Algorithms for Medical Image Processing

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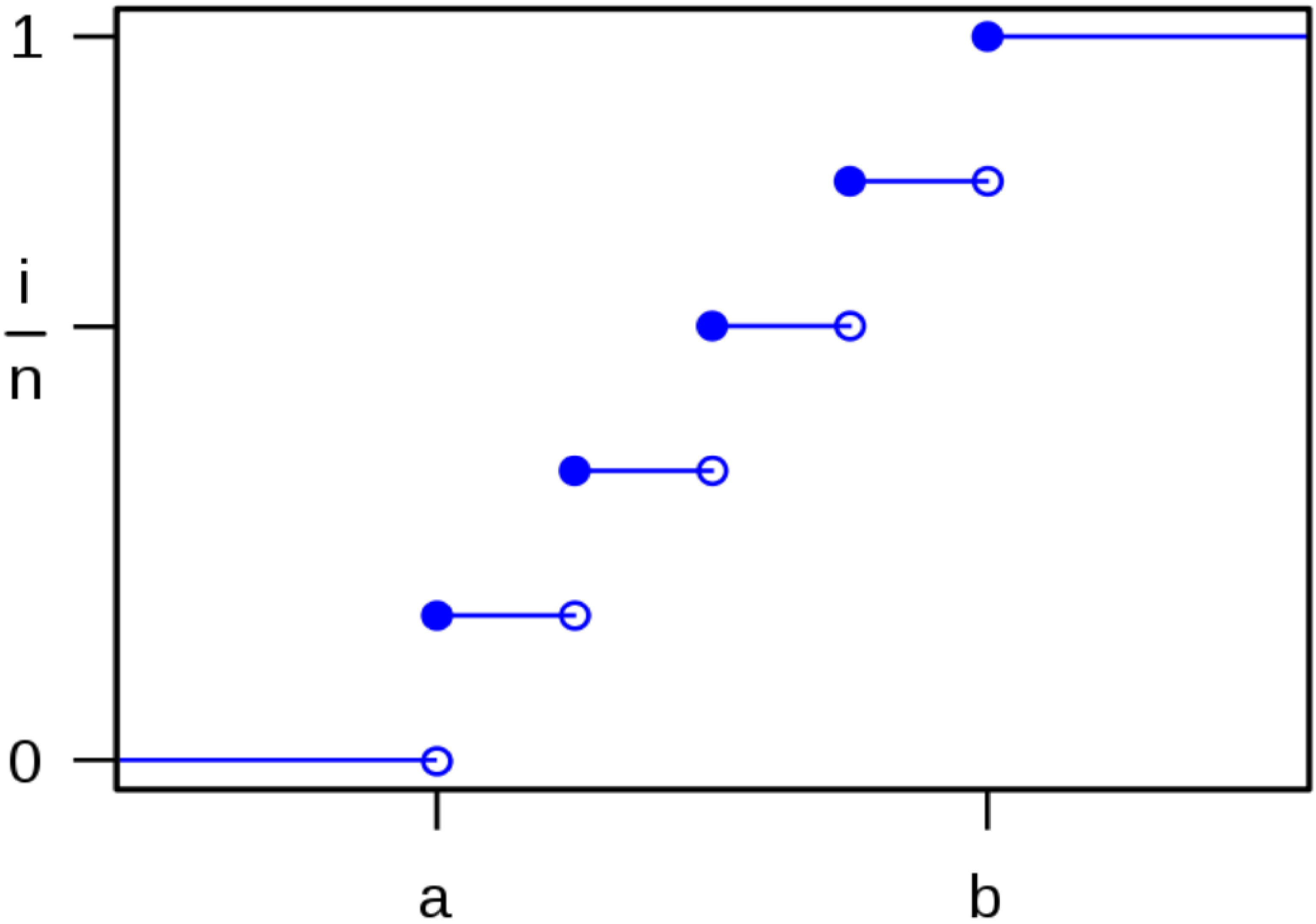
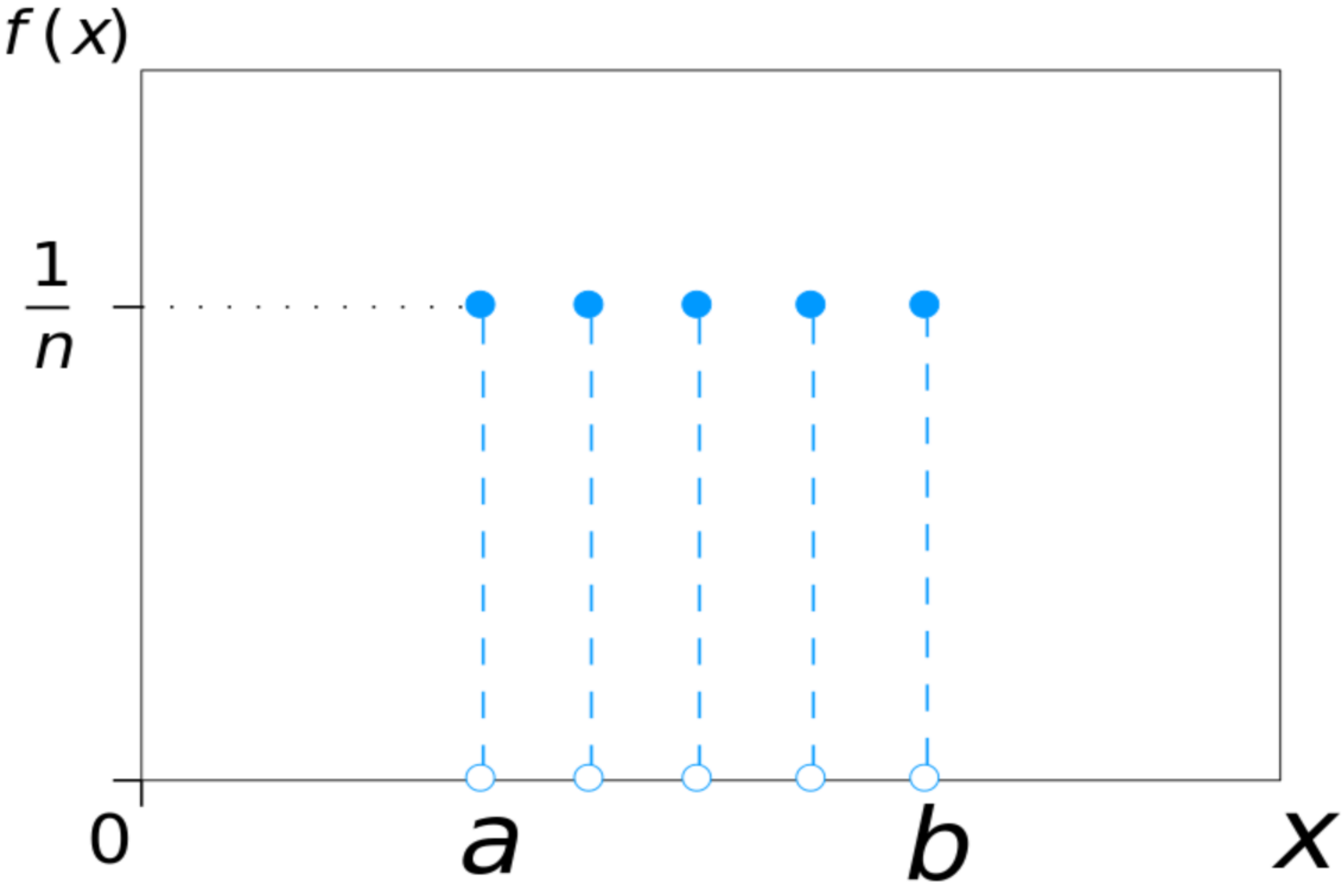
Random Experiment

Random Variable (Discrete, Continuous), CDF, PMF/PDF

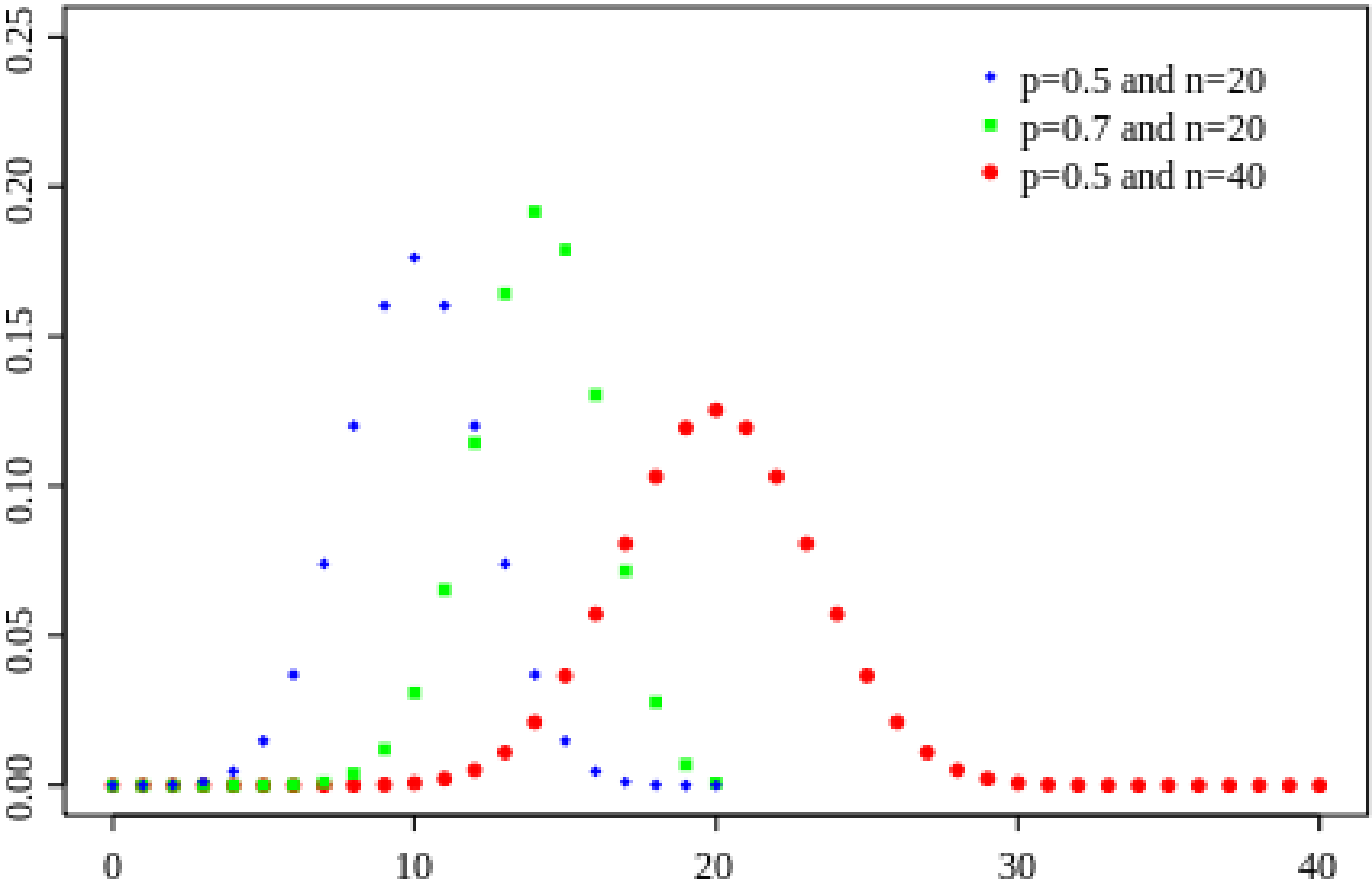
Uniform Distribution (CDF)



