unrestricted means $\hookrightarrow \begin{cases} -ve \\ +ve \\ 0 \end{cases}$

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Duality of LP Problem

Aka Rental Problem.

Definition: Duality of given LP problem is also the LP problem which is obtained by using the data of given LP problem. The given LP problem is known as primal and its transformation is called dual problem. Dual problem is also known as mirrored image of the primal.

Rules of obtaining the duality of ~~LP pro~~ primal

Rule 1:

standarize the problem by using slacks variables and surplus variables only.

Rule II:

If primal objective function is maximization type, and minimization type then its dual objective function would be minimization and maximization type respectively.

Rule III:

If dual function is to be maximized, then its constraints (exception to rule No: IV) will be in $\leq$ types, and if dual objective function is to be minimized then its constraints (exception to rule No: IV) will be in $\geq$ types.

Rule IV:

If i^{th} primal-variable is stated as "unrestricted", then i^{th} dual constraint will be $=$ type. Similarly, if i^{th} primal constraint is in "equal" type, then i^{th} dual variable will be stated as unrestricted.

Rule v:

Perform c_j is replaced by b_i ($c_j \leftrightarrow b_i$) and b_i is replaced by c_j, x_j are replaced by y_i ($x_j \rightarrow y_i$), a_{ij} are replaced by a_{ji} ($a_{ij} \longrightarrow a_{ji}$)

\# Find the duality of the following LP problem.

$$\text{Min } Z = 2x_1 + 3x_2 + 4x_3$$

$$\text{st: } 2x_1 + 3x_2 + 5x_3 \geq 2$$

$$3x_1 + x_2 + 7x_3 = 3$$

$$x_1 + 4x_2 + 6x_3 \leq 5$$

$$x_1, x_2 \geq 0, \; x_3 \text{ is unrestricted}$$

$\Rightarrow$ Here,

Standard form: $\text{Min } Z = 2x_1 + 3x_2 + 4x_3 + 0s_1 + 0s_3$

$$\text{st: } 2x_1 + 3x_2 + 5x_3 - s_1 = 2$$

$$3x_1 + x_2 + 7x_3 = 3$$

$$x_1 + 4x_2 + 6x_3 + s_3 = 5$$

$$s_1, s_3, x_1, x_2 \geq 0, \; x_3 \text{ is unrestricted}$$

Its dual is:

$$\text{Max } Z_y = 2y_1 + 3y_2 + 5y_3$$

$$\text{st: } 2y_1 + 3y_2 + y_3 \leq 2$$

$$3y_1 + y_2 + 4y_3 \leq 3$$

due to x_3 is unrestricted $\rightarrow$ $5y_1 + 7y_2 + 6y_3 = 4$ $\quad -y_1 \leq 0$ ie $y_1 \geq 0$, $y_3 \leq 0$

~~$y_1, y_3 \geq 0$, y_2 is unrestricted~~

$y_1 \geq 0$, $y_3 \leq 0$, y_2 is unrestricted due to the fact that second primal constraint is in "=" type.

Find the duality of following LP problem

$$\text{Min } Z = 2x_1 + 3x_2 + 4x_3$$

$$\text{st : } x_1 - 3x_2 + 4x_3 = 5$$

$$x_1 - 2x_2 \leq 3$$

$$2x_2 - x_3 \geq 4$$

$$x_1 x_2 \geq 0, \quad x_3 \text{ is unrestricted}$$

⇒ Here,

standard form:

$$\text{Min } Z = 2x_1 + 3x_2 + 4x_3 + 0.S_2 + 0.S_3$$

$$\text{st : } x_1 - 3x_2 + 4x_3 = 5$$

$$x_1 - 2x_2 + S_2 = 3$$

$$2x_2 - x_3 - S_3 = 4$$

$$x_1 x_2 S_2 S_3 \geq 0, \quad x_3 \text{ is unrestricted}$$

Its dual is: $\text{Max } Z_y = 5y_1 + 3y_2 + 4y_3$

$$\text{st : } y_1 + y_2 \leq 2$$

$$-3y_1 - 2y_2 + 2y_3 \leq 3$$

$4y_1 - y_3 = 4$ due to the fact that x_3 is unrestricted

$\cancel{0 \geq y_3 \Rightarrow}$ $y_2 \leq 0$ $-y_3 \leq 0$ ie $y_3 \geq 0$

$y_2 \leq 0$, $y_3 \geq 0$, y_1 is unrestricted.

Duality of Duality is primal itself.

Remark: while changing primal into its dual, if any primal variable is found to be -ve, then before going into standard form, it has to be made non negative.

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So set $y_2 = -y_2'$, $y_2' \geq 0$

Therefore, Max $z_y = 5y_1 - 3y_2' + 4y_3$

st : $y_1 - y_2' + S_1 = 2$

$-3y_1 + 2y_2' + 2y_3 + S_2 = 3$

$4y_1 - y_3 = 4$

$y_2', y_3 \geq 0$, y_1 is unrestricted

Its dual is :

$$\text{Min } z_w = 2w_1 + 3w_2 + 4w_3$$

st : $w_1 - 3w_2 + 4w_3 = 5$

$-w_1 + 2w_2 \geq -3$

ie $w_1 - 2w_2 \leq 3$

$2w_2 - w_3 \geq 4$

$w_1 \geq 0$, $w_2 \geq 0$ and w_3 is unrestricted

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\# Find optimal solutions of following LP problem with the help of optimal solution of its dual problem:

$$\text{Min } z = 7x_1 + 3x_2 + 8x_3$$

$$\text{st}: 8x_1 + 2x_2 + x_3 \geq 3$$

$$3x_1 + 6x_2 + 4x_3 \geq 4$$

$$4x_1 + x_2 + 5x_3 \geq 1$$

$$x_1 + 5x_2 + 2x_3 \geq 7$$

$$x_1 x_2 x_3 \geq 0$$

To convert into its dual the standard form is:

$$\text{Min } z = 7x_1 + 3x_2 + 8x_3 + 0S_1 + 0 \cdot S_2 + 0S_3 + 0 \cdot S_4$$

$$\text{st}: 8x_1 + 2x_2 + x_3 - S_1 = 3$$

$$3x_1 + 6x_2 + 4x_3 - S_2 = 4$$

$$4x_1 + x_2 + 5x_3 - S_3 = 1$$

$$x_1 + 5x_2 + 2x_3 - S_4 = 1$$

$$x_1 x_2 x_3 S_1 S_2 S_3 S_4 \geq 0$$

Its dual is:

$$\text{Max } z = 3y_1 + 4y_2 + y_3 + 7y_4$$

$$\text{st}: 8y_1 + 3y_2 + 4y_3 + y_4 \leq 7$$

$$2y_1 + 6y_2 + y_3 + 5y_4 \leq 3$$

$$y_1 + 4y_2 + 5y_3 + 2y_2 \leq 8$$

$$\cancel{-y_1 - y_2 -} \quad -y_1 \leq 0 \quad -y_2 \leq 0 \quad -y_3 \leq 0 \quad -y_4 \leq 0$$

$$y_1 \geq 0 \quad y_2 \geq 0 \quad y_3 \geq 0 \quad y_4 \geq 0$$

Standard form for the optimal solution:

$$\text{Max } z_y = 3y_1 + 4y_2 + y_3 + 7y_4 + 0S_1 + 0S_2 + 0S_3$$

$$\text{st}: 8y_1 + 3y_2 + 4y_3 + y_4 + S_1 = 7$$

$$2y_1 + 6y_2 + y_3 + 5y_4 + S_2 = 3$$

$$y_1 + 4y_2 + 5y_3 + 2y_2 + S_3 = 8$$

$$S_1, S_2, S_3, y_i \geq 0 \quad i = 1,2,3,4$$

IBFS table

Min r	C_B	B	X_{Bi}	c_j	3 y_1	4 y_2	1 y_3	7 y_4	0 S_1	0 S_2
7	0	S_1	7		8	3	4	1	1	0
0.6	0	S_2	3		2	6	1	5	0	1
4	0	S_3	8		1	4	5	2	0	0
		Z_j			0	0	0	0	0	0
		$Z_j - c_j$			-3	-4	-1	-7	0	0

Min r	C_B	B	X_{Bi}							
0.842	0	S_1	34/5	38/5	9/5	16/5	0	1	-1/5	0
1.5	7	y_4	3/5	2/5	6/5	1/5	1	0	1/5	0
34	0	S_3	34/5	1/5	8/5	23/5	0	0	-2/5	1
		$Z_j - c_j$		-1/5	22/5	2/5	0	0	7/5	0

C_B	B	X_{Bi}							
3	y_1	16/19	1	9/38	1/2	0	5/38	-1/38	0
7	y_4	5/19	0	21/19	0	1	-1/19	4/19	0
0	S_3	126/19	0	59/38	22/5	0	-1/38	-15/38	1
	$Z_j - c_j$		0	$\cancel{\frac{713}{160}}$ $\frac{169}{38}$	1/2	0	1/38	53/38	0

$\because Z_j - c_j \geq 0$, this implies problem has optimal solution

wh. $y_1 = 16/19$, $y_2 = 0$, $y_3 = 0$, $y_4 = 5/19$

and

$$\text{Max } z_y = 3 \times 16/19 + 0 \times 4 + 0 + 7 \times 5/19$$
$$= \frac{83}{19}$$

$\Rightarrow$ is the optimal solution table. From the optimal solution for primal with objective function value is.

dual solution $\text{Max } z_y = 3y_1 + 4y_2 + y_3 + 7 \cdot y_4$

$$= 3 \times 16/19 + 4 \times 0 + 0 + 7 \times 5/19$$
$$= 83/19$$

primal optimal solution:

~~Max Zx =~~ $\text{Min } Z_x = 7x_1 + 3x_2 + 8x_3$

$= 7 \times \frac{1}{38} + 3 \times \frac{53}{38} + 8 \times 0 = 83/19$

at $x_1 = |z_j - c_j^*$ element corresponding to $s_1|$

$= |\frac{1}{38}| = \frac{1}{38}$

$x_2 = |z_j - c_j^*$ element corresponding to $s_2|$

$= \left|\frac{53}{38}\right| = \frac{53}{38}$

$x_3 = |z_j - c_j^*$ element corresponding to $s_3|$

$= |0| = 0$

Duality Theorem

We know that dual of $\text{Min } Z_x = \sum_{j=1}^{n} c_j x_j$

$$\text{st: } \sum_{i=1}^{n} a_{ij} x_j \geq b_i \quad , x_j \geq 0 \qquad i = 1, 2 \ldots n$$

is $\text{Max } Z_y = \sum_{i=1}^{m} b_i y_i$

$$\text{st: } \sum a_{ji} y_i \leq c_j \qquad i = 1, 2 \ldots m$$
$$y_i \geq 0$$

State and prove the weak duality theorem

statement: let $x_1, x_2 \ldots x_n$ are the basic feasible solution of primal LP-problem and $y_1, y_2 \ldots y_m$ be the basic feasible solution of its dual LP problem. Then,

$$\sum_{j=1}^{n} c_j x_j \leq \sum_{i=1}^{m} b_i y_i$$

$\geq \leq$

Proof:

We know that dual of

$$\text{Min } z_x = \sum_{j=1}^{n} c_j x_j$$

$$\text{st: } \sum_{j=1}^{n} a_{ij} x_j \geq b_i, \quad x_j \geq 0 \quad i = 1, 2 \ldots n$$

Is:

$$\text{Max } z_y = \sum_{i=1}^{m} b_i y_i$$

$$\text{st: } \sum a_{ji} y_i \leq c_j, \quad i = 1, 2 \ldots m$$

$$y_i \geq 0$$

LHS of ~~constraint~~ objective function

$$\cancel{\sum a_{ij}} \sum_{j=}^{n} c_j x_j \geq \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ji} y_i x_j$$

$$= \sum_{i=1}^{m} y_i \sum_{j=1}^{n} a_{ij} x_j$$

$$= \sum_{j=1}^{m} y_i b_i$$

ie. $\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i$.

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State and prove strong Duality Theorem:

Statement: If $x_1^*, x_2^*, x_3^*, \ldots, x_n^*$ are optimal solution of primal LP problem and $y_1^*, y_2^*, \ldots, y_n^*$ be optimal solution of its dual then:

$$\text{Min } Z_x = \text{Max } Z_y$$

Proof:

We know that the dual of primal $\text{Min } Z = \sum c_j x_j$

st: $\sum a_{ij} x_j \geq b_i$ is:

$$\text{Max } Z_y = \sum b_i y_i$$
$$\text{st: } \sum a_{ij} y_i \leq c_j$$

let us take $\text{Min } Z_x = \sum c_j x_j$

$$= \sum x_j \sum a_{ij} y_i^* \quad \text{as } y_1^*, y_2^* \ldots \text{ are optimal}$$
$$= \sum y_i^* \sum a_{ij} x_j^*$$
$$= \sum y_i^* b_i$$

$$\text{Min } Z_x = \text{Max } Z_y$$

Hence the theorem

* Complementary Slackness *

Complementary slackness is the relation between number of primal constraints and dual variables

What happens if simplex method of solving LP problem has not been introduced?

eg: $\text{Max } Z = x_1 + 3x_2$

st: $x_1 + x_2 \leq 6$

$-x_1 + 2x_2 \leq 8$

$x_1, x_2 \geq 0$

$$\text{Max } z = x_1 + 3x_2$$

s.t: $x_1 + x_2 + x_3 = 6$ [x_3, x_4 taken as slack]

$$-x_1 + 2x_2 + x_4 = 8$$

$$x_1\, x_2\, x_3\, x_3 \geq 0$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{bmatrix}$$

Let us choose basis column: $B = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$

$$BX = b$$

ie $\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$

$$x_1 + x_2 = 6$$

$$-x_1 + 2x_2 = 8$$

Solving : $x_2 = \frac{14}{3}$ $x_1 = \frac{4}{3}$

Solution set $\left(\frac{14}{3}, \frac{4}{3}\right)$ is feasible

Let us again choose another basis : $B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

$$BX = b$$

ie $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$

$$x_1 = 6$$

$$2x_2 + x_4 = 8$$

$$\therefore x_4 = -4$$

Solution set $(0, 6, 0, -4)$, is not ~~not~~ feasible due to -ve in x_4

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* Transportation Model *

The special type of general linear programming problem in which the objective is to minimize the transportation cost of transporting the homogenous comodities from source S_i (origin) to the sink (or destination) D_j.

General mathematical formulation of Transportion problem is:

$$\text{Min } z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\text{st} : \sum_{i=1}^{m} x_{ij} = a_i, \quad j = 1, 2, \ldots, n$$

and $$\sum_{i=1}^{n} x_{ij} = b_j, \quad i = 1, 2 \ldots m$$

whose tabular form is:

	D_1	D_2	D_3	...	D_n	a_i
O_1	$x_{11}\, c_{11}$	$x_{12}\, c_{12}$	$x_{13}\, c_{13}$		$x_{1n}\, c_{1n}$	a_1
O_2	$x_{21}\, c_{21}$	$x_{22}\, c_{22}$	$x_{23}\, c_{23}$		$x_{2n}\, c_{2n}$	a_2
O_3	$x_{31}\, c_{31}$	$x_{32}\, c_{32}$	$x_{33}\, c_{33}$		$x_{3n}\, c_{3n}$	a_3
⋮						
O_m	$x_{m1}\, c_{m1}$	$x_{m2}\, c_{m2}$	$x_{m3}\, c_{m3}$		$x_{mn}\, c_{mn}$	a_m
demand amount b_j	b_1	b_2	b_3	...	b_n	$\underbrace{\sum a_i = \sum b_j}_{\text{Rim condition}}$

Its LP problem is:

$$\text{Min } z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n + c_{21} x_{21} + c_{22} x_{22} + c_{23} x_{23} + \ldots c_{2n} x_{2n} + c_{31} x_{31} + c_{32} x_{32} + c_{33} x_{33} + \ldots + c_{3n} x_{3n} + c_{m1} x_{m1} + c_{m2} x_{m2} + c_{m3} x_{m3} + \ldots + x_{mn} c_{mn}$$

st:

$$x_{11} + x_{12} + x_{13} + \cdots + x_{1n} = a_1$$
$$x_{21} + x_{22} + x_{23} + \cdots + x_{2n} = a_2$$
$$\vdots$$
$$x_{m1} + x_{m2} + x_{m3} + \cdots + x_{mn} = a_m$$

supply constraint

and

$$x_{11} + \cancel{x_{12}} + x_{21} + \cdots + x_{m1} = b_1$$
$$x_{12} + x_{22} + \cdots + x_{m2} = b_2$$
$$\vdots$$
$$x_{1n} + x_{2n} + \cdots + x_{mn} = b_n$$

if $x_{ij} \geq 0$

Variables = mn

Equations = $m+n$

Convert the following transportation problem into its linear programming problem:

	D_1	D_2	D_3	D_4	a_i
O_1	5	6	7	9	10
O_2	8	4	3	2	20
O_3	6	10	9	11	50
b_j	20	30	20	10	

Min Z: $5x_{11} + 6x_{12} + 7x_{13} + 9x_{14} + 8x_{21} + 4x_{22} + 3x_{23} + 2x_{24}$
$+ 6x_{31} + 10x_{32} + 9x_{33} + 11x_{34}$

st:

$$x_{11} + x_{12} + x_{13} + x_{14} = 10$$
$$x_{21} + x_{22} + x_{23} + x_{24} = 20$$
$$x_{31} + x_{32} + x_{33} + x_{34} = 50$$

and

$$x_{11} + x_{21} + x_{31} = 20$$
$$x_{21} + x_{22} + x_{32} = 30$$
$$x_{31} + x_{32} + x_{33} = 20$$
$$x_{41} + x_{42} + x_{43} = 10$$

if $x_{ij} \geq 0$

class-12-

\# A company has factories at A, B and C which supplies the product to warehouses D, E, F, G. The factories capabilities are: 230, 280 and 180 ~~of 180~~ respectively for regular production. The current Current ware house requirements are: 165, 175, 205 and 165 respectively. Unit shipping cost in Rs. between factories and warehouses are:

From \ To:	D	E	F	G
A	6	7	8	10
B	4	10	7	6
C	3	22	2	11

Find initial solution by using: (a) Northwest Corner Method (NWCM)
(b) Least Cost Method (CLCM)
(c) Vogel's Approximation Method (VAM)

Sol^n.

	D	E	F	G	Capabilities (a_i)
A	6	7	8	10	230
B	4	10	7	6	280
C	3	22	2	11	180
Requirement (b_j)	165	175	205	165	690 / 710

Balancing the problem we have:

A
B
C
Od

	D	E	F	G	a_i
A	(165) 6	(65) 7	8	10	~~230~~ 65
B	4	(110) 10	(170) 7	6	280 ~~170~~
C	3	22	(35) 2	(145) 11	~~180~~ ~~145~~
Od	0	0	0	(20) 0	20
b_j	165	~~175~~ ~~110~~	~~205~~ 35	~~165~~ 20	

By default ⇒ minimization

Min transportation cost $= 165 \times 6 + 65 \times 7 + 110 \times 10 + 170 \times 7 + 35 \times 2 + 145 \times 11 + 20 \times 0$

$= 5400$

at $x_{11} = 165$

$x_{12} = 65$

$x_{22} = 110$

$x_{23} = 170$

$x_{33} = 35$

$x_{34} = 145$

x_{23} means, 2nd origin ie B must provide 3rd destination ie F 170 units.

