Practical - 1

Limits and Continuity

1) $\lim_{x \to a} \left[\frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}} \right]$

2) $\lim_{y \to 0} \left[\frac{\sqrt{a+y} - \sqrt{a}}{y\sqrt{a+y}} \right]$

3) $\lim_{x \to \pi/6} \left[\frac{\cos x - \sqrt{3}\sin x}{\pi - 6x} \right]$

4) $\lim_{x \to 0} \left[\frac{\sqrt{x^2+5} - \sqrt{x^3-3}}{\sqrt{x^3+3} - \sqrt{x^2+1}} \right]$

5) Examine Continuity of following functions.

i) $f(x) = \frac{\sin 2x}{\sqrt{1-\cos 2x}}$ for $0 < x \leq \pi/2$
$= \frac{\cos x}{\pi - 2x}$ for $\frac{\pi}{2} < x \leq \pi$
at $x = \pi/2$

ii) $f(x) = \frac{x^2-9}{x-3}$ for $0 < x < 3$
$x + 3$ for $3 \leq x < 6$
$= \frac{x^2 + 9}{x+3}$ for $6 < x < 9$
at $x = 3$ and $x = 6$

6) Find value of k so that function $f(x)$ ~~in the~~ is the cts at ~~point~~

i) $f(x) = \frac{1-\cos 4x}{x^2}$ $x < 0$
$= k$ $x = 0$
at $x = 0$

2.) $f(x) = (\sec^2 x)^{\cot^2 x}$ $\quad x \neq 0$ $\Big\}$ at $x = 0$

$= k$ $\quad x = 0$

3.) $f(x) = \dfrac{\sqrt{3} - \tan x}{\pi - 3x}$ $\quad x \neq \pi/3$ $\Big\}$ at $x = \pi/3$

$= k$ $\quad x = \pi/3$

7.) Discuss continuity are those removable. Redefine function to remove discontinuity.

i.) $f(x) = \dfrac{1 - \cos 3x}{x \tan x}$ $\quad x \neq 0$ $\Big\}$ at $x = 0$

$= 9$ $\quad x = 0$

ii.) $f(x) = \dfrac{(e^{3x} - 1)\sin x^\circ}{x^2}$ $\quad x \neq 0$ $\Big\}$ at $x = 0$

$= \pi/60$ $\quad x = 0$

8.) If $f(x) = \dfrac{e^{x^2} - \cos x}{x^2}$ for $x \neq 0$ is continous at $x = 0$ find $f(0)$

9.) If $f(x) = \dfrac{\sqrt{2} - \sqrt{1 + \sin x}}{\cos^2 x}$ for $x \neq \frac{\pi}{2}$ is continuous at $x = \frac{\pi}{2}$, find $f\left(\frac{\pi}{2}\right)$

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1) $\lim_{x\to a}\left[\dfrac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}\right]$

$\Rightarrow \lim_{x\to a} \dfrac{a+2x-3x}{3a+x-4x} \cdot \dfrac{\sqrt{3a+x}+2\sqrt{x}}{\sqrt{a+2x}+\sqrt{3x}}$

$\lim_{x\to a} \dfrac{a-x}{3a-3x} \times \dfrac{\sqrt{3a+x}+2\sqrt{x}}{\sqrt{a+2x}+\sqrt{3x}}$

$\lim_{x\to a} \dfrac{1}{3} \cdot \dfrac{\sqrt{3a+x}+2\sqrt{x}}{\sqrt{a+2x}+\sqrt{3x}}$

$= \dfrac{1}{3} \times \dfrac{\sqrt{3a+a}+2\sqrt{a}}{\sqrt{a+2a}+\sqrt{3a}}$

$= \dfrac{1}{3} \dfrac{\sqrt{4a}+2\sqrt{a}}{\sqrt{3a}+\sqrt{3a}}$

$= \dfrac{1}{3} \cdot \dfrac{\cancel{2}\cancel{\sqrt{a}}\ 2\sqrt{a}}{\cancel{2}\times 3\sqrt{a}} = \dfrac{1}{3} \times \dfrac{2}{2} \cdot \dfrac{2\sqrt{a}}{\sqrt{3a}}$

$= \boxed{\dfrac{2\sqrt{a}}{3\sqrt{3a}}}$ //

2) $\lim_{y\to 0} \dfrac{\sqrt{a+y}-\sqrt{a}}{y\sqrt{a+y}}$

$= \lim_{y\to 0} \dfrac{a+y-a}{y\sqrt{a+y} \times \sqrt{a+y}-\sqrt{a}}$

$= \lim_{y\to 0} \dfrac{1}{\sqrt{a+y}\left(\sqrt{a+y}+\sqrt{a}\right)}$

$= \dfrac{1}{\sqrt{a}\times\left(\sqrt{a}+\sqrt{a}\right)}$

$= \boxed{\dfrac{1}{2a}}$ //

3) $\lim_{x \to \pi/6} \frac{\cos x - \sqrt{3}\sin x}{\pi - 6x}$

$\Rightarrow$ Put $x = \frac{\pi}{6} + h$, $x = \frac{\pi}{6} + h$

$\therefore x \to \pi/6$, $h \to 0$

$$\lim_{h \to 0} \frac{\cos(h + \pi/6) - \sqrt{3}\sin(h + \pi/6)}{\pi - 6(h + \pi/6)}$$

$$\lim_{h \to 0} \frac{\cos h \cdot \cos\pi/6 - \sin h \sin\pi/6 - \sqrt{3}\sin h \cos\pi/6 - \cos h \cdot \sin\pi/6}{\pi - 6h - \pi}$$

$$= \lim_{h \to 0} \frac{\cancel{\cos h \times \sqrt{3}/2} - \sin h \times \frac{1}{2} - \sqrt{3}\sin h \times \sqrt{3}/2 - \cancel{\cos h \times}}{-6h}$$

$$= \lim_{h \to 0} \frac{-\frac{\sin h}{2} - \sin h \times 3/2}{-6h}$$

$$= \lim_{h \to 0} \frac{-\frac{\cancel{4}^{2}\sin h}{\cancel{2}}}{-6h}$$

$$= \lim_{h \to 0} \frac{2\sin h}{\cancel{3}6h} = \lim_{h \to 0} \frac{1}{3}\frac{\sin h}{h}$$

$$= \frac{1}{3}\lim_{h \to 0}\frac{\sin h}{h}$$

$$\boxed{\frac{1}{3}}$$

4.) $$\lim_{x \to \infty} \left[\frac{\sqrt{x^2+5} - \sqrt{x^2-3}}{\sqrt{x^2+3} - \sqrt{x^2+1}} \right]$$

By rationalising numerator & denomenator both

$$\lim_{x \to \infty} \left[\frac{\sqrt{x^2+5} - \sqrt{x^2-3}}{\sqrt{x^2+3} - \sqrt{x^2-1}} \times \frac{\sqrt{x^2+5} + \sqrt{x^2-3}}{\sqrt{x^2+5} + \sqrt{x^2+5}} \times \frac{\sqrt{x^2+3} + \sqrt{x^2-1}}{\sqrt{x^2+3} + \sqrt{x^2+5}} \right]$$

$$\lim_{x \to \infty} \left[\frac{(x^2+5-x^2+3)}{(x^2+3-x^2-1)} \frac{(\sqrt{x^2+3} + \sqrt{x^2+1})}{(\sqrt{x^2+5} + \sqrt{x^2-3})} \right]$$

$$\lim_{x \to \infty} \frac{8(\sqrt{x^2+3} + \sqrt{x^2+1})}{2(\sqrt{x^2+5} + \sqrt{x^2-3})}$$

$$4 \lim_{x \to \infty} \frac{\sqrt{x^2\left(1+\frac{3}{x^2}\right)} + \sqrt{x^2\left(1+\frac{1}{x^2}\right)}}{\sqrt{x^2\left(1+\frac{5}{x^2}\right)} + \sqrt{x^2\left(1-\frac{3}{x^2}\right)}}$$

After applying limit we get

$= 4.$

5) i) $F(x) = \dfrac{\sin 2x}{\sqrt{1-\cos 2x}}$ $0 < x < \frac{\pi}{2}$

$= \dfrac{\cos x}{\pi - 2x}$ $\frac{\pi}{2} < x < \pi$ } at $x = \pi/2$

Sol^n. a) $\because F\left(\frac{\pi}{2}\right) = \dfrac{\sin 2(\pi/2)}{\sqrt{1-\cos 2\left(\frac{\pi}{2}\right)}} = \dfrac{0}{\sqrt{1-(-1)}} = \dfrac{0}{\sqrt{2}}$

$F\left(\frac{\pi}{2}\right) = 0$

$\therefore$ F at $x = \frac{\pi}{2}$ define

b) $\lim\limits_{x \to \frac{\pi}{2}^+} F(x) = \lim\limits_{x \to \frac{\pi}{2}^+} \dfrac{\cos x}{\pi - 2x}$

Put $x - \frac{\pi}{2} = h$ $x = \frac{\pi}{2} + h$ as $x \to \frac{\pi}{2}$ $h \to 0^+$

$= \lim\limits_{h \to 0^+} \dfrac{\cos(\pi/2 + h)}{\pi - 2(\pi/2 + h)}$

$= \lim\limits_{h \to 0^+} \dfrac{-\sin h}{\pi - \pi - 2h}$

$= \lim\limits_{h \to 0^+} \dfrac{-\sin h}{-2h} = \dfrac{1}{2}.$

$\lim\limits_{x \to \frac{\pi}{2}^-} F(x) = \lim\limits_{x \to \frac{\pi}{2}^-} \dfrac{\sin 2x}{\sqrt{1-\cos 2x}}$

$= \lim\limits_{x \to \frac{\pi}{2}^-} \dfrac{2\sin x \cdot \cos x}{\sqrt{2\sin^2 x}}$

$= \lim\limits_{x \to \frac{\pi}{2}^-} \dfrac{2\sin x \cdot \cos x}{\sqrt{2}\ \sin x}$

$= \dfrac{2}{\sqrt{2}} \cdot \lim\limits_{x \to \pi/2} \cos x$

$= 0$

$\therefore$ L.H.L $\neq$ R.H.L

$\therefore$ f is not continuous at $x = \frac{\pi}{2}$

i) $f(x) = \begin{cases} \dfrac{x^2-9}{x-3}, & 0 < x < 3 \\ x+3, & 3 \leq x < 6 \\ \dfrac{x^2-9}{x+3}, & 6 \leq x < 9 \end{cases}$ at $x = 3$ & $x = 6$.

