I + T z_b + F_d + \delta \tag{1}$$

$$\dot{\mathbf{q}} = \tfrac{1}{2}\mathbf{q} \otimes \mathrm{P}(\omega) \tag{2}$$

where p is the position of O_b with respect to O_I, and $\mathbf{q} = [q_0\ \bar{q}^\top]^\top$ is the unit quaternion describing the orientation of O_b with respect to O_I. The angular velocity of O_b represented in O_b is denoted as ω. The terms g and T are gravitational constant and mass normalized collective thrust, respectively. Without loss of generality, the axis z_I is defined as $e_3 = [0\ 0\ 1]^\top$. The rotor drag applied on the multirotor is denoted as F_d, and it is expressed as follows [IFAC]:

$$F_d = c_d \mathbf{q} \odot \pi_z \mathbf{q}^{-1} \odot \dot{p} \tag{3}$$

where c_d is the rotor drag coefficient, and $\pi_z = I_3 - e_3^\top e_3$ is the matrix projecting a vector onto the $x_b y_b$ plane. The external forces and model uncertainties excluding the rotor drag term are lumped in $\delta \in \mathbb{R}^{3\times 1}$.

In eq. (2), the angular velocity ω is used as the input term. It is possible based on the assumption that the multirotor body angular rate ω could be directly controlled as also supposed in [DANDREA]. Therefore, the input terms of the multirotor dynamics in eqs. (1) and (2) are T and ω.

Uncertainty and disturbances In the free flight scenario which is not incorporating physical interaction with environment such as pushing or pulling, rotor and fuselage drag could be considered as the biggest external disturbances [DRAG]. XXXXX Conclusion is that the rotor drag is not ignorable and proportional to the body velocity. Other disturbances are small and can be assumed to be bounded to certain values.

B. Multirotor control

Let us assume that the reference trajectory of the differential flat output[MEL] $\{p(t)^r, \psi^r(t)\}$ is given. Here, $p^r(t)$ is the reference trajectory of the multirotor, and $\psi^r(t)$ represents the reference rotation of the multirotor about the z_b body axis, i.e. yaw angle in the Euler angle representation.

To control the translational motion of a multirotor, we utilized a geometric control method [TYL] which is widely utilized in the researches on multirotors [RPG][UPENN][MIT]. Let $e_p = p - p^r$ and $e_v = \dot{p} - \dot{p}^r$ be the position and velocity error terms. With the above definitions, the mass normalized thrust T and desired thrust direction of a multirotor z_b^d could be computed as the following procedure:

$$\begin{aligned} \ddot{p}^d &= -K_p e_p - K_v e_v + g e_3 + \ddot{p}^r \\ z_b^d &= \ddot{p}^d / |\ddot{p}^d| \\ T &= \langle \ddot{p}^d, z_b\rangle = |\ddot{p}^d| \langle z_b^d, z_b\rangle \end{aligned} \tag{4}$$

where K_p and K_v are gain matrices with positive diagonal entries. By substituting the terms T and $\ddot{p}^d$ in (1), the error dynamics of the translational motion could be derived as follows:

$$\begin{aligned} \ddot{e}_p &= -g e_3 + T z_b + F_d + \delta - \ddot{p}^r + \ddot{p}^d - \ddot{p}^d \\ &= -K_p e_p - K_v e_v + F_d + \delta + T z_b - \ddot{p}^d \\ &= -K_p e_p - K_v e_v + F_d + \delta + |\ddot{p}^d| \{ \langle z_b^d, z_b\rangle z_b - z_b^d \} \\ &= -K_p e_p - K_v e_v + F_d + \delta + |\ddot{p}^d| \mathrm{s}_\Phi u \end{aligned} \tag{5}$$

where Φ is the angle between the axes z_b and z_b^d, and u is the unit vector indicating the direction of the term inside of the curly bracket.

Similarly, the rotational motion of a multirotor could be analyzed by combining it with a control law that generates the desired angular rate ω^d. First, we compute the desired attitude of the multirotor with $\psi^r(t)$ and z_b^d as follows: [MEL]

$$\begin{aligned} \bar{y}_b &= [\ -\mathrm{s}_{\psi^r} \quad \mathrm{c}_{\psi^r} \quad 0\ ]^\top \\ x_b^d &= \mathrm{S}(\bar{y}_b) z_b^d / |\mathrm{S}(\bar{y}_b) z_b^d| \\ y_b^d &= \mathrm{S}(z_b^d) x_b^d / |\mathrm{S}(z_b^d) x_b^d|. \end{aligned}$$

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Then, based on the axes $\{x_b^d, y_b^d, z_b^d\}$, the desired coordinate could be represented with the unit quaternion $\mathbf{q}^d$. The attitude error between $\mathbf{q}^d$ and $\mathbf{q}$ is denoted as $\mathbf{q}_e = \mathbf{q}^{-1} \otimes \mathbf{q}^d$. To decrease the attitude error $\mathbf{q}_e = [q_e \ \bar{q}_e^\top]^\top$, the angular velocity is set as

$$\omega = \begin{cases} \omega_d^d - k_p \bar{q}_e & \text{if } q_e \geq 0, \\ \omega_d^d + k_p \bar{q}_e & \text{if } q_e < 0. \end{cases} \tag{6}$$

where ω_d^d is the desired angular velocity of $\mathbf{q}^d$ represented in $\mathbf{q}^d$. According to [ETH], the attitude error $\mathbf{q}_e$ is globally asymptotically stable equilibrium point. Hence, it is possible to assume that the error Φ is also asymptotically stable, and bounded by a known value. NEEDS MORE EXPLANATIONS.