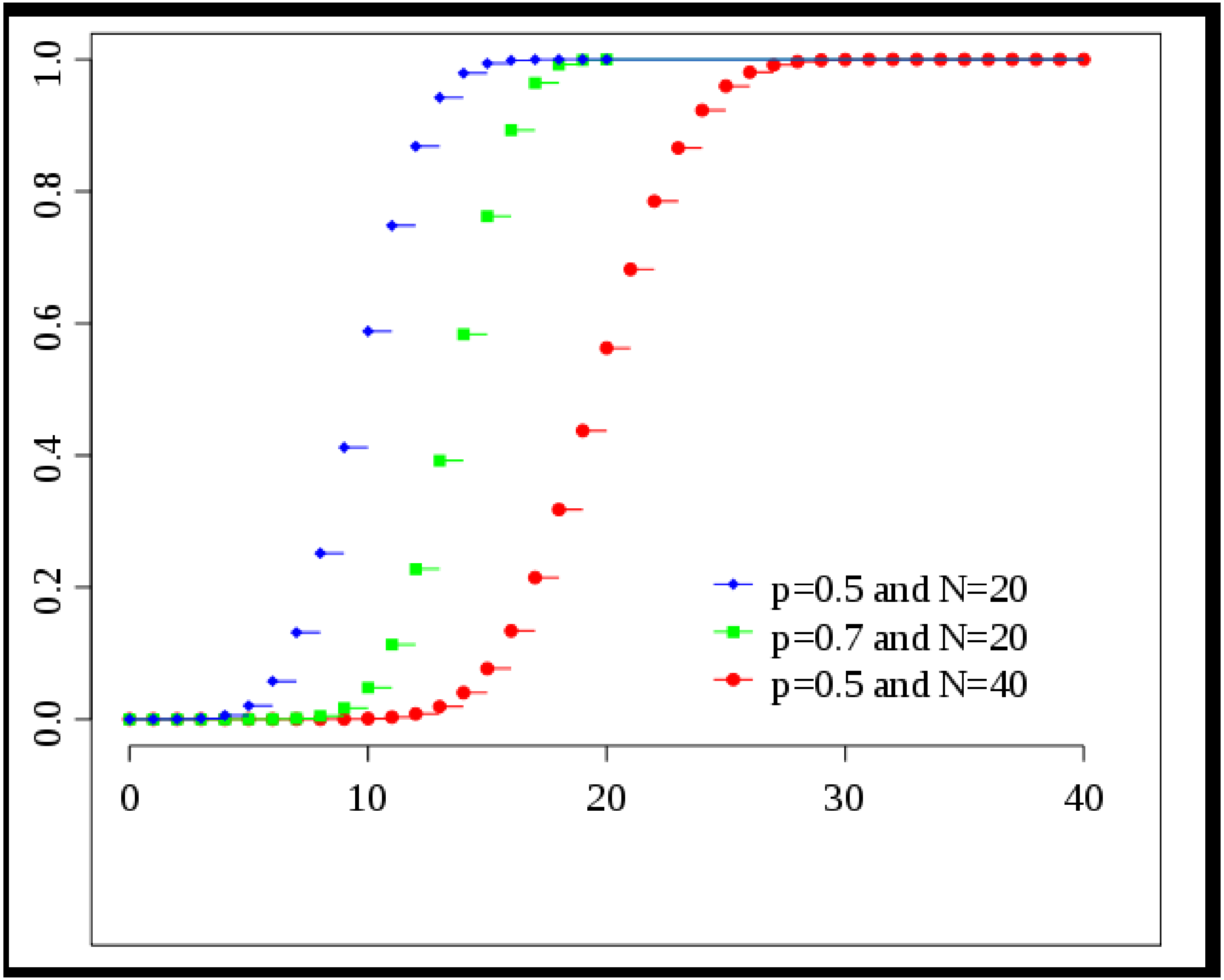
Uniform Distribution (PMF)



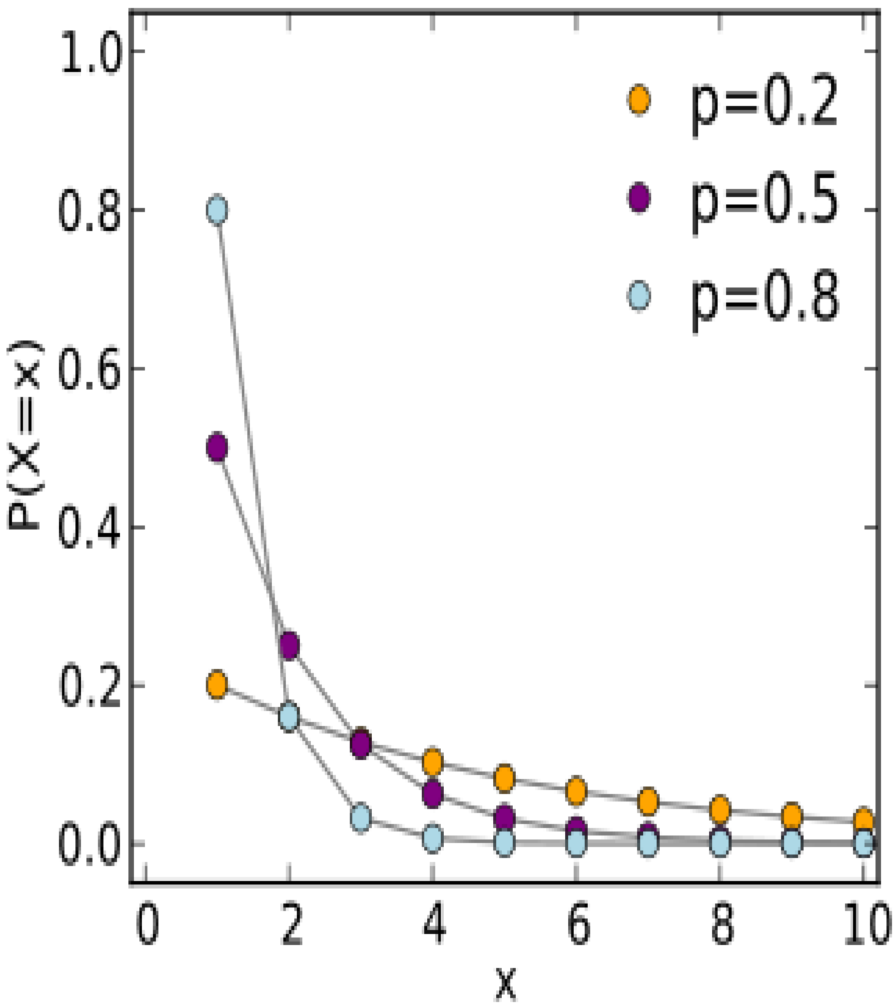
Binomial Distribution (PMF)

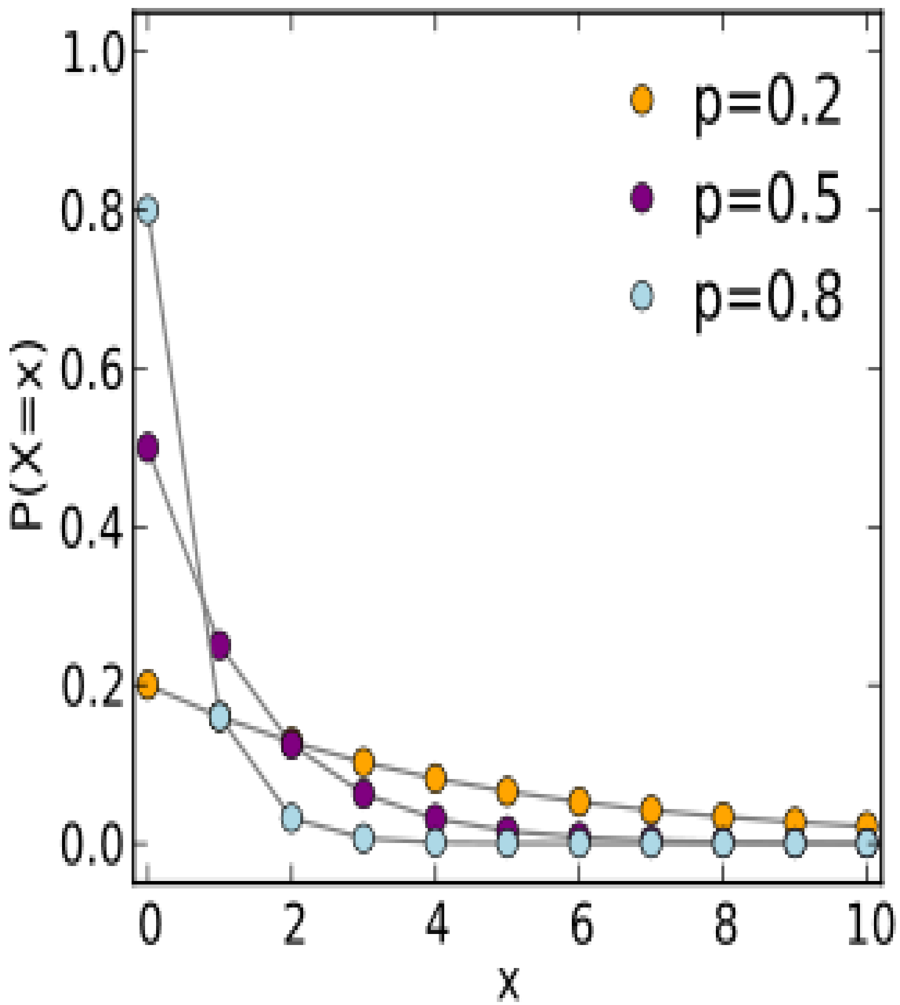


Binomial Distribution (CDF)

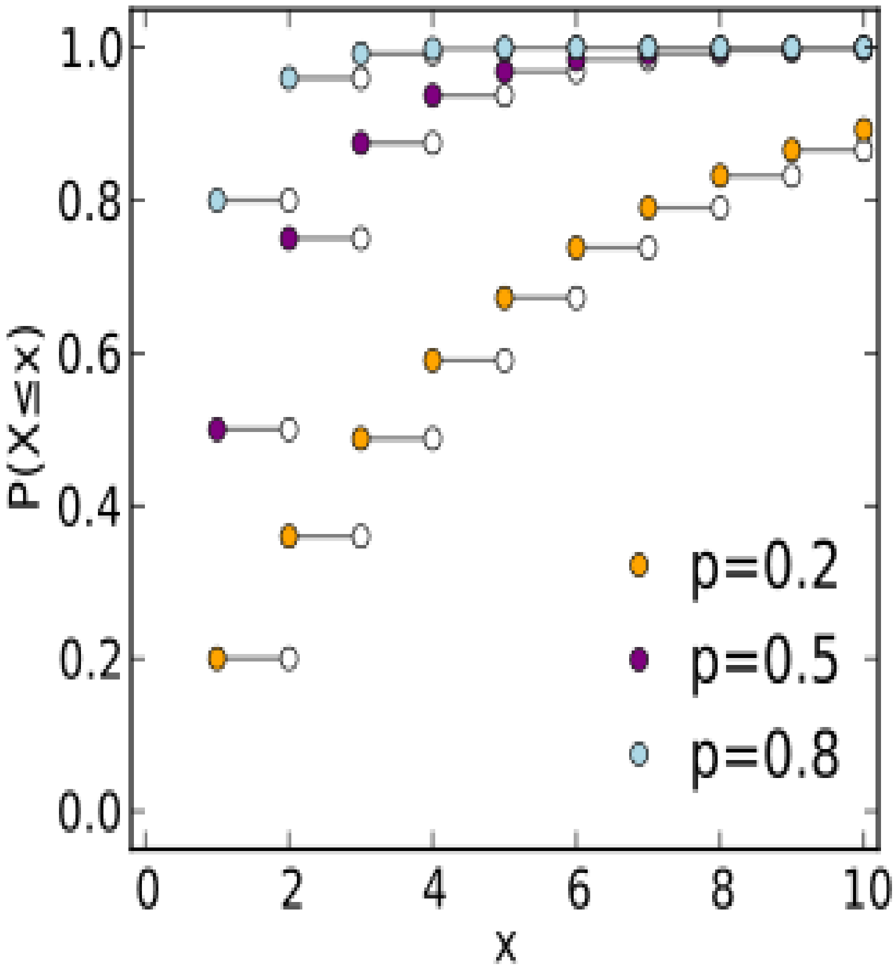


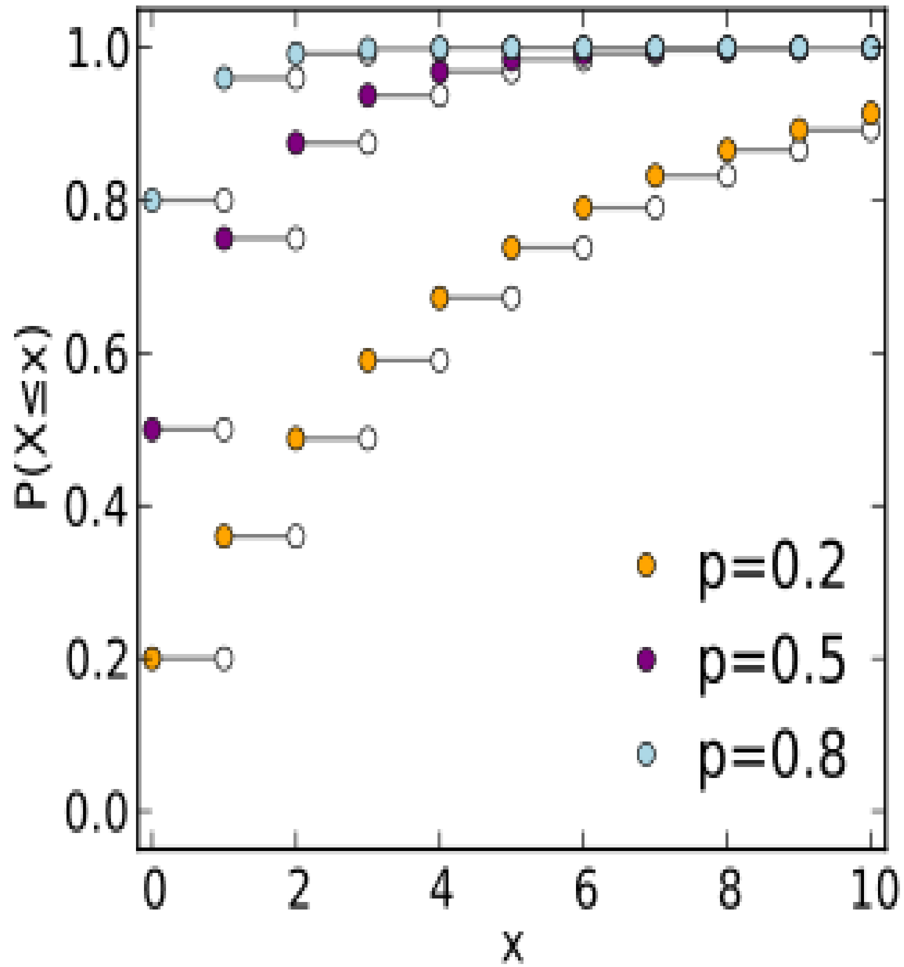
Geometric Distribution (PMF)