Sol^n:- a) For $x = 3$

$F(3) = 3+3 = 6$

$\therefore$ f is defined at $x = 3$

b) $\lim\limits_{x \to 3^+} F(x) = \lim\limits_{x \to 3^+} (x+3)$

$= 6.$

$\lim\limits_{x \to 3^-} f(x) = \lim\limits_{x \to 3^-} \dfrac{x^2-9}{x-3}$

$= \lim\limits_{x \to 3^-} \dfrac{(x-3)(x+3)}{\cancel{(x-3)}}$

$= \lim\limits_{x \to 3^-} 6.$

$\lim\limits_{x \to 3^-} F(x) = \lim\limits_{x \to 3^+} F(x) = F(3)$

$\therefore$ from (a) and (b)

F is continuous at $x = 3$.

for $x = 6$

a) $F(x) = \dfrac{x^2-9}{x+3}$

$= \dfrac{36-9}{6+3}$

$= \dfrac{27}{\ldots}$

b.) $\lim_{x \to 6^+} f(x) = \lim_{x \to 6^+} \frac{x^2 - 9}{x+3}$

$= 3$

$\lim_{x \to 6^-} f(x) = x + 3$

$= 6 + 3$

$= 9$

from (a) and (b)

f is discontinuous at $x = 6$

6) i.) $f(x) = \frac{1 - \cos 4x}{x^2}$, $x \neq 0$

$= k$, $x = 0$ } at $x = 0$

Sol^n:- $\because$ f is continuous at $x = 0$

$\therefore \lim_{x \to 0} f(x) = f(0)$

$\therefore \lim_{x \to 0} \frac{1 - \cos 4x}{x^2} = k$

$\therefore \lim_{x \to 0} \frac{2\sin^2 2x}{x^2} = k$

$\therefore 2 \lim_{x \to 0} \left(\frac{\sin 2x}{x} \right)^2 = k$

$\therefore 2 (2)^2 = k$

$\boxed{\therefore k = 8}$

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ii) $f(x) = (\sec^2 x)^{\cot^2 x}$, $x \neq 0$
$= k$, $x = 0$ } at $x = 0$

$\Rightarrow$ f is continuous at $x = 0$

$$\lim_{x \to 0} f(x) = k$$

$$\lim_{x \to 0} (\sec^2 x)^{\cot^2 x} = k$$

$$\lim_{x \to 0} (1 + \tan^2 x)^{\frac{1}{\tan^2 x}} = k$$

$$\lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e$$

$$e = k$$

$$k = e$$

iii) $f(x) = \dfrac{\sqrt{3} - \tan x}{\pi - 3x}$, $x \neq \pi/3$
$= k$, $x = \pi/3$ } at $x = \pi/3$

$\Rightarrow$ tan (A+B)

$\because$ ~~f is~~ continuous at $x = \pi/3$

$$\therefore \lim_{x \to \pi/3} f(x) = f\left(\frac{\pi}{3}\right)$$

$$\lim_{x \to \pi/2} \frac{\sqrt{3} - \tan x}{\pi - 3x} = k$$

Put, $x - \frac{\pi}{3} = h$, $x = \frac{\pi}{3} + h$, as $x \to \frac{\pi}{3}$
$\Rightarrow h \to 0$

$$\lim_{h\to 0} \frac{\sqrt{3} - \tan(\pi/3 + h)}{\pi - 3(\pi/3 + h)}$$

$$= \lim_{h\to 0} \frac{\sqrt{3} - \left(\dfrac{\tan\frac{\pi}{3} + \tan h}{1 - \tan\frac{\pi}{3}\tan h}\right)}{\pi - \pi - 3h}$$

$$= \lim_{h\to 0} \frac{\sqrt{3}\,(1 - \tan\pi/3 \tanh) - \tan\pi/3 - \tanh}{(-3h)(1 - \tan\pi/3 \tanh)}$$

$$= \lim_{h\to 0} \frac{\sqrt{3}\,(1 - \sqrt{3}\tanh) - \sqrt{3} - \tanh}{-3h(1 - \sqrt{3}\tanh)}$$

$$= \lim_{h\to 0} \frac{\cancel{\sqrt{3}} - 3\tanh - \cancel{\sqrt{3}} - \tanh}{-3h(1 - \sqrt{3}\tanh)}$$

$$= \lim_{h\to 0} \frac{-4\tanh}{-3h(1 - \sqrt{3}\tanh)}$$

$$= \cancel{\lim_{h\to}}\ \frac{4}{3} \lim_{h\to 0} \left(\frac{\tanh}{h}\right)\left(\frac{1}{1 - \sqrt{3}\tanh}\right)$$

$$= \frac{4}{3}\left(\frac{1}{1 - \sqrt{3}(0)}\right)$$

$$= \frac{4}{3}.$$

7) i.) $f(x) = \dfrac{1 - \cos 3x}{x \tan x}$ $\quad x \neq 0$

$\quad\quad 9 \quad\quad x = 0$

Sol^n:- $\lim_{x\to 0} f(x) = \frac{9}{2} \neq 9 = f(0)$

$\therefore f$ is not continuous at $x = 0$

8.8

Redefine function

$$f(x) = \begin{cases} \dfrac{1-\cos x}{x \tan x} & x \neq 0 \\ \dfrac{9}{2} & x = 0 \end{cases}$$

Now, $\lim_{x \to 0} f(x) = f(0)$

F has removable discontinuity at $x = 0$

ii) $f(x) = \dfrac{(e^{3x} - 1)\sin x^\circ}{x^2}$, $x \neq 0$

$= \dfrac{\pi}{60}$, $x = 0$

at $x = 0$

Consider,

$$\lim_{x \to 0} \frac{(e^{3x}-1)\sin x^\circ}{x^2} = \frac{\lim_{x \to 0} (e^{3x}-1) \sin\left(\frac{\pi x}{180}\right)}{x^2}$$

$$= 3 \lim_{x \to 0} \left(\frac{e^{3x}-1}{3x}\right) \lim_{x \to 0} \frac{\sin\left(\frac{\pi x}{180}\right)}{x}$$

$$= 3 \log e \frac{\pi}{180} = \frac{\pi}{60} = f(0)$$

$\therefore$ F is at $x = 0$.

viii) $f(x) = \dfrac{e^{x^2} - \cos x}{x^2}$, $x = 0$

is continous at $x = 0$

$\therefore$ Given,

$$\lim_{x \to 0} \frac{e^{x^2} - \cos x}{x^2} = f(0)$$

$$\lim_{x \to 0} \frac{e^{x^2} - \cos x - 1 + 1}{x^2}$$

$$\lim_{x\to 0} \frac{(e^{x^2}-1)+(1-\cos x)}{x^2}$$

$$\lim_{x\to 0} \frac{e^{x^2}-1}{x^2} + \lim_{x\to 0} \frac{1-\cos x}{x^2}$$

$$\log e + \lim_{x\to 0} \frac{2\sin^2 x/2}{x^2}$$

$$\log e + 2\lim_{x\to 0} \left(\frac{\sin x/2}{x}\right)^2$$

Multiply with 2 on Num & Deno.

$$= 1 + 2 \times \frac{1}{4} = \frac{3}{2} = f(0)$$ //

ix) $F(x) = \frac{\sqrt{2} - \sqrt{1+\sin x}}{\cos^2 x}$ $\quad x \neq \pi/2$

$f(0)$ is continuous at $x = \pi/2$

$$\lim_{x\to\pi/2} \frac{\sqrt{2}-\sqrt{1+\sin x}}{\cos^2 x} \times \frac{\sqrt{2}+\sqrt{1+\sin x}}{\sqrt{2}+\sqrt{1+\sin x}}$$

$$\lim_{x\to\pi/2} \frac{2-1+\sin x}{\cos^2 x(\sqrt{2}+\sqrt{1+\sin x})}$$

$$= \lim_{x\to\pi/2} \frac{1+\sin x}{1-\sin^2 x(\sqrt{2}+\sqrt{1+\sin x})}$$

$$= \lim_{x \to \pi/2} \frac{\cancel{1+\sin x}}{(1-\sin x)\,(\cancel{1+\sin x})\,(\sqrt{2} + \sqrt{1+\sin x})}$$

$$= \lim_{x \to \pi/2} \frac{1}{(1-\sin x)\,(\sqrt{2} + \sqrt{1+\sin x})}$$

$$= \frac{2}{2(\sqrt{2}+\sqrt{2})} = \frac{1}{2(2\sqrt{2})}$$

$$= \frac{1}{4\sqrt{2}}$$

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Practical No. 2

Topic : Derivative

Q1. Show that the following function defined from $\mathbb{R}$ to $\mathbb{R}$ are differentiable

i) $\cot x$ ii) $\operatorname{cosec} x$ iii) $\sec x$

Q2. If $f(x) = 4x+1, \; x \leq 2$

$= x^2+5, \; x>0$ at $x=2$

Then find f is differentiable or not?

Q3. If $f(x) = 4x+7, \; x<3$

$= x^2+3x+1, \; x \geq 3$, at $x=3$

Then find f is differentiable or not?

Q4. If $f(x) = 8x-5, \; x \leq 2$

$= 3x^2-4x+7, \; x>2$ at $x=2$

Then find f is differentiable or not?

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Answers

1) i) $F(x) = \cot x$

consider,

$$D(F(a)) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{\cot x - \cot a}{x - a}$$

Put $(x - a) = h$

$\therefore \quad x = a + h$

As, $x \to a$; $h \to a$

$$= \lim_{h \to 0} \frac{\cot(a+h) - \cot a}{(a+h) - a}$$

$$= \frac{\lim_{h \to 0} \frac{\cos(a+h)}{\sin(a+h)} - \frac{\cos a}{\sin a}}{\lim_{h \to 0} h}$$

$$= \frac{1}{h} \lim_{h \to 0} \frac{\cos(a+h)\sin a - \cos a \sin(a+h)}{\sin(a+h)\sin a}$$

$$= \frac{1}{h} \lim_{h \to 0} \frac{\sin(a - (a+h))}{\sin(a+h)\sin a}$$

$$= -\lim_{h \to 0} \frac{\sin h}{h} \cdot \lim_{h \to 0} \operatorname{cosec}(a+h) \operatorname{cosec} a$$

$$= -1 \cdot \operatorname{cosec}(a+0) \operatorname{cosec} a$$

$$= -\operatorname{cosec}^2 a$$

$\therefore \quad D f(a) = -\operatorname{cosec}^2 a$

$\therefore \quad f$ is differentiable $\forall a \in R$.

ii) $f(x) = \operatorname{cosec} x$

Consider,

$$D(F(a)) = \lim_{x \to a} \frac{F(x) - F(a)}{x - a}$$

$$= \lim_{x \to a} \frac{\operatorname{cosec} x - \operatorname{cosec} a}{x - a}$$

Let,

$x - a = h$, $x = a + h$

as $x \to a$, $h \to 0$

$$= \lim_{h \to 0} \frac{\operatorname{cosec}(a+h) - \operatorname{cosec}(a)}{a + h - a}$$

$$= \frac{\lim_{h \to 0} \frac{1}{\sin(a+h)} - \frac{1}{\sin a}}{\lim_{h \to 0} h}$$

$$= \frac{1}{h} \lim_{h \to 0} \frac{\sin a - \sin(a+h)}{\sin(a+h) \sin a}$$

$$= \frac{1}{h} \lim_{h \to 0} \frac{2\cos\left(\frac{a + a + h}{2}\right) \sin\left(\frac{a - a - h}{2}\right)}{\lim_{h \to 0} \sin^2 a \cosh + \cos a \sin a \sinh}$$

$$= \frac{1}{h} \cancel{\lim_{h \to 0}} \frac{2\cos\left(\frac{2a+h}{2}\right) \sin\left(\frac{-h}{2}\right)}{\sin^2 a}$$

$$= -\lim_{h \to 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \cdot \frac{\lim_{h \to 0} \cos\left(\frac{2a+h}{2}\right)}{\sin^2 a}$$

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$$= \frac{-1 \cdot \cos\left(\frac{2a}{2}\right)}{\sin^2 a}$$

$$= \frac{-1 \cdot \cos a}{\sin^2 a}$$

$$= -\cot a \cdot \operatorname{cosec} a$$

$\therefore D f(a) = -\cot a \cdot \operatorname{cosec} a$

$\therefore$ f is differentiable at $\forall x \in R$

iii) $f(x) = \sec x$

consider,

$$D(f(a)) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{\sec x - \sec a}{x - a}$$

Let,

$x - a = h$, $x = a + h$

as, $x \to a$, $h \to 0$

$$= \lim_{h \to 0} \frac{\sec(a+h) - \sec a}{a + h - a}$$

$$= \frac{1}{h} \lim_{h \to 0} \frac{1}{\cos(a+h)} - \frac{1}{\cos a}$$

$$= \frac{1}{h} \lim_{h \to 0} \frac{\cos a - \cos(a+h)}{\cos(a+h)\cos a}$$

$$= \frac{\frac{1}{h} \lim_{h \to 0} -2\sin\left(\frac{a + a + h}{2}\right) \sin\left(\frac{a - a \cdot h}{2}\right)}{\lim_{h \to 0} (\cos a \cosh - \sin a \sin h) \cos a}$$

$$= \frac{-2}{h} \lim_{h \to 0} \frac{\sin\left(\frac{2a+h}{2}\right) \sin\left(\frac{-h}{2}\right)}{\cos^2 a}$$

$$= \frac{-2}{h} \lim_{h \to 0} \sin\left(\frac{2a+h}{2}\right) \cdot \lim_{h \to 0} \sin\left(\frac{h}{2}\right) / h$$

$$= \frac{\cancel{2}}{\cos^2 a} \sin\left(\frac{2a+0}{2}\right) \frac{1}{\cancel{2}}$$

$$= \frac{\sin a}{\cos^2 a}$$

$= \tan a \cdot \sec a.$

$\therefore Df(a) = \tan a . \sec a$

$\therefore f$ is differentiable at $\forall x \in R$.