Assumption : In this work, we assume that the norm of the external disturbance is bounded as $|\delta| \leq \bar{\delta}$. Also, the norm of the nominal collective thrust is bounded as $|ge_3 + \ddot{p}^r| \leq \bar{T}$ given any $\mathcal{C}^2$ reference trajectory x^r. Furthermore, the norm of the thrust axis error is also bounded such as $|\mathrm{s}_\Phi| \leq \bar{\mathrm{s}}_\Phi$ with the attitude controller.

C. Stability analysis

In Sec. II-B, it is assumed that the attitude error is bounded for all flight duration. In this subsection, we investigate the stability of the translational dynamics in eq. (5).

As the first step, the dynamics system is summarized as follows:

$$\begin{aligned} \dot{e}_p &= e_v \\ \dot{e}_v &= -K_p e_p - K_v e_v + F_d + \delta + \mathrm{s}_\Phi |\ddot{p}^d| u. \end{aligned} \tag{7}$$

To analyze the stability of the error dynamics conveniently, we assume that $K_p = k_p I_3$ and $K_v = k_v I_3$ with positive scalar gain values k_p and k_v. Then, we define the Lyapunov candidate function as $\bar{V} = \frac{1}{2} e^\top \bar{P} e$ where $e = [e_p^\top \ e_v^\top]^\top$ and

$$\bar{P} = \begin{bmatrix} (k_p + k_d) I_3 & I_3 \\ I_3 & I_3 \end{bmatrix}.$$

The Lyapunov function is positive definite when $k_p + k_v > 1$. Furthermore, the directional derivative of $\bar{V}$ can be summarized as follows:

$$\begin{aligned} \dot{\bar{V}} &= -k_p e_p^\top e_p - k_v e_v^\top e_v + e_v^\top e_v \\ &\quad + (e_p + e_v)^\top \{\delta + F_d + |\ddot{p}^d| \mathrm{s}_\Phi u\}. \end{aligned} \tag{8}$$

Then, by having $\ddot{p}^d$ in eq. (4), $|\delta| \leq \bar{\delta}$, $|F_d| \leq c_d |\dot{p}^r + e_v|$, and $|\mathrm{s}_\Phi u| < \bar{\mathrm{s}}_\Phi$, the above relation is rearranged further as

$$\begin{aligned} \dot{\bar{V}} &\leq -k_p |e_p|^2 - (k_v + 1)|e_v|^2 + (|e_p| + |e_v|) \\ &\quad \times \{\bar{\delta} + c_d |\dot{p}^r + e_v| + \bar{s}_\Phi (| - k_p e_p - k_v e_v| + \bar{T})\} \\ &\leq -\bar{e}^\top \bar{Q} \bar{e} + \Delta |\bar{e}| \end{aligned} \tag{9}$$

where $\bar{e} = [|e_p| \ |e_v|]^\top$, $\Delta = \bar{\delta} + c_d |\dot{p}^r| + \bar{s}_\Phi \bar{T}$ and

$$\bar{Q} = \begin{bmatrix} k_p(1 - \bar{s}_\Phi) & -\frac{1}{2}\{c_d + \bar{s}_\Phi (k_p + k_d)\} \\ -\frac{1}{2}\{c_d + \bar{s}_\Phi (k_p + k_d)\} & k_v(1 - \bar{s}_\Phi) - 1 - c_d \end{bmatrix}.$$

Here, the matrix $\bar{Q}$ can be considered as the summation of the two separated parts such as $\bar{Q} = Q_A + \bar{\mathrm{s}}_\Phi Q_B$ where

$$\begin{aligned} Q_A &= \begin{bmatrix} k_p & -\frac{1}{2} c_d \\ -\frac{1}{2} c_d & k_v - 1 - c_d \end{bmatrix} \\ Q_B &= \begin{bmatrix} -k_p & -\frac{1}{2}(k_p + k_v) \\ -\frac{1}{2}(k_p + k_v) & -k_v \end{bmatrix}. \end{aligned} \tag{10}$$