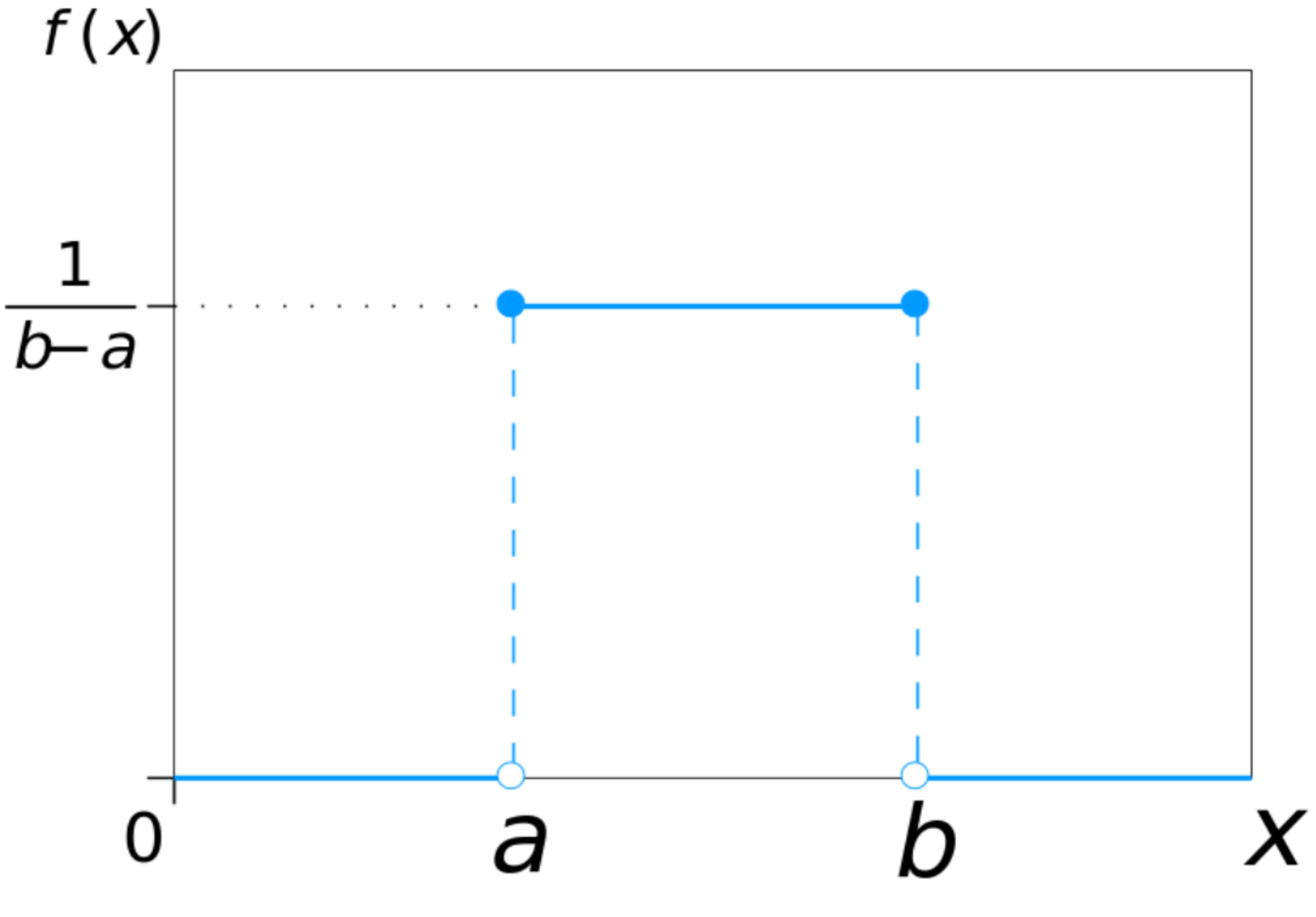


Geometric Distribution (CDF)

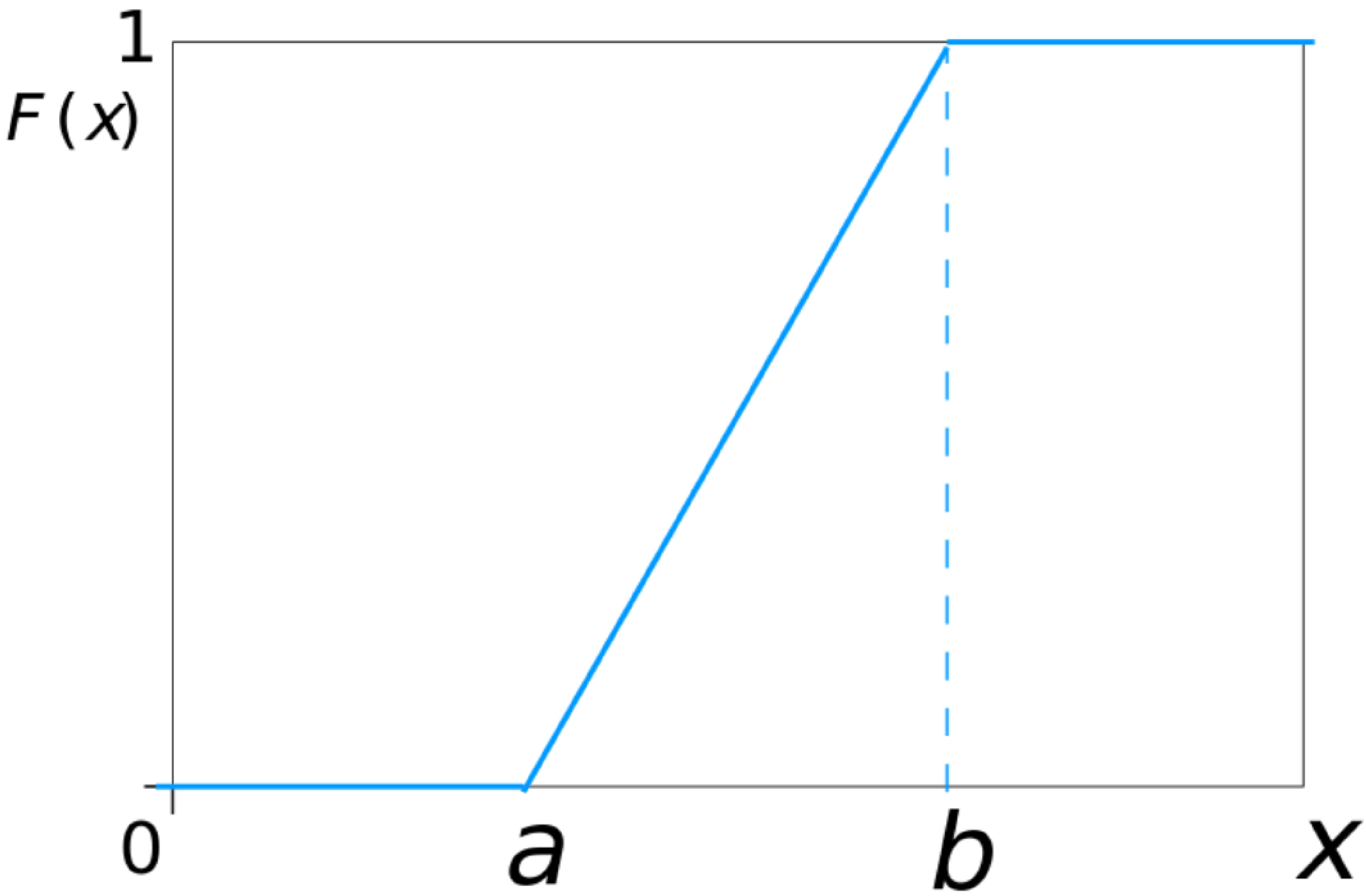




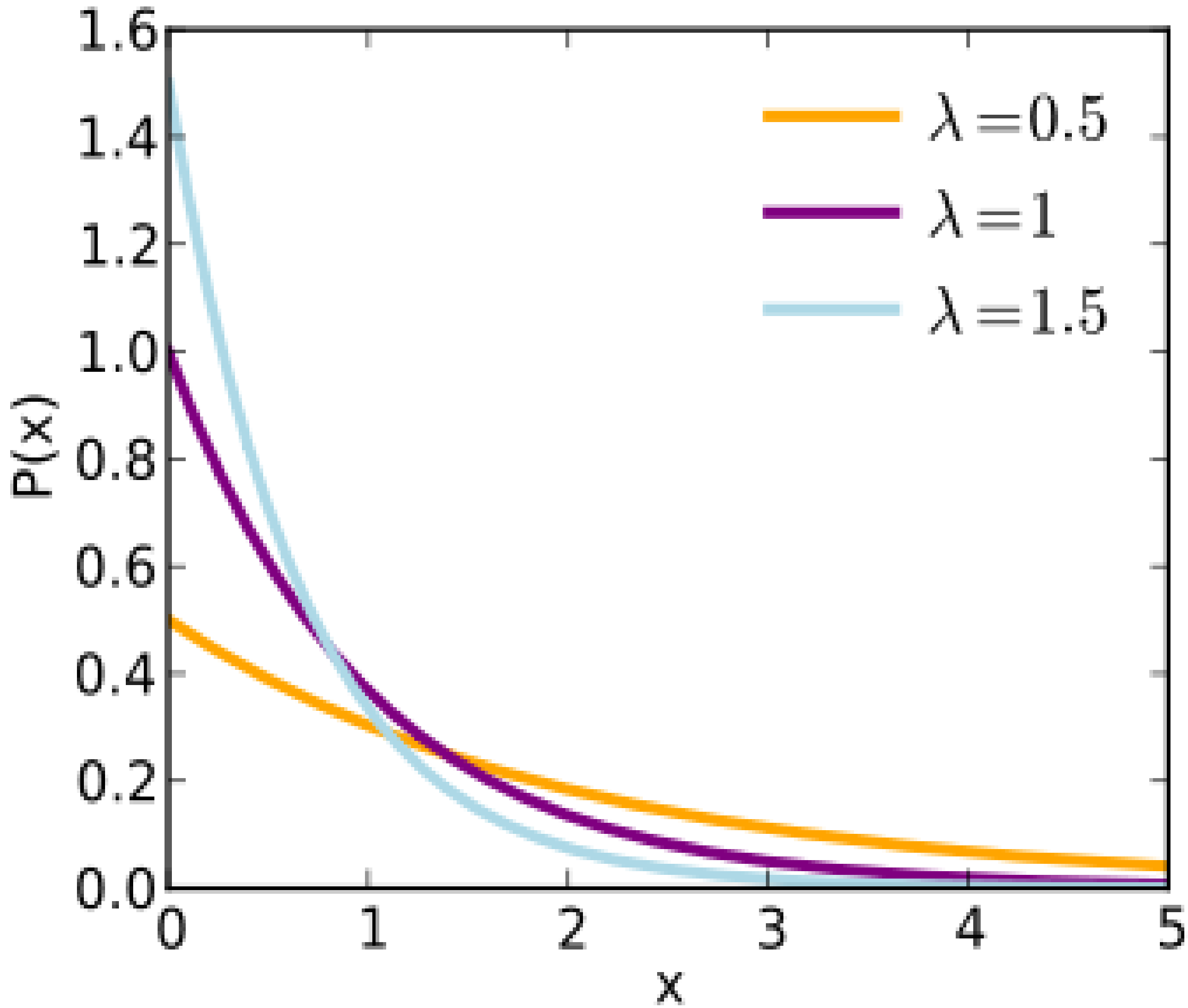
Uniform Distribution (PDF)