Algorithmic steps of northwest Corner Method (NCM) are:

Step I:

Balance the transportation problem if it is not so and minimize the problem if it has been given as maximization type.

Step II:

Allocate to the cell (1,1) with $x_{11} =$ Minimum (a_1, b_1)

Step III:

Check whether:

ⓐ $a_1 < b_1$

ⓑ $b_1 < a_1$

ⓒ $a_1 = b_1$

if $a_1 < b_1$, then next allocation be made on the cell (2,1) with $x_{21} =$ Minimum ($b_1 - a_1$, a_2)

if $b_1 < a_1$ then next allocation we made on the cell $(1,2)$ with $x_{12} = \text{minimum}(a_1 - b_1, b_2)$

if $a_1 = b_1$ then next allocation we made on the cell $(2,2)$ with $x_{22} = \text{minimum}(a_2, b_2)$

Step-IV

Continue the process until we get to South East Corner Cell. Then find the objective function value as well as the solution

* Nature of the Solution *

→ Number of basic cells = 7 (occupied cells)

→ value of $m+n-1 = 4+4-1 = 7$

→ If number of basic cells $= m+n-1$, then solution obtained is called non degenerate otherwise it is called degenerate solution.

~~II. Least Cost~~

II. Least Cost Method:

	D	E	F	G	Capabilities (a_i)
A	6	7	8	10	230
B	4	10	7	6	280
C	3	22	2	11	180
Requirement (b_j)	165	175	205	165	690 / 710

	D	E	F	G	a_i
A	6	(175) 7	(25) 8	(30) 10	~~230~~ 30 ~~55~~
B	(165) 4	10	7	(115) 6	~~280~~ ~~115~~
C	3	22	(180) 2	11	~~180~~
Od	0	0	0	(20) 0	~~690~~ ~~20~~
b_j	165	175	~~205~~ 25	~~165~~ ~~145~~ ~~30~~	

$$\text{Min } C = 175\times7+25\times8+30\times10+165\times4+115\times6+180\times2+20\times0$$
$$= 3495$$

$X_{12} = 175$

$X_{13} = 25$

$X_{14} = 30$

$X_{21} = 165$

$X_{24} = 115$

$X_{33} = 180$

~~So~~ Solution is non degenerate

Algorithm steps for Least Cost Method:

Step - I :

Balance the problem and convert into minimization type

Step - II :

Identify the least cost in the cost matrix and allocate it at the cell (i, j) with $x_{ij} = \min(a_i, b_j)$

Step - III :

Line out either row or column in the case when $(a_i < b_j)$ or $(b_j < a_i)$ respectively.

Step - IV :

Continue the process until all the a_i and b_j are introduced into the matrix

Vogel's Approximation Method

This method of finding initial solution of transportation problem is superior to NWCM and LCM in the sense that solution obtained by this method is very close to the solution obtained by optimal solution method.

eg:

	D	E	F	G								
A	(50) 6	(155) 7	(25) 8	10	230	(1)	(1)	(1)	(1)	(1)	(1)	(1)
B	(115) 4	10	7	(165) 6	~~280~~ 115	(2)	(2)	(2)	(2)	(3)	(-)	(-)
C	3	22	(180) 2	11	~~180~~	(1)	(1)	(-)	(-)	(-)	(-)	(-)
Od	0	(20) 0	0	0	~~20~~	(0)	(-)	(-)	(-)	(-)	(-)	(-)
	50 ~~165~~	~~175~~ 155	~~205~~ 25	165								
	(3)	(7)	(2)	(6)								
	(1)	(3)	(5)	(4)								
	(2)	(3)	(1)	(4)								
	(2)	(3)	(1)	(-)								
	(-)	(3)	(-)	(-)								

Step 1:

In balanced minimization problem go row-wise and In first Row R_1 identify the ~~lea~~ ~~last~~ least cost and next to least cost. Find their difference and display it alongside in parenthesis.

Same procedure in columns also.

Step 2:

Identify the column or row, corresponding to largest difference among all the differences. The allocation is made in this row or column ~~or~~ in the cell which has least element. ~~with allocation~~ After allocation line out column or row

Step 3:

Repeat ~~of~~ step 1 and 2 in remaining elements (costs)

$$\text{Min cost} = 50\times6 + 155\times7 + 25\times8 + 115\times4 + 165\times6 + 180\times2$$
$$= 3595$$

At $x_{11} = 60$

$x_{12} = 155$

$x_{13} = 25$

$x_{21} = 115$

$x_{23} = 165$

$x_{83} = 180$

It is non degenerate solution as: No of basic cells = 7
Value of $m+n-1 = 7$

Here non degenerate.

Optimal Solution of Transportation Problem

	D	E	F	G	u_i
A	50 6	155 7	25 8	10	0
B	115 4	10	7	165 6	-2
C	3	22	15 2	11	-6
Ad	0	25 0	0	10 0	-7
v_j	6	7	8	8	

Optimal solution of Transportation Problem is obtained by using modified distribution method (MODI - method). This method is also known as ~~uv~~ u-v Method where u and v are penalties

set $u_1 = 0$ and use the following formula ~~to~~ to basic cells.

basic cell = incircled cells.

to ~~not~~ find values of u_i and v_j

$c_{ij} = u_i + v_j$

$c_{11} = u_1 + v_1$

$6 = 0 + v_1$

$\therefore v_1 = 6$

$c_{12} = u_1 + v_2$

$7 = 0 + v_2$

$\therefore v_2 = 7$

$c_{13} = u_1 + v_3$

or, $8 = 0 + v_3$

$\therefore v_3 = 8$

$c_{21} = u_2 + v_1$

$4 = u_2 + 6$

$\therefore u_2 = -2$

$c_{2\cancel{4}} = u_2 + v_3$

or, $6 = -2 + v_3$

$\therefore v_3 = 8$

$c_{3\cancel{3}} = u_3 + v_{\cancel{4}3}$

or, $\cancel{8}2 = u_3 + 8$

$\therefore u_3 = -6$

$c_{42} = u_4 + v_2$

or, $0 = u_4 + 7$

$\therefore u_4 = -7$

For non basic cells that opportunity cost d_{ij} by using the formula

$d_{ij} = u_i + v_j - c_{ij}$ ~~$c_{ij} = u_i + v_j$~~

$d_{14} = u_1 + v_4 - c_{14}$

$= 0 + 8 - 10$

$= -2$

$d_{22} = -5$

$d_{23} = -1$

$d_{31} = -3$

$d_{32} = -21$

$d_{\cancel{34}} = -9$

$d_{41} = -1$

$\Rightarrow$ if $d_{ij} \nleq 0$, then identify the largest ~~diff~~ d_{ij} non basic cell. Here we take (4,4) and allocate $+\theta$ and form a closed loop that turns through basic cells at 90°.

$\rightarrow$ If all $d_{ij} \leq 0$, it indicates table attains optimal solution but this is not case in our example above.

Set Min$(X_{ij} - \theta) = 0$

Min$(50-\theta, 165-\theta, 20-\theta) = 0$

ie $20 - \theta = 0$

$\therefore \theta = 20$

	D	E	F	G	u_i
A	30 6	175 7	25 8	10	0
B	135 4	10	7	145 6	-2
C	3	22	180 2	11	-6
Od	0	0	0	20 0	-8
v_j	6	7	8	8	

$C_{ij} = u_i + v_j$

or, ~~$C_{22} = u_2 + v_2$~~ $\qquad C_{14} = u_1 + v_4$

or, ~~$10 =$~~ $\qquad 10 = 0 + v_4$

$\therefore v_4 = 10$

Non basic cells:

$d_{14} = -2$

$d_{22} = -5$

$d_{23} = -1$

$d_{31} = -3$

$d_{32} = -21$

$d_{34} = -9$

$d_{41} = -7$

$d_{42} = -1$

$d_{43} = 0$

$\because$ All $d_{ij} < 0$

Above table attains optimal solution:

min TC $= 30 \times 6 + 175 \times 7 + 25 \times 8 + 135 \times 4 + 145 \times 6 + 180 \times 2$

$=$ Rs. 3375

<u>Remark:</u> If initial solution obtained by any one of 3 methods gives degenerate ie no of basic cells $\neq m+n-1$, then to start optimal solution procedure we need to make it non degenerate by assigning small +ve number $\epsilon \to 0$ to the cell (non-basic) which has least cost.

class-14-

Find the maximum profit of the following ~~tran~~ transportation problem with initial solution obtained by LCM:

To

	A	B	C	a_i
From X	7	3	4	2
Y	2	1	3	3
Z	3	4	6	5
b_j	4	3 5	5	10 / 12

To

	A	B	C	a_i
X	7	3	4	2
Y	2	1	3	3
From Z	3	4	6	5
Od	0	0	0	2
b_j	4	3	5	

This is being maximization problem, we need to change it into minimization problem by subtracting all the profit from largest profit (7).

