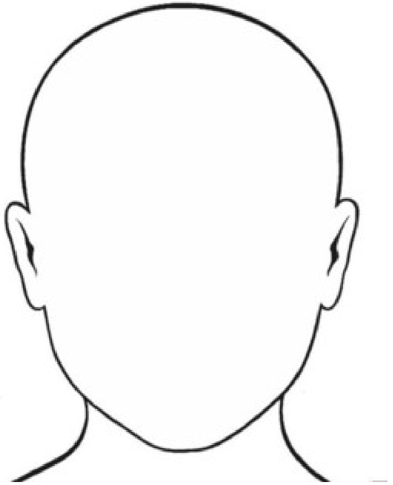
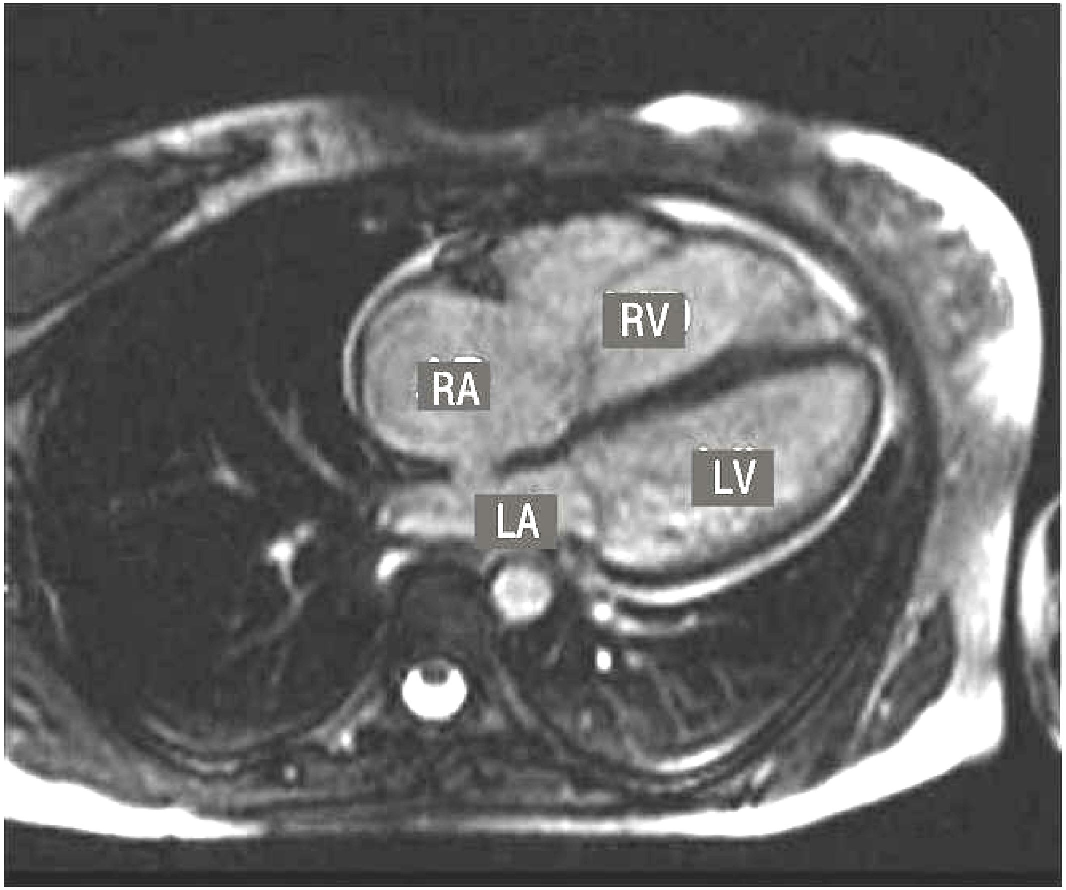
Statistical Shape Analysis