By setting $k_v > 1 + c_d$ and $k_p > \frac{c_d^2}{4(k_v - 1 - c_d)}$, we can make the matrix Q_A positive definite. On the other hand, the matrix Q_B cannot be positive definite since $\det(Q_B) = -\frac{1}{4}(k_p - k_v)^2 \leq 0$, i.e. $\lambda_{\max}(Q_B) \geq 0$ and $\lambda_{\min}(Q_B) \leq 0$. However, even though Q_B is not positive definite, we can assure that the matrix $\bar{Q}$ is positive definite with the condition $\bar{\mathrm{s}}_\Phi < -\lambda_{\min}(Q_A)/\lambda_{\min}(Q_B)$ which could readly be achieved since $\bar{\mathrm{s}}_\Phi \approx 0$ with a well designed attitude controller. Therefore, with the gain values k_p and k_v fulfilling $\lambda(\bar{Q}) \geq 0$, eq. (9) becomes

$$\begin{aligned} \dot{V} &\leq -\lambda_{\min}(\bar{Q})|\bar{e}|^2 + \Delta |\bar{e}| \\ &\leq -\lambda_{\min}(\bar{Q})(1 - \theta)|\bar{e}|^2 - \lambda_{\min}(\bar{Q})\theta |\bar{e}|^2 + \Delta |\bar{e}| \end{aligned}$$

where $0 < \theta < 1$. Finally,

$$\dot{\bar{V}} \leq -\lambda_{\min}(\bar{Q})(1 - \theta)|\bar{e}|^2, \text{ where } |\bar{e}| \geq \frac{\Delta}{\lambda_{\min}(\bar{Q})\theta}.$$

Therefore, the error dynamics of the translational system of a multirotor is uniformly ultimately bounded. Furthermore, the ultimate bound b could be computed as

$$b = \sqrt{\frac{\lambda_{\max}(\bar{P})}{\lambda_{\min}(\bar{P})} \mu^2}, \quad \mu = \frac{\Delta}{\lambda_{\min}(\bar{Q})\theta}. \tag{11}$$

In this analysis, we can see that any error will eventually enter into the ultimate bound with radius b within finite time. The ultimate bound is the function of Δ and it is the combination of $\bar{\delta}$, $|\dot{p}^r|$, and $\bar{T}$. Also, the drag coefficient c_d and the maximum attitude error $\bar{\mathrm{s}}_\Phi$ could be interpreted as the weight terms. Hence, the radius of the bound will get bigger by having larger values of disturbances, reference velocity, and reference acceleration ($\bar{T} = |ge_3 + \ddot{p}^r|$).

However, even though we could get the general sense on how the error will behave, it is difficult to know the tight ultimate bound that the error states actually resides. Also, given initial set of error states, it is hard to see how the error will evolve before getting into the ultimate bound.

III. Computing Funnels

In Sec. II-C, we checked the general behavior of the translational error dynamic system via classical Lyapunov stability analysis. Furthermore, with sum of squares optimization tools, we can compute the reachable set of the state e. The concept of funnel and numerical method for computing funnel can be referred to [MAJ1,MAJ2].

A. Concept of funnel

Funnel $\mathcal{F}(t) \subset \mathbb{R}^{6\times 1}$ represents the reachable set of error $e(t) \in \mathbb{R}^{6\times 1}$ given the initial set of error $e(0) \in \xi_0$ and a closed-loop dynamic equation. In other words, the error $e(0)$ contained in the set ξ_0 will evolve only inside of the set $\mathcal{F}(t)$ for $t \geq 0$. These property of the funnel could be written as follows:

$$e(0) \in \xi_0,\ \xi_0 \subset \mathcal{F}(0) \ \Rightarrow \ e(t) \in \mathcal{F}(t), \ \ \forall t \geq 0. \tag{12}$$

Define a time varying positive definite matrix $P(t) \in \mathbb{R}^{6\times 6}$, a positive scalar variable $\alpha(t)$, and $V(t,e) = e(t)^\top P(t) e(t)$. In the following analysis, $V(t,e)$ will be the Lyapunov function, and $\alpha(t)$ will be the parameter indicating the level of $V(t,e)$. Then, we represent the set $\mathcal{F}(t)$ as

$$\mathcal{F}(t) = \{e(t) \in \mathbb{R}^{6\times 1} | V(t,e) \leq \alpha(t)\} \tag{13}$$

where V and α are constrained such that

$$\begin{aligned} &\dot{V}(t,\hat{e}) < \dot{\alpha}(t) \\ &\text{for} \quad \hat{e}(t) = \{e(t) | V(t,e) = \alpha(t),\ t \in [0,\infty)\}. \end{aligned} \tag{14}$$

From the constraint, it is obvious that the state $e(t) \in \mathcal{F}(t)$ cannot escape the sublevel set described by $V(t,e) \leq \alpha(t)$. Therefore, the set $\mathcal{F}(t)$ defined in eqs. (13) and (14) complies with the definition of funnel in eq. (12) [MAJ].