	A	B	C	a_i
X	2 ~~0~~	~~4~~	~~3~~	~~2~~
Y	~~5~~ 2	~~6~~ 1	~~4~~	~~3~~ ~~1~~
Z	~~4~~	~~3~~	5 ~~1~~	~~5~~
Od	~~7~~	~~7~~ 2	~~7~~	~~2~~
b_j	~~4~~ ~~2~~	~~3~~ ~~2~~	~~5~~	

Max ~~profit~~

Now to find optimal solution, we have

	A	B	C	u_i
X	2 0	4	ε 3	0
Y	2 5	1 6	+θ 4	5
Z	4	3	5 1	-2
Od	7	2 7	7	6
v_j	0	1	3	

Here, basic cells = 5

value of $m+n-1 = 4+3-1 = 6$

$\because$ basic cells $\neq m+n-1$, it is degenerate.

To make it non degenerate, we assign $\epsilon \to 0$ to the non basic cell which has least cost. (1, 3)

For basic cells. $c_{11} = u_1 + v_1$ [Set $u_1 = 0$]

or $0 = 0 + v_1$

$\therefore v_1 = 0$

$c_{13} = u_1 + v_3$

or, $3 = 0 + v_3$

$\therefore v_3 = 3$

$c_{21} = u_2 + v_1$

or, $5 = u_2 + 0$

$\therefore u_2 = 5$

$c_{22} = u_2 + v_2$

or, $6 = 5 + v_2$

$\therefore v_2 = 1$

$c_{33} = u_3 + v_3$

or $1 = u_3 + 3$

$\therefore u_3 = -2$

$c_{42} = u_4 + v_2$

or, $7 = u_4 + 1$

$\therefore u_4 = 6$

For Non-basic cells.

$d_{12} = -3 \quad [1+0-3]$

$d_{23} = 4 \quad [3+5-4] > 0$

$d_{31} = -6 \quad [0+(-2)-4]$

$d_{32} = -4 \quad [1-2-3]$

$d_{41} = -1$

$d_{44} = 2 \quad > 0$

$\because$ all $d_{ij} \neq 0$, so from the closed loop from the non-basic cell which has greatest non -ve d_{ij} (most +ve d_{ij}) ie $d_{ij} = d_{23}$

$\text{Min}(\varepsilon-\theta, 2-\theta) = 0$

or $\varepsilon - \theta = 0$

$\therefore \theta = \varepsilon$

	A	B	C	u_i
X	(2+ε) 0	4	3	0
Y	(2-ε) 5	(1) 6	(ε) 4	5
Z	4	3	(5) 1	2
Od	7	(2) 7	7	6
v_j	0	1	-1	

For basic cells, $c_{ij} = u_i + v_j$ and set $u_i = 0$.

$c_{23} = u_2 + v_3$, $4 = 5 + v_3$, $\therefore v_3 = -1$

$c_{11} = u_1 + v_1$, $0 = 0 + v_1$, $\therefore v_1 = 0$

$c_{21} = u_2 + v_1$, $5 = u_2 + 0$, $\therefore u_2 = 5$

$c_{22} = u_2 + v_2$, $6 = 5 + v_2$, $\therefore v_2 = 1$

$c_{33} = u_3 + v_3$, $\therefore u_3 = 2$

$c_{42} = u_4 + v_2$ or $7 = u_4 + 1$ $\therefore u_4 = 6$

Home node = point where loop starts.

For non basic cells:

$d_{12} = -3$

$d_{13} = -4$

$d_{31} = -2$

$d_{32} = 0$

$d_{41} = -1$

$d_{43} = -2$

$\because$ All $d_{ij} \leq 0$

$\Rightarrow$ The problem gets optimal solution

$\therefore$ Maximization profit $= (2+\varepsilon)\times 7 + (2-\varepsilon)\times 2 + 1\times 1 + \varepsilon\times 3 + 5\times 6$

$= 14+4+1+30+8\varepsilon$

$= 49 \quad [\because \varepsilon = 0]$

at $x_{11} = 2$, $x_{21} = 2$, $x_{22} = 1$, $x_{23} = 0$, $x_{33} = 5$

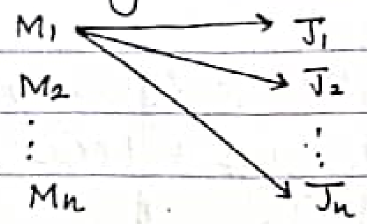
Does your optimal solution possess alternative optimal solution? Give reason

$\Rightarrow$ The given problem posses the alternative optimal solution because the non basic cell has $d_{32} = 0$.

Unit-3   class-15

⁎# Assignment Problem:

Definition: Special case of transportation problem in which supply amount and demand taken to be is called "assignment problem". The main objective of assignment problem is to minimize. The cost of assignment of n machines to n jobs.



General mathematical formulation of Assignment Problem is:

$$\text{Minimize } Z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} x_{ij}$$

subject to the constraints:

$$\sum_{i=1}^{n} x_{ij} = 1, \quad j = 1 \ldots n$$

and

$$\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, 2 \ldots n$$

which has tabular form as:

Jobs

Machines	J_1	J_2	$\ldots$	J_n	a_i
M_1	$c_{11}x_{11}$	$c_{12}x_{12}$	$\ldots$	$c_{1n}x_{1n}$	1
M_2	$c_{21}x_{21}$	$c_{22}x_{22}$	$\ldots$	$c_{2n}x_{2n}$	1
$\vdots$	$\vdots$				$\vdots$
M_n	$c_{n1}x_{n1}$	$c_{n2}x_{n2}$	$\ldots$	$c_{nn}x_{nn}$	1
b_j	1	1	$\ldots$	1	

provided: $x_{ij} = 1$ if i^{th} machine is assigned to j^{th} job
$= 0$, otherwise

Convert the following assignment problem into linear programming pr.

	J_1	J_2	J_3
M_1	5	6	7
M_2	9	2	4
M_3	~~8~~7	8	10

$$\text{Minimize } Z = 5x_{11} + 6x_{12} + 7x_{13} + 9x_{21} + 2x_{22} + 4x_{23} + 7x_{31} + 8x_{32} + 10x_{33}$$

Subject to constraints:

$$x_{11} + x_{12} + x_{13} = 1$$

$$x_{21} + x_{22} + x_{23} = 1$$

$$x_{31} + x_{32} + x_{33} = 1$$

Also,

$$x_{11} + x_{21} + x_{31} = 1$$

$$x_{12} + x_{22} + x_{32} = 1$$

$$x_{13} + x_{23} + x_{33} = 1$$

For all $x_{ij} \geq 0 \quad i, j = 1, 2, 3$

* Optimization solution of Assignment Problem *

Hungarian Method

For the optimization of Assignment Problem, the well known Hungarian method is used and it has the following steps to proceed:

Step I

Assignment Problem always has to be square ie it should have square cost matrix. If it is not so, then make it square either by introducing dummy row or dummy column, with the cost called 0.

eg:

	J_1	J_2	J_3
M_1	5	7	9
M_2	6	8	10

$\Rightarrow$

	J_1	J_2	J_3
M_1	5	7	9
M_2	6	8	10
M_d	0	0	0

	J_1	J_2
M_1	10	12
M_2	12	13
M_3	13	16

$\Rightarrow$

	J_1	J_2	J_d
M_1	10	12	0
M_2	12	13	0
M_3	13	16	0

Step-II:

Change the problem into minimization type, after the Step I, If it is maximization type. The method is same as in Transportation Problem.

Step-III:

Go row-wise, identify least cost in R1 and subtract it from the costs of corresponding row. Continue the process until you get to last row.

Step-IV:

Go column wise in the reduced matrix obtained in step-III.

step 3:

	J_1	J_2	J_3
M_1	0	2	4
M_2	0	2	4
M_3	0	0	0

step 4:

	J_1	J_2	J_3
M_1	0	2	4
M_2	0	2	4
M_3	0	0	0

Step-V:

Encircle any ~~row~~ zero in R1, any other zeros present in corresponding row and circle are crossed out.

eg:

	J_1	J_2	J_3
M_1	0	~~0~~	4
M_2	~~0~~	2	4
M_3	~~0~~	0	0

Go to R2, encircle any zero in R2 and cross out all the zeros in corresponding rows and columns.
Repeat the process until you get to last row

	J_1	J_2	J_3
M_1	0	~~0~~	4
M_2	~~0~~	2	4
M_3	~~0~~	0	~~0~~

Check whether each row and column has encircled 0 or not.
If yes, then it is the indication that assignment problem gets optimal solution. Then find the solution as well as objective function solution.

eg:

	J_1	J_2	J_3
M_1	5	7	9
M_2	6	8	6
M_3	0	0	0

$\Rightarrow$

	J_1	J_2	J_3
M_1	0	2	4
M_2	~~0~~	2	0
M_3	~~0~~	0	~~0~~

$\text{Min } z = 5 + 6 = 11$

$x_{13} = 1$

$x_{23} = 1$

Also, If not then go to step 6.

Step VI:

Draw minimum number of horizontal and vertical lines that cover all the zeros

	J_1	J_2	J_3
M_1	0	0	4
M_2	0	2	4
M_3	0	0	0

Step VII:

Identify the least cost which is non lying on the line. Subtract it from all the cost which are not on the line. And add it to the cost which lies in the intersection of horizontal and vertical lines. While ~~By~~ doing so, all the elements dying on the line are taken as they are.

	J_1	J_2	J_3
M_1	2	0	4
M_2	0	0	2
M_3	2	0	0

← wrong example.

This suggests optimize solution.

Alpha corporation has 4 plants each of which can manufacture any one of 4 products : A, B, C, ~~and~~ or D. Production cost differ from one plant to another and ~~cold~~ so do the sales revenue. The revenue and cost data are given below. Determine which product should each plant produce in order to maximize the profit.