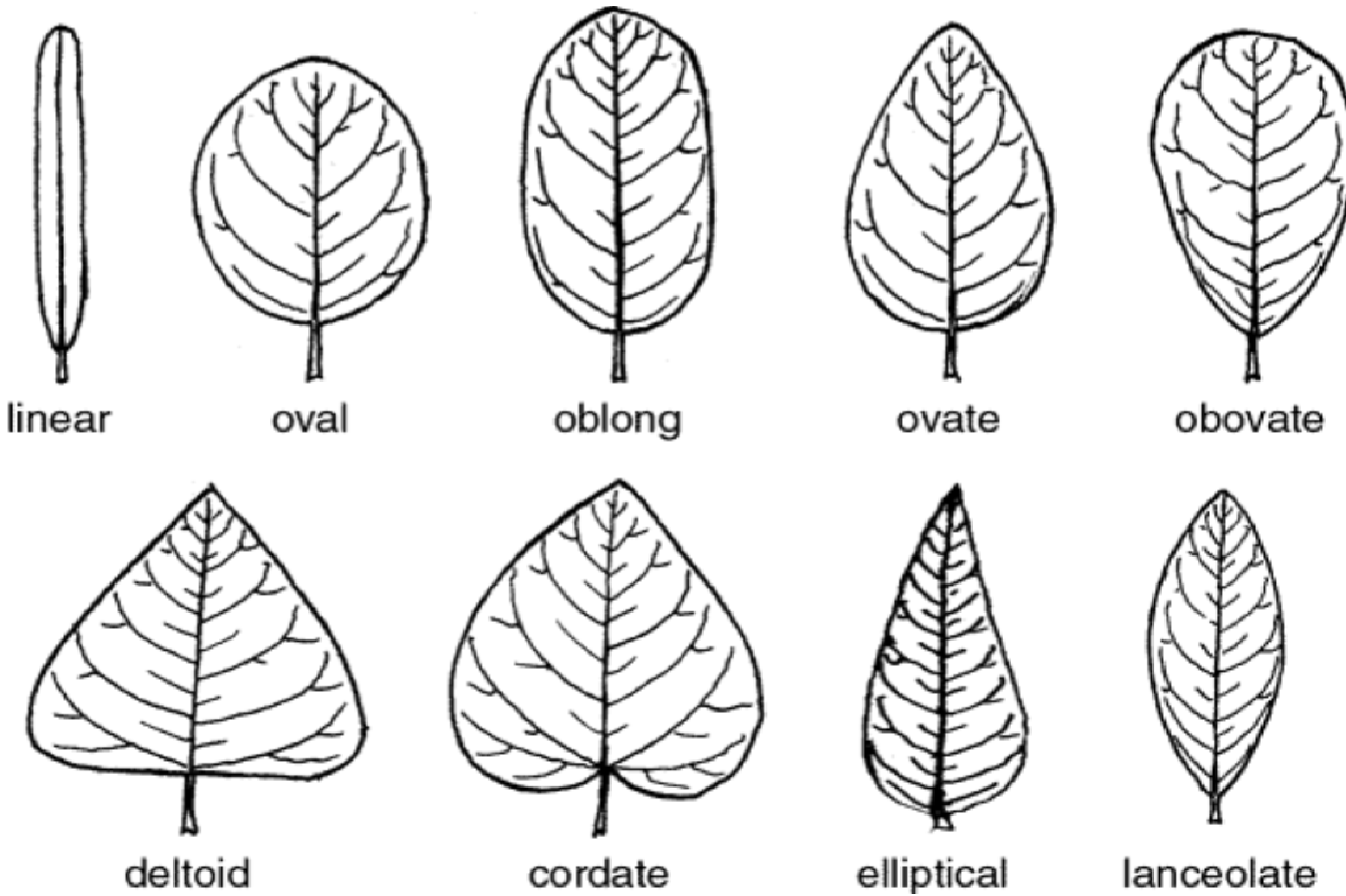
Suyash P. Awate

Shape

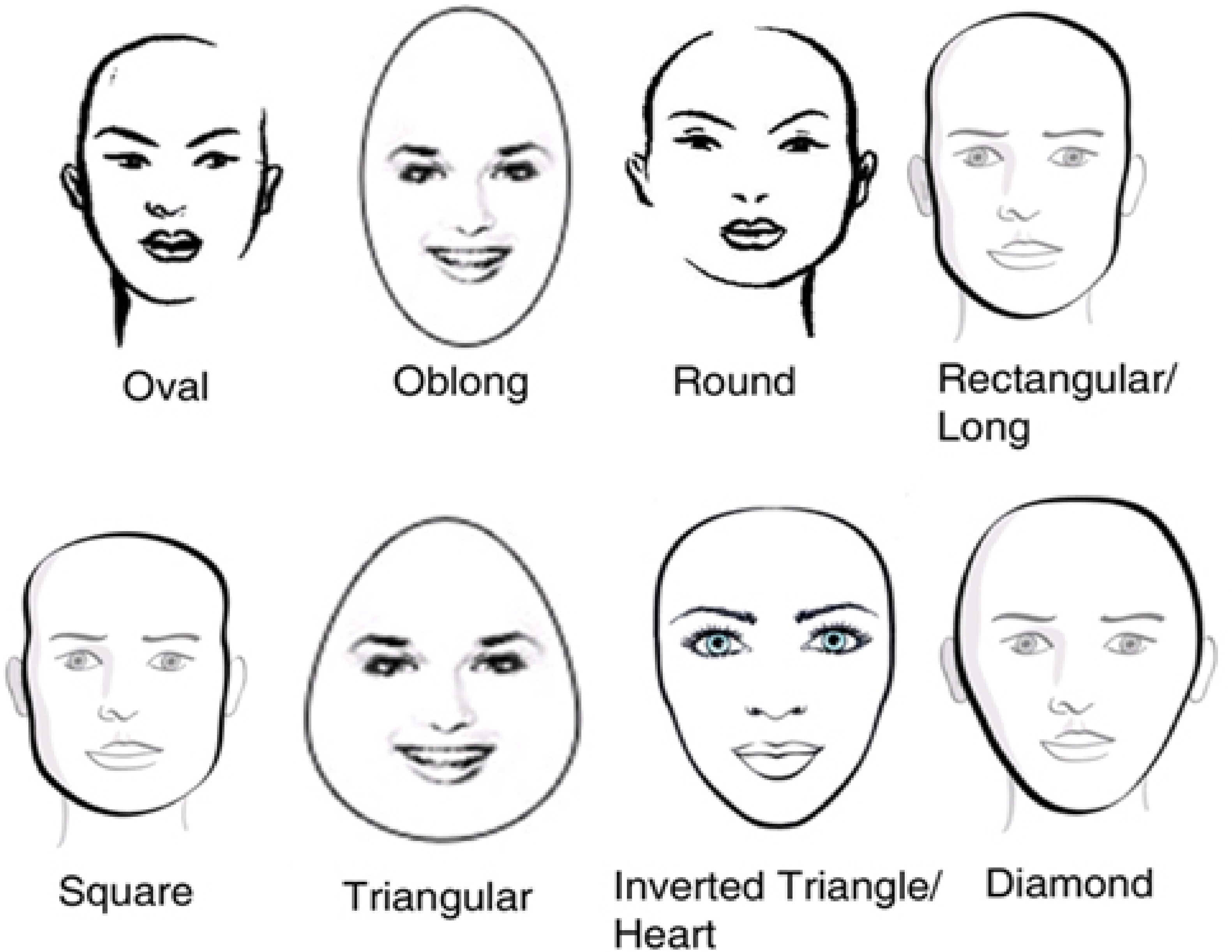
- What is shape ?
  - Shape [noun]: the external form, contours, or outline of someone or something

Shape Variability



Shape Variability



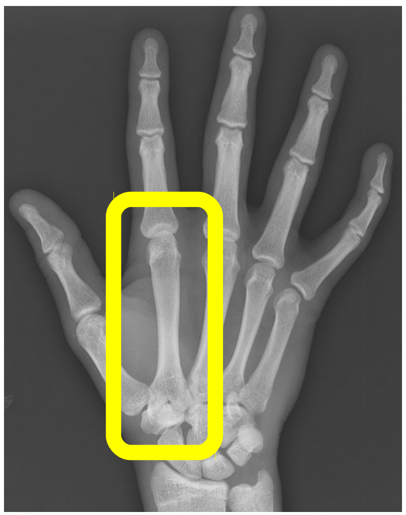
Shape

- Are these same shapes ?   Yes.
  - Shape versus size
  - Shape versus pose

Shape

Statistical Shape Analysis

- Normal / abnormal variation of shape of structure



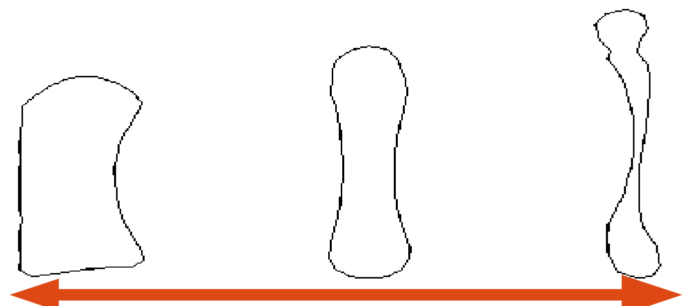
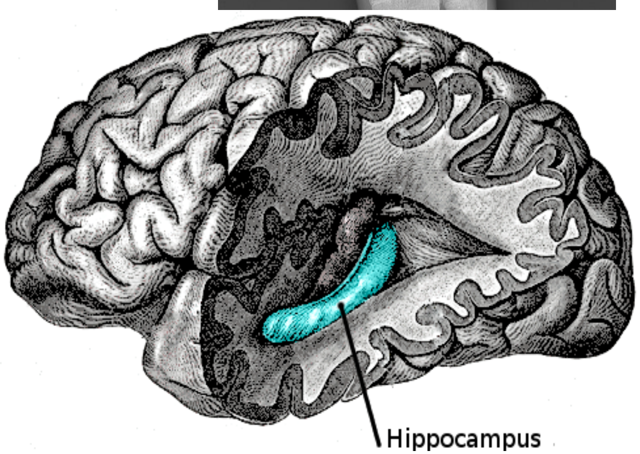
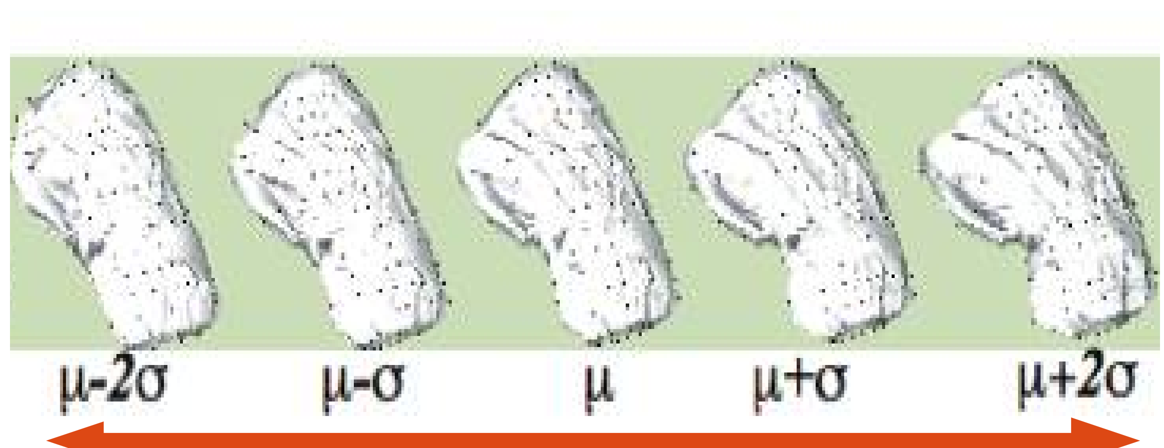


Figure 8.14 *The first mode of variation; $-2.5\lambda_1$, mean shape, $2.5\lambda_1$. Courtesy N.D. Efford, School of Computer Studies, University of Leeds.*