Another constraint to fulfill the property of funnel defined in eq. (12) is related to the initial condition of a funnel. The initial set of $e(0)$, i.e. ξ_0, should be the subset of $\mathcal{F}(0)$. To enforce the constraint, we define $\xi_0 = \{e | e^\top R e \leq 1\}$ with a positive definite matrix R. Then, the constraint can be written as follows:

$$V(0,\hat{e}) \leq \alpha(0) \ \text{ for } \ \hat{e} = \{e | e^\top R e \leq 1\}. \tag{15}$$

Furthermore, we want to find V and α that represent the reachable set of e small to make the funnel as less consertative as possible. Since P is a positive definite matrix and V is a quadratic function of e, it is possible to define an ellipsoid that enclosing the set of error $e = \{e | V(e,t) = \alpha(t)\}$. Then, by minimizing the volume of the enclosing ellipsoid, the size of the funnel could be minimized consequently. The outer ellipsoid can be formulated such as

$$\begin{aligned} &\hat{e}^\top(t) S(t) \hat{e}(t) \leq 1 \\ &\text{for} \quad \hat{e}(t) = \{e(t) | V_{\mathcal{F}}(t,e) = \alpha(t),\ t \in [t_0, t_f]\}. \end{aligned}$$

where $S(t) \in \mathbb{R}^{6\times 6}$ is the positive definite matrix representing the shape of the outer shell ellipsoid. Note that the volume of the ellipsoid is proportional to the determinant of $S(t)$.

Based on these observations, we can formulate the optimization problem to find the funnel as follows:

$$\begin{aligned} \inf_{P,\alpha,S} \quad & \int_{t_0}^{t_f} \det S(t) \mathrm{d}t \\ \text{s.t.} \quad & \dot{V}(t,\hat{e}) < \dot{\alpha}(t) \text{ for } \hat{e} = \{e(t) | V(t,e) = \alpha(t)\}, \\ & \hat{e}^\top S(t) \hat{e} \leq 1 \text{ for } \hat{e} = \{e(t) | V(t,e) = \alpha(t)\}, \\ & V(t_0,\hat{e}) \leq \alpha(t_0) \text{ for } \hat{e} = \{e | e^\top R e \leq 1\}. \end{aligned}$$

B. Computing funnel

We compute the funnel for the multirotor translational error dynamic system. First of all, we define the Lyapunov function in the quadratic form such as $V = e(t)^\top P(t) e(t)$ with the time varying positive definite matrix

$$P(t) = \begin{bmatrix} P_p(t) & P_{pv}(t) \\ P_{pv}(t) & P_v(t) \end{bmatrix} \tag{16}$$

where $P_p(t)$, $P_{pv}(t)$, and $P_v(t)$ are diagonal matrices with positive entries. Then, with eq. (7), the directional derivative of the Lyapunov function V is rearranged as follows:

$$\begin{aligned} \dot{V} = &-e^\top (A^\top P + PA) e + e^\top \dot{P} e \\ &+ 2e^\top PB(\delta + \mathrm{s}_\Phi |\ddot{p}^d| u + F_d) \end{aligned}$$

where

$$A = \begin{bmatrix} 0_{33} & I_3 \\ -K_p & -K_d \end{bmatrix}, \quad B = \begin{bmatrix} 0_{33} \\ I_3 \end{bmatrix}.$$

In the right hand side of the above equation, the third term further developed as

$$\begin{aligned} & e^\top PB(\delta + \mathrm{s}_\Phi | - K_p e_p - K_v e_v + g e_3 + \ddot{p}^r | \bar{u} + F_d) \\ & \leq |e^\top PB| \{\bar{\delta} + \bar{\mathrm{s}}_\Phi (k_p |e_p| + k_v |e_v| + \bar{T}) + c_d |\dot{p}^r + e_v|\} \\ & \leq (p_{pv}|e_p| + p_v |e_v|)\{\bar{\mathrm{s}}_\Phi (k_p |e_p| + k_v |e_v|) + c_d |e_v| + \Delta\} \end{aligned} \tag{17}$$

where p_{pv}, p_v, k_p, and k_v are maximum eigenvalues of P_{pv}, P_v, K_p, and K_v. Also, $\Delta = \bar{\delta} + c_d |\dot{p}^r| + \bar{\mathrm{s}}_\Phi \bar{T}$. To handle the error states with the norm, i.e. $|e_p|$ and $|e_v|$, we define two variables $\bar{e}_p \in \mathbb{R}$ and $\bar{e}_v \in \mathbb{R}$ satisfying the following constraints:

$$\begin{aligned} \bar{e}_p^2 = e_p^\top e_p = |e_p|^2, \quad \bar{e}_p \geq 0 \\ \bar{e}_v^2 = e_v^\top e_v = |e_v|^2, \quad \bar{e}_v \geq 0. \end{aligned}$$

Then, $\dot{V}$ is further developed as follows:

$$\begin{aligned} \dot{V} \leq & -e^\top (PA + A^\top P) e + e^\top \dot{P} e \\ & + 2(p_{pv}\bar{e}_p + p_v \bar{e}_v)\{\bar{\mathrm{s}}_\Phi (k_p \bar{e}_p + k_v \bar{e}_v) + c_d \bar{e}_v + \Delta\}. \end{aligned} \tag{18}$$

