Dynamics of a multirotor

$$\begin{aligned}
\dot{p} &= v \\
m\dot{v} &= mge_3 + TRe_3 \\
\dot{R} &= R\hat{\omega} \\
J\dot{\omega} &= -\omega \times J\omega + \tau.
\end{aligned}$$

Define $X_1 = p$, $X_2 = v$, and errors

$$\begin{aligned}
e_p &= X_1 - p^r \\
e_v &= X_2 - v^r.
\end{aligned}$$

Error dynamics

$$\begin{aligned}
\dot{e}_p &= v - \dot{p}^r \\
&= e_v \\
\dot{e}_v &= ge_3 + \frac{T}{m}Re_3 - \dot{v}^r.
\end{aligned}$$

The collective thrust T is designed as

$$\begin{aligned}
n_d &= m(-K_p e_p - K_v e_v - ge_3 + \dot{v}^r) \\
R_d e_3 &= \frac{n_d}{|n_d|} \\
T &= n_d^T R e_3 \\
&= |n_d|(e_3^T R_d^T R e_3).
\end{aligned}$$

The error dynamics for e_v with the collective thrust

$$\begin{aligned}
\dot{e}_v &= ge_3 - \dot{v}^r + \frac{n_d}{m} + \frac{T}{m}Re_3 - \frac{n_d}{m} \\
&= ge_3 - \dot{v}^r - K_p e_p - K_v e_v - ge_3 + \dot{v}^r + \frac{T}{m}Re_3 - \frac{n_d}{m} \\
&= -K_p e_p - K_v e_v + \frac{1}{m}(n_d^T R e_3)Re_3 - \frac{n_d}{m} \\
&= -K_p e_p - K_v e_v + \frac{|n_d|}{m}\{(e_3^T R_d^T R e_3)Re_3 - R_d e_3\}.
\end{aligned}$$

The attitude error function is defined as $\Psi = \frac{1}{2}\mathrm{tr}(I_3 - R_d^T R)$, and the dynamics of the error function is

$$
\begin{aligned}
\Psi &= \frac{1}{2}\mathrm{tr}(I_3 - R_d^T R) \\
\dot{\Psi} &= -\frac{1}{2}\mathrm{tr}(R_d^T R\hat{e}_\omega) \\
&= -\frac{1}{4}\mathrm{tr}(-\hat{e}_\omega R^T R_d + R_d^T R\hat{e}_\omega) \\
&= -\frac{1}{4}\mathrm{tr}(\hat{e}_\omega(R_d^T R - R^T R_d)) \\
&= -\frac{1}{2}\mathrm{tr}(\hat{e}_\omega \hat{e}_R) \\
&= e_R^T e_\omega
\end{aligned}
$$

where $e_R = \frac{1}{2}(R_d^T R - R^T R_d)^\vee$ and $e_\omega = \omega - R^T R_d \omega_d = \omega - \omega_d^R$.

Actually, e_R could be interpreted in the following way

$$
\begin{aligned}
R^T R_d &= I_3 + \sin\theta\hat{K} + (1-\cos\theta)\hat{K}^2 \\
\hat{e}_R &= \frac{1}{2}(I_3 - \sin\theta\hat{K} + (1-\cos\theta)\hat{K}^2 - I_3 - \sin\theta\hat{K} - (1-\cos\theta)\hat{K}^2) \\
&= -\sin\theta\hat{K} \\
e_R &= -\sin\theta K
\end{aligned}
$$

If the angular velocity is designed to be $\omega = -K_R e_R + \omega_d^R$, the error e_ω becomes as $e_\omega = -K_R e_R$. Also, the attitude error dynamics becomes as

$$
\dot{\Psi} = -e_R^T K_R e_R.
$$

The error dynamics $\dot{e}_R$ is derived as

$$
\begin{aligned}
\dot{e}_R &= \frac{1}{2}(R_d^T R\hat{e}_\omega + \hat{e}_\omega R^T R_d)^\vee \\
&= \frac{1}{2}(\mathrm{tr}(R^T R_d)I_3 - R^T R_d)e_\omega \\
&= C(R^T R_d)e_\omega
\end{aligned}
$$

where $C = \frac{1}{2}(\mathrm{tr}(R^T R_d)I_3 - R^T R_d)$, $\dot{R}_d = R_d \mathrm{S}(\omega_d)$ and $\omega_d^R = R^T R_d \omega_d$. The matrix $C(R^T R_d)$ is further analyzed as follows:

$$
\begin{aligned}
C(R^T R_d) &= \frac{1}{2}\{(1+2\cos\theta)I_3 - (I_3 + \sin\theta\hat{K} + (1-\cos\theta)\hat{K}^2)\} \\
&= \cos\theta I_3 + \frac{1}{2}\{-\sin\theta\hat{K} - (1-\cos\theta)\hat{K}^2\} \\
&= \cos\theta I_3 + \frac{1}{2}\{\hat{e}_R - \frac{1-\cos\theta}{\sin^2\theta}\hat{e}_R^2\} \\
&= \cos\theta I_3 + \frac{1}{2}\{\hat{e}_R - \frac{1}{1+\cos\theta}\hat{e}_R^2\}
\end{aligned}
$$

In addition, the determinant $\det(C(R^T R_d)) = 4\cos\theta(\cos\theta + 1)$ where θ is the rotation angle from R_d to R.