Sales revenue (in 1000 Rs) To Plant

Product	1	2	3	4
A	50	68	49	62
B	60	70	51	74
C	52~~42~~	62	49	68
D	55	~~48~~64	48	66

cost production

Product	1	2	3	4
A	49	60	45	61
B	55	63	45	49
C	55	67	53	70
D	58	65	54	68

⇒ The profit matrix of the problem is : Profit = sales revenue − cost production

Max

Product	1	2	3	4
A	1	8	4	1
B	5	7	6	25
C	~~2~~−3	−5	−4	−2
D	−3	−1~~4~~	−6	−2

Relative cost matrix is :

Min :

	1	2	3	4
A	24	17	21	24
B	20	18	19	0
C	28	30	29	27
D	28	26	31	27

	1	2	3	4
A	7	0	4	7
B	20	18	19	0
C	1	3	2	X
D	2	X	5	1

~~∵ Not all 0 are encircled it is not optimal.~~

	1	2	3	4
A	7	0	4	7
B	20	18	19	0
C	1	3	2	X
D	2	0	5	1

⇒

	1	2	3	4
A	6	0	2	7
B	19	18	17	0
C	0	3	X	X
D	1	X	3	1

∵ Not all 0 are in row, assigned it is not optimal

⇒

	1	2	3	4
A	5	0	1	7
B	18	18	16	0
C	0	4	0	1
D	0	X	2	1

⇒ $Max = 8 + 25 + (-4) + (-3)$

$= 26$

At $x_{12} = 1$

$x_{24} = 1$

$x_{33} = 1$

$x_{41} = 1$

Assignment of product to plants are:

product A → Plant 2

product B → Plant 4

product C → Plant 3

product D → Plant 1

class-16

* Travelling Stalesman Problem *

Defn: Travelling Salesman Problem is the special case of assignment problem which has the objective to minimize the cost of travelling for distance of travelling or time of travelling as well as the optimal route scheduling. The general mathematical formulation of travelling salesman problem is:

$$\text{Minimize } Z = \sum_{i=1}^{n-1} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Subject to constraints

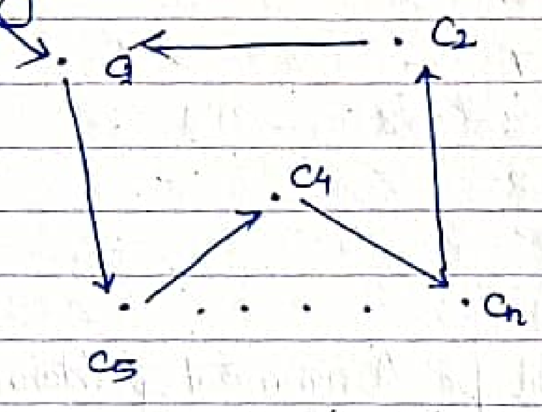
$$\sum_{i=1}^{n-1} x_{ij} = 1, \; j = 1, 2 .. n$$

and

$$\sum_{j=1}^{n} x_{ij} = 1, \; i = 1, 2 ... n-1$$

where $x_{ij} = \infty$ when $i = j$

Starting:



The assumption under which traveling salesman problem is designed:

Assumption 1: The traveller salesman should know distance between pair of city or time betn pair of city or cost betn pa of city

Assumption 2: The travelling salesman should visit all cities and return to home city.

Assumption 3: While travelling no city is visited twice and no backtracking.

Assumption 4: Travelling salesman is not allowed to give proxy visit.

Assumption 5: Without visiting all the cities, we cannot comeback to home city.

\# A salesman has to visit four cities A, B, C and D. The distance is 100 km between the pair of cities are given below. Find the route (optimal) so that total distance travelled will be minimum

From city	A	B	C	D
A	–	1	6	8
B	7	–	8	5
C	6	8	–	9
D	8	5	9	–

Remark:

Make the problem optimal for Assignment problem. And go ahead taking this optimal problem for the solution of travelling salesman problem.

→ Using Hungarian Method, we have

	A	B	C	D
A	–	0	5	7
B	2	–	3	40
C	50	2	–	83
D	3	0	4	–

	A	B	C	D
A	–	0	2	7
B	2	–	0	~~0~~
C	0	2	–	3
D	3	~~0~~	1	–

∴ Not all row ~~and~~ or column has encircled 0, it is not optimal.

	A	B	C	D
A	–	0	2	7
B	2	–	0	0
C	0	2	–	3
D	3	0	1	–

	A	B	C	D
A	–	0	1	6
B	3	–	~~0~~	0
C	0	2	–	2
D	3	~~0~~	0	–

It shows the problem gives optimal solution for assignment problem. Let us check whether it is optimal for travelling salesman problem or not.

check: A → B → D → C → A

Problem is attaining the optimal solution for travelling salesman ~~solu~~ problem also. Hence the optimal routine schedule is A → B → D → C → A and total distance travelled is =
Min = 1 + 5 + 6 + 9 = 21 km

A salesman has to visit 5 cities A, B, C, D and E. The distances in 100km are given below. Then find the total minimum distance travelled solution as well as optimal routine schedule

	A	B	C	D	E
A	–	1	6	8	4
B	7	–	8	5	6
C	6	8	–	9	7
D	8	5	9	–	8
E	4	6	7	8	–

→

	A	B	C	D	E
A	–	0	5	7	3
B	2	–	3	0	1
C	0	2	–	3	1
D	3	0	4	–	3
E	0	2	3	4	–

	A	B	C	D	E
A	–	0	2	7	2
B	2	–	$\cancel{0}$	0	$\cancel{0}$
C	0	2	–	3	$\cancel{0}$
D	3	$\cancel{0}$	1	–	2
E	$\cancel{0}$	2	0	4	–

The solution is not optimal;

	A	B	C	D	E
A	–	0	1	6	2
B	3	–	0	0	1
C	0	2	–	2	0
D	3	0	0	–	2
E	1	3	0	4	–

	A	B	C	D	E
A	–	0	1	5	1
B	3	–	1	0	1
C	~~0~~	3	–	2	0
D	2	~~0~~	0	–	1
E	0	3	~~0~~	3	–

It shows the problem gives optimal solution for assignment problem. Let us check whether it optimal for travelling salesman:

check: A → B → D → C → E → A

Problem is attaing optimal solution for travelling salesman problem too.
Hence optimal routine schedule is: A → B → D → C → E → A

$$Min = 1 + 5 + 7 + 9 + 4 = 2600 \text{ km}$$

Find the optimal route schedule of the following travelly problem

	A	B	C	D	E
A	–	17	16	18	14
B	17	–	18	15	16
C	16	18	–	19	17
D	18	15	19	–	18
E	14	16	17	18	–

	A	B	C	D	E
A	–	3	2	4	0
B	2	–	3	0	1
C	0	2	–	3	~~0~~ 1
D	3	0	4	–	3
E	0	2	3	4	–

	A	B	C	D	E
A	–	3	0	4	0
B	2	–	1	0	1
C	0	2	–	3	~~0~~ 1
D	3	0	2	–	3
E	0	2	1	4	–

It shows problem gives optimal solution for assignment problem let us check whether it is optimal or not.

check: A → C → ~~D~~ E → ~~B~~ A ~~D~~

This table gives optimal solution for assignment problem but not for travelly salesman.

CS CamScanner

	A	B	C	D	E
A	–	4	0	4	0
B	3	–	1	0	1
C	0	2	–	2	0
D	3	0	1	–	2
E	0	2	0	3	–

lets assign cell (B, C)

	A	B	D	E
A	–	4	4	0
C	0	2	2	0
D	3	0	–	2
E	0	2	3	–

Using Hungarian method:

	A	B	D	E
A	–	4	4	0
C	0	2	2	0
D	3	0	–	2
E	0	2	3	–

	A	B	D	E
A	–	4	2	0
C	0	2	0	0
D	3	0	–	2
E	0	2	1	–

Unit: 4 class-17

Project Network Analysis

Project Network Analysis

CPM PERT

Project:

Project is the task that has so many activities which consumes time, effort, and resources and is to be completed within a time limit. The project network analysis can be studied by two methods:

a. Critical Path Method (CPM)

b. Project Evaluation and Review Technique (PERT)

Some Terminologies:

ⓐ **Event:**

Event is the point in time at which the activity starts or ends and it is denoted by : ○

i — ith starting event

j — jth finishing event.

ⓑ **Activities:**

Activities are the tasks which consumes time, effort and resource and it is denoted by directed line segment between the events.

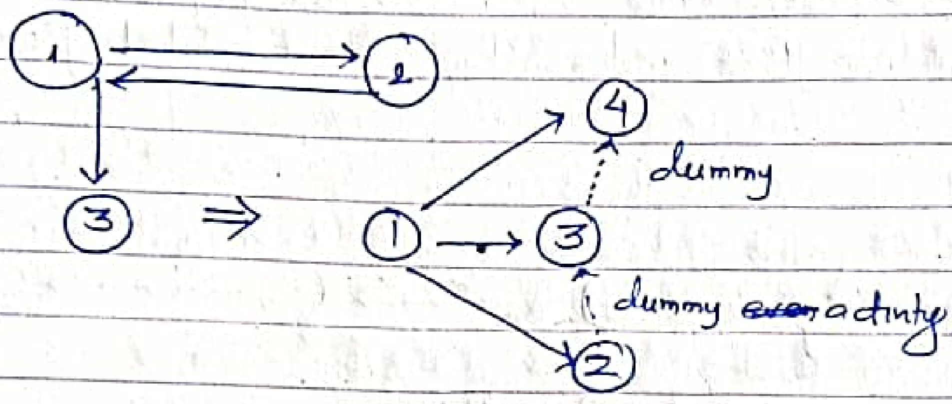
(tail event) $i \xrightarrow{A} j$ (head event)

It is denoted by capital / upper case letter over the arrow lines

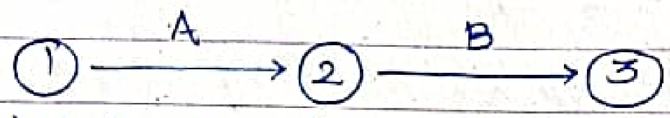
Tail node should have lower ~~b~~ number than head node.