Q2. $F(x) = 4x+1 \quad , x \leq 2 \quad$ at $x = 2$.
$\quad = x^2+5 \quad ; x > 2$

→ R.H.D

$$DF(2^+) = \lim_{x \to 2^+} \frac{F(x) - f(a)}{x-a}$$

$$= \lim_{x \to 2^+} \frac{x^2+5-9}{x-2}$$

$$= \lim_{x \to 2} \frac{x^2-4}{x-2}.$$

$$= \lim_{x \to 2} \frac{(x-\cancel{2})(x+2)}{(\cancel{x-2})}$$

$= 2+2$

$= 4.$

$\therefore$ R.H.D $= 4$.

→ L.H.D

$$DF(2^-) = \lim_{x \to 2^-} \frac{F(x) - f(a)}{x-a}$$

$$= \lim_{x \to 2} \frac{4x+1-9}{x-2}$$

$$= \lim_{x \to 2} \frac{4x-8}{x-2}$$

$$= \lim_{x \to 2} \frac{4(x-2)}{x-2}$$

$= 4.$

$\therefore$ L.H.D = 4

$\therefore$ L.H.D = R.H.D

$\therefore$ f is differentiable at $x = 2$

3.) $f(x) = 4x+7$, $x < 3$ at $x = 3$

$= x^2 + 3x + 1$, $x \geq 3$

R.H.D

$$Df(3^+) = \lim_{x \to 3^+} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to 3} \frac{x^2 + 3x + 1 - 19}{x - 3}$$

$$= \lim_{x \to 3} \frac{x^2 + 3x - 18}{x - 3}$$

$$= \lim_{x \to 3} \frac{\cancel{(x-3)}(x+6)}{\cancel{(x-3)}}$$

$$= 3 + 6$$

R.H.D = 9

L.H.D

$$Df(3^-) = \lim_{x \to 3^-} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to 3} \frac{4x + 7 - 19}{x - 3}$$

$$= \lim_{x \to 3} \frac{4x - 12}{x - 3}$$

$$= 4 \lim_{x \to 3} \frac{\cancel{x-3}}{\cancel{x-3}}$$

L.H.D = 4

R.H.D $\neq$ L.H.D

$\therefore$ f is not differentiable at $x = 3$

4) $f(x) = 8x-5 \qquad x \leq 2 \qquad$ at $x=2$

$\quad = 3x^2-4x+7 \qquad x>2$

→ R.H.D

$$Df(2^+) = \lim_{x \to 2^+} \frac{f(x)-f(a)}{x-a}$$

$$= \lim_{x \to 2} \frac{3x^2-4x+7-11}{x-2}$$

$$= \lim_{x \to 2} \frac{3x^2-4x-4}{x-2}$$

$$= \lim_{x \to 2} \frac{3x^2-6x+2x-4}{x-2}$$

$$= \lim_{x \to 2} \frac{3x(x-2)+2(x-2)}{x-2}$$

$$= \lim_{x \to 2} \frac{(3x+2)\cancel{(x-2)}}{\cancel{(x-2)}}$$

$$= 3(2)+2$$

$$= 8$$

R.H.D

L.H.D

$$Df(2^-) = \lim_{x \to 2^-} \frac{f(x)-f(a)}{x-a}$$

$$= \lim_{x \to 2} \frac{8x-5-11}{x-2}$$

$$= \lim_{x \to 2} \frac{8x-16}{x-2}$$

$$= 8 \lim_{x \to 2} \frac{(x-2)}{(x-2)}$$

[illegible]

[illegible]

[illegible] differentiable [illegible]

Practical No. 3

Application of Derivative

1. Find the intervals in which function is increasing or decreasing

i) $f(x) = x^3 - 5x - 11$

ii) $f(x) = x^2 - 4x$

iii) $f(x) = 2x^3 + x^2 - 20x + 4$

iv) $f(x) = x^3 - 27x + 5$

v) $f(x) = 69 - 24x - 9x^2 + 2x^3$

2. Find the intervals in which function is concave upwards and concave downwards

i) $y = 3x^2 - 2x^3$

ii) $y = x^4 - 6x^3 + 12x^2 + 5x + 7$

iii) $y = x^3 - 27x + 5$

iv) $y = 69 - 24x - 9x^2 + 2x^3$

v) $y = 2x^3 + x^2 - 20x + 4$

Answers

Q1 i) $f(x) = x^3 - 5x - 11$

$\Rightarrow f'(x) = 3x^2 - 5$

f is increasing iff $f'(x) > 0$

$3x^2 - 5 > 0$

$3x^2 > 5$

$x^2 > \frac{5}{3}$

$x > \pm\sqrt{\frac{5}{3}}$

$+ \quad -\sqrt{\frac{5}{3}} \quad - \quad \sqrt{\frac{5}{3}} \quad +$

$\therefore x \in \left(-\infty, -\sqrt{\frac{5}{3}}\right) \cup \left(\sqrt{\frac{5}{3}}, \infty\right)$

$\therefore f$ is decreasing iff $f'(x) < 0$

$\therefore 3x^2 - 5 < 0$

$\therefore 3x^2 < 5$

$\therefore x^2 < \frac{5}{3}$

$x < \pm\sqrt{\frac{5}{3}}$.

$+ \quad -\sqrt{\frac{5}{3}} \quad - \quad \sqrt{\frac{5}{3}} \quad +$

$\therefore x \in \left(-\sqrt{\frac{5}{3}}, \sqrt{\frac{5}{3}}\right)$

i) $f(x) = x^2 - 4x$

$\Rightarrow f'(x) = 2x - 4$

$\therefore f$ is increasing iff $f'(x) > 0$

$\therefore 2x - 4 > 0$

$\therefore 2(x-2) > 0$

$x - 2 > 0$

$x > 2$

$\therefore x \in (2, \infty)$

$\therefore f$ is decreasing iff $f'(x) < 0$

$2x - 4 < 0$

$2(x-2) < 0$

$x - 2 < 0$

$x < 2$

$\therefore x \in (-\infty, 2)$

iii) $f(x) = 2x^3 + x^2 - 20x + 4$

$f'(x) = 6x^2 + 2x - 20$

$\therefore f$ is increasing iff $f'(x) > 0$

$\therefore 6x^2 + \cancel{2x} - 20 > 0$

$\therefore 6x^2 + 12x - 10x - 20 > 0$

$\therefore 6x(x+2) - 10(x+2) > 0$

$(x+2)(6x-10) > 0$

$\therefore x \in (-\infty, -2) \cup \left(\frac{10}{6}, \infty\right)$

F is decreasing iff $f'(x) < 0$

$6x^2 + 2x - 20 < 0$

$6x^2 + 12x - 10x - 20 < 0$

$6x(x+2) - 10(x+2) < 0$

$\therefore (6x-10)(x+2) < 0$

$-2 \quad \frac{10}{6}$

$\therefore x \in (-2, 10/6)$

iv.) $f(x) = x^3 - 27x + 5$

$\Rightarrow f'(x) = 3x^2 - 27$

$= 3(x^2 - 9)$

F is increasing iff $f'(x) > 0$

$\therefore 3(x^2-9) > 0$

$\therefore x^2 - 9 > 0$

$(x-3)(x+3) > 0$

$-3 \quad 3$

$\therefore x \in (-\infty, -3) \cup (3, \infty)$

f is decreasing iff $f'(x) < 0$

$3(x^2-9) < 0$

$x^2-9 < 0$

$(x-3)(x+3) < 0$

$\therefore x \in (-3, 3)$

v) $f(x) = 69 - 24x - 9x^2 + 2x^3$

$\Rightarrow f'(x) = -24 - 18x + 6x^2$

i.e.) $6x^2 - 18x - 24$

$6(x^2 - 3x - 4)$

$\therefore f$ is increasing iff $f'(x) > 0$

$\therefore 6(x^2 - 3x - 4) > 0$

$x^2 - 3x - 4 > 0$

$x^2 - 4x + x - 4 > 0$

$x(x-4) + 1(x-4) > 0$

$(x+1)(x-4) > 0$

$x \in (-\infty, 1) \cup (4, \infty)$

f is decreasing iff $f'(x) < 0$

$\therefore 6(x^2 - 3x - 4) < 0$

$\therefore x^2 - 3x - 4 < 0$

$x^2 - 4x + x - 4 < 0$

$x(x-4) + 1(x-4) < 0$

$(x+1)(x-4) < 0$

Q2

i) $y = 3x^2 - 2x^3$

Let,

$f(x) = y = 3x^2 - 2x^3$

$\therefore f'(x) = 6x - 6x^2$

$\therefore f''(x) = 6 - 12x$

$= 6(1-2x)$

$f''(x)$ is concave upwards iff,

$f''(x) > 0$

$6(1-2x) > 0$

$1-2x > 0$

$-2x > -1$

$2x < 1$

$x < \frac{1}{2}$

$\therefore x \in (-\infty, \frac{1}{2})$

$f(x)$ is concave downwards iff,

$f''(x) < 0$

$6(1-2x) < 0$

$1-2x < 0$

$1 < 2x$

$2x > 1$

$x > \frac{1}{2}$

$\therefore x \in (\frac{1}{2}, \infty)$

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i) $y = x^4 - 6x^3 + 12x^2 + 5x + 7$

Let,

$f(x) = y = x^4 - 6x^3 + 12x^2 + 5x + 7$

$\therefore f'(x) = 4x^3 - 18x^2 + 24x + 5$

$\therefore f''(x) = 12x^2 - 36x + 24$

$= 12(x^2 - 3x + 2)$

$f''(x)$ is concave upwards iff

$f''(x) > 0$

$12(x^2 - 3x + 2) > 0$

$x^2 - 3x + 2 > 0$

$x^2 - x - 2x + 2 > 0$

$x(x-1) - 2(x-1) > 0$

$(x-2)(x-1) > 0$

$x \in (-\infty, 1) \cup (2, \infty)$

$f''(x)$ is concave downwards iff

$f''(x) < 0$

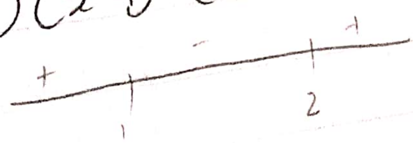
$12(x^2 - 3x + 2) < 0$

$x^2 - 3x + 2 < 0$

$x^2 - x - 2x + 2 < 0$

$x(x-1) - 2(x-1) < 0$

$(x-2)(x-1) < 0$



$x \in (1, 2)$

iii) $y = x^3 - 27x + 5$

=) Let,

$F(x) = y = x^3 - 27x + 5$

$F'(x) = 3x^2 - 27$

$F''(x) = 6x$

$F''(x)$ is concave upwards iff,

$F''(x) > 0$

$6x > 0$

$x > 0$

$\therefore x \in (0, \infty)$

$F''(x)$ is concave downwards iff,

$F''(x) < 0$

$6x < 0$

$x < 0$

$\therefore x \in (-\infty, 0)$

iv) $y = 69 - 24x - 9x^2 + 2x^3$

=) Let,

$F(x) = y = 69 - 24x - 9x^2 + 2x^3$

$F'(x) = -24 - 18x + 6x^2$

$F''(x) = -18 + 12x$

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$f''(x)$ is concave upwards iff,

$f''(x) > 0$

$-18 + 12x > 0$

$12x > 18$

$x > \frac{18}{12}$

$x \in (\frac{3}{2}, \infty)$

$f''(x)$ is concave downwards iff,

$f''(x) < 0$

$-18 + 12x < 0$

$12x < 18$

$x < \frac{18}{12}$

$x \in (-\infty, \frac{3}{2})$

v) $y = 2x^3 + x^2 - 20x + 4$

let,

$f(x) = y = 2x^3 + x^2 - 20x + 4$

$f'(x) = 6x^2 + 2x - 20$

$f''(x) = 12x + 2$

$= 2(6x + 1)$

$\therefore$ $f''(x)$ is concave upwards iff.