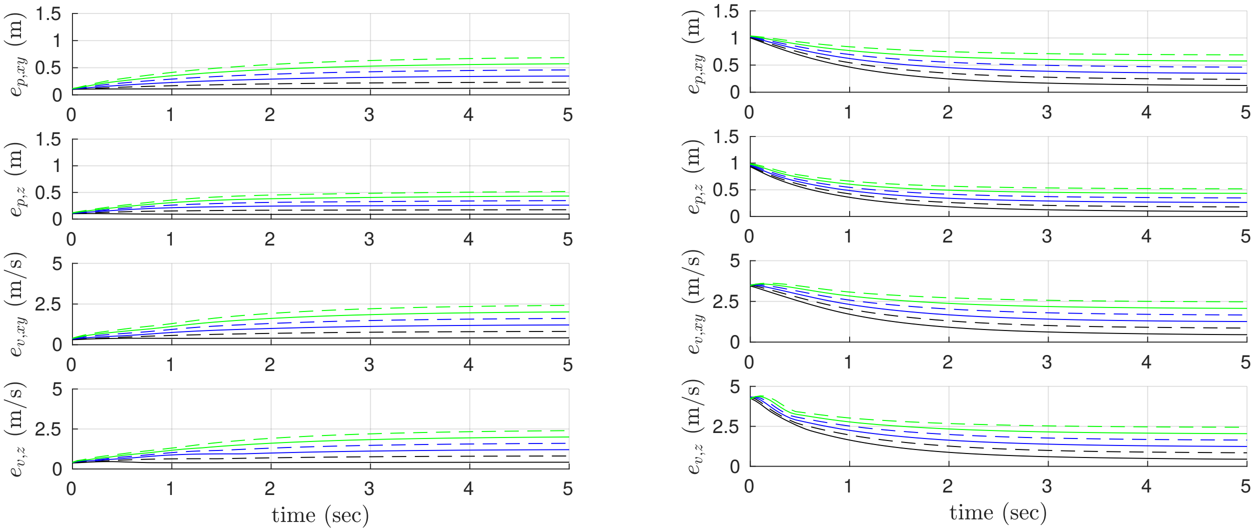
In V and $\dot{V}$, the variables of the optimization problem are continuous in t. To solve the problem in more efficient manner, we discretized variables P, α, and S in t. The discretized time is denoted as t_n $(n = 0, \cdots, N)$. Then, the optimization problem can be reformulated as the following:

$$\inf_{P(t_n),\alpha(t_n),S(t_n),p_{pv}(t_n),p_v(t_n)} \quad \sum_{n=0}^{N} \det(S(t_n)) \tag{19}$$

$$\begin{aligned} \text{s.t.} \quad & \dot{\alpha}(t_n) - \dot{V}(t_n) \geq 0 \text{ with constraints } c_1 \text{ to } c_9, \\ & 1 - e^\top S(t_n) e \geq 0 \text{ with constraints } c_9 \text{ and } c_{10}, \\ & \alpha(t_0) - V(t_0) \geq 0 \text{ with constraint } c_{11}. \end{aligned}$$

where

$$\begin{aligned} & c_1 : \ \bar{e}_p^2 = e_p^T e_p, && c_2 : \ \bar{e}_p \geq 0 \\ & c_3 : \ \bar{e}_v^2 = e_v^T e_v, && c_4 : \ \bar{e}_v \geq 0 \\ & c_5 : \ p_{pv}(t_n) I_3 \geq P_{pv}(t_n), && c_6 : \ p_{pv}(t_n) I_3 \geq -P_{pv}(t_n) \\ & c_7 : \ p_v(t_n) I_3 \geq P_v(t_n), && c_8 : \ p_v(t_n) I_3 \geq -P_v(t_n) \\ & c_9 : \ \alpha(t_n) - V(t_n) = 0, && c_{10} : \ S(t_n) > 0 \\ & c_{11} : \ 1 - e^\top R e = 0. \end{aligned}$$



(a) Funnels computed with the initial region with ξ^s.

(b) Funnels computed with the initial region with ξ^b.

Fig. 1: Funnels computed with various values of Δ_k and ξ_0. The detailed parameters and settings are available in table. I. The black solid and dotted lines are with $\Delta_l = 0.5$ and 1.0, the blue solid and dotted lines are with $\Delta_l = 1.5$ and 2.0, and the green solid and dotted lines are $\Delta_l = 2.5$ and 3.0.

The constraints c_5 to c_8 is added to ensure $p_{pv}(t_n) \geq \|P_{pv}(t_n)\|$ and $p_v(t_n) \geq \|P_v(t_n)\|$ with diagonal matrices $P_{pv}(t_n)$ and $P_v(t_n)$. In addition, $\dot{P}(t_n)$ and $\dot{\alpha}(t_n)$ could be implemented as

$$\dot{P}(t_n) = \tfrac{P(t_{n+1})-P(t_n)}{\mathrm{d}t}, \quad \dot{\alpha}(t_n) = \tfrac{\alpha(t_{n+1})-\alpha(t_n)}{\mathrm{d}t}. \tag{20}$$

The optimization problem in eq. (19) could be transformed into the form of sum of squares problem and solved efficiently by referring [MAJ].