ⓒ Dummy Activity:
The activity which does not consume time effort and resource but it is used to avoid the dangling of the network of the project.

Eg.



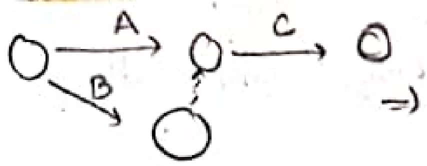
ⓓ Predecessor:



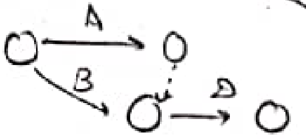
Activity A is the predecessor of activity B.
Activity B is the successor of activity B.

An architect has been awarded a contact to prepare ~~plants~~ Plans for an urban renewal project. The job consists of following activities and their estimated times.

Activity	Description of Activity	Immediate Predecessor.	Time in days.
A	Prepare preliminary sketch	–	2
B	outline specification	–	1
C	prepare drawings	A	3
D	write specification	A, B	2
E	Run off prints	C, D	1
F	Have specification	B, D	3
G	Assemble bid package	E, F	1



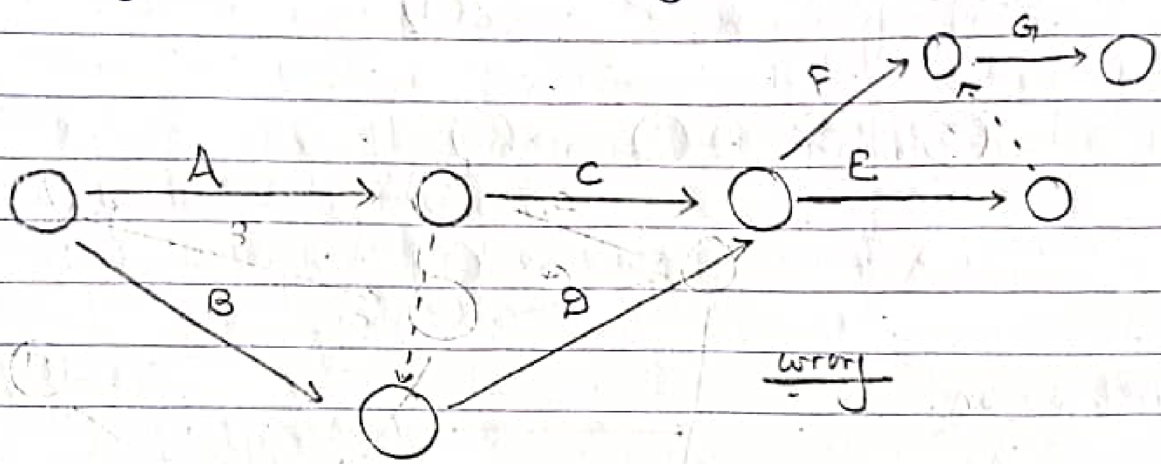
⇒ c starts only after finishing of A and B



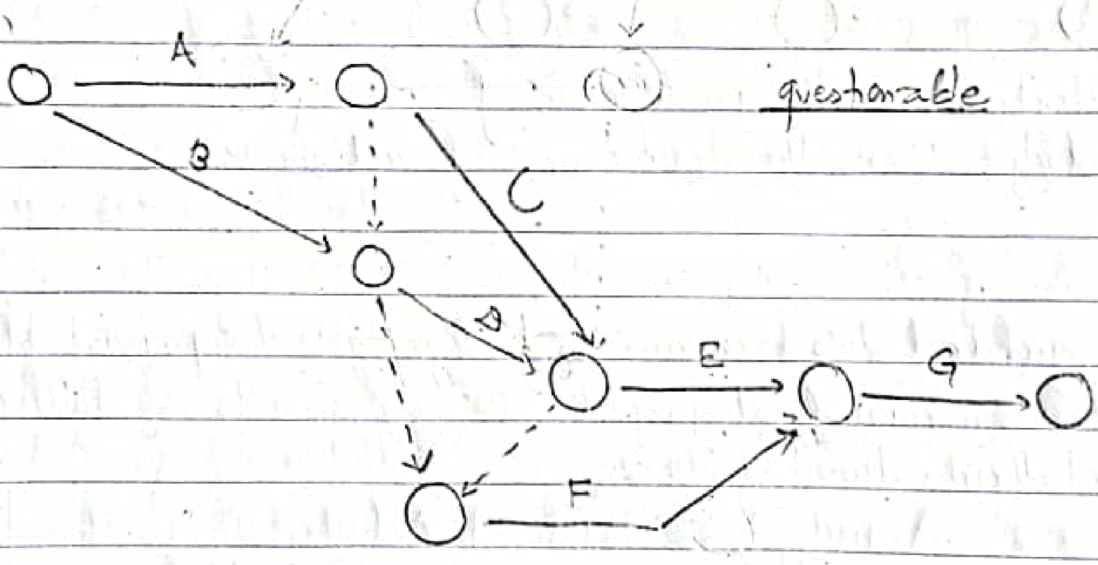
⇒ D starts only after finishing of A and B

(a) Draw the Network diagram of the project
(b) By using critical path method, ~~ee~~ (CPM) find forward pass, backward pass, critical activities, critical path.
(c) Find expected time to complete the project.
(d) Find free float, independent float, Total float.

(a) ⇒



wrong



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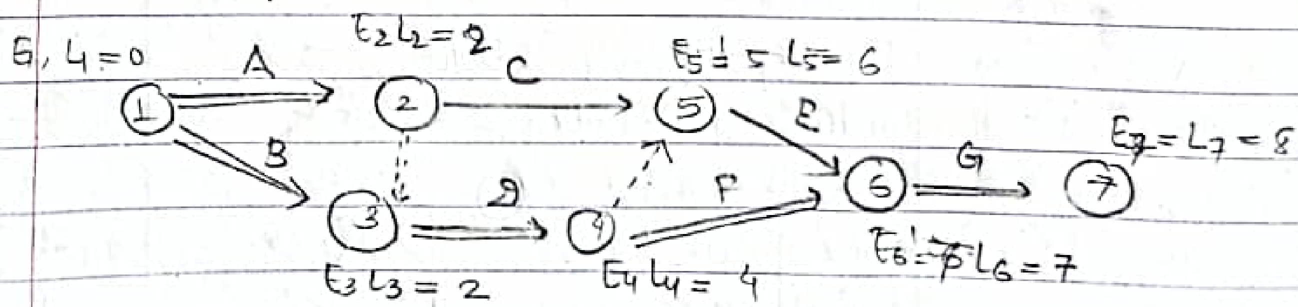

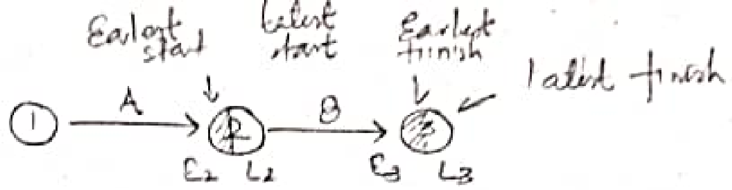

(b) Forward Pass: [computes earliest start time and earliest finish time]

let $E_1 = 0$

$E_2 = E_1 + t_{12} = 0 + 2 = 2$

$E_3 = Max(E_1 + t_{12}, E_2 + t_{23})$

$E_4 = E_3 + t_{34}$

$= 2 + 2 = 4$

$E_5 = Max(E_2 + t_{25}, E_4 + t_{45})$

$= Max(2+3, 4+0)$

$= 5$

$E_6 = Max[E_4 + t_{46}, E_5 + t_{56}]$

$= Max[4+3, 5+1]$

$= 7$

$E_7 = E_6 + t_{67}$

$= 7 + 1 = 8$

Backward pass: computes latest start and latest finish.

set $L_7 = E_7 = 8$

$L_6 = L_7 - t_{67} = 8 - 1 = 7$

$L_5 = L_6 - t_{56} = 6$

$L_4 = Min[L_5 - t_{45}, L_6 - t_{46}]$

$= Min[6-0, 7-3] = 4$

$L_3 = L_4 - t_{4} = 4 - 2 = 2$

$L_2 = Min[L_3 - t_{23}, L_5 - t_{25}]$

$= Min[2-0, 6-3] = 2$

$L_1 = Min[L_2 - t_{12}, L_3 - t_{13}]$

$= Min[2-2, 2-1] = 0$

L .

Critical Activities are: A, B, d_1, D, F and G.
Non Critical Activities are: C, d_2, E.

Critical Path is:

$A \rightarrow d_1 \rightarrow D \rightarrow F \rightarrow G$

$B \rightarrow D \rightarrow F \rightarrow G$

The expected time to complete the project is: the time taken by critical path.

