

Dynamics of a multirotor

$$\begin{aligned}\dot{p} &= v \\ m\dot{v} &= mge_3 + TRe_3 \\ \dot{R} &= R\hat{\omega} \\ J\dot{\omega} &= -\omega \times J\omega + \tau.\end{aligned}$$

Define  $X_1 = p$ ,  $X_2 = v$ , and errors

$$\begin{aligned}e_p &= X_1 - p^r \\ e_v &= X_2 - v^r.\end{aligned}$$

Error dynamics

$$\begin{aligned}\dot{e}_p &= v - \dot{p}^r \\ &= e_v \\ \dot{e}_v &= ge_3 + \frac{T}{m}Re_3 - \dot{v}^r.\end{aligned}$$

The collective thrust  $T$  is designed as

$$\begin{aligned}n_d &= m(-K_p e_p - K_v e_v - ge_3 + \dot{v}^r) \\ R_d e_3 &= \frac{n_d}{|n_d|} \\ T &= n_d^T R e_3 \\ &= |n_d|(e_3^T R_d^T R e_3).\end{aligned}$$

The error dynamics for  $e_v$  with the collective thrust

$$\begin{aligned}\dot{e}_v &= ge_3 - \dot{v}^r + \frac{n_d}{m} + \frac{T}{m}Re_3 - \frac{n_d}{m} \\ &= ge_3 - \dot{v}^r - K_p e_p - K_v e_v - ge_3 + \dot{v}^r + \frac{T}{m}Re_3 - \frac{n_d}{m} \\ &= -K_p e_p - K_v e_v + \frac{1}{m}(n_d^T R e_3)Re_3 - \frac{n_d}{m} \\ &= -K_p e_p - K_v e_v + \frac{|n_d|}{m}\{(e_3^T R_d^T R e_3)Re_3 - R_d e_3\}.\end{aligned}$$

The attitude error function is defined as  $\Psi = \frac{1}{2}\text{tr}(I_3 - R_d^T R)$ , and the dy-

namics of the error function is

$$\begin{aligned}
\Psi &= \frac{1}{2} \text{tr}(I_3 - R_d^T R) \\
\dot{\Psi} &= -\frac{1}{2} \text{tr}(R_d^T R \hat{e}_\omega) \\
&= -\frac{1}{4} \text{tr}(-\hat{e}_\omega R^T R_d + R_d^T R \hat{e}_\omega) \\
&= -\frac{1}{4} \text{tr}(\hat{e}_\omega (R_d^T R - R^T R_d)) \\
&= -\frac{1}{2} \text{tr}(\hat{e}_\omega \hat{e}_R) \\
&= e_R^T e_\omega
\end{aligned}$$

where  $e_R = \frac{1}{2}(R_d^T R - R^T R_d)^\vee$  and  $e_\omega = \omega - R^T R_d \omega_d = \omega - \omega_d^R$ .

Actually,  $e_R$  could be interpreted in the following way

$$\begin{aligned}
R^T R_d &= I_3 + \sin \theta \hat{K} + (1 - \cos \theta) \hat{K}^2 \\
\hat{e}_R &= \frac{1}{2}(I_3 - \sin \theta \hat{K} + (1 - \cos \theta) \hat{K}^2 - I_3 - \sin \theta \hat{K} - (1 - \cos \theta) \hat{K}^2) \\
&= -\sin \theta \hat{K} \\
e_R &= -\sin \theta K
\end{aligned}$$

If the angular velocity is designed to be  $\omega = -K_R e_R + \omega_d^R$ , the error  $e_\omega$  becomes as  $e_\omega = -K_R e_R$ . Also, the attitude error dynamics becomes as

$$\dot{\Psi} = -e_R^T K_R e_R.$$

The error dynamics  $\dot{e}_R$  is derived as

$$\begin{aligned}
\dot{e}_R &= \frac{1}{2}(R_d^T R \hat{e}_\omega + \hat{e}_\omega R^T R_d)^\vee \\
&= \frac{1}{2}(\text{tr}(R^T R_d) I_3 - R^T R_d) e_\omega \\
&= C(R^T R_d) e_\omega
\end{aligned}$$

where  $C = \frac{1}{2}(\text{tr}(R^T R_d) I_3 - R^T R_d)$ ,  $\dot{R}_d = R_d S(\omega_d)$  and  $\omega_d^R = R^T R_d \omega_d$ . The matrix  $C(R^T R_d)$  is further analyzed as follows:

$$\begin{aligned}
C(R^T R_d) &= \frac{1}{2}\{(1 + 2 \cos \theta) I_3 - (I_3 + \sin \theta \hat{K} + (1 - \cos \theta) \hat{K}^2)\} \\
&= \cos \theta I_3 + \frac{1}{2}\{-\sin \theta \hat{K} - (1 - \cos \theta) \hat{K}^2\} \\
&= \cos \theta I_3 + \frac{1}{2}\{\hat{e}_R - \frac{1 - \cos \theta}{\sin^2 \theta} \hat{e}_R^2\} \\
&= \cos \theta I_3 + \frac{1}{2}\{\hat{e}_R - \frac{1}{1 + \cos \theta} \hat{e}_R^2\}
\end{aligned}$$

In addition, the determinant  $\det(C(R^T R_d)) = 4 \cos \theta (\cos \theta + 1)$  where  $\theta$  is the rotation angle from  $R_d$  to  $R$ .