$f''(x) > 0$

$2(6x - 1) > 0$

$6x - 1 > 0$

$6x > 1$

$x > \frac{1}{6}$

c)

$f''(x)$ = concave down ...

$f''(x) < 0$

$2(6x+1) < 0$

$6x+1 < 0$

$6x < -1$

$x < -\frac{1}{6}$

$x \in (-\infty, -\frac{1}{6})$

Practical 4

1) Find maxima and minima.

i) $f(x) = x^2 + \frac{16}{x^2}$ ii) $f(x) = 3 - 5x^3 + 3x^5$

iii) $f(x) = x^3 - 3x^2 + 1$ in $[-\frac{1}{2}, 4]$ iv) $f(x) = 2x^3 - 3x^2 - 12x + 1$ in $[-2, 3]$

2) Find the root of following equation by Newton's Method (Take 4 iteration only). Correct upto 4 decimal.

i) $f(x) = x^3 - 3x^2 - 55x + 9.5$ (take $x = 0$)

ii) $f(x) = x^3 - 4x - 9$ in $[2, 3]$

iii) $f(x) = x^3 - 1.8x^2 - 10x + 17$ in $[1, 2]$

Answers.

1) $f(x) = x^2 + \frac{16}{x^2}$

$f'(x) = 2x - \frac{32}{x^3}$.

Now, $f'(x) = 0$

$\therefore 2x - \frac{32}{x^3} = 0$

$2x = \frac{32}{x^3}$

$x^4 = \frac{32}{2}$

$x^4 = 16$

$x = \pm 2$

$f''(x) = 2 + \frac{96}{x^4}$

$f''(2) = 2 + \frac{96}{2^4}$

$= 2 + \frac{96}{16}$

$= 2 + 6$

$= 8 > 0$

$\therefore f$ has minimum value at $x = 2$.

$\therefore f(2) = 2^2 + \frac{16}{2^2}$

$= 4 + \frac{16}{4}$

$= 4+4$

$= 8$

$f''(-2) = 2 + \frac{96}{-2^4}$

$= 2 + \frac{96}{16}$

$= 2 + 6$

$= 8 > 0$

$\therefore$ f has maximum value at x

$\therefore$ function reaches minimum value at $x = 2$, $x = -2$

ii) $f(x) = 3 - 5x^3 + 3x^5$

$\therefore f'(x) = -15x^2 + 15x^4$

Consider,

$f(x) = 0$

$-15x^2 + 15x^4 = 0$

$15x^4 = 15x^2$

$x^2 = 1$

$x = \pm 1$

$\therefore f''(x) = -30x + 60x^3$

$\therefore f(1) = -30 + 60$

$= 30 > 0$

$\therefore$ f has minimum value at $x = 1$.

$\therefore f(1) = 3 - 5(1)^3 + 3(1)^5$

$= 6 - 5$

$= 1$

$\therefore f''(-1) = -30(-1) + 60(-1)^3$

$= 30 - 60$

$= -30 < 0$

$\therefore$ f has maximum value at $x = -1$

$\therefore f(-1) = 3 - 5(-1)^3 + 3(-1)^5$

$= 3 + 5 - 3$

$= 5$

$\therefore$ f has the maximum value 5 at $x = -1$ and has the minimum value at $x = 1$

$f(x) = x^3 - 3x^2 + 1$

$f'(x) = 3x^2 - 6x$

Consider,

$f'(x) = 0$

$\therefore 3x^2 - 6x = 0$

$3x(x-2) = 0$

$\therefore 3x = 0 \,/\, x - 2 = 0$

$\therefore x = 0 \,/\, x = 2.$

$f''(x) = 6x - 6$

$\therefore f''(0) = 6(0) - 6$

$= 0 - 6$

$= -6 < 0$

$\therefore$ f has maximum value at $x = 0$

$\therefore f''(0) = (0)^3 - 3(0)^2 + 1$

$= 1$

$f''(2) = 6(2) - 6$

$= 12 - 6$

$= 6 > 0$

$\therefore$ f has minimum value at $x = 2$

$f(2) = (2)^3 - 3(2)^2 + 1$

$= 8 - 3(4) + 1$

$= 9 - 12$

$= -3$

$\therefore$ f has maximum value 1 at $x = 0$ and

f has maximum value -3 at $x = 2$.

iii) $f(x) = 2x^3 - 3x^2 - 12x + 1$

$\therefore f'(x) = 6x^2 - 6x - 12$

Consider,

$f'(x) = 0$

$\therefore 6x^2 - 6x - 12 = 0$

$\therefore 6(x^2 - x - 2) = 0$

$x^2 - x - 2 = 0$

$x^2 + x - 2x - 2 = 0$

$x(x+1) - 2(x+1) = 0$

$(x+1)(x-2) = 0$

$\therefore x = 2 \,/\, x = -1$

$\therefore f''(x) = 12x - 6$

$\therefore f''(2) = 12(2) - 6$

$= 24 - 6$

$= 18 > 0$

$\therefore$ f has minimum value

$\therefore f(2) = 2(2)^3 - 3(2)^2 - 12(2) + 1$

$= 2(8) - 3(4) - 24 + 1$

$= 16 - 12 - 24 + 1$

$= -19$

$\therefore f''(-1) = 12(-1) - 6$

$= 12 - 6$

$= -18 < 0$

$\therefore$ f has maxim value at $x = -1$

$\therefore f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) + 1$

$= -2 - 3 + 12 + 1$

$= 8$

$\therefore$ f has maximum value 8 at $x = -1$

and

f has minimum value ~~at~~ 8 at

$x = 2.$

$x_0 = 0 \rightarrow$ Given

Q2. i) $f(x) = x^3 - 3x^2 - 55x + 9.5$

$f'(x) = 3x^2 - 6x - 55$

By Newton's method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$\therefore x_1 = 0 + \frac{9.5}{55}$

$\therefore x_1 = 0.1727$

$\therefore f(x_1) = (0.1727)^3 - 3(0.1727)^2 - 55(0.1727) + 9.5$

$= 0.0051 - 0.0895 - 9.4985 + 9.5$

$= -0.0829.$

$\therefore f'(x_1) = 3(0.1727)^2 - 6(0.1727) - 55$

$= 0.0895 - 1.0362 - 55$

$= -55.9467.$

$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= \frac{0.1727 - 0.0829}{55.9467}$

$= 0.1712$

$\therefore f(x_2) = (0.1712)^3 - 3(0.1712)^2 - 55(0.1712) + 9.5$

$= 0.0050 - 0.0879 - 9.416 + 9.5$

$= 0.0011$

$\therefore f'(x_2) = 3(0.1712)^2 - 6(0.1712) - 55$

$= 0.0879 - 1.0272 - 55$

$= -55.9393.$

$\therefore x^3 = x_2 - \frac{f(x_2)}{f'(x_1)}$

$= 0.1712 + \frac{0.0011}{55.9393}$

$= 0.1712.$

∴ The root of equation is 0.1712.

$f(x) = x^3 - 4x - 9$ [2,3]

⇒ $f'(x) = 3x^2 - 4$

$f(2) = 2^3 - 4(2) - 9$
$= 8 - 8 - 9$
$= -9$

∴ $f(3) = 3^3 - 4(3) - 9$
$= 27 - 12 - 9$
$= 6$

let $x_0 = 3$ be the initial approximation.

∴ By Newton's Method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 3 - \frac{6}{23}$$

$$= 2.7392.$$

$f(x_1) = (2.7392)^3 - 4(2.7392) - 9.$
$= 20.5528 - 10.9568 - 9$
$= 0.596$

∴ $f'(x_1) = 3(2.7392)^2 - 4.$
$= 22.5096 - 4.$
$= 18.5096$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 2.7892 - \frac{0.896}{18.6096}$$

$$= 2.7071$$

$$F(x_2) = (2.7071)^3 - 4(2.7071) - 9$$

$$= 19.8386 - 10.8284 - 9$$

$$= 0.0102$$

$$F'(x_2) = 3(2.7071)^2 - 4$$

$$= 21.9851 - 4$$

$$= 17.9851$$

$$\therefore x_3 = x_2 - \frac{F(x_2)}{F'(x_2)}$$

$$= 2.7071 - \frac{0.0102}{17.9851}$$

$$= 2.7071 - 0.0056$$

$$= 2.7015$$

$$F(x_3) = (2.7015)^3 - 4(2.7015) - 9$$

$$= 19.7158 - 10.806 - 9$$

$$= -0.0901$$

$$\therefore F'(x_3) = 3(2.7015)^2 - 4$$

$$= 21.8943 - 4$$

$$= 17.8943$$

$$x_4 = 2.7015 + \frac{0.0901}{17.8943}$$

$$= 2.7015 + 0.0050$$

$$= 2.7065$$

$f(x) = x^3 - 1.8x^2 - 10x + 17$ [1, 2]

$f'(x) = 3x^2 - 3.6x - 10$

$f(1) = (1)^3 - 1.8(1)^2 - 10(1) + 17$

$= 1 - 1.8 - 10 + 17$

$= 6.2$

$f(2) = (2)^3 - 1.8(2)^2 - 10(2) + 17$

$= 8 - 7.2 - 20 + 17$

$= -2.2$

let, $x_0 = 2$ be initial approximation

By Newton's Method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = x_0 - \frac{f(x_2)}{f'(x_2)}$$

$$= 2 - \frac{2.2}{5.2}$$

$$= 2 - 0.4230$$

$$= 1.577$$

$f(x_1) = (1.577)^3 - 1.8(1.577)^2 - 10(1.577) + 17$

$= 3.9219 - 4.4764 - 15.77 + 17$

$= 0.6755$

$f'(x_1) = 3(1.577)^2 - 3.6(1.577) - 10$

$= 7.4608 - 5.6772 - 10$

$= -8.2164$

$$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1.577 + \frac{0.6755}{8.2164}$$

$$= 1.577 + 0.0822$$

$$= 1.6592$$

$F(x_2) = (1.6592)^3 - 1.8(1.6592)^2 - 10(1.6592) + 1.7$

$= 4.5677 - 4.9553 - 16.592 + 17$

$= 0.0204.$

$F'(x_2) = 3(1.6592)^2 - 3.6(1.6592) - 10$

$= 8.2588 - 5.97312 - 10$

$= -7.743.$

$$x_3 = x_2 - \frac{F(x_2)}{F'(x_2)}$$

$$= 1.6592 + \frac{0.0204}{7.7143}$$

$= 1.6592 + 0.0026$

$= 1.6618$

$F(x_3) = (1.6618)^3 - 1.8(1.6618)^2 - 10(1.6618) + 17$

$= 4.5892 - 4.9708 - 16.618 + 17.$

$= 0.0004.$

$F'(x_3) = 3(1.6618)^2 - 3.6(1.6618) - 10$

$= 8.2847 - 5.9824 - 10$

$= -7.6977$

$$\therefore x_4 = x_3 - \frac{F(x_3)}{F'(x_3)}$$

$$= 1.8618 + \frac{0.0004}{7.6977}$$

$= 1.6618$

$\therefore$ The root of equation is 1.6618.