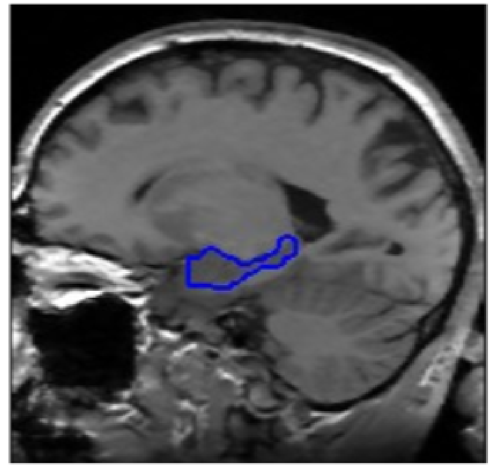


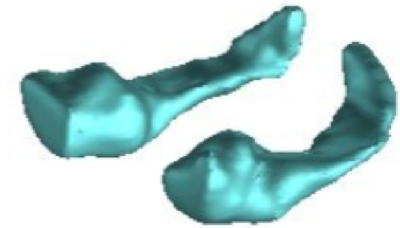


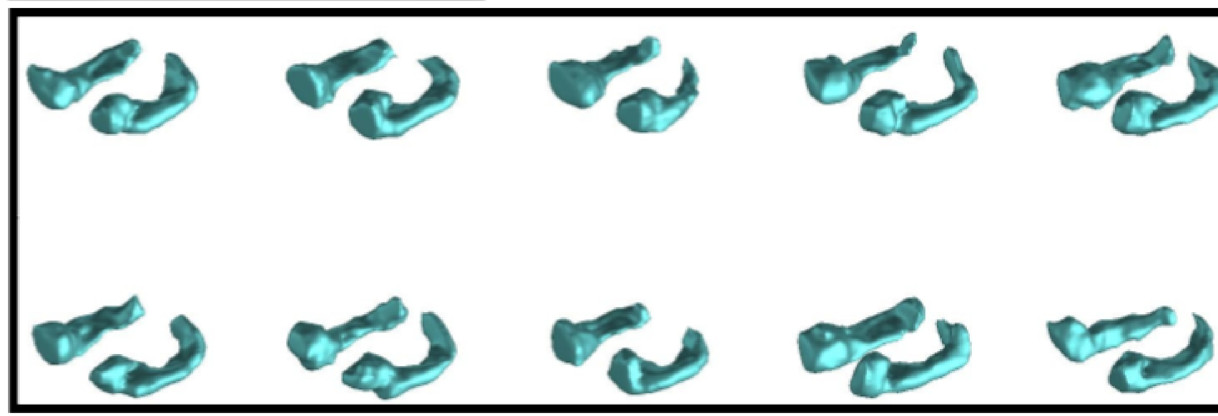
rope.ucdavis.edu/~owenc/research/loca.htm