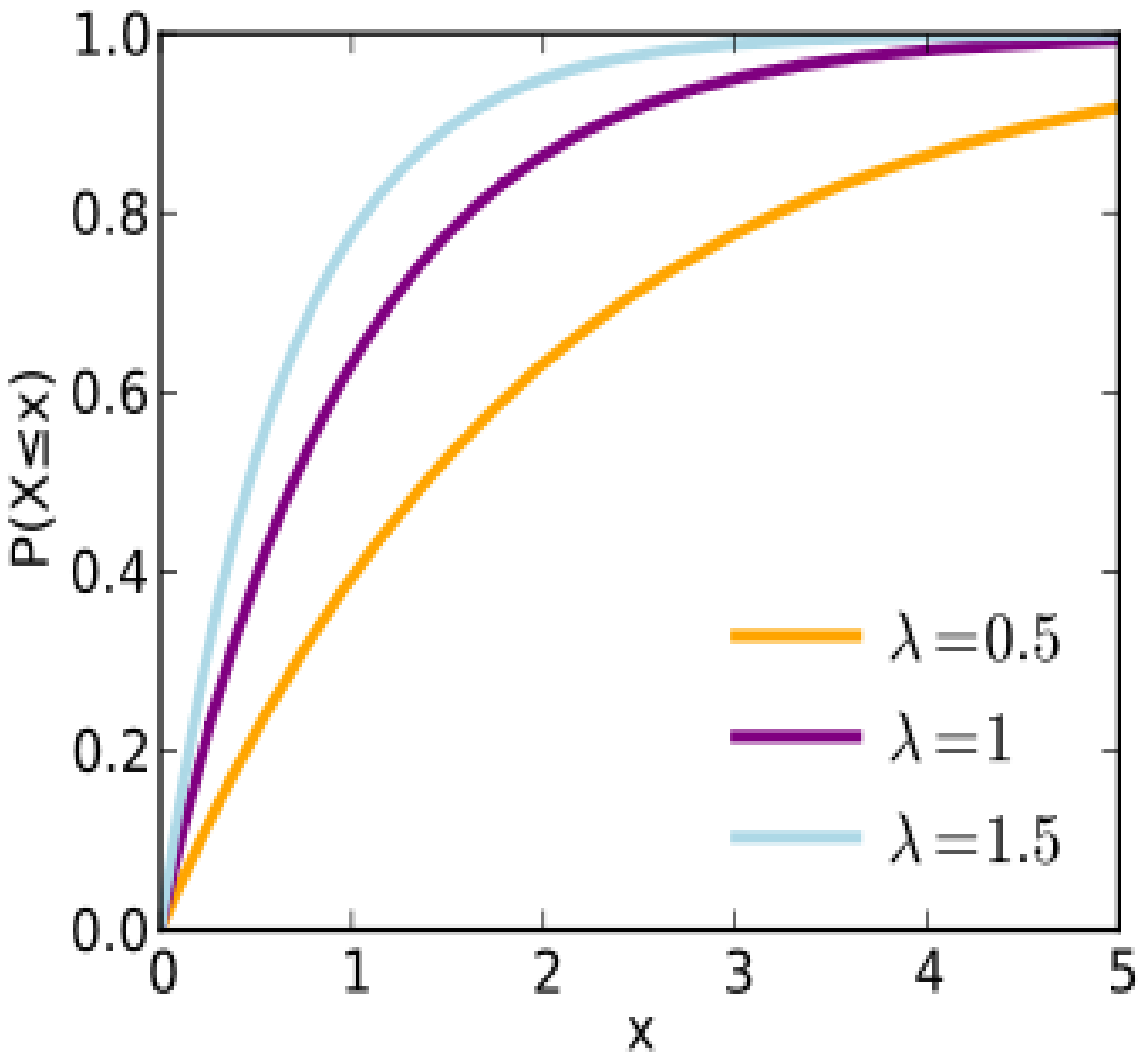
Uniform Distribution (CDF)



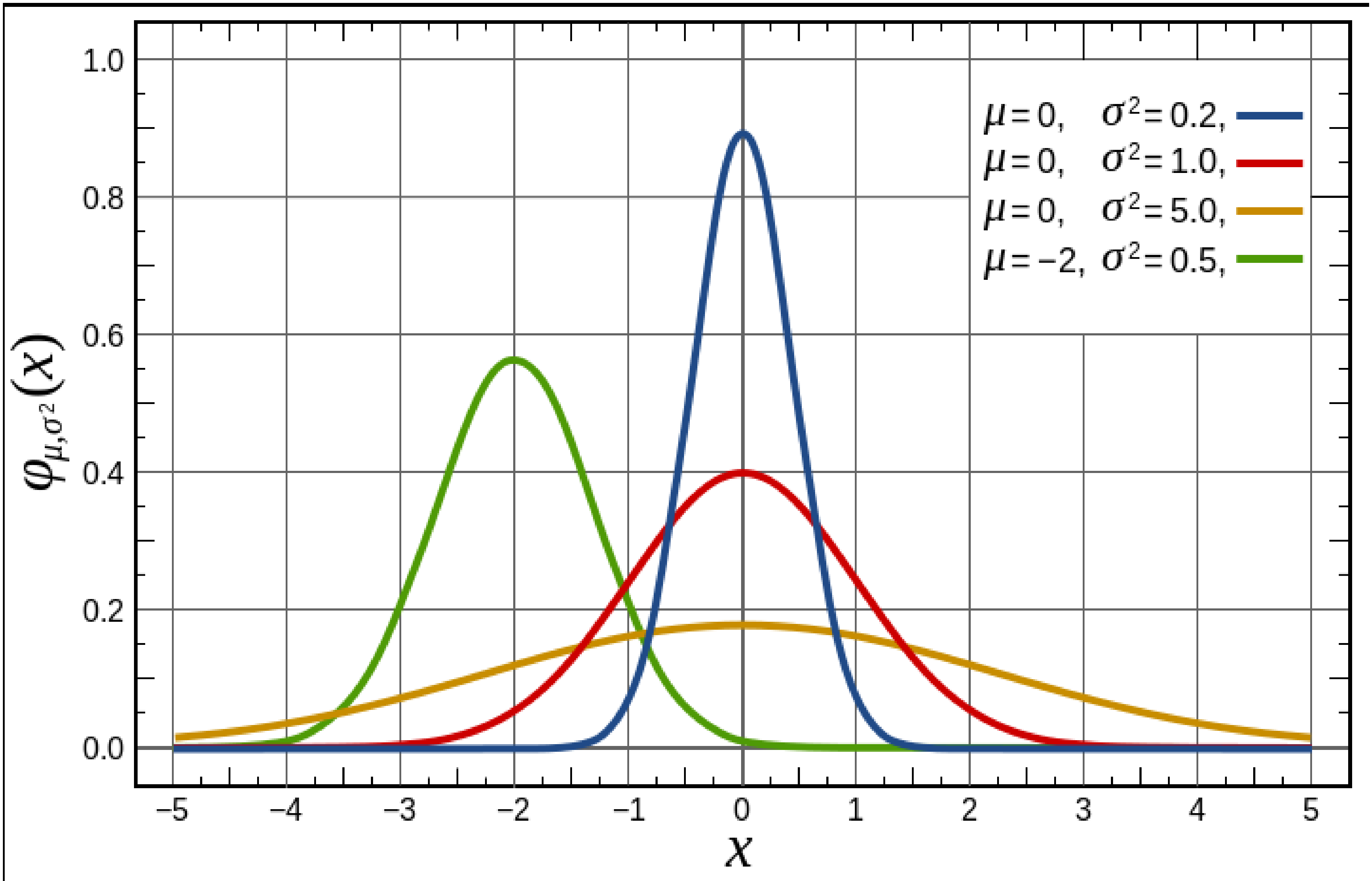
Exponential Distribution (PDF)



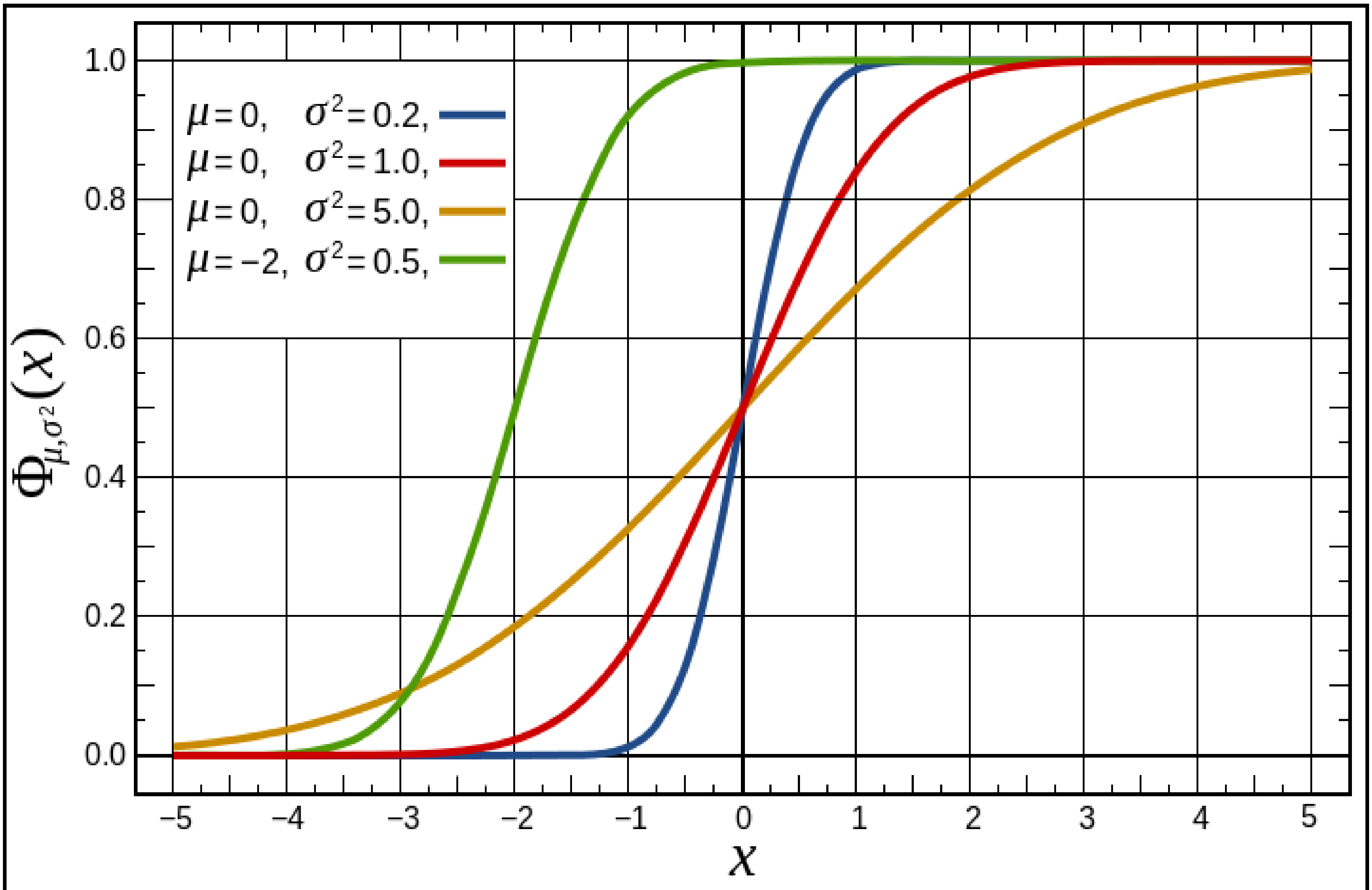
Exponential Distribution (CDF)