Answers

i) $\int \frac{dx}{\sqrt{x^2+2x-3}}$

$$I = \int \frac{dx}{\sqrt{x^2+2x-3}}$$

$$= \int \frac{dx}{\sqrt{x^2+2x+1-4}}$$

$$= \int \frac{dx}{\sqrt{(x+1)^2-(2)^2}}$$

Comparing with $\int \frac{dx}{\sqrt{x^2-a^2}}$; $x^2 = (x+1)^2$, $a^2 = (2)^2$

$$I = \log\left|x+\sqrt{x^2-a^2}\right| + c$$

$$= \log\left|x+1+\sqrt{(x+1)^2-(2)^2}\right| + c$$

ii) $\int (4e^{3x}+1)\,dx$

$$I = \int (4e^{3x}+1)\,dx$$

$$= \int 4e^{3x}\,dx + \int 1\,dx$$

$$= \frac{4e^{3x}}{3} + x + c$$

iii) $\int (2x^2 - 3\sin x + 5\sqrt{x})\,dx$

$$I = \int (2x^2 - 3\sin x + 5\sqrt{x})\,dx$$

$$= 2\int x^2\,dx - 3\int \sin x\,dx + 5\int \sqrt{x}\,dx$$

$$= \frac{2}{3}x^3 + 3\cos x + 5 \times \frac{2}{3}x^{3/2} + c$$

$$= \frac{2}{3}x^3 + 3\cos x + \frac{10}{3}x^{3/2} + c.$$

$\therefore I = \frac{-1}{8} t^4 \cos(2t^4) + \frac{1}{16}\sin(2t^4) + c$

vi) $\int \sqrt{x}\,(x^2-1)\,dx$

$I = \int \sqrt{x}\,(x^2-1)\,dx$

$= \int (\sqrt{x}\cdot x^2 - \sqrt{x})\,dx$

$= \int (x^{5/2} - \sqrt{x})\,dx$

$= \int x^{5/2}\,dx - \int \sqrt{x}\,dx$

$= \frac{2}{7}x^{7/2} - \frac{2}{3}x^{3/2} + c$

vii) $\int \frac{1}{x^3}\sin\left(\frac{1}{x^2}\right)dx$

$= \int \frac{1}{x^3}\sin\left(\frac{1}{x^2}\right)dx$

let, $\frac{1}{x^2} = t$

$x^{-2} = t$

$\frac{-2}{x^3}\,dx = dt$

$\therefore I = \frac{1}{-2}\int \frac{-2}{x^3}\sin\left(\frac{1}{x^2}\right)dx$

$= \frac{-1}{2}\int \sin t\,dt$

$= \frac{-1}{2}\left[-\cos t\right] + c$

$= \frac{1}{2}\cos t + c$

Resubstituting $t = \frac{1}{x^2}$

$\therefore I = \frac{1}{2}\cos\left(\frac{1}{x^2}\right) + c$

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viii) $\int \frac{\cos x}{\sqrt[3]{\sin^2 x}}$

$I = \int \frac{\cos x}{\sqrt[3]{\sin^2 x}} dx$

let, $\sin x = t$

$\cos x \, dx = dt$

$\therefore I = \int \frac{dt}{\sqrt[3]{t^2}}$

$= \int \frac{dt}{t^{2/3}}$

$= \int t^{-2/3} dt$

$= 3 t^{1/3} + c$

$= 3 (\sin x)^{1/3} + c$

$= 3 \sqrt[3]{\sin x} + c$

ix) $\int e^{\cos^2 x} \cdot \sin 2x \, dx$

$I = \int e^{\cos^2 x} \sin 2x \, dx$

let, $\cos^2 x = t$

$-2\cos x \sin x \, dx = dt$

$-2\sin 2x \, dx = dt$

$\therefore I = -\int -\sin 2x \, e^{\cos^2 x} dx$

$= -\int e^t dt$

$= -e^t + c$

Resubstituting $t = \cos^2 x$

$\therefore I = -e^{\cos^2 x} + c$

x.) $\int \left(\frac{x^2 - 2x}{x^3 - 3x^2 + 1} \right) dx$

$I = \int \left(\frac{x^2 - 2x}{x^3 - 3x^2 + 1} \right) dx$

Let,

$x^3 - 3x^2 + 1 = t$

$(3x^2 - 6x)\, dx = dt$

$3(x^2 - 2x)\, dx = dt$

$(x^2 - 2x)\, dx = \frac{dt}{3}$

$\therefore \quad I = \int \frac{1}{t} \frac{dt}{3}$

$= \frac{1}{3} \int \frac{dt}{t}$

$= \frac{1}{3} \log t + c$

Resubstituting $t = x^3 - 3x^2 + 1$,

$I = \frac{1}{3} \log (x^3 - 3x^2 + 1) + c$

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Practical No. 6

Topic: Application of Integration & Numerical Integration.

Q1. Find the length of the following curve.

1) $x = t\sin t$, $y = 1 - \cos t$ $\quad t \in [0, 2\pi]$
2) $Y = \sqrt{4-2^2}$ $\quad x \in [-2, 2]$.
3) $Y = x^{3/2}$ in $[0, 4]$
4) $x = 3\sin t$, $y = 3\cos t$. $\quad t \in [0, 2\pi]$
5) $x = \frac{1}{6}y^3 + \frac{1}{2y}$ on $y \in [1, 2]$

Q2. Using Simpson's rule. Solve the following:-

1) $\int_0^2 e^{x^2}\,dx$ with $n = 4$.
2) $\int_0^4 x^2\,dx$ with $n = 4$.
3) $\int_0^{\pi/3} \sqrt{\sin x}\,dx$ with $n = 6$.

8 Answers

4) $y = \sqrt{4-x^2}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{4-x^2}}$$

$$= \frac{-x}{\sqrt{4-x^2}}$$

$$I = \int_{-2}^{2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_{-2}^{2} \sqrt{1+\frac{x^2}{4-x^2}}\, dx$$

$$= \int_{-2}^{2} \sqrt{\frac{4-x^2+x^2}{4-x^2}}\, dx$$

$$= \int_{-2}^{2} \sqrt{\frac{4}{4-x^2}}\, dx$$

$$= 2\int_{-2}^{2} \frac{1}{\sqrt{2^2-x^2}}\, dx$$

$$= 2\left[\sin^{-1}(x/2)\right]_{-2}^{2}$$

$$= 2\left[\sin^{-1}(1) - \sin^{-1}(-1)\right]$$

$$= 2\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right]$$

$y = x^{3/2}$ $\quad x \in [0,4]$

$\frac{dy}{dx} = \frac{3}{2} x^{1/2}.$

$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$

$= \int_0^4 \sqrt{1 + \frac{9}{4}x} \, dx.$

$= \frac{1}{2} \int_0^4 4 + 9x \, dx.$

$= \frac{1}{2} \left[\frac{[4+9x]^{3/2}}{3/2} \times \frac{1}{9} \right]_0^4$

$= \frac{1}{27} \left[(4+9x)^{3/2} \right]$

$= \frac{1}{27} \left[(4+0)^{7/2} - (4+31)^{7/5} \right]$

$= \frac{1}{27} (70^{3/2} - 8)$ units

$x = 3\sin t \qquad y = 3\cos t$

$\frac{dx}{dt} = 3\cos t \qquad \frac{dy}{dt} = -3\sin t.$

$I = \int_0^{2\pi} \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} \, dt.$

$= \int_0^{2\pi} \sqrt{(3\cos t)^2 + (-3\sin t)^2} \, dt.$

$= \int_0^{2\pi} \sqrt{9\sin^2 t + 9\cos^2 t} \, dt$

$= \int_0^{2\pi} 3 \, dt$

$$= 3\int_0^{2\pi} dt$$

$$= 3\,[x]_0^{2\pi}$$

$$= 3[2\pi - 0]$$

$$I = 6\pi \text{ units.}$$

v.) $x = \frac{1}{6}y^3 + \frac{1}{2y}$

$\Rightarrow \frac{dx}{dy} = \frac{y^2}{2} - \frac{1}{2y^2}$

$\frac{dx}{dy} = \frac{y^4-1}{2y^3}$

$$I = \int_0^2 \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy.$$

$$= \int_0^2 \sqrt{1+\frac{(y^4-1)^2}{4y^4}}\, dy.$$

$$= \int_0^2 \sqrt{\frac{(y^4-1) - 4\times y^4 \times 1}{4y^4}}\, dy.$$

$$= \int_0^2 \sqrt{\frac{(y^4+1)^2}{(2y^2)^2}}\, dy.$$

$$= \int_0^2 \frac{y^4+1}{2y^2}\, dy$$

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$= \frac{1}{2}\int_1^2 y^2\,dy + \frac{1}{2}\int_1^2 y^{-2}\,dy$

$= \frac{1}{2}\left[\frac{y^3}{3} - \frac{y^{-1}}{1}\right]_1^2$

$= \frac{1}{2}\left[\frac{8}{3} - \frac{1}{2} \quad \frac{-1}{3} - 1\right]$

$= \frac{1}{2}\left[\frac{7}{3} + \frac{1}{2}\right]$

$= \frac{1}{2}\left[\frac{17}{6}\right]$

$= \frac{17}{12}$ units.

Q2. $\int_0^2 e^{x^2}\,dx$ with $n = 4$

$\Rightarrow$ $h = \frac{b-a}{n} = \frac{2-0}{4} = 0.5$

x	0	0.5	1	1.5	2
y	1	1.284	2.7183	9.4877	54.5982
	y_0	y_1	y_2	y_3	y_4

$\int_0^2 e^{x^2}\,dx = \frac{2}{3}\left[(y_0 + y_4) + 4(y_1 + y_3) + 2(y_2)\right]$

$= \frac{0.5}{3}\left[(1 + 54.5982) + 5(1.284 + 9.487) + 2 \cdot [illegible]\right]$

$= \frac{0.5}{3}\left[55.5982 + 43.0866 + 5.436\right]$

$\int_0^2 e^{x^2}\,dx = 17.3535.$

$$\frac{\pi}{24}\left[1.3483 + 4(1.9948) + 2(1.8846)\right]$$

$$\frac{\pi}{24}\left[1.3483 + 7.9792 + 3.7692\right]$$

$$= \frac{\pi}{24} \times 13.0963$$

$$\int \sqrt{6 - x}\, dx = 0.9044$$

Practical No 7

Topic: Differential Equation

Q1.

i) $x \frac{dy}{dx} + y = e^x$

$\frac{dy}{dx} + \frac{1}{x} y = \frac{e^x}{x}$

$P(x) = \frac{1}{x}$ & $Q(x) = \frac{e^x}{x}$

$I.F = e^{\int 1/x \, dx}$

$= e^{\int 1/x \, dx}$

$= e^{\ln x}$

$I.F = x$

$y(I.F) = \int Q(x)(I.F)dx + c$

$= \int \frac{e^x}{x} \cdot x \, dx + c$

$= \int e^x \, dx + c$

$xy = e^x + c$

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$$e^{x}\frac{dy}{dx} + 2e^{x}y = 1.$$

$$\frac{dy}{dx} + \frac{2e^{x}}{e^{x}}y = \frac{1}{e^{x}} \qquad (\div \text{ by } e^{x})$$

$$\frac{dy}{dx} + 2y = \frac{1}{e^{x}}.$$

$$\frac{dy}{dx} + 2y = e^{-x}.$$

$$P(x) = 2 \qquad Q(x) = e^{-x}.$$

$$\int P(x)\,dx$$

$$IF = e^{\int 2dx}$$
$$= e^{2x}$$

$$y = (I.F.) = \int Q(x)\,(I.F.)\,dx + c$$

$$y \cdot e^{2x} \int e^{-x} + 2x\,dx + c$$
$$= \int e^{x}\,dx + c$$
$$y \cdot e^{2x} = e^{x} + c.$$

$$x \cdot \frac{dy}{dx} = \frac{\cos x}{x} - 2y$$

$$x\frac{dy}{dx} = \frac{\cos x}{x} - 2y.$$

$$\therefore \frac{dy}{dx} + \frac{2y}{x} = \frac{\cos x}{x^{2}}.$$

$$P(x) = 2(x) \qquad Q(x) = \frac{\cos x}{x^{2}}.$$

$$I.f = e^{\int P(x)\,dx}$$
$$= e^{\int 2/x\,dx}.$$

$= e^{2/x} x.$

$= \ln x^2$

$I.F = x^2$

$y(I.F) = \int Q(x)\,(I.F)\,dx + C$

$= \int \cos x + C.$

$x^2 y = \sin x + C.$

iv) $x \cdot \frac{dy}{dx} + 3y = \frac{\sin x}{x^2}.$

$\frac{dy}{dx} + \frac{3y}{x} = \frac{\sin x}{x^3}$ ($\div$ by x on both side).