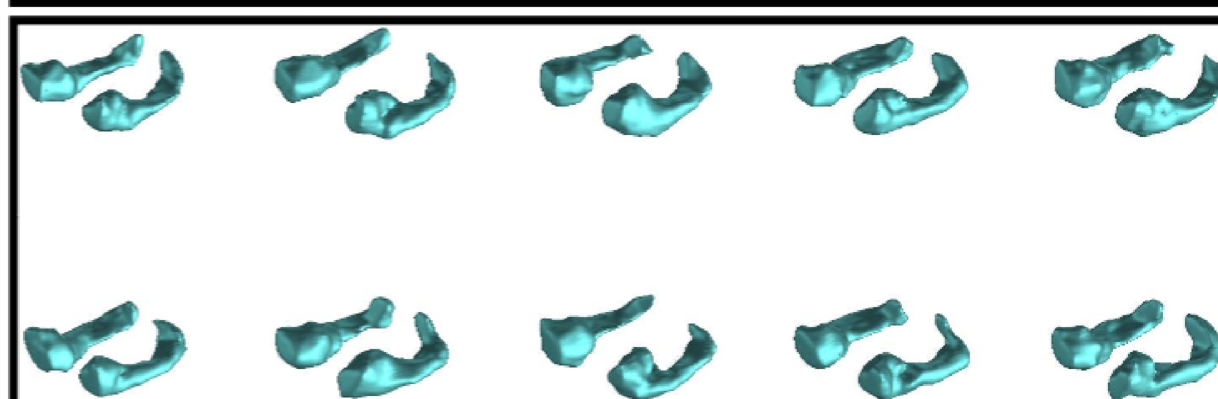
Statistical Shape Analysis

- Dementia
  - Shape of hippocampus





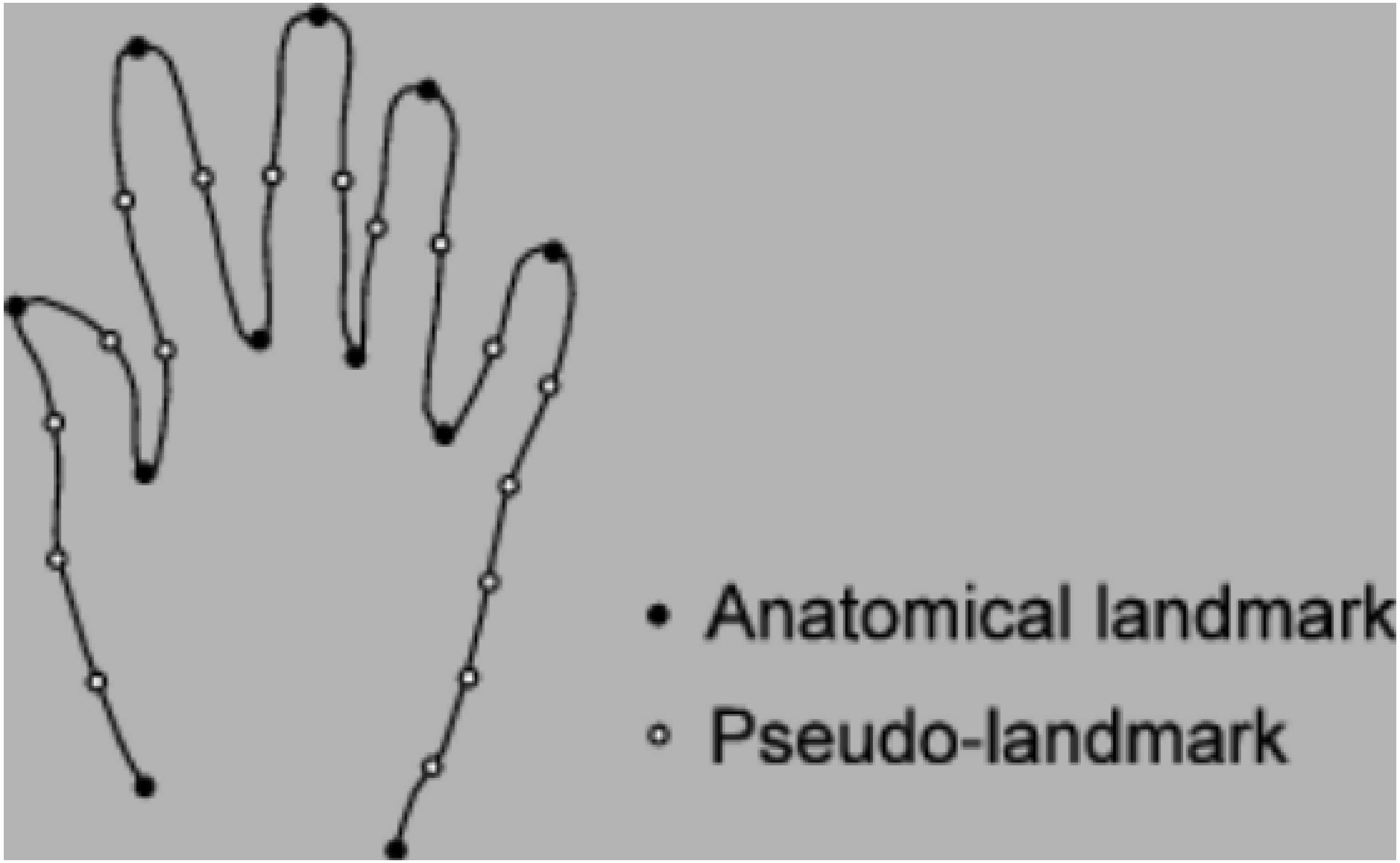




P. Golland. MIT Doctoral Dissertation

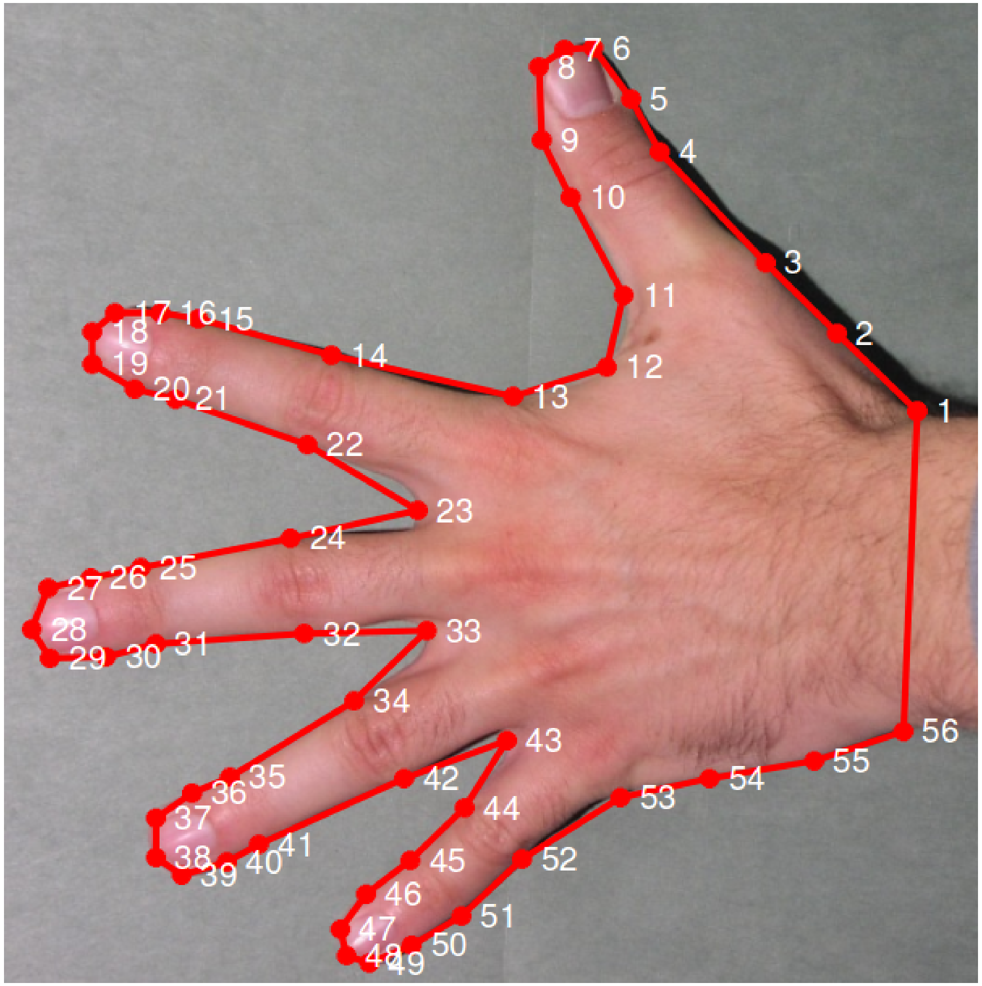
Shape Representation

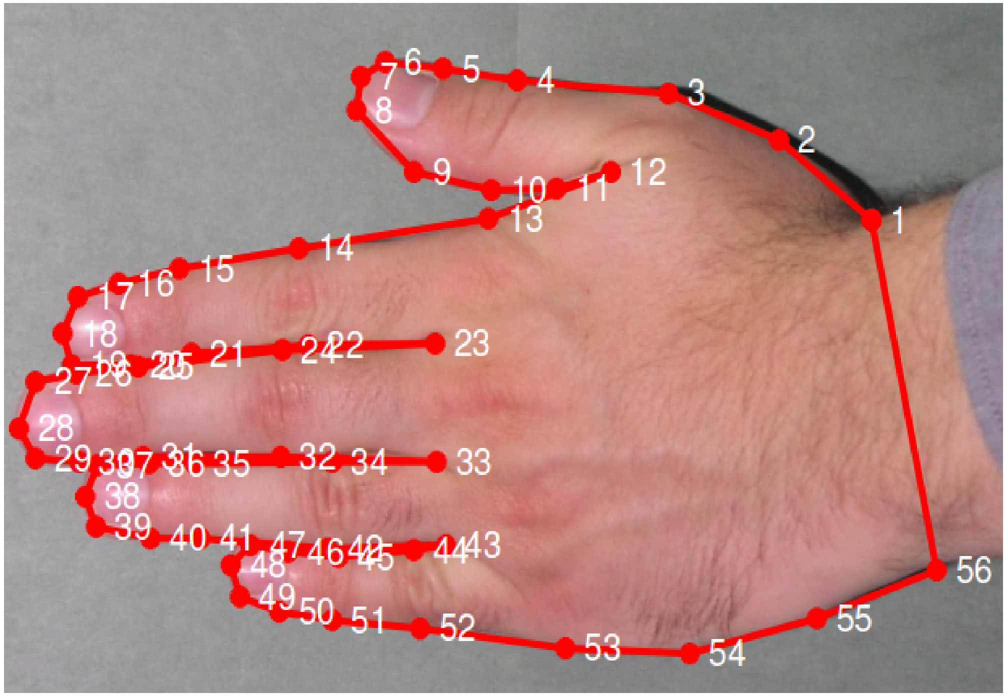
- Outline represented as a pointset
  - Where to place the points ?



Shape Representation

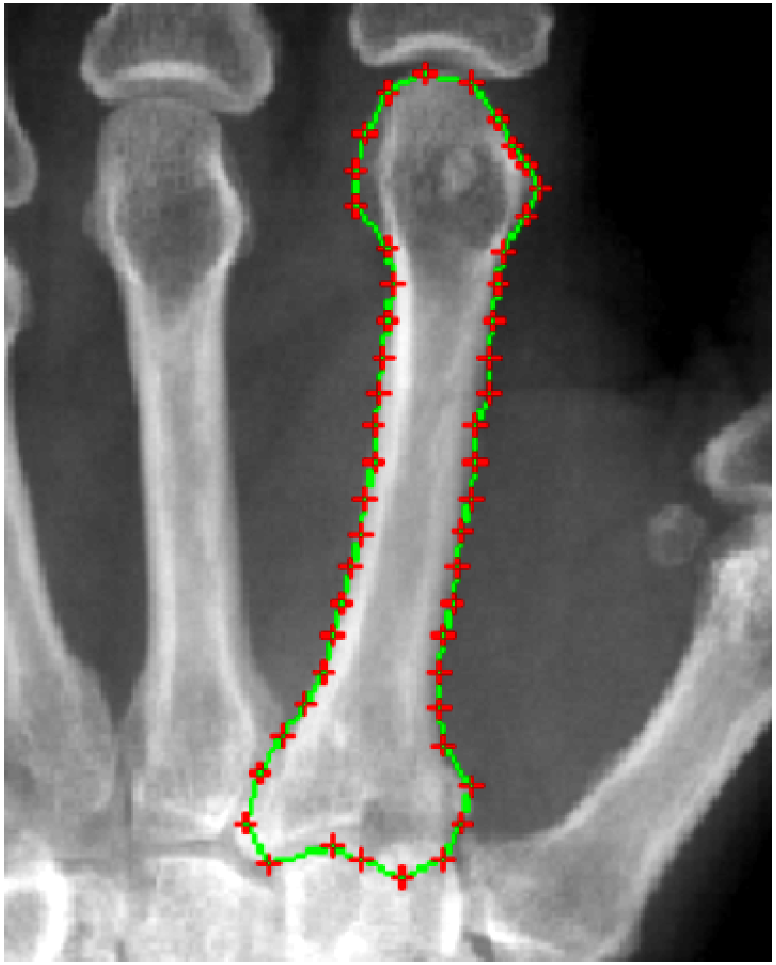
- Outline represented as a pointset
  - Where to place the points ?





Shape Representation

- Pointsets
  - Where to place the points ?