$\therefore$ Expected time $= 1 + 2 + 3 + 1 = 7$ days

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Floating

(a) Free float:

FF_{ij} $E_j - E_i - t_{ij}$

Eg: Non-critical activities $FF_{ij} = E_j - E_i - t_{ij}$

(2)→(5) c $FF_{25} = E_5 - E_2 - t_{25} = 5-2-3 = 0$

d_2 $FF_{45} = E_5 - E_4 - t_{45} = 5-4-0 = 1$

E

(b) Independent float

It is the amount of time which can be used without affecting the head and tail events and is driven by the formula:

$$IF_{ij} = E_j - L_i - t_{ij}$$

Eg: Non critical activities $IF_{ij} = E_j - L_i - t_{ij}$

(2)→(5) C $IF_{25} = E_5 - L_2 - t_{25} = 5-2-3 = 0$

d_2 $IF_{45} = E_5 - L_4 - t_{45} = 5-4-0 = 1$

E $IF_{56} = E_6 - L_5 - t_{56} = 7-6-1 = 0$

(c) Total Float

is the amount of time by which an activity can be delayed or extended from its early start date without delaying the project completion time. And it is given by the formula:

$$TF_{ij} = L_j - E_i - t_{ij}$$

Eg. Non critical activities $TF_{ij} = L_j - E_i - t_{ij}$

(2)→(5) c $TF_{25} = 6-2-3 = 1$

d_2 $TF_{45} = L_5 - E_4 - t_{45} = 6-4-0 = 2$

E $TF_{56} = L_6 - E_5 - t_{56} = 7-5-1 = 1$

* USE of CPM *

When the project's all activities have deterministic time of comple and each activities has repeatitive activities in nature

* PERT *

It is the ~~deleeeeea~~ probabilistic approach of project network in the sence, for completion of an activity. It has 3 types of uncertain ~~are~~ times of completion:

(i) Optimistic time (to)
(ii) Pessimistic time (tp)
(iii) Most likely time (tm)

I. Optimistic Time (to)

It is the shortest time of completion of an activity under over confidence of the situation.

II. Pessimistic Time (tp)

It is the longest time of completion of an activity under the assumption that market situation is ~~to~~ hitting hard.

III. Most likely Time (tm)

It is the time in between optimistic and pessimistic times under the assumption that everything goes in normal ways.

Some formula

(a) The expected time of an activity is given by:

$$t_e = \frac{t_o + 4t_m + t_p}{6}$$

(b) The variants of an activity time is:

$$\sigma_e^2 = \left[\frac{t_p - t_o}{6}\right]^2$$

(c) The project can be completed in the scheduled time T_s is given by:

$$\text{Probability } (z) = \frac{T_s - T_e}{\sigma_e}$$

where t_e is expected completion time of project

{expected time = mean time in probability}

σ_e is the squareroot of sum of variants of critical activities.

The following network diagram represents the activities associated with a project.

Activities	1-2	1-3	1-4	2-5	2-6	3-6	4-7	5-7	6-7
	A	B	C	D	E	F	G	H	I
t_o	5	18	26	16	15	6	7	7	3
t_p	10	22	40	20	25	12	12	9	5
t_m	8	20	33	18	20	9	10	8	4

(a) Draw the network diagram

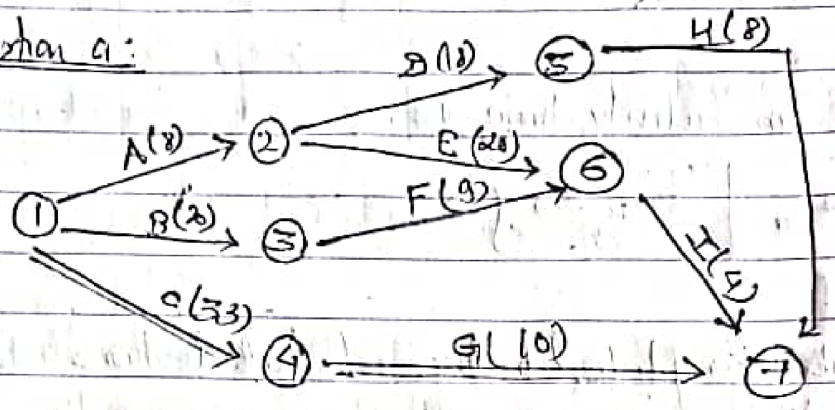
(b) Find critical path

(c) Find expected completion time of project.

d. Find the probability that the project can be completed in 41.5 w

e. Find the duration of the project that will have 35% chance of being completed

Question a:



$$t_e = \frac{t_p + 4t_m + t_p}{6} \qquad \sigma_e = \left(\frac{t_p - t_o}{6}\right)^2$$

Question b

Activity	t_e	σ_e
1-2	8	
1-3	20	
$\boxed{1-4}$	33 →	5.44
2-5	18	
2-6	20	
3-6	9	
$\boxed{4-7}$	10 →	0.69
5-7	8	
6-7	4	

A → D → H = 8 + 18 + 8 = 34 (weeks)

A → E → I = 20 + 20 + 4 = 32 (weeks)

B → F → I = 20 + 9 + 4 = 33 (weeks)

C → G = 33 + 10 = 43 45 (weeks)

∴ Critical ~~time is~~ path is : C → G

Question c

Expected time to complete the project is:

$$t_e = \frac{t_o + 4t_m + t_p}{6} = 43 \text{ weeks}$$

ie time consumed by critical path.

Question d

~~Probability (z)~~ $\sigma_e = \sqrt{5.44 + 0.69}$

$\sigma_e = 2.47$

$$\text{Prob}(z) = \frac{41.5 - 43}{2.47} = -0.607$$

$= \text{Prob}(z \le -0.6073)$

$= 0.2514$

Question e

$Z_{0.95} = 1.65$

The expected time during which project can be completed in the ~~schedule~~ 95% chance ~~time~~ $T(s)$ ~~41.5~~

$$\frac{x - 43}{2.47} = 1.65$$

$\therefore x = 47.075$

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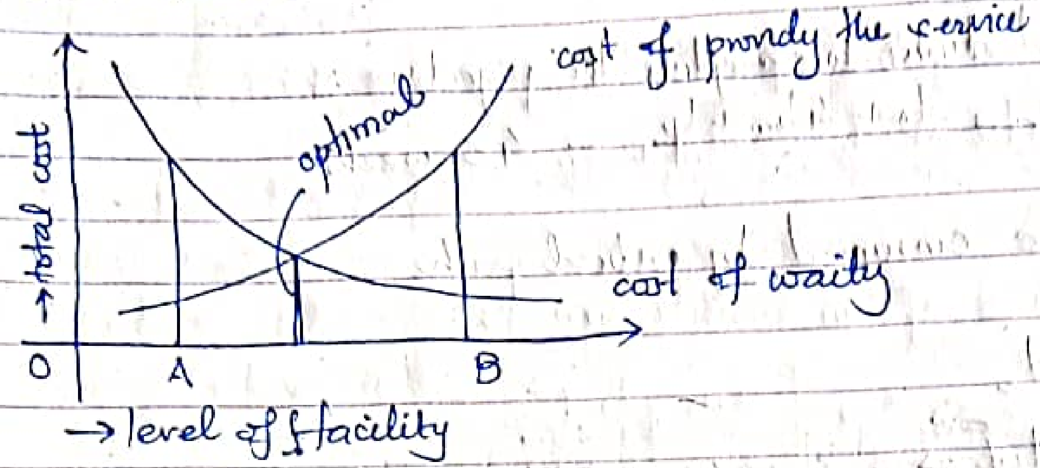
Unit 5: Queuing Model



Queuing Model:

→ It is the science of waiting line. The queue forms when the customer comes to the service facility and cannot find the service immediately upon its arrival. The main objective of study of Queuing Theory is to reduce waiting time, either by adding additional server or by making provision of additional service facility. The study of Queuing theory enable us to make balance between the cost of waiting time and the cost of providing the service. It also enables us to understand the mechanism of the system where we find bottleneck problem.

Queuing Theory tackles the problem either by physically or by psychologically. Customers are the entities which seeks the service and it may be tangible or intangible. For example: vehicle waiting in petrol station and ~~decision~~ case waiting in court to be decided.

The elements of Queuing Model

a. Arrival pattern
b. Service pattern
c. Service discipline
d. Stages of service
e. Channel of service
f. Types of customer

a. Arrival Pattern:

Arrival pattern is characterised by the probability distribution that costumer arrive in the system and it is given by the arrival rate (λ), and it is the number of costumers arrived per unit time. In the poisson probability distribution fashion

$$P(n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

↳ what is the probability that there are n costumer arrived in time t?

Arrival rate is also defined by the mean ~~time~~ inter arrival time to arrive a costumer to the system. And it is denoted by $\left(\frac{1}{\lambda}\right)$

b. Service Pattern:

Service Pattern is characterised by the number of costumers served per unit time and is denoted by service rate (μ). and service rate is also given by the mean time that a server takes to serve a costumer. And it is denoted by: $\left(\frac{1}{\mu}\right)$

c. Service Discipline:

The manner by which ~~cos~~ customer are selected by the service is called service disciplines. The general service disciplines are:

I. First Come First Serve (FCFS)

II. Last Come First Serve (LCFS)

III. SIRO : Service in Random Order

IV. RR : Round Robin.

Unless and otherwise it is mentioned, the service discipline is taken to be FCFS.

CS CamScanner

d. Stages of Service

The service in series is defined by stages of server.

e. Channel of Service

Service channel is parallel queuing system

f. Types of ~~Custom~~ Customer

Depending upon the nature of service, customer are of following types:

i. Bulking Customer
ii. Reneging Customer
iii. Jockeying customer

i. Bulking customer

→ The customer comes to the system and observe the waiting line and decides not to join the queue and leave the system is said to be bulking customer.

ii. Reneging Customer

→ Customer who joins the queue, wait for a while and after some time he/she looses the patience and decides not to wait further and leave the system

iii. Jockeying Customer

→ In parallel queuing system, the customer shifts from one queue to another queue so as to reduce the waiting time

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** Queuing Notation *

P_n = Prob that there are n costumers in the system

λ = mean arrival rate

$\frac{1}{\lambda}$ = mean inter-arrival time

μ = Mean service rate

$\frac{1}{\mu}$ = mean service rate

N = Number customer in the system = Number of customer waiting ≠ Number of customers in the service.