Therefore, the error dynamics  $\dot{e}_R$  could be further analyzed by setting  $\omega = -K_R e_K + \omega_d^R$  as follows:

$$\begin{aligned}\dot{e}_R &= C(R^T R_d) e_\omega \\ &= -C(R^T R_d) K_R e_R \\ &= -(\cos \theta I_3 + \frac{1}{2} \{ \hat{e}_R - \frac{1}{1 + \cos \theta} \hat{e}_R^2 \}) K_R e_R\end{aligned}$$

If the Lyapunov candidate is set as  $V_R = \frac{1}{2} e_R^T e_R$ , the directional time derivative of the function becomes:

$$\begin{aligned}\dot{V}_R &= e_R^T \dot{e}_R \\ &= -e_R^T (\cos \theta I_3 + \frac{1}{2} \{ \hat{e}_R - \frac{1}{1 + \cos \theta} \hat{e}_R^2 \}) K_R e_R \\ &= -\cos \theta \cdot e_R^T K_R e_R\end{aligned}$$

Let me start Lyapunov stability analysis for the whole system.

$$V = \frac{1}{2} \{ e_p^T K_p e_p + e_v^T e_v + e_R^T e_R \} + \alpha e_p^T e_v$$

$$\begin{aligned}\dot{V} &= e_p^T K_p \dot{e}_p + e_v^T \dot{e}_v + e_R^T \dot{e}_R + \alpha e_p^T \dot{e}_v + \alpha e_v^T \dot{e}_p \\ &= e_p^T K_p e_v + e_v^T \left\{ -K_p e_p - K_v e_v + \frac{|n_d|}{m} [(e_3^T R_d^T R e_3) R e_3 - R_d e_3] \right\} - \cos \theta \cdot e_R^T K_R e_R \\ &\quad + \alpha e_v^T e_v + \alpha e_p^T \left\{ -K_p e_p - K_v e_v + \frac{|n_d|}{m} [(e_3^T R_d^T R e_3) R e_3 - R_d e_3] \right\} \\ &< -e_v^T K_v e_v + s^2 \frac{|e_v| |n_d|}{m} - c^2 e_R^T K_R e_R + \alpha e_v^T e_v - \alpha e_p^T K_p e_p - \alpha e_p^T K_v e_v + \alpha s^2 \frac{|e_p| |n_d|}{m} \\ &< -\alpha e_p^T K_p e_p - e_v^T (K_v - \alpha I_3) e_v - c^2 e_R^T K_R e_R - \alpha e_p^T K_v e_v + s^2 \frac{|e_v| + \alpha |e_p|}{m} |n_d| \\ &< -\alpha e_p^T K_p e_p - e_v^T (K_v - \alpha I_3) e_v - c^2 e_R^T K_R e_R - \alpha e_p^T K_v e_v \\ &\quad + s^2 (|e_v| + \alpha |e_p|) (K_p |e_p| + K_v |e_v| + B) \\ &< -\alpha e_p^T K_p e_p - e_v^T (K_v - \alpha I_3) e_v - c^2 e_R^T K_R e_R - \alpha e_p^T K_v e_v \\ &\quad + \alpha s^2 k_p |e_p|^2 + s^2 k_v |e_v|^2 + s^2 (\alpha k_v + k_p) |e_p| |e_v| + s^2 (|e_v| + \alpha |e_p|) B \\ &< e_p^T (-\alpha K_p + \alpha s^2 k_p I_3) e_p + e_v^T (-K_v + (\alpha + s^2 k_v) I_3) e_v - c^2 e_R^T K_R e_R \\ &\quad - \alpha e_p^T K_v e_v + s^2 (\alpha k_v + k_p) (\frac{a_1}{2} |e_p|^2 + \frac{a_2}{2} |e_v|^2) + s^2 B (\frac{a_3 + a_5}{2} + \frac{a_4}{2} |e_v|^2 + \frac{\alpha a_6}{2} |e_p|^2)\end{aligned}$$

where  $c$  and  $s$  are parameters satisfying  $c^4 = 1 - e_R^T e_R$  and  $s^4 = e_R^T e_R$ . Also,  $k_v$  and  $k_p$  are maximum eigenvalues of  $K_v$  and  $K_p$ , respectively. To satisfy Young's inequality,  $a_1 a_2 = 1$ ,  $a_3 a_4 = 1$ ,  $a_5 a_6 = 1$ .