Shape Analysis

- How to measure similarity / distance between shapes ?
  - Assume representation = pointset
  - Must account for
    - Translation (coordinate-frame origin)
    - Rotation (pose)
    - Scale (size)
      - Uniformly / isotropic scaling; enlarging / shrinking

Shape Analysis

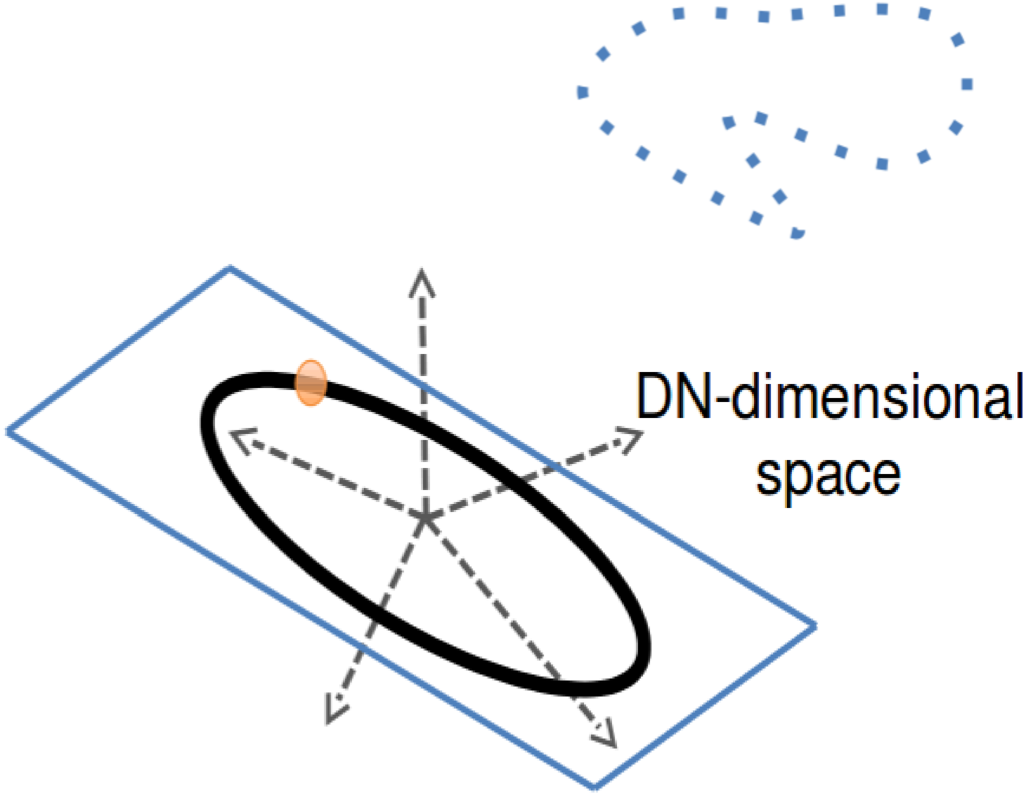
- Shape representation
  - Choose an ordering on the points → Vector of N point coordinates
    - $Shape_1 = z_1 = [\ z_{11} \ldots z_{1N}\ ] = [\ x_{11}\ y_{11}\ \ x_{12}\ y_{12}\ \ \ldots\ \ldots\ \ x_{1N}\ y_{1N}\ ]$
    - $Shape_2 = z_2 = [\ z_{21} \ldots z_{2N}\ ] = [\ x_{21}\ y_{21}\ \ x_{22}\ y_{22}\ \ \ldots\ \ldots\ \ x_{2N}\ y_{2N}\ ]$

Shape Analysis

- Pointset transformations
  - (Special) Similarity Transform
    - Linear coordinate transformation comprising :
      - Translation : T = [tx ty]'
      - Scaling (identical scaling of all coordinates; NO shear) : “s”
      - Rotation
        - $M_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$
        - Orthogonal matrix of determinant +1
      - NO reflection allowed (hence, “special”)
        - Reflection matrix has determinant -1
        - $M_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$ (reflection about line at angle θ/2)
      - Let point a = [a1 a2]' and point b = [b1 b2]'
      - Then, similarity transformation is : $b \leftarrow s\, M_\theta\, a + T$

Shape Analysis

- Shape space
  - Pointset
    - { x_n in R^D : n=1,...,N }
    - Vector of length DN
  - Space dimension < DN
    - Translation reduces D
    - Scale reduces 1
    - Rotation reduces D(D-1)/2
      - Degrees of freedom in an orthogonal matrix
  - Preshape space = hypersphere resulting from factoring out translation and scale



Shape Analysis

- Shape distance (squared) between pointsets z_1 , z_2

  $= \min_\theta \; d^2 (z_1, \mathrm{SimTrans} (z_2; tx, ty, s, \theta))$

  - Translation (tx, ty) is chosen s.t. centroids match
    - Puts pointsets on same hyperplane
  - Scale (s) is chosen s.t. norms match
    - Puts pointsets on same hypersphere
  - Rotation is explicitly optimized (in terms of θ)
  - d(.,.) is measured in preshape space (modulo rotation)
    - Geodesic distance on hypersphere
    - Approximated by Euclidean (chord) distance (Procrustes distance)

Shape Analysis

- Procrustes Analysis for shape matching

  (1) Align w.r.t. Location
  - Compute the centroid of each pointset
  - Subtract centroid from each point

  (2) Align w.r.t. Scale
  - Re-scale each shape to have equal size
  - Size = norm of vector

  (3) Align w.r.t. Rotation
  - We'll see next

  (4) Procrustes Distance = Euclidean distance between transformed shape vectors

Shape Analysis

- Procrustes Analysis for shape matching
  - Given : N data points in 2 shapes (say, z_1 and z_2) in 2D
    - z_1 , z_2 lie in preshape space
  - Goal : Find transformation parameters
    - Rotation : 1 variable
    - Cootes et al. 1995 (Active Shape Models) solved also for :
      - Translation : 2 variables
      - Scale : 1 variable
  - Strategy : Minimize $\sum_{n = 1,\dots,N} \| z_{1n} - s M_\theta z_{2n} - T \|^2$
    - How will you optimize ?

Shape Analysis

- Similarity / distance between shapes
  - Strategy : Minimize $\sum_{n = 1,...,N} || z_{1n} - s M_\theta z_{2n} - T ||^2$
    - Optimization algorithm
      - First solve for { tx, ty, a, b } where $a = s \cos \theta$, $b = s \sin \theta$
      - Quadratic in { tx, ty, a, b } → Closed-form solutions
      - Then, solve for s and θ, given a and b

Shape Analysis

- Similarity / distance between shapes
  - Strategy : Minimize $\sum_{n = 1,\ldots,N} \| z_{1n} - s M_\theta z_{2n} - T \|^2$
    - Optimal translation T turns out to be the same as before !
      - For any given s, θ : optimal T is the difference between centroids
      - Can apply in the beginning because, once centroid is at origin, changing scale and rotation doesn't change centroid
    - Optimal scale “s” is slightly different !
      - Previously, transformed pointset was ensured to be unit norm
      - If unconstrained minimization over “s”, that guarantee is lost !
    - Optimization algorithm : Alternating minimization
      - How will you initialize ?

Shape Analysis

- What is the mean shape ?
  - How to define the mean ?
    - Karcher mean, Frechet mean
      - Mean = centroid
    - Assume a Gaussian distribution on shapes
      - Distance between shapes = Procrustes distance
      - For simplicity, assume the distribution is isotropic
        - No covariance matrix
        - Just variance (scalar)
    - Define the mean as the ML estimate !
      - What does this produce ?

Shape Analysis

- What is the mean shape ?
  - How to define the mean ?
    - Mean shape := pointset that minimizes the sum of squared distances to all given pointsets
  - How to find the mean ?
    - Given : M pointsets (say, z_m), each having N points
    - Goal : Find the mean shape
      - But, distances depend on optimal transformations between each data pointset and mean !
      - So, must find transformation parameters !
    - Strategy
      - Optimize over mean + N sets of transformation parameters

Shape Analysis

- What is the mean shape ?
  - Optimize
    - Minimize : $\sum_{n = 1,\ldots,N} (z - s_m M_{m\theta} z_{mn} - T_m)^2$
      - z = Mean pointset
      - $\{ s_m, M_m, T_m \}$ = Transformation parameters for each data shape
    - How to optimize ?
      - Alternating minimization

        (1) Given mean, find optimal transformations
        - Can be solved independently for each data shape

        (2) Given all transformations, find optimal mean pointset
        - How to do that ?