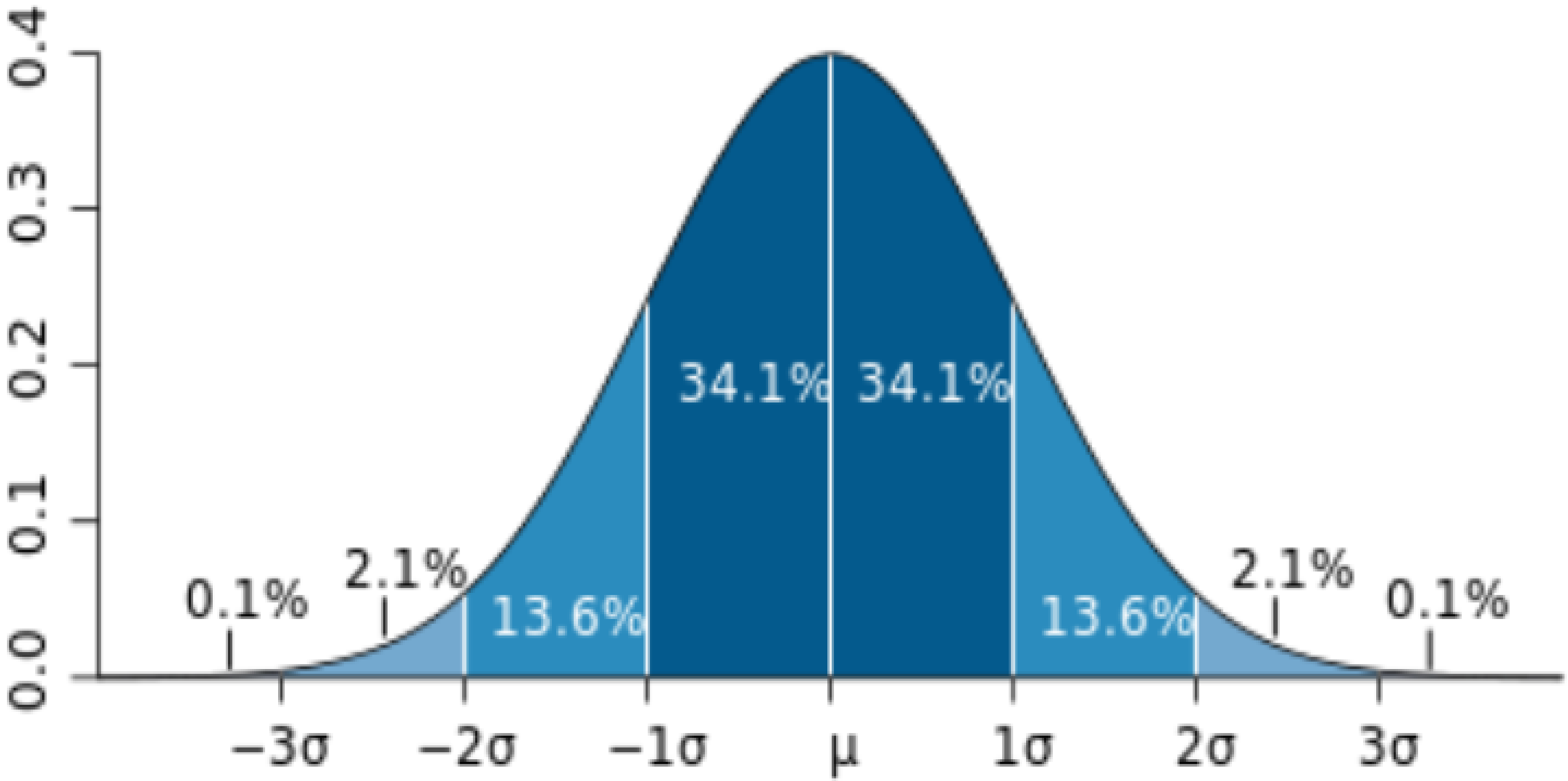
Normal Distribution (PDF)



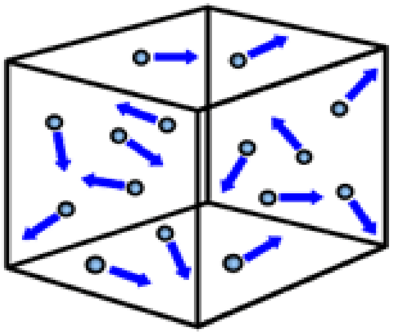
Normal Distribution (CDF)



Normal Distribution (PDF): Areas Under Curve

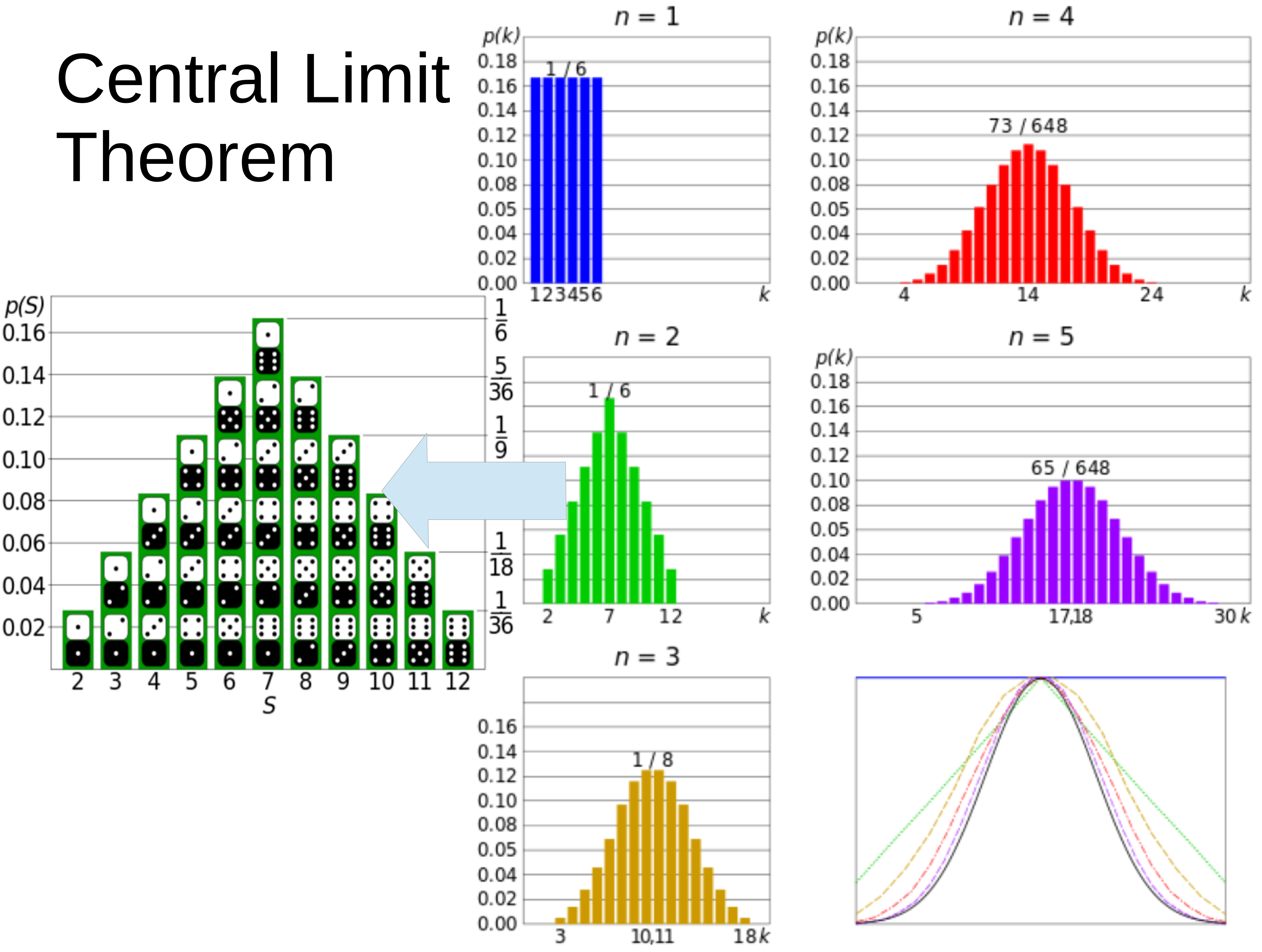


Normal Distributions in Real World

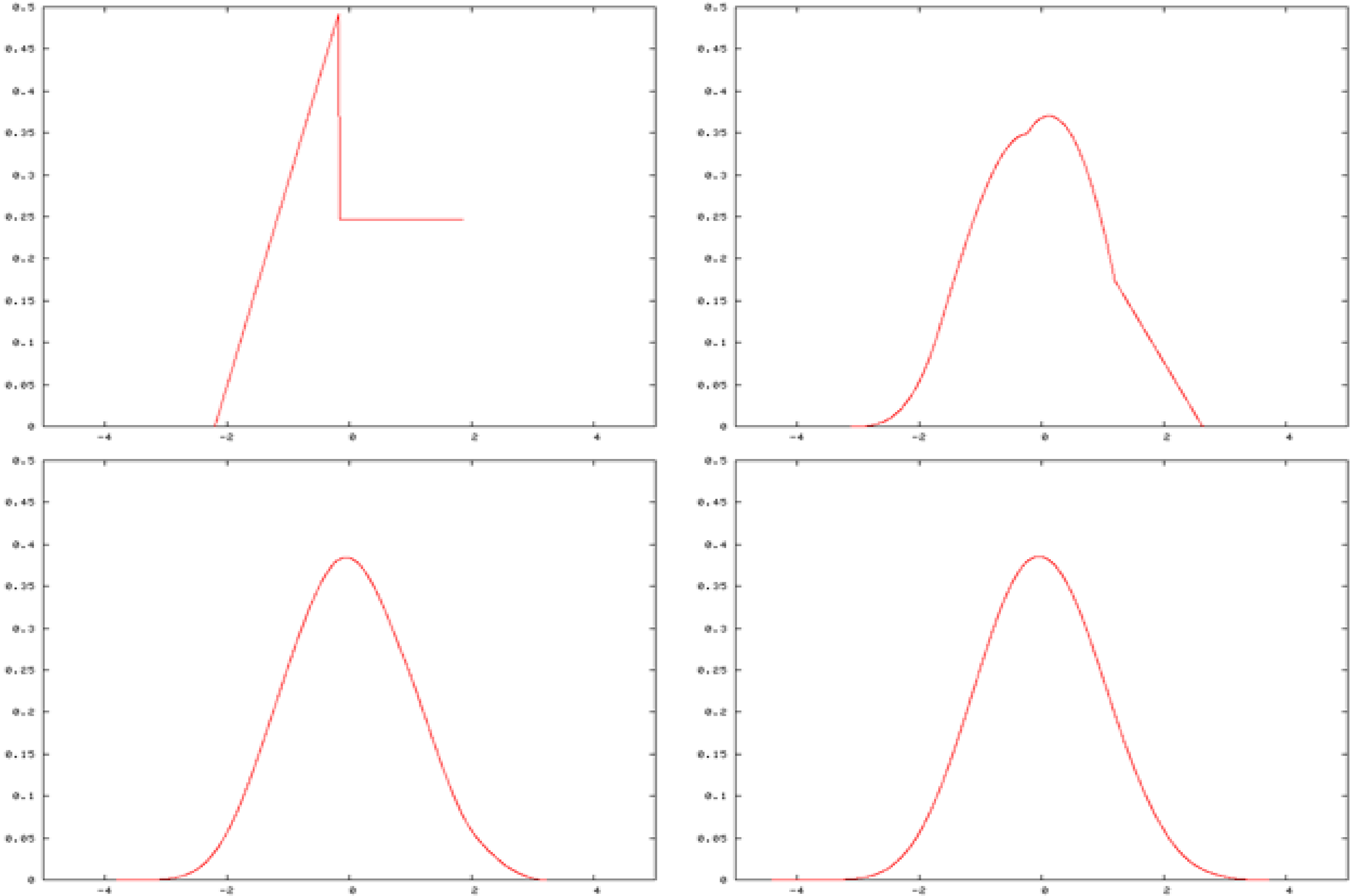


- Velocities of molecules in ideal gas
  - Gas at high temperature and low pressure
- Diffusion of dye in agar
  - http://www.youtube.com/watch?v=AIZ2ji10Rk0
- Diffusion of dye
  - http://en.wikipedia.org/wiki/Diffusion
- Consequence of the Central Limit Theorem
  - Next slide