Therefore, the error dynamics $\dot{e}_R$ could be further analyzed by setting $\omega = -K_R e_K + \omega_d^R$ as follows:

$$\begin{aligned}
\dot{e}_R &= C(R^T R_d)e_\omega \\
&= -C(R^T R_d)K_R e_R \\
&= -(\cos\theta I_3 + \frac{1}{2}\{\hat{e}_R - \frac{1}{1+\cos\theta}\hat{e}_R^2\})K_R e_R
\end{aligned}$$

If the Lyapunov candidate is set as $V_R = \frac{1}{2}e_R^T e_R$, the directional time derivative of the function becomes:

$$\begin{aligned}
\dot{V}_R &= e_R^T \dot{e}_R \\
&= -e_R^T(\cos\theta I_3 + \frac{1}{2}\{\hat{e}_R - \frac{1}{1+\cos\theta}\hat{e}_R^2\})K_R e_R \\
&= -\cos\theta \cdot e_R^T K_R e_R
\end{aligned}$$

Let me start Lyapunov stability analysis for the whole system.

$$V = \frac{1}{2}\{e_p^T K_p e_p + e_v^T e_v + e_R^T e_R\} + \alpha e_p^T e_v$$

$$\begin{aligned}
\dot{V} &= e_p^T K_p \dot{e}_p + e_v^T \dot{e}_v + e_R^T \dot{e}_R + \alpha e_p^T \dot{e}_v + \alpha e_v^T \dot{e}_p \\
&= e_p^T K_p e_v + e_v^T \left\{ -K_p e_p - K_v e_v + \frac{|n_d|}{m}\left[(e_3^T R_d^T R e_3)R e_3 - R_d e_3\right]\right\} - \cos\theta \cdot e_R^T K_R e_R \\
&\quad + \alpha e_v^T e_v + \alpha e_p^T \left\{ -K_p e_p - K_v e_v + \frac{|n_d|}{m}\left[(e_3^T R_d^T R e_3)R e_3 - R_d e_3\right]\right\} \\
&< -e_v^T K_v e_v + s^2 \frac{|e_v||n_d|}{m} - c^2 e_R^T K_R e_R + \alpha e_v^T e_v - \alpha e_p^T K_p e_p - \alpha e_p^T K_v e_v + \alpha s^2 \frac{|e_p||n_d|}{m} \\
&< -\alpha e_p^T K_p e_p - e_v^T (K_v - \alpha I_3) e_v - c^2 e_R^T K_R e_R - \alpha e_p^T K_v e_v + s^2 \frac{|e_v| + \alpha |e_p|}{m}|n_d| \\
&< -\alpha e_p^T K_p e_p - e_v^T (K_v - \alpha I_3) e_v - c^2 e_R^T K_R e_R - \alpha e_p^T K_v e_v \\
&\quad + s^2(|e_v| + \alpha|e_p|)(K_p|e_p| + K_v|e_v| + B) \\
&< -\alpha e_p^T K_p e_p - e_v^T (K_v - \alpha I_3) e_v - c^2 e_R^T K_R e_R - \alpha e_p^T K_v e_v \\
&\quad + \alpha s^2 k_p |e_p|^2 + s^2 k_v |e_v|^2 + s^2(\alpha k_v + k_p)|e_p||e_v| + s^2(|e_v| + \alpha|e_p|)B \\
&< e_p^T(-\alpha K_p + \alpha s^2 k_p I_3)e_p + e_v^T(-K_v + (\alpha + s^2 k_v)I_3)e_v - c^2 e_R^T K_R e_R \\
&\quad - \alpha e_p^T K_v e_v + s^2(\alpha k_v + k_p)(\frac{a_1}{2}|e_p|^2 + \frac{a_2}{2}|e_v|^2) + s^2 B(\frac{a_3 + a_5}{2} + \frac{a_4}{2}|e_v|^2 + \frac{\alpha a_6}{2}|e_p|^2)
\end{aligned}$$

where c and s are parameters satisfying $c^4 = 1 - e_R^T e_R$ and $s^4 = e_R^T e_R$. Also, k_v and k_p are maximum eigenvalues of K_v and K_p, respectively. To satisfy Young's inequality, $a_1 a_2 = 1$, $a_3 a_4 = 1$, $a_5 a_6 = 1$.