Shape Analysis

- What is the mean shape ?
  - Optimize

    (1) Given mean, find optimal transformations

    (2) Given all transformations, find optimal mean pointset

    - Remember : the mean pointset must be in **same** preshape shape !
      - Centroid at origin + Unit norm
    - Exact optimization is possible, but a bit complicated
    - Lets look at a popular approximate algorithm
      - 2 issues
        - Convergence
          - Theoretically, convergence isn't guaranteed
          - Practically, convergence is observed (almost always)
        - Correctness
          - Practically, converged solution close to correct solution

Shape Analysis

- What is the mean shape ?
  - Optimize

    (1) Given mean, find optimal transformations

    (2) Given all transformations, find optimal mean pointset

    - Popular approximate algorithm :

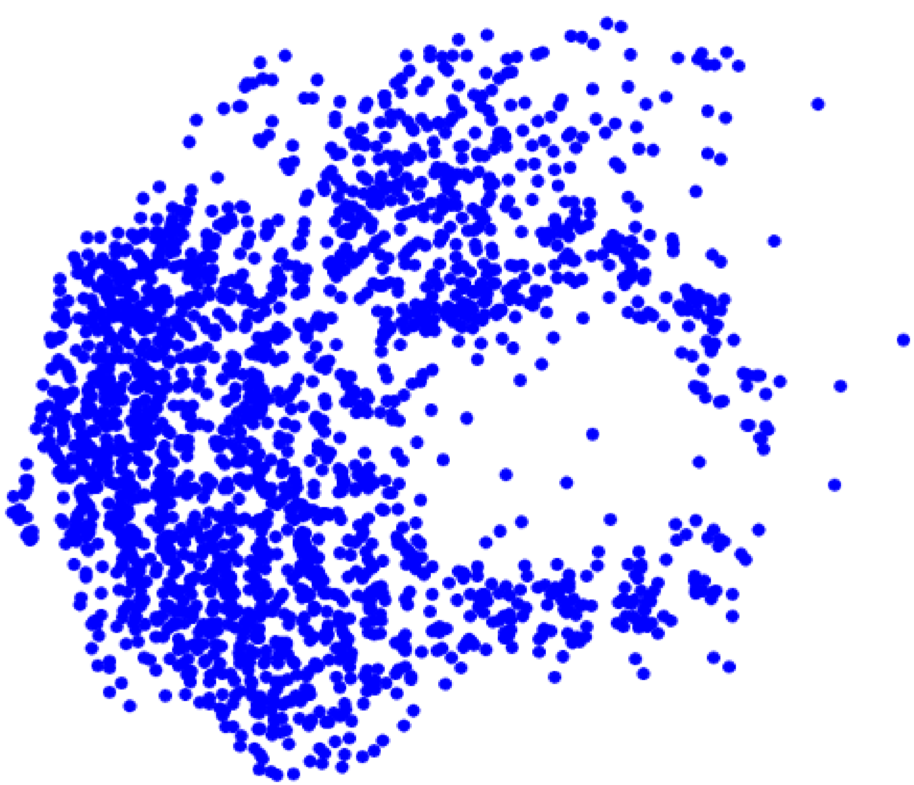
      (1) Average all (transformed) pointsets

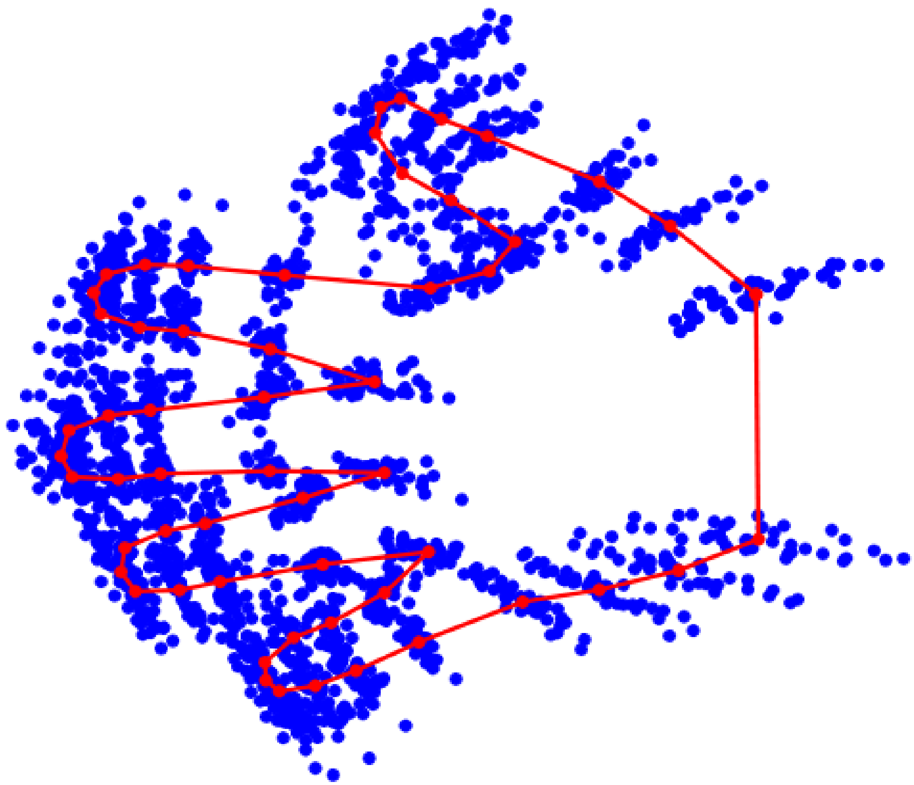
      - Resulting pointset guaranteed to have centroid at origin !
      - Why ?

      (2) Take resulting pointset and rescale (divide) by the norm

Shape Analysis

- What is the mean shape ?
  - Left : unaligned pointsets
  - Right : aligned pointsets + mean shape
  - http://graphics.stanford.edu/courses/cs164-09-spring/Handouts/paper_shape_spaces_imm403.pd



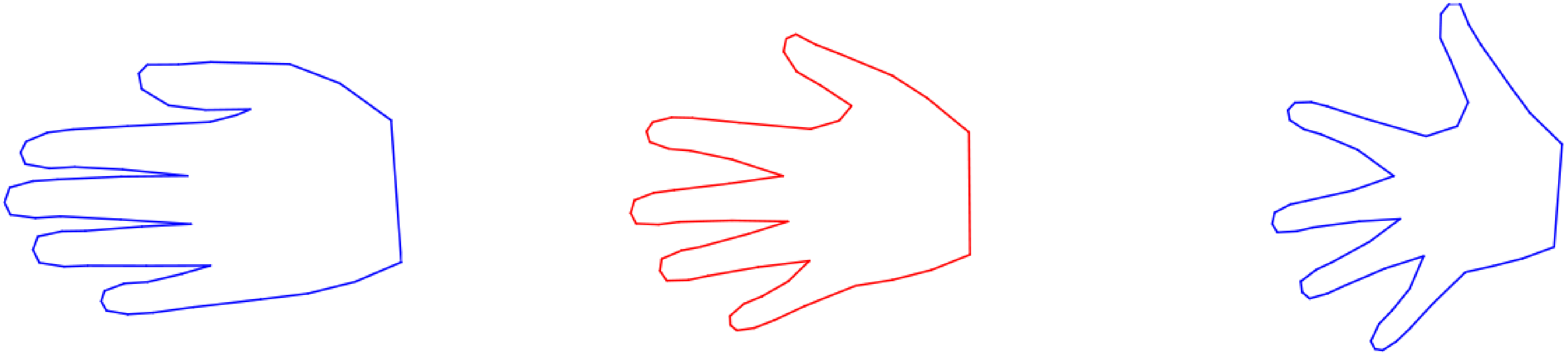


Shape Analysis

- How to learn shape variability ?
  - Assume : Shape mean is given
    - Optimized using previous approach
  - Estimate a covariance matrix
    - ML estimation
    - Now, assuming a Gaussian model with a covariance
    - This is just the sample covariance !