C. Funnel library

In the optimization problem, the list of parameters are : the initial set of error states ξ_0; the gain matrices K_p and K_d; the norm of maximum attitude angle error $\bar{s}_\Phi$; the rotor drag coefficient c_d; and the lumped disturbance term δ. Among these parameters, the gain matrices are manually set by an operator to satisfy the required flight performance in accordance with applications. The attitude error $\bar{s}_\Phi$ and drag coefficient c_d terms should be seleceted to be comparable with respect to the actual values of the multirotor. Accordingly, the terms K_p, K_d, c_d, and $\bar{s}_\Phi$ can be set with the current setting of the multirotor. However, the lumped disturbance term $\Delta(=\bar{\delta}+c_d|\dot{p}^r|+\bar{s}_\Phi|ge_3+\ddot{p}^r|)$ keeps changing while a multirotor is in flight. For example, the external disturbance, e.g. wind condition, could be different location to location. Also, the reference velocity $\dot{p}^r$ and accelerations $\ddot{p}^r$ are derived from the reference trajectory so that the terms $|\dot{p}^r|$ and $|ge_3 + \ddot{p}^r|$ will change while the multirotor is in maneuver. According to the stability analysis in eq. (11), the radius of the ultimate bound varies proportionally with Δ, and the behavior of the error states will also be altered with respect to Δ. Therefore, we generate *library of funnels* with various values of Δ, which will be denoted as Δ_l with $l(=1,\cdots,L)$ to use them depending on the magnitude of $|\Delta|$.

Let U_{Δ_l} represent the set of ultimate bound when the norm of disturbance is Δ_l. For each Δ_l, we evaluate the funnel with two different settings of initial sets such that $U_{\Delta_l} \subset \xi_b$ and $\xi_s \subset U_{\Delta_l}$. By doing this, we can see how the error states behave until they reach their ultimate bound from both smaller and larger regions.

In summary, a funnel library is organized as the set such as $F_{\text{lib}} = \{F_{\Delta_1}, \cdots, F_{\Delta_{l_M}}\}$. Each element is evaluated with different value of disturbance Δ_l, and it is constructed with funnels two different set of initial conditions as $F_{\Delta_l} = \{F^b_{\Delta_l}, F^s_{\Delta_l}\}$ where the superscript $j \in \{b, s\}$ indicates the initial condition $\{\xi_b, \xi_s\}$. The elements are the sequence of compact sets such as $F^j_{\Delta_l} = \{F^j_\Delta(t_0), \cdots, F^j_\Delta(t_N)\}$ where

$$F^j_{\Delta_l}(t_n) = \{e | e^\top P^j_{\Delta_l}(t_n) e \leq \alpha^j_{\Delta_l}(t_n)\}. \tag{21}$$

Here, $P^j_{\Delta_l}(t_n)$ and $\alpha^j_{\Delta_l}(t_n)$ are the parameters of the Lyapunov function eq. (18) optimized in the funnel generation. Then, a piece of funnel in the library is quoted as $F^j_{\Delta_l}(i)$

param	value	param	value
$\mathrm{d}t$	0.05 [s]	N	100
$K_{p,x}$	10 [s^{-2}]	$K_{v,x}$	4 [s^{-1}]
$K_{p,y}$	10 [s^{-2}]	$K_{v,y}$	4 [s^{-1}]
$K_{p,z}$	15 [s^{-2}]	$K_{v,z}$	6 [s^{-1}]
$\bar{s}_\Phi$	0.0532[$\cdot$] $\approx \sin 3^\circ$	R_b (ξ_b)	blockdiag$(1.56 I_3, 0.11 I_3)$ [m^{-2}, $m^{-2}s^2$]
c_d	0.32 [s^{-1}]	R_s (ξ_s)	blockdiag$(100 I_3, 11.1 I_3)$

TABLE I: parameters used for computing funnel library

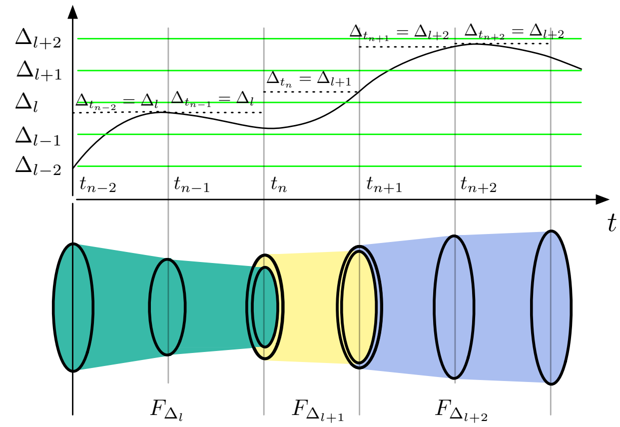


Fig. 2: Concept figure to explain how to connect funnels.

where i indicates the sequence number, i.e. t_n, of funnel generated with the setting Δ_l and j.