$P(x) = 3/x \quad Q(x) = \sin x/x^3$

$= e^{\int P(x)\,dx}$

$= e^{\int 3/x\,dx}.$

$= e^{3/x}\,dx$

$= e^{\ln x^3}$

$I.F = x^3.$

$= \frac{3v+3}{v+2}$

$= \frac{3(v+1)}{v+2}$

$\cancel{\int \frac{(v+2)}{(v+1)}\,dv = 3dx}$

~~v + log~~ |

$\int \frac{v+1}{v}\,dx + \int \frac{1}{v+1}\,dv = 3x.$

$v + \log|x| = 3x + C.$

$2x + 3y + \log|2x+3y+1| = 3x + C$

$3y = x - \log|2x + 3 \ldots|$

$e^{x}\frac{dy}{dx} + 2e^{x}y = 1.$

$\frac{dy}{dx} + \frac{2e^{x}}{e^{x}}y = \frac{1}{e^{x}}$ $\quad (\div \text{ by } e^{x})$

$\therefore \frac{dy}{dx} + 2y = \frac{1}{e^{x}}.$

$\frac{dy}{dx} + 2y = e^{-x}.$

$P(x) = 2. \qquad Q(x) = e^{-x}.$

$I.F = e^{\int P(x)\,dx} = e^{\int 2dx.}$

$= e^{2x}.$

$y \cdot (I.F.) = \int Q(x)\,(I.F.)\,dx + c$

$y \cdot e^{2x} = \int e^{-x + 2x}dx + c.$

$= \int e^{x}\,dx + c.$

$y \cdot e^{2x} = e^{x} + c.$

$\frac{dy}{dx} = \frac{2x + 3y - 1}{6x + 9y + 6}$

put $2x + 3y = v$

$2 + 3\frac{dy}{dx} = \frac{dv}{dx}$

$\frac{dy}{dx} = \frac{1}{3}\left(\frac{dv}{dx} - 2\right)$

$\frac{1}{3}\left(\frac{dv}{dx} - 2\right) = \frac{1}{3}\left(\frac{v - 1}{v + 2}\right).$

$\frac{dv}{dx} = \frac{v-1}{v+2} + 2.$

$$\frac{dv}{dx} = \frac{v-1+2v+4}{v+2}.$$

vi.) $\sec^2 x \cdot \tan y \, dx + \sec^2 y \tan x \, dy = 0$

$\Rightarrow$) $\sec^2 x \cdot \tan y \, dx = -\sec^2 y \cdot \tan x \, dy$.

$$\frac{\sec^2 x \, dx}{\tan x} = -\frac{\sec^2 y}{\tan y} dy.$$

$$\int \frac{\sec^2 x \, dx}{\tan x} = -\int \frac{\sec^2 y \, dy}{\tan y}.$$

$\therefore \log|\tan x| = -\log|\tan y| + c$

$\log|\tan x \cdot \tan y| = c.$

$\tan x \cdot \tan y = e^{c}$

vii.) $\frac{dy}{dx} = \sin^2(x-y+1)$

$\Rightarrow$) put $x-y+1 = v.$

differentiating on both sides

$x-y+1 = v$

$$1-\frac{dy}{dx} = \frac{dv}{dx}$$

$$\cancel{1-\frac{dv}{dx} = \frac{dy}{dx}}.$$

$$1-\frac{dv}{dx} = \sin^2 v.$$

$$\frac{dv}{dx} = 1-\sin^2 v.$$

$$\frac{dv}{dx} = \cos^2 v.$$

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$$\frac{dv}{\cos^2 v} = dx.$$

$$\int \sec^2 v \, dv = \int dx.$$

$$\tan v = x + c.$$

$$\tan(x+y-1) = x + c.$$

iii) $$\frac{dy}{dx} = \frac{2x+3y-1}{6x+9y+6}$$

Put, $2x + 3y = v.$

$$2 + \frac{3dy}{dx} = \frac{dv}{dx}.$$

$$\frac{dy}{dx} = \frac{1}{3}\left(\frac{dv}{dx} - 2\right)$$

$$\frac{1}{3}\left(\frac{dy}{dx} - 2\right) = \frac{1}{3}\left(\frac{v-1}{v+2}\right)$$

$$\frac{dv}{dx} = \frac{v-1}{v+2} + 2.$$

$$\frac{dv}{dx} = \frac{v-1+2v+4}{v+2}.$$

$$= \frac{3v+3}{v+2}.$$

$$= \frac{3(v+1)}{v+2}$$

$$\int \left(\frac{v+2}{v+1}\right) dv = 3dx$$

$$\int \frac{v+1}{v} dx + \int \frac{1}{v+1} dv = 3x.$$

$$v + \log|x| = 3x + c.$$

$$2x + 3y + \log|2x+3y+1| = 3x + C$$

$$3y = x - \log|2x+3y+1| + c.$$

10/1/2022

Practical No. 8

Topic:- Euler's Method.

1.) $\frac{dy}{dx} = y + e^{x} - 2$, $y(0) = 2$, $h = 0.5$, find $y(2)$

2.) $\frac{dy}{dx} = 1 + y^2$, $y(0) = 0$, $h = 0.2$, find $y(1)$

3.) $\frac{dy}{dx} = \sqrt{\frac{x}{y}}$ $y(0) = 1$, $h = 0.2$ find $y(1)$

4.) $\frac{dy}{dx} = 3x^2 + 1$ $y(1) = 2$, find $y(2)$
for $h = 0.5$, $h = 0.25$.

5.) $\frac{dy}{dx} = \sqrt{xy} + 2$ $y(1) = 1$, Find $y(1.2)$ with
$h = 0.2$.

Answers

$\frac{dy}{dx} = y + e^x - 2.$

$f(x,y) = y + e^x - 2$, $y_0 = 2$, $x_0 = 0$, $h = 0.5$.

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	2	1	2.5
1	0.5	2.5	2.87	3.57435
2	1	3.5743	4.2925	5.3615

$$y_{n+1} = y_n + h f(x_n, y_n)$$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
3	1.5	5.3615	7.8431	9.28305
4	2	9.2831		

$\therefore$ By Euler's formula,

$y(2) = 9.2831$

2.) $\frac{dy}{dz} = 1+y^2$

$f(x,y) = 1+y^2$, $y_0 = 0$, $x_0 = 0$, $h = 0.2$.

using Euler's Iteration Formula,

$$y_{n+1} = y_n + hf(x_n, y_n)$$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	0	1	0.2
1	0.2	0.2	1.04	0.408
2	0.4	0.408	1.1665	0.6413
3	0.6	0.6413	1.4113	0.9236
4	0.8	0.9236	1.8530	1.2942
5	1	1.2942		

$\therefore$ By Euler's Formula,

$y(1) = 1.2942.$

i) $\frac{dy}{dx} = \sqrt{\frac{x}{y}}$ $y(0) = 1, x_0 = 0, h = 0.2$

Using Euler's iteration formula,

$$y_{n+1} = y_n + hF(x_n, y_n).$$

n	x_n	y_n	$F(x_n, y_n)$	y_{n+1}
0	0	1	0	0
1	0.2	0		
2	0.4			
3	0.6			
4	0.8			
5	1			

4.) $\frac{dy}{dx} = 3x^2+1$ $\quad y_0 = 2, \quad x_0 = 1, \quad h = 0$

$\Rightarrow$ For $h = 0.5$

Using Euler's iteration formula,

$y_{n+1} = y_n + hf(y x_n, y_n)$.

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	1	2	4	4
1	1.5	4	49	28.5
2	2	28.5		

By Euler's formula,

$y(2) = 28.5$.

for $h = 0.25$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	1	2	4	3
1	1.25	3	5.6875	4.4219
2	1.5	4.4219	7.75	6.3594
3	1.75	6.3694	10.1815	8.9048
4	2	8.9048		

$\therefore$ By Euler's Formula,

$y(2) = 8.9048$.

$\frac{dy}{dx} = \sqrt{xy} + 2 \qquad y_0 = 1, \; x_0 = 1, \; h = 0.2$

Using Euler's iteration formula,

$$y_{n+1} = y_n + hf(x_n, y_n)$$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	1	1	3	1.6
1	1.2	1.6		

By Euler's formula,
$y(1.2) = 1.6$

AW
17/01/2020

Practical - 9

Limits and Partial Order Derivatives

1.) Evaluate the following limits :-

i.) $\lim_{(x,y)\to(-4,-1)} \dfrac{x^3 - 3y + y^2 - 1}{xy + 5}$.

$\Rightarrow$ At $(-4,-1)$, denominato $\neq 0$.

$\therefore$ By applying limit

$$= \frac{(-4)^3 - 3(-1) + (-1)^2 - 1}{-4(-1) + 5}$$

$$= \frac{-64 + 3 + 1 - 1}{4 + 5}$$

$$= \frac{-61}{9}.$$

ii.) $\lim_{(x,y)\to(2,0)} \dfrac{(y+1)(x^2+y^2-4x)}{x+3y}$.

$\Rightarrow$ At $(2,0)$, denominator $\neq 0$

$\therefore$ By applying limit,

$$= \frac{(0+1)\left((2)^2 + 0 - 4(2)\right)}{2+0}$$

$$= \frac{1(4+0-8)}{2}$$

$$= \frac{-4}{2}$$

$$\lim_{(x,y,z)\to(1,1,1)} \frac{x^2-y^2z^2}{x^3-x^2yz}$$

At $(1,1,1)$, Denominator $= 0$

$$\lim_{(x,y,z)\to(1,1,1)} \frac{x^2-y^2z^2}{x^3-x^2yz}$$

$$\lim_{(x,y,z)\to(1,1,1)} \frac{\cancel{(x-yz)}(x+yz)}{x^2\cancel{(x-yz)}}$$

$$= \lim_{(x,y,z)\to(1,1,1)} \frac{x+yz}{x^2}.$$

On Applying limit

$$= \frac{1+1(1)}{1^2}$$

$= 2.$

Q2. Find f_x, f_y from each of the following :-

1) $f(x,y) = xye^{x^2}+y^2$

$$\therefore f_x = \frac{\partial}{\partial x}(f(x,y))$$

$$= \frac{\partial(xye^{x^2}+y^2)}{\partial x}$$

$$= ye^{x^2+y^2}(2x)$$

$$= 2xye^{x^2+y^2}$$

$$\therefore f_y = \frac{\partial(f(x,y))}{\partial y}$$

$$= \frac{\partial(xye^{x^2}+y^2)}{\partial y}$$

$$= xe^{x^2+y^2}(2y)$$

$$\therefore f_y = 2yxe^{x^2+y^2}$$

ii.) $f(x,y) = e^x \cos y$

$$f_x = \frac{\partial (f(x,y))}{\partial x}$$

$$= \frac{\partial (e^x \cos y)}{\partial x}$$

$$\therefore f_x = e^x \cos y$$

$$f_y = \frac{\partial (f(x,y))}{\partial y}$$

$$= \frac{\partial (e^x \cos y)}{\partial y}$$

$$\therefore f_y = -e^x \sin y.$$

iii.) $f(x,y) = x^3y^2 - 3x^2y + y^3 + 1.$

$$F_x = \frac{\partial (f(x,y))}{\partial x}$$

$$= \frac{\partial (x^3y^2 - 3x^2y + y^3 + 1)}{\partial x}$$

$$\therefore f_x = 3x^2y^2 - 6xy.$$

$$f_y = \frac{\partial (f(x,y))}{\partial y}$$

$$= \frac{\partial (x^3y^2 - 3x^2y + y^3 + 1)}{\partial y}$$

$$\therefore f_y = 2x^3y - 3x^2 + 3y^2$$

Using definition find values of f_x, f_y at $(0,0)$ for $f(x,y) = \frac{2x}{1+y^2}$.