Shape Analysis

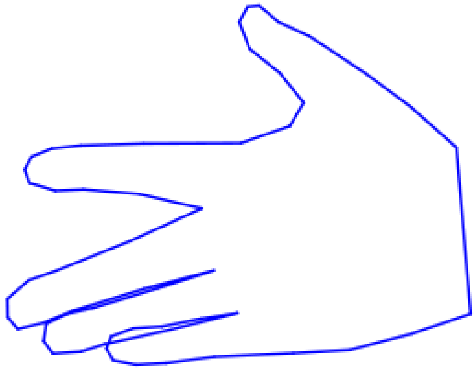
- How to learn modes of variation ?
  - Perform eigen analysis of covariance matrix
    - Principal eigen vectors give principal modes of variation



(a) $b_1 = -3\sqrt{\lambda_1}$ **(b)** $b_1 = 0$ **(c)** $b_1 = +3\sqrt{\lambda_1}$

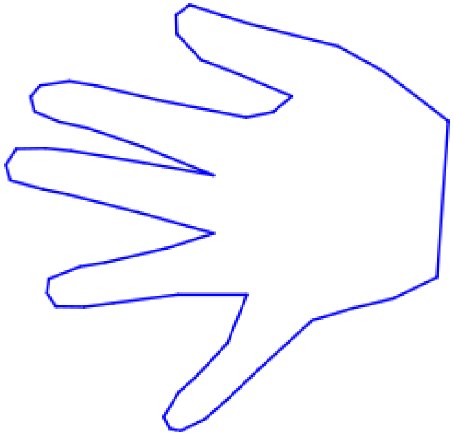
Shape Analysis

- How to learn modes of variation ?
  - Perform eigen analysis of covariance matrix
    - Principal eigen vectors give principal modes of variation



(d) $b_2 = -3\sqrt{\lambda_2}$

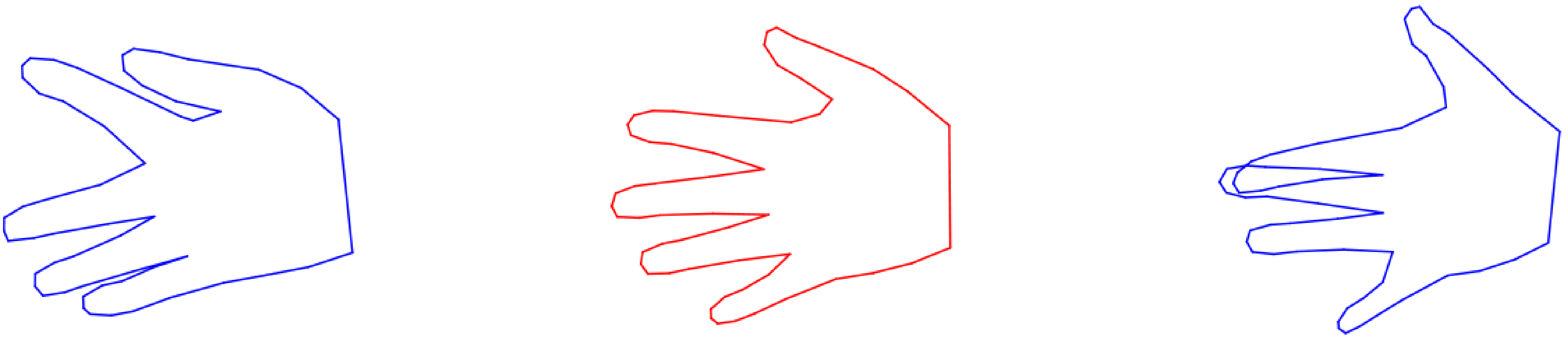
(e) $b_2 = 0$



(f) $b_2 = +3\sqrt{\lambda_2}$

Shape Analysis

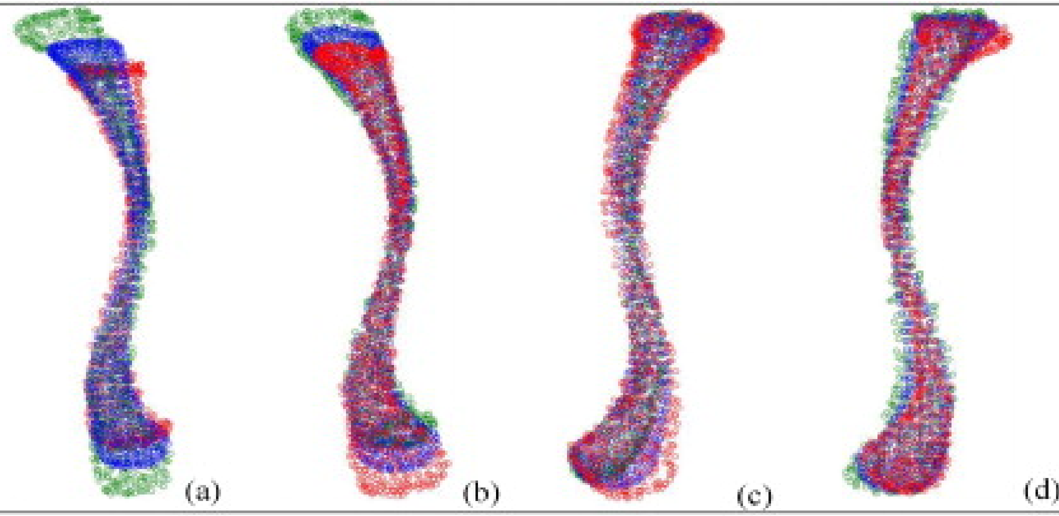
- How to learn modes of variation ?
  - Perform eigen analysis of covariance matrix
    - Principal eigen vectors give principal modes of variation



(g) $b_3 = -3\sqrt{\lambda_3}$ **(h)** $b_3 = 0$ **(i)** $b_3 = +3\sqrt{\lambda_3}$

Shape Analysis

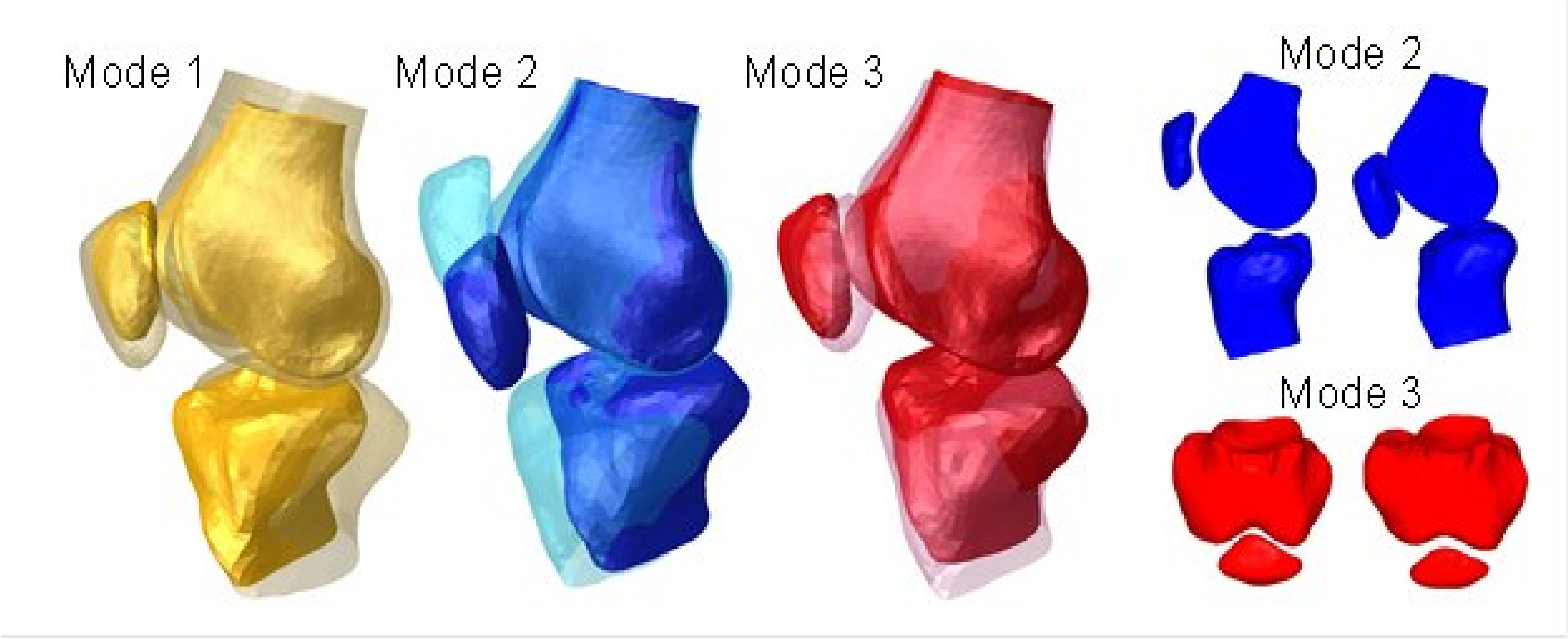
- How to learn modes of variation ?
  - Perform eigen analysis of covariance matrix
    - Principal eigen vectors give principal modes of variation
      - http://www.sciencedirect.com/science/article/pii/S0169260713001740