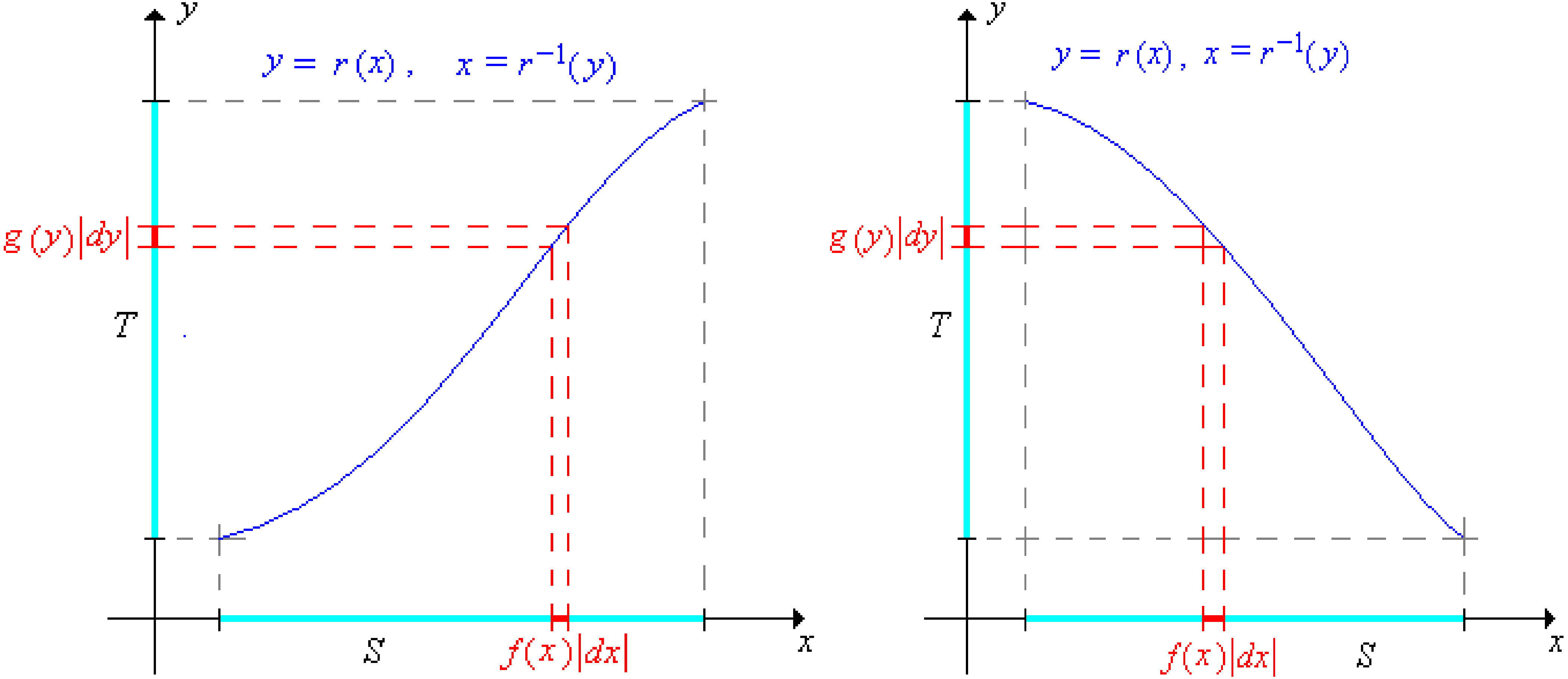
Central Limit Theorem



Central Limit Theorem



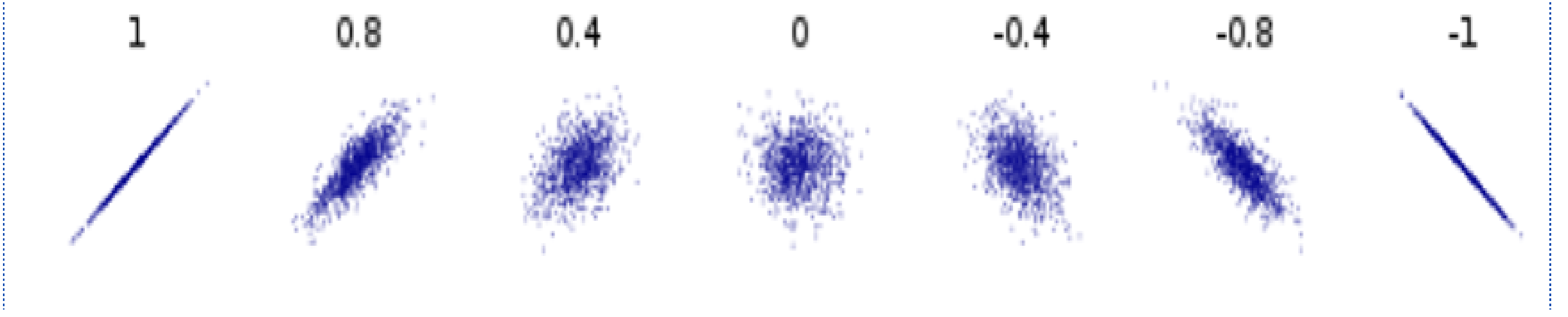
Transformation of a RV



- Covariance, Correlation

Correlation

- Noise content in the joint PDF of X, Y



Correlation

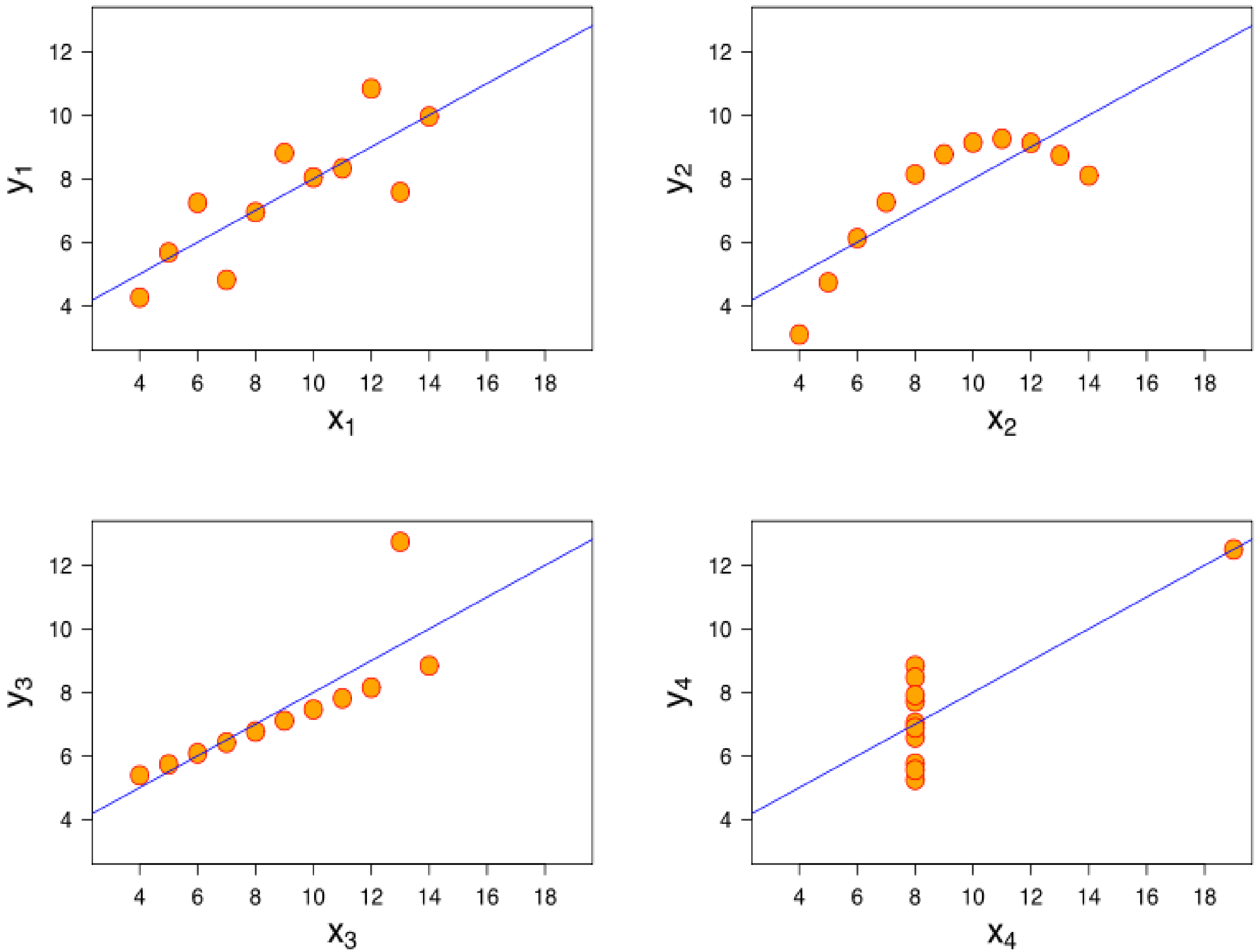
- Slope of the relationship between X, Y

Correlation

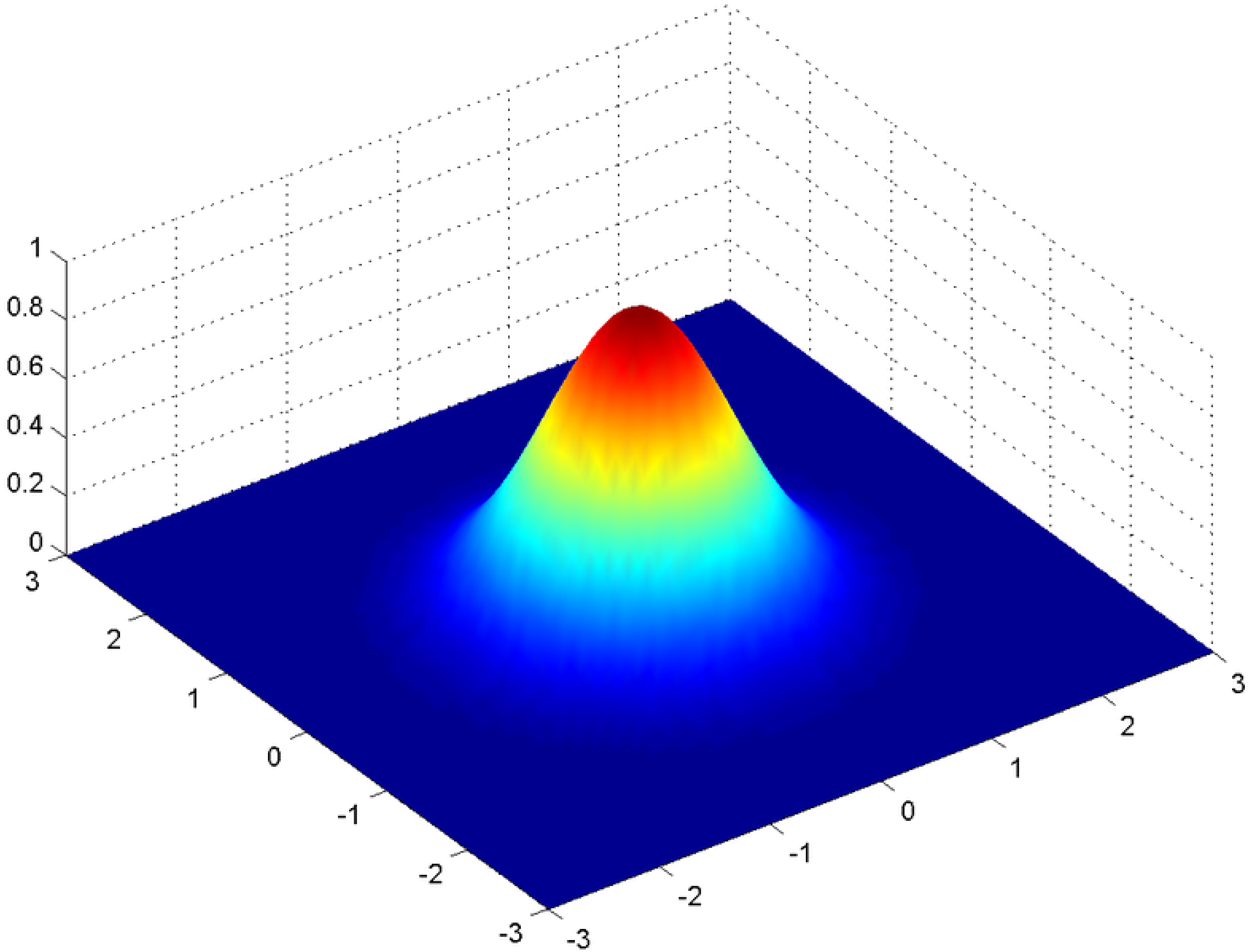
- Nonlinearity of the relationship between X, Y
  - Uncorrelated X,Y doesn't imply independent X,Y