L_q = mean number customers waiting in Line.

L_s = mean number of customers in the system = Number of customer in queue + In service.

W_q = mean waiting time of a customer

W_s = mean time spent by a customer in a system = waiting time + service time s or c = Number of server

$p = \frac{\lambda}{\mu} < 1$ traffic intensity or server

utilization factor, % of time server is busy

* Kendall's notation *

a|b|c : d|e|f

where :

a: Arrival pattern

b: Service Pattern

c or s : Number of servers.

d: system capacity

e: service discipline

f: pool of population = population size

If not mentioned then d|e|f are taken default as:

$d = \infty$

$e = FCFS$

$f = \infty$

Remark ⇒ If $a = M$ i.e. Arrival is poisson distribution
$b = M$ i.e. service time distribution is exponential
then queuing system is called "Markovian Queuing System".

In a service department manned by one server on an average one customer arrives every 10 minutes. It has been found that each customer requires 6 minutes to be served. Then:

a. Find the percentage of the time server being busy.
b. Find probability that server being either
c. Find the number of ~~costumes~~ customer waiting in queue
d. Find number of customer in the system
e. Find the min time to wait for customer in a queue.
f. Find the total min time that a customer has to spend until his departure
g. Find the probability that there are more than 6 customers in the system.

The arrival and service time distribution are poissoned and exponential distribution respectively.

~~ans~~ Accordy to ~~model~~ problem the queuing model is:

$$M|M|1 : \{\infty | FCFS | \infty$$

Given that,

$$\frac{1}{\lambda} = 10\,\text{min} \quad \therefore \lambda = \frac{1}{10}\ \text{customer}/\text{min}$$

and $\frac{1}{\mu} = 6 \text{min}$

$\therefore \mu = \frac{1}{6}$ customer/min

where server is single.

a. $\rho = \frac{\lambda}{\mu} = \frac{1}{10} \times \frac{6}{1}$

$= \frac{3}{5} = 0.6$

$\therefore$ Server is ~~60%~~ busy 60% of ~~their~~ its total time

b. $P_0 = 1 - \rho = 1 - 0.6 = 0.4$

40% of time, server remains idle

c. $L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{(0.1)^2}{\frac{1}{6} \times \left(\frac{1}{6} - \frac{1}{10}\right)}$

$= 0.9$

d. $L_s = \frac{\lambda}{\mu - \lambda} = \frac{1/10}{\frac{1}{6} - 1/10} = 1.5 \approx 2$

e. $W_q = \frac{L_q}{\lambda}$

$= \frac{1}{1/10} = 10$ minute

f. $W_s = \frac{L_s}{\lambda} = \frac{2}{1/10} = 20$ minute

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g. $P(n>k) = \left(\frac{\lambda}{\mu}\right)^{k+1}$

$p(n>3) = (0.6)^4 = 0.1296$

Patients arrive at a clinic according to a poisson distribution at the rate of 30 patient per hour. Waiting room does not accomodate more than 14 patients. The examination time per patient is exponential with the mean time 20 mins

a. Find the effective arrival rate at the clinic.

b. What is the probability that arriving patient will not wait.

c. What is the expected waiting time until a patient is discharged from the clinic.

d. Find mean waiting time in queue.

⇒ The model is :

$M|M|1 : N|FCFS|\infty$

where $N = 14+1 = 15$

(a) Effective Arrival rate:

$\lambda_{eff} = \lambda(1-P_N)$

$P_N = P_0 \rho^N$

where $\lambda = 30/hr$

$\frac{1}{\mu} = 20$ min

$= \frac{20}{60} = \frac{1}{3}$ hr

$\therefore \mu = 5$ patient per hour

$$P_0 = \frac{1-\rho}{1-\rho^{N+1}}$$

$$= \frac{1-10}{1-(10)^{15+1}} = 9 \times 10^{-16}$$

$$\lambda_{eff} = 30\left[1-0.9\right]$$

$$\therefore \lambda_{eff} = 3$$

(b) Probability of any patient will not wait.

$$P_0 = 1 - P_{eff} = 1 - \frac{\lambda_{eff}}{\mu}$$

$$= 1 - \frac{3}{3} = 0$$

(c) Expected waiting time until a patient is discharged.

$$W_s = \frac{L_s}{\lambda}$$

and $L_s = \dfrac{\lambda}{\mu - \lambda} = \dfrac{3}{3-\cancel{5}} = \dfrac{\cancel{5}}{0}$

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Unit-6 Simulation

Types : I] Iconic Simulation
II] Mathematical Simulation

→ The simulation is one of the operation research techniques for the solution of real life problem which cannot be solved by any conventional mathematical formula.

→ Simulation is not the optimization technique and various definitions given by operation research scientists are as follows:

→ Simulation is the process of designing a model of a real system and conducting experiments with this model for the purpose of understanding the behaviour for the operation of the system

↳ This definition was given by Shannon

→ Simulation is a numerical technique for conducting experiment on a digital computer, which involves certain types of mathematical and logical relationships necessary to describe the behaviour and structure of a complex real world system over extended periods of time.

↳ This definition is given by Naylor et al.

→ X simulated Y is true if and only if:

I> X and Y are formal systems

II> Y is taken to be a real system

III> The rules of validity ~~are~~ in X are:
~~non-free error.~~ non-error-free, otherwise X will be real system.

↳ This is given by: churchman

Depending upon the variable used in the simulation, The simulations are of the following types:

a) Deterministic

↳ In which variable taken to be non probabalistic

b. Probabelistic simulation (Stochastic simulation)

↳ In which variable is taken to be random variable.

c. Synamic Simulation

↳ In which time variable is taken in the simulation.

Random Number→

The sequence of natural numbes in which probabelity of picking up of:

① Has the same probabeldy that the picking up of other number.

Random Number Generation

→ Random Number can be generated in following 2 ways:

a] Arithmetic or Random Number Generation

b] Computer Random Number Generation Method

a] Arithmetic Random Number Generation Method:

By using the formula:

$$r_n = p \cdot r_{n-1} \ (\text{modulo } m)$$

$$= p \cdot r_{n-1} \Big|_m$$

where r_0 is the seed of random numbers and ~~the~~ $p, m \in N$

With the following information, generate two random number by using arithmetic random number generation.

$r_0 = 35$

$p = 97$

$m = 100$

from the formula: $r_n = p \cdot r_{n-1} |_m$

$$\text{or, } r_1 = p\, r_0 |_{100}$$

$$\text{or } r_1 = 97 \times 35 |_{100}$$

$$\therefore r_1 = 3395 |_{100} = 95$$

Now,

$$r_2 = 97 \times 95 |_{100}$$

$$= 15$$

* Monte Carlo Simulation *

It is probabilistic approach of ~~simulating of~~ problem using the random numbers and probabilistic distribution

* Queueing Simulation *

A firm has single channeled service station with the following arrival and service time ~~probs~~ distribution. The random numbers for arrival patterns are 02, 87, 98, 10, 47 and random number for service pattern are : 89, 18, 85, 08, 90.

Inter Arrival Time (min)	Probability	Service Time (min)	Probability
10	0.35	5	0.20
15	0.25	10	0.35
20	0.20	15	0.25
25	0.12	20	0.15
30	0.08	25	0.05

a. Find the expected time during which server remains either idle.
b. Find expected number of costumers waiting in line.
c. Find expected time that a customer has to wait.
d. Find the mean time that a customer has to spend in the system.
e. Find the mean inter arrival time.
f. Find the % of the time server is busy.

Arrival RN	Int. Arrival t	Prob	Cummulative Probability	Arrival Interval	Service RN	Service Time	Prob	Cum. Prob	Service Interval
02	10	0.35	0.35	00-34	89	5	0.20	00-19	
87	15	0.25	0.60	35-59	18	10	0.55	20-54	
98	20	0.20	0.80	60-79	83	15	0.80	55-79	
10	25	0.12	0.92	80-91	08	20	~~0.95~~ 1.00	80-99	
47	30	0.08	1.00	92-99	90	25	1.00	~~95-99~~	

Arrival RN	Int Time	Arrived At	Service starts At:	SRN	Service Time	Service ends at	Service idle Time	No. of customer waiting	Customer waiting Time
02	10	7:10	7:10	89	20	7:30	10	–	–
87	25	7:35	7:35	18	5	7:40	5	–	–
98	30	8:05	8:05	83	20	8:25	25	–	–
10	10	8:15	8:25	08	5	8:30	–	1	10
47	15	8:30	8:30	90	20	8:50	–	–	–
$\Sigma X_1 = 90$					$\Sigma X_2 = 70$		$\Sigma X_3 = 40$	$\Sigma X_4 = 1$	$\Sigma X_5 = 10$

$\therefore N = 5.$

(a) $\frac{\Sigma X_3}{5} = \frac{40}{5} = 8$ min

(b) $\frac{\Sigma X_4}{5} = \frac{1}{5} = 0.2 \approx 0$

(c) $\frac{\Sigma X_5}{5} = \frac{10}{5} = 2$ min

(d) $\frac{\Sigma X_2}{5} + \frac{\Sigma X_5}{5} = \frac{70}{5} + \frac{10}{5} = 16$ min

(e) $\frac{\Sigma X_1}{5} = \frac{90}{5} = 18$ min

(f) $\cancel{\frac{70}{110}}\times \frac{70}{70+40} \times 100 = 63.63 \Rightarrow \frac{\text{mean service time}}{\text{total service time}} \times 100$