As an example, we have generated funnels with the setting in table I. Funnels are displayed with the Δ_l for every 0.5 from 0.5 to 3.0 $[ms^{-2}]$. The computed funnels are displayed in Fig. 1. The funnel, which is ellipsoid, is projected to each coordinate for visualization purposes. There are two notable characteristics from the generated funnels shown in fig. 1. First, if the disturbance Δ_l is set to the same value, the funnels converges to the similar values regardless of the initial condition j. The second one is that the ultimate bounds of the each axis are proportional to the value of Δ_l. These are the expected result from the analysis in eq. (11).

D. Combining funnels

In sec. III-B, we described the method how to generate funnels with various fixed disturbance norm values. However, the disturbance $\Delta(t)$ will keep varying with respect to time in most of flight operation scenarios. Let us assume that we use only one funnel computed with Δ_l satistying $|\Delta(t)| \leq \Delta_l$ for $t > 0$. In this case, the funnel analysis will indicate that set of error will eventually converge to the ultimate bound with $|\Delta| = \Delta_l$. Then, the selected funnel will be too conservative since the size of the ultimate bound is proportional to the norm of disturbance. Hence, in this subsection, we present the method for applying the funnel library computed with the fixed values of Δ_l to the general cases with time varying $\Delta(t)$.

In accordance with the funnel computation, we discretized the flight time with the same dt used in eq. (20). Let denote the descritezed time as t_m $(m = 0, \cdots, M)$. Then, our approach for generating funnel with the time varying disturbance is assigning i_m, j_m, and Δ_m to quote $F^{j_m}_{\Delta_m}(i_m)$ for all discretized time t_m during the flight.

First of all, we explain the method for deciding the disturbance level Δ_l for $t \in [t_m, t_{m+1}]$ which will be denoted as Δ_m. Among all Δ_l values, the obvious choice would be using the minimum one satisfying $|\Delta(t)| \leq \Delta_l$ during $t \in [t_m, t_{m+1}]$. It is because that Δ_l is the closest to the actual $|\Delta(t)|$ as well as the condition required in eq. (17)

Algorithm 1 Assigning funnels along reference trajectory

function COMBINE FUNNELS ($\Delta(t)$, ξ_o, $\mathcal{F}$)
  find Δ_l **s.t.** $\Delta_{l-1} < \Delta(0) \leq \Delta_l$
  $\Delta_0 \leftarrow \Delta_l$
  find t_n **and** j **with smallest volume of** $\bar{F}$ **s.t.**
  $\bar{F} = \{F | \xi_0 \subset F,\ F \in F_{\Delta_0}\}$
  $i_0 \leftarrow t_n$ (index of $\bar{F}$)
  $j_0 \leftarrow j$ (initial condition of $\bar{F}$)
  for $m \leftarrow 1$ to M **do**
    find Δ_l **s.t.** $\Delta_{l-1} < \hat{\Delta} \leq \Delta_l$
    where $\hat{\Delta} = \max(\Delta(t)),\ \forall t \in [t_m, t_{m+1}]$
    $\Delta_m \leftarrow \Delta_l$
    find t_n **and** j **with smallest volume of** $\bar{F}$ **s.t.**
    $\bar{F} = \{F | F^{j_{m-1}}_{\Delta_{m-1}}(i_{m-1}+1) \subset F,\ F \in F_{\Delta_m}\}$
    $i_m \leftarrow t_n$, $j_m \leftarrow j$
  end for
  return Δ_m, j_m, i_m for $m = \{1, \cdots, M\}$
end function

is satisfied. If the step size between Δ_l and time intervals are discretized fine enough, the actual disturbance will be well reflected. This procedure is described in the first row of the fig. 2.

Secondly, with the selected disturbance level Δ_m, we present the method for picking the indices i_m and j_m. For further explanations, for now, we assume that Δ_{m-1}, i_{m-1}, and j_{m-1} are given. Once we have $F^{j_{m-1}}_{\Delta_{m-1}}(i_{m-1})$, the set of error states at t_m can be directly accessible from the funnel library by referring $F^{j_{m-1}}_{\Delta_{m-1}}(i_{m-1}+1)$. Then, we can find $\bar{F} \in F_{\Delta_m}$ satisfying $F^{j_{m-1}}_{\Delta_{m-1}}(i_{m-1}+1) \subset \bar{F}$. It means that the set of error defined with $F^{j_{m-1}}_{\Delta_{m-1}}(i_{m-1})$ will evolve and be the subset of $\bar{F}$ at t_m. Accordingly, it is possible to assign any combination of i_m and j_m indicating elements of $\bar{F}$. However, because we are interested in evaluating the reachable set as tight as possible, we choose i_m and j_m indicating $F^{j_m}_{\Delta_m}(i_m)$ with the smallest volume. Selecting the smallest volume funnel could be easily done since the volume of an ellipsoid is proportional to the determinant of the ellipsoid defined in eq. (21).

So far, we explained the method how to find $F^{j_m}_{\Delta_m}(i_m)$ for t_m. In actual scenario, at t_0, Δ_0 can be computed in the same manner and the initial set of error ξ_0 can act a role same as $F^{j_{m-1}}_{\Delta_{m-1}}(i_{m-1}+1)$. Therefore, we can select funnels in the library from t_0 to t_M in order. The overall procedure is summarized in the algorithm 1.