Correlation

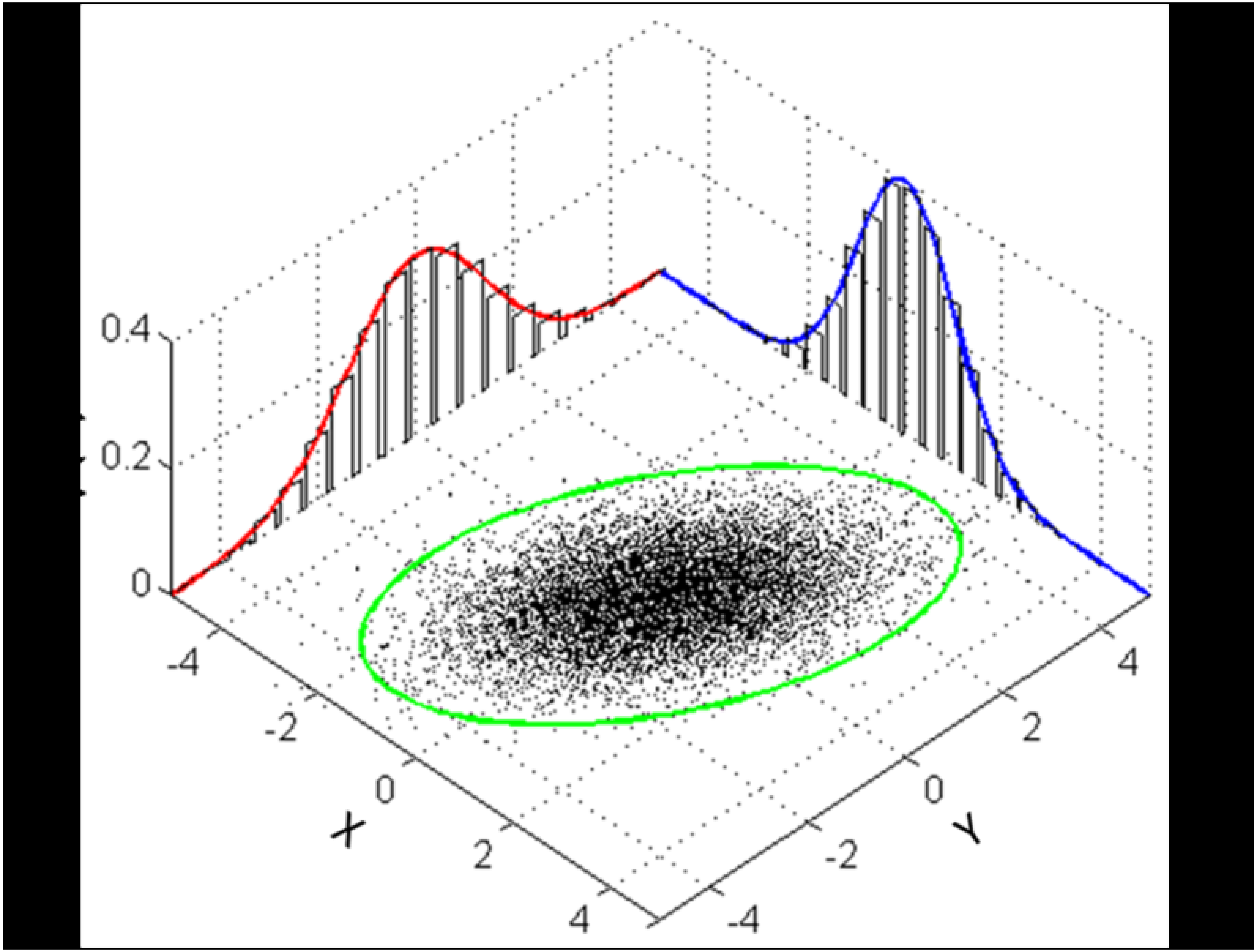
- "Strange" effects of noise, outliers, and nonlinearity
  - For all plots correlation = 0.86



Multivariate Gaussian (PDF)



Multivariate Gaussian (Marginal PDFs)



Statistical Model

- Probabilistic description of real-world phenomena
- Involves a PDF; which involves parameters
- Examples
  - Distribution of heights of a 1-year old baby
    - Gaussian
  - Distribution of complex data in MRI, within the same tissue
    - Gaussian
  - Distribution of complex data in MRI, involving multiple tissues
    - Each tissue PDF Gaussian

Model Fitting

- Fit a model to data
  - Estimate parameters of model using data
- Example
  - Distribution of heights of a 1-year old baby (Gaussian)
    - Data = ?
    - Parameters = ?
  - Distribution of complex data in MRI, involving multiple tissues (Gaussian)
    - Data = ?
    - Parameters = ?

Estimator and Estimate

- Estimator = Rule / Formula / Equation for calculating the value of a given quantity, given data
  - Example
    - A formula for Gaussian mean, given data
- Estimate = Value given by the estimator
  - Example
    - Mean = 5.2, given data/measurements A
    - Mean = 4.9, given data/measurements B

Noise

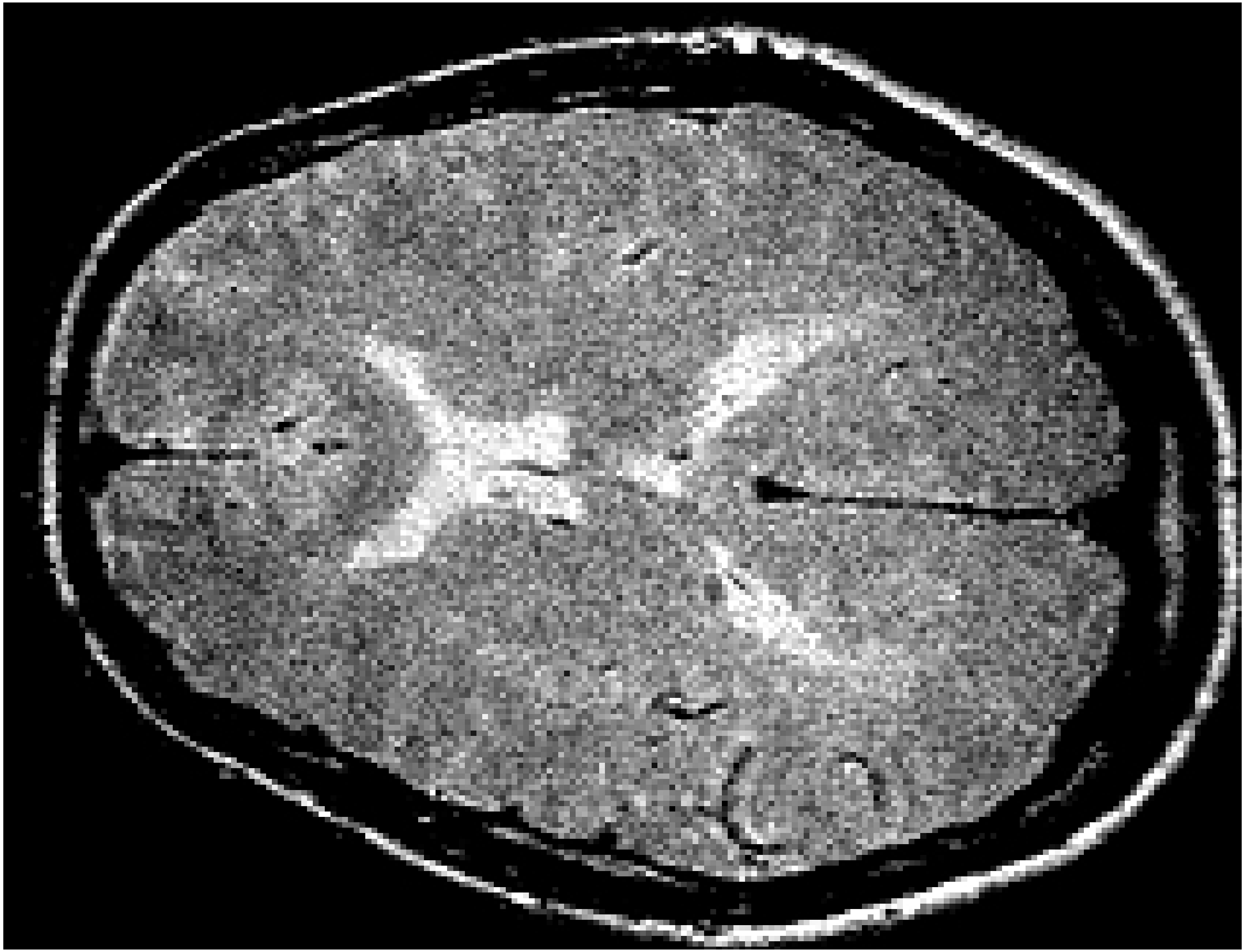
- Undesirable and unknown modifications to the measured data during :
  - Acquisition
  - Storage
  - Transmission
  - Processing
  - Conversion

Noise (video)

Noise (image)

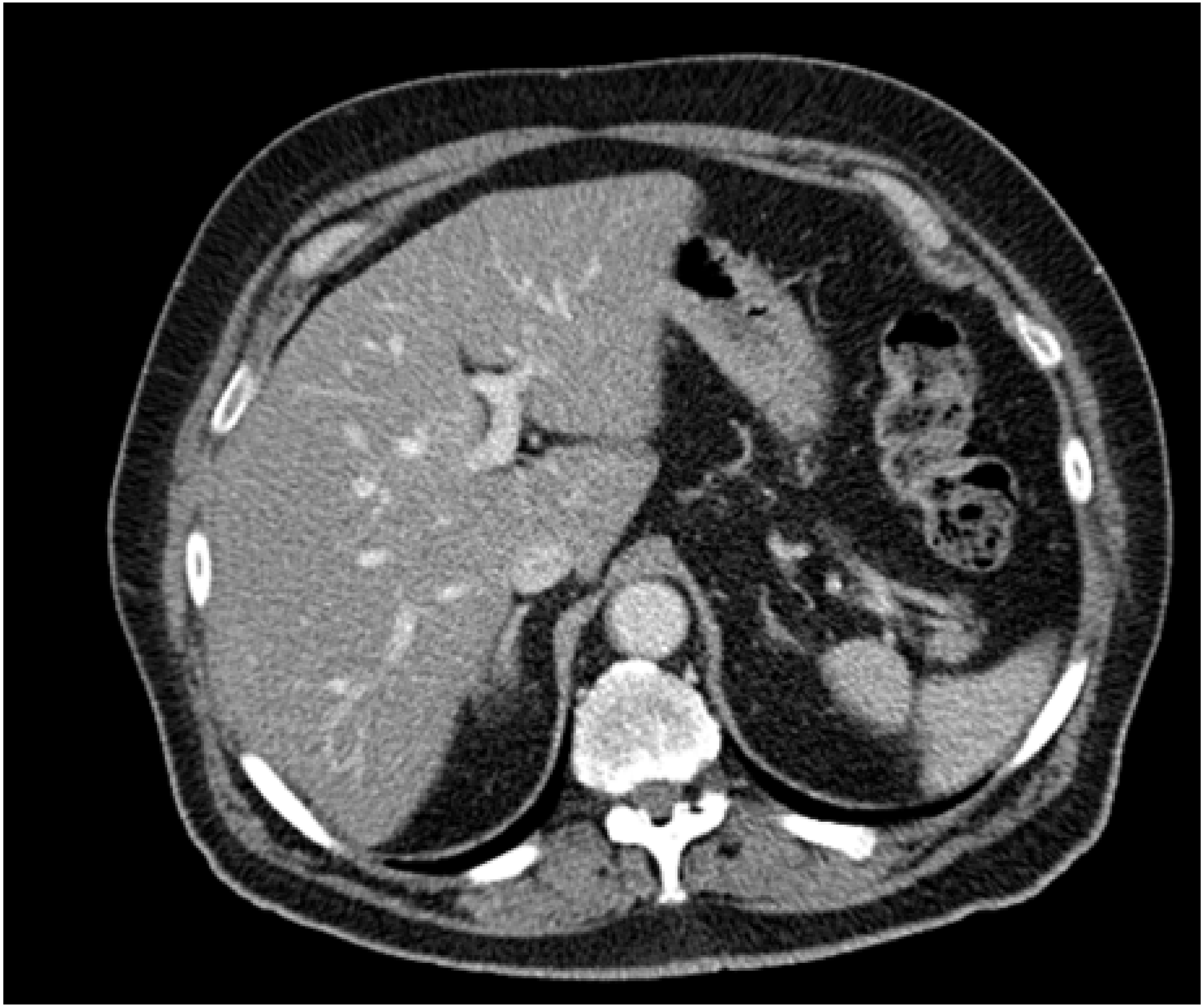
Noise (Image)