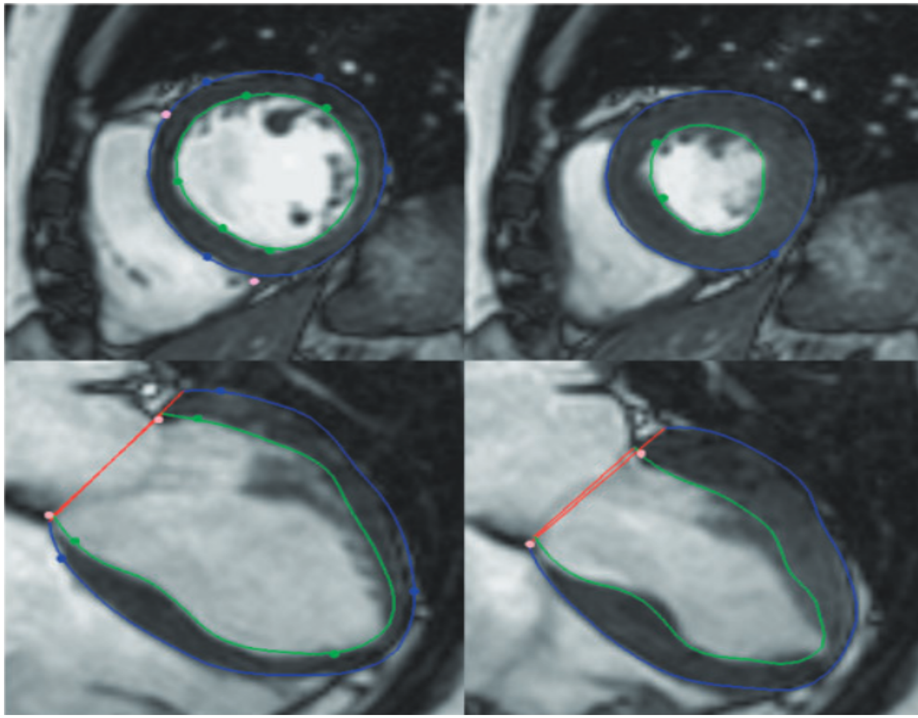
Mode	1	2	3	4
(-3SD, Mean Shape, +3SD)	 (a)	(b)	(c)	(d)
Percentage of Variance (%)	31.9	19.0	10.6	5.0
Description	Overall length (T_1)	Sternal diameter (T_2) and acromial diameter (T_3)	Medial curve radius (R_1) and inferior curve radius (R_2)	Overall diameter (T_4)

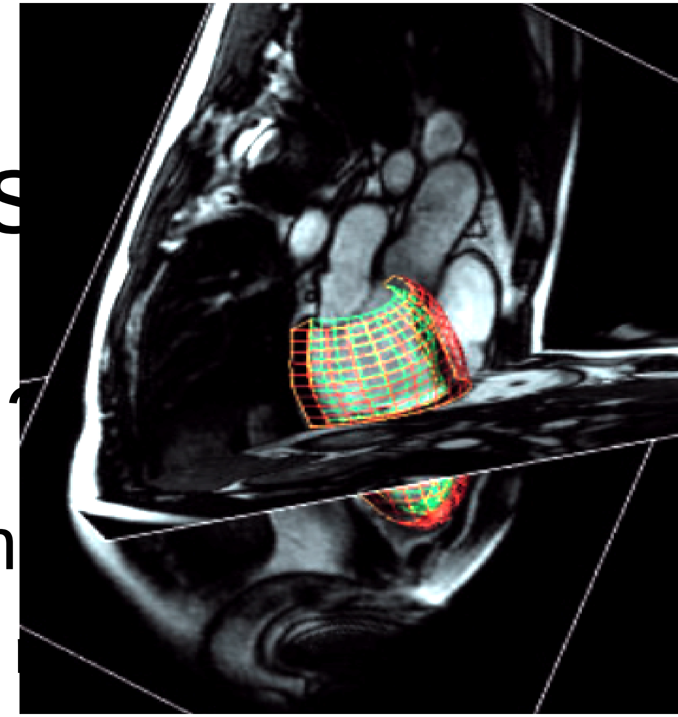
* The green, blue and red models represent the -3 SD, the median and the +3SD models

Shape Analysis

- How to learn modes of variation ?
  - Perform eigen analysis of covariance matrix
    - Principal eigen vectors give principal modes of variation
      - http://www.du.edu/rsecs/departments/mme/biomechanics/research/statisticalshapemodeling.







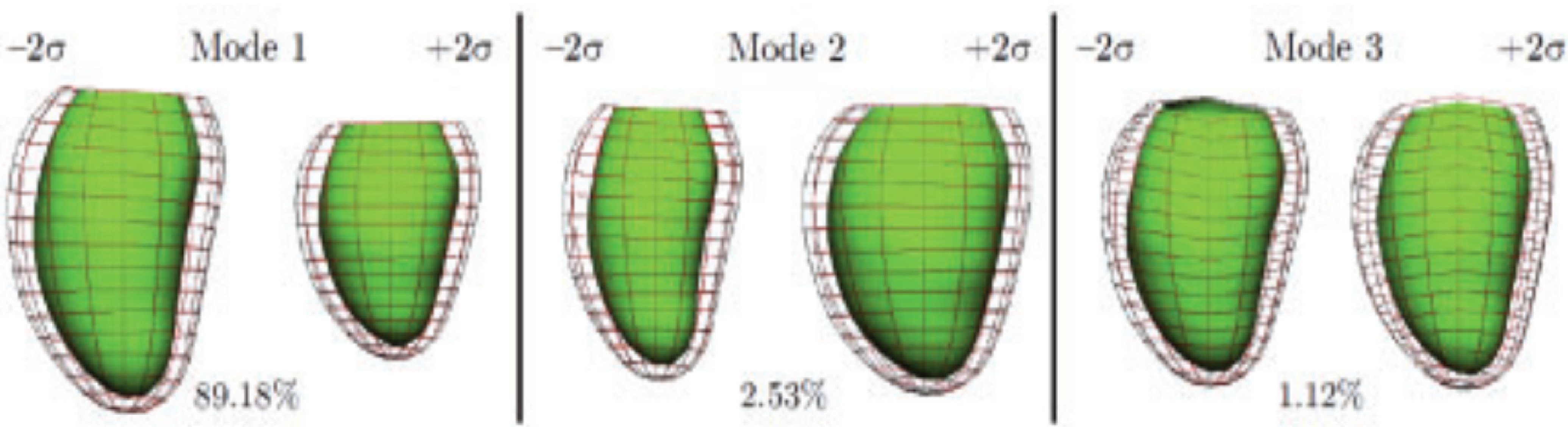
e Analysis

s of variation ?

ysis of covarian

tors give principal

– The Cardiac Atlas Project (http://www.cardiacatlas.org)
http://openi.nlm.nih.gov/detailedresult.php?img=3150036_btr360f4&req=4



Shape Analysis

- Medical applications of learning statistical shape models

  (1) Scientific study to understand variability

  (2) Hypothesis testing to test for shape differences in a specific disorder / disease

  (3) Classification of a patient based on shape

  - e.g., autism spectrum disorder

  (4) Shape priors for segmentation