$$f_x = \frac{\partial}{\partial x}\left(\frac{2x}{1+y^2}\right)$$

$$= \frac{1+y^2 \frac{\partial}{dx}(2x) - 2x \frac{\partial}{\partial x}(1+y^2)}{(1+y^2)^2}$$

$$= \frac{2+2y^2-0}{(1+y^2)^2}$$

$$= \frac{2\cancel{(1+y^2)}}{(1+y^2)\cancel{(1+y^2)}}$$

$$= \frac{2}{1+y^2}$$

At $(0,0)$

$$= \frac{2}{1+0}$$

$$= 2.$$

$$f_y = \frac{\partial}{\partial y}\left(\frac{2x}{1+y^2}\right)$$

$$= \frac{1+y^2 \frac{\partial}{\partial x}(2x) - 2x \frac{\partial}{dx}(1+y^2)}{(1+y^2)^2}$$

$$= \frac{1+y^2(0) - 2x(2y)}{(1+y^2)^2}$$

$$= \frac{-4xy}{(1+y^2)^2}$$

∂

At $(0,0)$, $= \frac{-4(0)(0)}{(1+0)^2}$

$= 0$

4.) Find all second order partial derivatives of f. Also [illegible] whether $f_{yx} = f_{xy}$.

i) $f(x,y) = \frac{y^2 - xy}{x^2}$.

$$f_x = \frac{x^2 \frac{\partial}{\partial x}(y^2 - xy) - (y^2 - xy)\frac{\partial}{\partial x}(x^2)}{(x^2)^2}.$$

$$= \frac{x^2(-y) - (y^2 - xy)(2x)}{x^4}$$

$$= \frac{-x^2y - 2x(y^2 - xy)}{x^4}.$$

$$F_y = \frac{2y - x}{x^2}$$

$$f_{xx} = \frac{\partial}{dx}\left(\frac{-x^2y - 2x(y^2 - xy)}{x^4}\right)$$

$$= \frac{\cancel{x^4}\left(\frac{\partial}{\partial x}(-x^2y - 2xy^2 + 2x^2y)\right) - (-x^2y - 2xy + 2x^2y)\frac{\partial (x^4)}{\partial x}}{(x^4)^2}$$

$$= \frac{x^4(-2xy - 2y^2 + 4xy) - 4x^3(-x^2y - 2xy + 2x^2y)}{x^6}$$

$$f_{yy} = \frac{\partial y}{\partial y}\left(\frac{-x^2y - 2xy^2 + 2x^2y}{x^4}\right) \quad \text{——} \quad (2)$$

$$= \frac{-x^2 - 4xy + 2x^2}{\cancel{x}\,x^4.} \quad \text{——} \quad (3)$$

$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{2y-x}{x^2}\right)$$

$$= \frac{x^2\frac{\partial}{\partial x}(2y-x) - (2y-x)\frac{\partial}{\partial x}(x^2)}{(x^2)^2}$$

$$= \frac{-x^2 - 4xy + 2x^2}{x^4.} \quad \text{——} \quad (4)$$

from 3 & 4.;

$$f_{yy} = f_{yx}$$

i) $f(x,y) = x^3 + 3x^2y^2 - \log(x^2+1)$

$$f_x = \frac{\partial}{\partial x}\left(x^3 + 3x^2y^2 - \log(x^2+1)\right) \qquad f_y = \frac{\partial}{\partial y}\left(x^3 + 3x^2y^2 - \log(x^2+1)\right)$$

$$= 3x^2 + 6xy^2 - \frac{2x}{x^2+1.} \qquad = 0 + 6x^2y - 0$$

$$= 6x^2y$$

$$f_{xx} = 6x + 6y^2 - \left(\frac{\cancel{x^2+1}\frac{d(2x)}{dx} - 2x\frac{\partial(x^2+1)}{\partial x}}{(x^2+1)^2}\right).$$

$$= 6x^2 + 6y^2 - \left(\frac{2(x^2+1) - 4x^2}{(x^2+1)^2}\right) \quad \text{——} \quad (1)$$

$$f_{yy} = \frac{\partial (6x^2y)}{\partial y}$$
$$= 6x^2 \qquad \text{———— (2)}$$

$$f_{xy} = \frac{\partial}{\partial y}\left(3x^2 + 6xy^2 - \frac{2x}{x^2+1}\right)$$
$$= 0 + 12xy - 0$$
$$= 12xy \qquad \text{———— (3)}$$

$$f_{yx} = \frac{\partial (6x^2y)}{\partial x}$$
$$= 12xy$$

from (3) & (4),

$$f_{xy} = f_{yx}$$

iii) $F(x,y) = \sin(x_* y) + e^{x+y}$.

$$\Rightarrow F_x = y\cos(xy) + e^{x+y}(1)$$
$$= y\cos(xy) + e^{x+y}$$

$$F_y = x\cos(xy) + e^{x+y}(1)$$
$$= x\cos(xy) + e^{x+y}$$

$$f_{xx} = \frac{\partial}{\partial x}\left(y\cos(xy) + e^{x+y}\right)$$
$$= -y\sin(xy).(y) + e^{x+y}(1)$$
$$= -y^2\sin(xy) + e^{x+y} \qquad \text{———— (1)}$$

$$f_{yy} = \frac{\partial}{\partial y}\left(x\cos(xy) + e^{x+y}\right)$$
$$= -x\sin(xy)(x) + e^{x+y}(1)$$
$$= -\sin(xy) + e^{x+y} \qquad \text{———— (2)}$$

$f_{xy} = \frac{\partial}{\partial y}\left(y\cos(xy) + e^{x+y}\right)$

$= -y^2 \sin(xy) + \cos(xy) + e^{x+y}$ ——— (3)

$f_{yx} = \frac{\partial}{\partial x}\left(x\cos(xy) + e^{x+y}\right)$

$= -x^2 \sin(xy) + \cos(xy) + e^{x+y}$ ——— (4)

$\therefore$ from 3 & 4.

$f_{xy} \neq f_{yx}$.

Find the linearization of $F(x,y)$ at given point.

$f(x,y) = \sqrt{x^2+y^2}$ at $(1,1)$

$f(1,1) = \sqrt{1^2+1^2} = \sqrt{2}$.

$f_x = \frac{1}{2\sqrt{x^2+y^2}}(2x)$

$= \frac{x}{\sqrt{x^2+y^2}}$

f_x at $(1,1) = \frac{1}{\sqrt{2}}$

$f_y = \frac{1}{2\sqrt{x^2+y^2}}(2y)$

$= \frac{y}{\sqrt{x^2+y^2}}$

f_y at $(1,1) = \frac{1}{\sqrt{2}}$.

$\therefore L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$= \sqrt{2} + \frac{1}{\sqrt{2}}(x-1) + \frac{1}{\sqrt{2}}(y-1)$

$= \sqrt{2} + \frac{1}{\sqrt{2}}(x-1+y-1)$

$= \sqrt{2} + \frac{1}{\sqrt{2}}(x+y-2)$

$= \sqrt{2} + \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y - \frac{2}{\sqrt{2}}$

$= \frac{x+y}{\sqrt{2}}$.

ii) $F(x,y) = 1-x+y\sin x$ at $(\pi/2, 0)$

$\Rightarrow F(\frac{\pi}{2}, 0) = 1 - \frac{\pi}{2} + 0 = 1 - \frac{\pi}{2}$.

$f_x = 0 - 1 + y\cos x$.

f_x ~~at~~ $(\frac{\pi}{2}, 0) = -1 + 0$

$= -1$

$f_y = 0 - 0 + \sin x$.

f_y at $(\frac{\pi}{2}, 0) = \sin\frac{\pi}{2} = 1$.

$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$= 1 - \frac{\pi}{2} + (-1)(x - \frac{\pi}{2}) + 1(y-0)$

$= 1 - \frac{\pi}{2} - x + \frac{\pi}{2} + y$

$= 1 - x + y$.

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$f(x,y) = \log x + \log y$ [illegible] 90

$f(\;) = \log(\;) + \log(\;) = 0$

$f_x = \frac{1}{x} = 0$

f_x at $(\;,\;) = 1$ $\qquad f_y = 0 + \frac{1}{y}$

f_y at $(\;,\;) = 1$

$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$= 0 + 1(x-1) + 1(y-1)$

$= x - 1 + y - 1$

$= x + y - 2$

Practical - 10

Q1. Find the directional derivative of following function at given points & in the direction of given vector.

i) $F(x,y) = x + 2y - 3$ $\quad a = (1,-1) \quad u = 3i - j$

$\Rightarrow$ Here, $u = 3i - j$ is not a unit vector

$\therefore |\bar{u}| = \sqrt{(3)^2 + (1)^2} = \sqrt{9+1} = \sqrt{10}$.

Unit vector along u is $\frac{u}{|u|} = \frac{1}{\sqrt{10}}(3,-1)$.

$= \left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$.

$f(a+hu) = f(1,-1) + h\left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$.

$f(a) = f(1,-1) = (1) + 2(-1) - 3 = 1 - 2 - 3 = -4$.

$F(a+hu) = f(1,-1) + h\left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$

$= f\left(1 + \frac{3}{\sqrt{10}}\right), \left(-1 - \frac{h}{\sqrt{10}}\right) - 3$.

$F(a+hu) = \left(1 + \frac{3}{\sqrt{10}}\right) + 2\left(-1 - \frac{h}{\sqrt{10}}\right) - 3$

$= \frac{1+3}{\sqrt{10}} - 2 - \frac{2h}{\sqrt{10}} - 3$.

$\therefore f(a+hu) = -4 + \frac{h}{\sqrt{10}}$

$$D_u f(a) = \lim_{h \to 0} \frac{f(a+hu) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{-4} + \cancel{h}/\sqrt{10} + \cancel{4}}{\cancel{h}}$$

$$D_u f(a) = \frac{1}{\sqrt{10}}$$

$f(x) = y^2 - 4x + 1$ $\quad a = (3, 4) \quad u = i + 5j$

Here, $u = i + 5j$ is not a unit vector

$$|\bar{u}| = \sqrt{(1)^2 + (5)^2} = \sqrt{26}.$$

Unit vector along u is $\frac{u}{|u|} = \frac{1}{\sqrt{26}}(1, 5)$

$$= \left(\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\right).$$

$f(a) = f(3,4) = (4)^2 - 4(3) + 1 = 5.$

$$f(a + hu) = f(3, 4) + h\left(\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\right)$$

$$= f\left(3 + \frac{h}{\sqrt{26}}, 4 + \frac{5h}{\sqrt{26}}\right).$$

$$f_{x,y}(a + hu) = \left(4 + \frac{5h}{\sqrt{26}}\right)^2 - 4\left(3 + \frac{h}{\sqrt{26}}\right) + 1$$

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$$= 16 + \frac{26h^2}{26} + \frac{40h}{\sqrt{26}} - 12 - \frac{4h}{\sqrt{26}} + 1$$

$$= \frac{25h^2}{26} + \frac{40h - 4h}{\sqrt{26}} + 5$$

$$= \frac{25h^2}{26} + \frac{36h}{\sqrt{26}} + 5$$

$$D_u f(a) = \lim_{h \to 0} \frac{\frac{25h^2}{26} + \frac{36h}{\sqrt{26}} + 5 - 5}{h}$$

$$= \frac{\cancel{h}\left(\frac{25h}{26} + \frac{36}{\sqrt{26}}\right)}{\cancel{h}}$$

$$\therefore D_u f(a) = \frac{25h}{26} + \frac{36}{\sqrt{26}} \;//$$

iii) $2x + 3y \qquad a = (1, 2), \; u = (3i + 4j)$

Here $u = 3i + 4j$ is not a unit vector

$|\bar{u}| = \sqrt{(3)^2 + (4)^2} = \sqrt{25} = 5$

Unit vector along u is $\frac{u}{|u|} = \frac{1}{5}(3, 4)$

$$= \left(\frac{3}{5}, \frac{4}{5}\right)$$

$F(a) = F(1, 2) = 2(1) + 3(2) = 8$

$F(a + hu) = F(1, 2) + h\left(\frac{3}{5}, \frac{4}{5}\right)$

$$= F\left(1 + \frac{3h}{5}, 2 + \frac{4h}{5}\right)$$

$$f(a+hu) = 2\left(1+\frac{3h}{5}\right) + 8\left(1+\frac{4h}{5}\right)$$
$$= 2 + \frac{6h}{5} + 6 + \frac{12h}{5}$$
$$= \frac{18h}{5} + 8$$

$$D_u f(a) = \lim_{h\to 0} \frac{\frac{18h}{5} + \cancel{8} - \cancel{8}}{h}$$
$$= \frac{18}{5} //$$

Find gradient vector for following function at given point.