Note. To implement the algorithm, we have to make the routine to check whether the condition $F_1 \subset F_2$ is true or false where the the sets $F_1 = \{e | e^\top R_1 e \leq 1\}$ and $F_2 = \{e | e^\top R_2 e \leq 1\}$ and R_1 and R_2 are positive definite matrices. In F_1 and F_2, the center of the ellipsoids are coincident at zero. Since the sets are defined as quadratic functions, we can show that $F_1 \subset F_2$ is satisfied when $\lambda_{\max}(D_1^{-1} R_2 D_1^{-\top}) \leq 1$ where D_1 is the Cholesky decomposition of $R_1 (= D_1 D_1^\top)$.

E. Funnel for checking robustness of reference trajectories

Once we generate the funnel library and combination of funnel pieces around the reference trajectory, the reachable set of multirotor position and velocity could be computed. Since we are particularly interested in checking the robustness of the reference trajectory, we focus on computing the reachable set of position. As in eq. (21), a piece of funnel could be represented with the inequality

$$F^{j_m}_{\Delta_m}(i_m) = \{e | e^\top P^{j_m}_{\Delta_m}(i_m) e \leq \alpha^{j_m}_{\Delta_m}(i_m)\}.$$

Let $P_m = P^{j_m}_{\Delta_m}(i_m)/\alpha^{j_m}_{\Delta_m}$, and, from eq. (16), the structure of P_m is

$$P_m = \begin{bmatrix} P_{m,p} & P_{m,pv} \\ P_{m,pv} & P_{m,v} \end{bmatrix}.$$

Then, the set of translational error, including position and velocity erros, can be projected into the position error ellipsoid such as

$$e_p^\top (P_{m,p} - P^2_{m,pv} P^{-1}_{m,v}) e_p = 1. \tag{22}$$

Therefore, for each n, the set of position can be evaluated with e_p satisfying eq. (22) such as $p_m = p^r(t_m) + e_p$.

For example, let us define the position of a point on an obstacle as p_{obs} and the parameter η as

$$\eta = (p_{\text{obs}} - p^r(t_m))^\top (P_{m,p} - P^2_{m,pv} P^{-1}_{m,v})(p_{\text{obs}} - p^r(t_m)).$$

If $\eta \leq 1$, the point on the obstacle is inside of the reachable set of the position., Consequently, there will be a possibility of collision between the multirotor and the obstacle. On the other hands, if $\eta > 1$, the funnel analysis guarantees that the collision will not happen.

IV. Experiments

Experimental scenario is the following. We want to find the *safe* velocity that a multirotor can go through a pipe type object as shown in fig. 3.

A. Experimental setup

B. Experimental results

V. CONCLUSIONS

ACKNOWLEDGMENT

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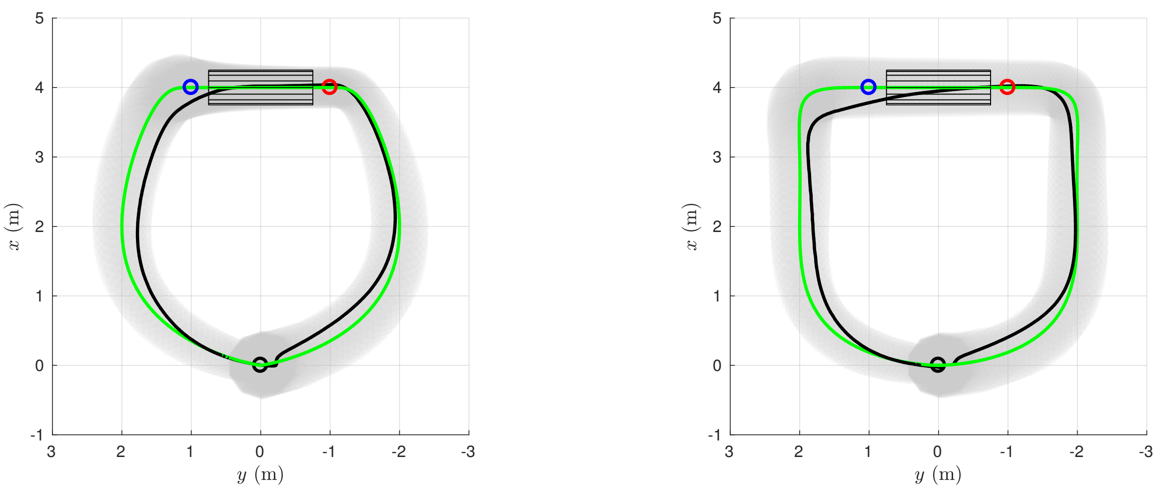

(a) The reference speed between the blue and red points is 1.0 m/s. (b) The reference speed between the blue and red points is 2.6 m/s.

Fig. 3: The green and black lines are the reference and measured trajectories, respectively. The shaded regions represent the funnels around the reference trajectory. The goal of this experiment is to move inside of the pipe between the blue and red points safely.