$f(x,y) = x^y + y^x \quad a = (1,1)$

$F_x = y \cdot x^{y-1} + y^x \log y$

$F_y = x^y \log x + x y^{x-1}$

$\nabla f(x,y) = (F_x, F_y)$

$= (y x^{y-1} + y^x \log y,\ x^y \log x + x y^{x-1})$

$f(1,1) = (1+0,\ 1+0)$

$= (1, 1) //$

$f(x,y) = (\tan^{-1} x) \cdot y^2 \quad a = (1,-1)$

$f_x = \frac{1}{1+x^2} \cdot y^2$

$f_y = 2y \tan^{-1} x$

$\nabla f(x,y) = (F_x, F_y)$

$= \left(\frac{y^2}{1+x^2},\ 2y \tan^{-1} x\right)$

$$F(1,-1) = \left(\frac{1}{2}, \tan^{-1}(1)(-2)\right)$$

$$= \left(\frac{1}{2}, \frac{\pi}{42}(-2)\right)$$

$$= \left(\frac{1}{2}, -\frac{\pi}{2}\right)$$

iii) $F(x,y,z) = xyz - e^{x+y+z}$ $a = (1,-1,0)$

$F_x = yz - e^{x+y+z}$

$F_y = xz - e^{x+y+z}$

$F_z = xy - e^{x+y+z}$

$\nabla F(x,y,z) = F_x, F_y, F_z$

$= yz - e^{x+y+z}, xz - e^{x+y+z}, xy - e^{x+y+z}$

$F(1,-1,0) = \left((-1)(0) - e^{1+(-1)+0}, (1)(0) - e^{1+(-1)+0}, (1)(-1) - e^{1+(-1)+0}\right)$

$= (0 - e^0, 0 - e^0, -1 - e^0)$

$(-1, -1, -2)$

Q3 Find the equation of tangent & normal to the following using curves at given points

i) $x^2 \cos y + e^{xy} = 2$ at $(1,0)$

$F_x = \cos y \, 2x + e^{xy} y$

$F_y = x^2(-\sin y) + e^{xy} . x$

$(x_0, y_0) = (1,0)$ $\therefore x_0 = 1, y_0 = 0$

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eq^n of tangent

$f_x(x-x_0) + f_y(y-y_0) = 0$

$f_x(x_0, y_0) = \cos 0 \; 2(1) + e^0 0$
$= 1(2) + 0$
$= 2$

$f_y(x_0, y_0) = (1)^2(-\sin 0) + e^0 . 1$
$= 0 + 1.1$
$= 1$

Eq^n of

$2(x-1) + 1(y-0) = 0$
$2x - 2 + y = 0$
$2x + y - 2 = 0$ //

$\therefore$ The eq^n of tangent is $2x + y - 2$.

Eq^n of normal
$= ax + by + c = 0$
$\therefore bx + ay + d = 0$

$\therefore 1(1) + 2(y) + d = 0$ at $(1,0)$
$\therefore 1 + 2y + d = 0$
$0 = 1 + 2(0) + d$
$\therefore d + 1 = 0$
$\therefore d = -1$ //.

ii) $x^2 + y^2 - 2x + 3y + 2 = 0$ at $(2, -2)$

$F_x = 2x + 0 - 2 + 0 + 0$

$= 2x - 2$

$F_y = 0 + 2y - 0 + 3 + 0$

$= 2y + 3$

$(x_0, y_0) = (2, -2)$ $\therefore x_0 = 2,\ y_0 = -2$

$\therefore F_x(x_0, y_0) = 2(2) - 2 = 2$

$F_y(x_0, y_0) = 2(-2) + 3 = -1$

eq^n of tangent

$F_x(x - x_0) + F_y(y - y_0) = 0$

$2(x-2) + (-1(y+2)) = 0$

$2x - 2 - y - 2 = 0$

$2x - y - 4 = 0$ //

$\therefore$ The required eq^n of tangent is $2x - y - 4 = 0$

eq^n of Normal :

$ax + by + c = 0$

$bx + ay + d = 0$

$\therefore -1(x) + 2(y) + d = 0$

$-x + 2y + d = 0$

$-2 + 2(-2) + d = 0$ at $(2, -2)$

$-2 - 4 + d = 0$

$-6 + d = 0$

$d = 6$ //

Find the eq^n of tangent & normal line to each of the following surface.

$x^2 - 2yz + 3y + xz = 7$ at $(2,1,0)$

$f_x = 2x - 0 + 0 + z$

$f_x = 2x + z$.

$f_y = 0 - 2z + 3 + 0$

$= 2z + 3$

$f_z = 0 - 2y + 0 + x$

$= -2y + x$.

$(x_0, y_0, z_0) = (2,1,0)$ $\therefore$ $x_0 = 2, y_0 = 1, z_0 = 0$

$f_x(x_0, y_0, z_0) = 2(2) + 0 = 4$.

$f_y(x_0, y_0, z_0) = 2(0) + 3 = 3$

$f_z(x_0, y_0, z_0) = -2(1) + 2 = 0$

Equation of tangent

$f_x(x - x_0) + f_y(y - y_0) + f_z(z - z_0) = 0$

$\therefore$ $4(x-2) + 3(y-1) + 0(z-0) = 0$

$4x - 8 + 3y - 3 + 0 = 0$

$4x + 3y - 11 = 0$ //

Equation of normal at $(4, 3, -11)$

$$\frac{x - x_0}{f_x} = \frac{y - y_0}{f_y} = \frac{z - z_0}{f_z}$$

$$\frac{x-2}{4} = \frac{y-1}{3} = \frac{z+11}{0}$$ //

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ii) $3xy^2 - x - y + z = -4.$ at $(1,-1,2)$

$3xyz - x - y + z + 4 = 0$ at $(1,-1,2)$

$f_x = 3yz - 1 - 0 + 0 + 0$

$= 3yz - 1$

$f_y = 3xz - 0 - 1 + 0 + 0$

$= 3xz - 1$

$f_z = 3xy - 0 - 0 + 1 + 0$

$= 3xy + 1$

$(x_0, y_0, z_0) = (1,-1,2) \quad \therefore x_0 = 1, \; y_0 = -1, \; z_0 = 2$

$\therefore F_x(x_0, y_0, z_0) = 3(-1)(2) - 1 = -7$

$\therefore F_y(x_0, y_0, z_0) = 3(+1)(2) - 1 = 5$

$\therefore F_z(x_0, y_0, z_0) = 3(1)(-1) + 1 = -2.$

Eq^n of tangent:

$-7(x-1) + 5(y+1) - 2(z-2) = 0$

$-7x + 7 + 5y + 5 - 2z + 4 = 0$

$-7x + 5y - 2z + 16 = 0$ //

Eq^n of normal:

$$\frac{x - x_0}{f_x} = \frac{y - y_0}{f_y} = \frac{z - z_0}{F_z}$$

$$\frac{x-1}{-7} = \frac{y+1}{5} = \frac{z-2}{-2}$$ //

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Find the local maxima & minima for the following functions

$f(x,y) = 3x^2 + 3y\ 3y^2 - 3xy + 6x - 4y.$

$\therefore f_x = 6x + 0 - 3y + 6 - 0$

$= 6x - 3y + 6$

$f_y = 0 + 2y - 3x + 0 - 4$

$= 2y - 3x - 4$

$f_x = 0$

$6x - 3y + 6 = 0$

$3(2x - y + 2) = 0$

$2x - y + 2 = 0$

$2x - y = -2$ —— 1

$f_y = 0$

$2y - 3x - 4 = 0$

$2y - 3x = 4$ —— 2

Multiply eq^n 1 with 2

$\therefore 4x - 2y = -4$

$2y - 3x = 4$

$x = 0$

Substitute value of x in eq^n 1

$2(0) - y = -2$

$-y = -2$ $\therefore y = 2$

$\therefore$ Critical points are $(0,2)$

$r = f_{xx} = 6$

$t = f_{yy} = 2$

$S = f_{xy} = -3$

Here $r > 0$

$= rt - s^2$

$= 6(2) - (-3)^2$

$= 12 - 9$

$= 3 > 0$

$\therefore$ f has maximum at $(0,2)$

$\therefore 3x^2 + y^2 - 3xy + 6x - 4y$ at $(0,2)$.

$3(0)^2 + (2)^2 - 3(0)(2) + 6(0) - 4(2)$

$0 + 4 - 0 + 0 - 8$

$= -4$ //.

ii.) $F(x,y) = 2x^4 + 3x^2y^2 - y^2$

$f_x = 8x^3 + 6xy$.

$f_y = 3x^2 - 2y$

$f_x = 0$

$\therefore 8x^3 + 6xy = 0$

$2x(4x^2 + 3y) = 0$

$4x^2 + 3y = 0$ —— ①

$f_y = 0$

$3x^2 - 2y = 0$ —— ②

Multiply eqn ① with 3

② with 4.

$$12x^2 + 9y = 0$$
$$-12x^2 \mp 8y = 0$$
$$17y = 0$$
$$\therefore y = 0 \,//$$

Substitute value of y in eq^n (1)

$$4x^2 + 3(0) = 0$$
$$4x^2 = 0$$
$$x = 0$$

Critical point is $(0,0)$

$r = f_{xx} = 24x^2 + 6x$

$t = f_{yy} = 0 - 2 = -2$

$s = f_{xy} = 6x - 0 = 6x = 6(0) = 0$

r at $(0,0)$

$= 24(0) + 6(0)$

$= 0$

$\therefore r = 0$

$f(x,y)$ at $0,0)$

$= 2(0)^4 + 3(0)^2(0) - (0)$

$= 0 + 0 - 0$

$= 0$

$\therefore rt - s^2 = 0(-2) - (5)^2$

$= 0 - 0$

$= 0$

$\therefore r = 0$ & $rt - s^2 = 0$

(nothing to say)

iii) $f(x,y) = x^2 - y^2 + 2x + 8y - 70$

$f_x = 2x + 2$

$f_y = -2y + 8$

$f_x = 0 \quad \therefore \quad 2x + 2 = 0$

$\therefore x = -1$

$f_y = 0 \quad -2y + 8 = 0$

$y = \frac{-8}{2}$

$\therefore y = 4$

Critical point is $(-1, 4)$

$r = f_{xx} = 2$

$t = f_{yy} = -2$

$s = f_{xy} = 0$

$r > 0$

$rt - s^2 = 2(-2) - (0)^2$

$= -4 - 0$

$= -4 < 0$

$F(x,y)$ at $(-1, 4)$

$(-1)^2 - (4)^2 + 2(-1) + 8(4) - 70$

$1 + 16 - 2 + 32 - 70$

$= 17 + 30 - 70$

$= 37 - 70$

$= 33$

AK

07/12/202[illegible]