Noise (Brain MRI)

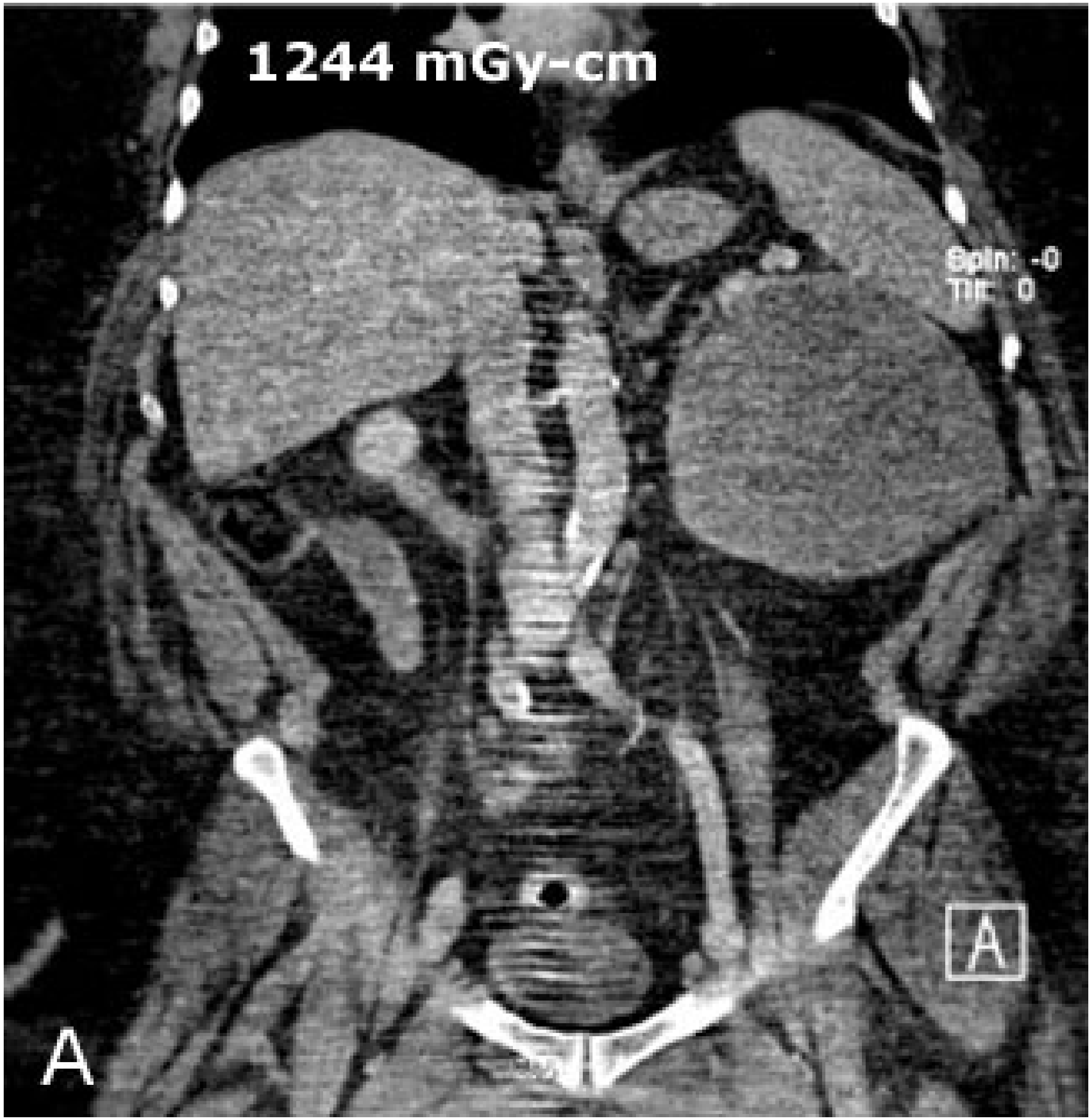


Noise (Knee MRI)

Noise (Abdomen CT)



Noise (Abdomen CT)

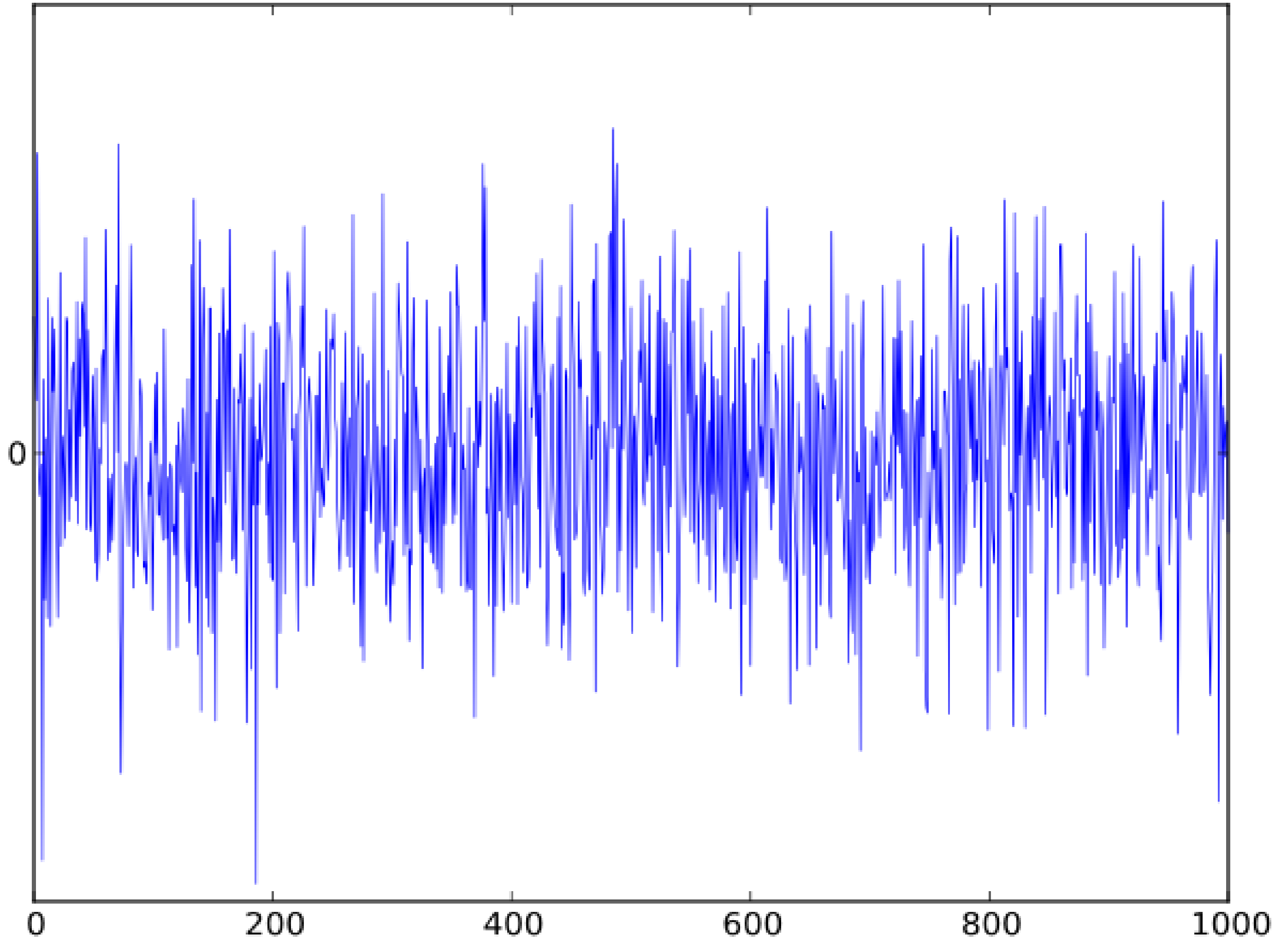


Noise (Fetal Ultrasound)

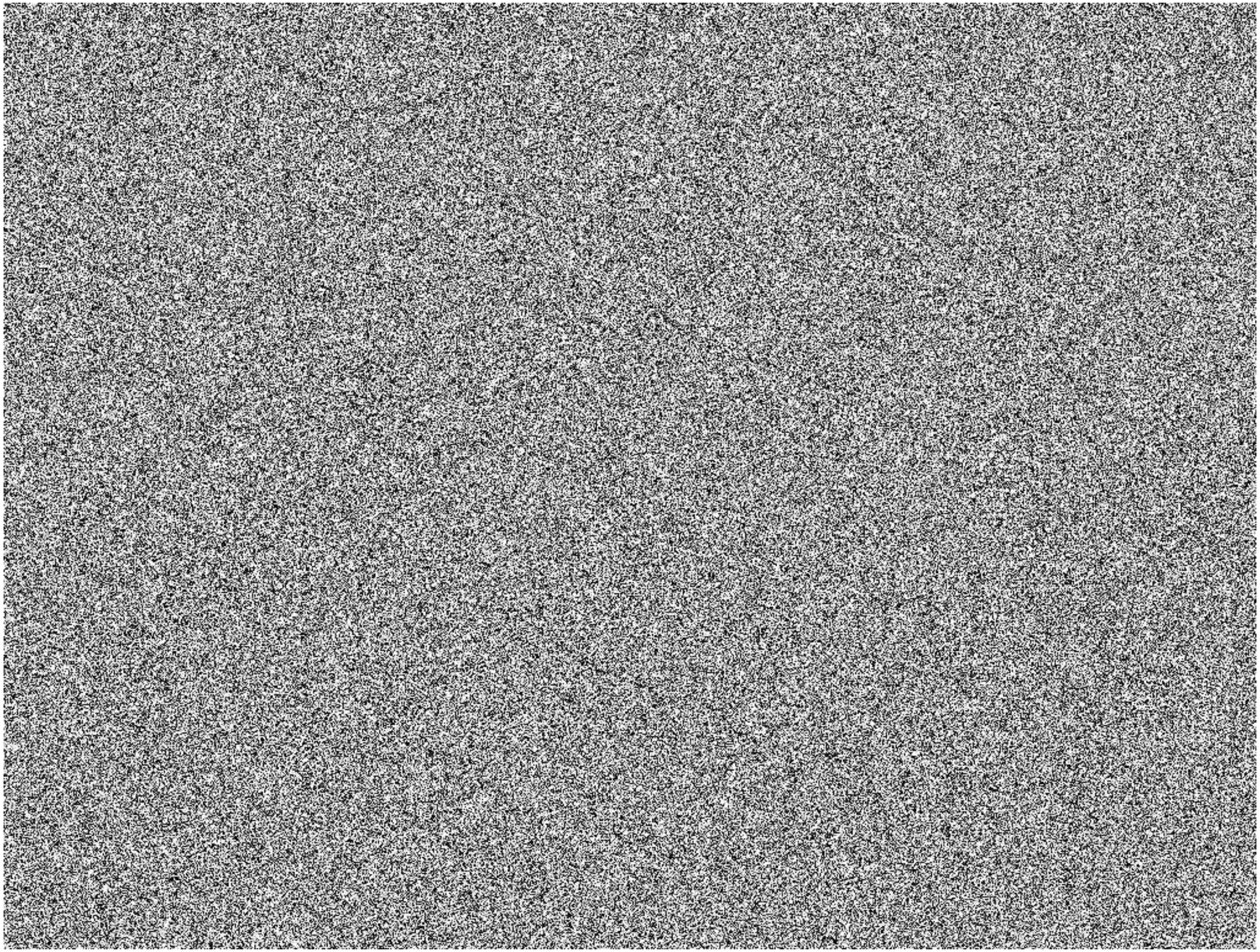
Noise Model

- Per-pixel model
  - P (Data | TrueSignal, PixelLocation)
- Spatial model (spatial correlations)
  - Examples
    - (Weakly) White noise
      - Uncorrelated RVs
      - Zero mean : E [Data | TrueSignal] = TrueSignal
      - Finite variance
    - (Strongly) White noise
      - Independent RVs
      - Zero mean + finite variance
      - Independent & Identically Distributed = i.i.d

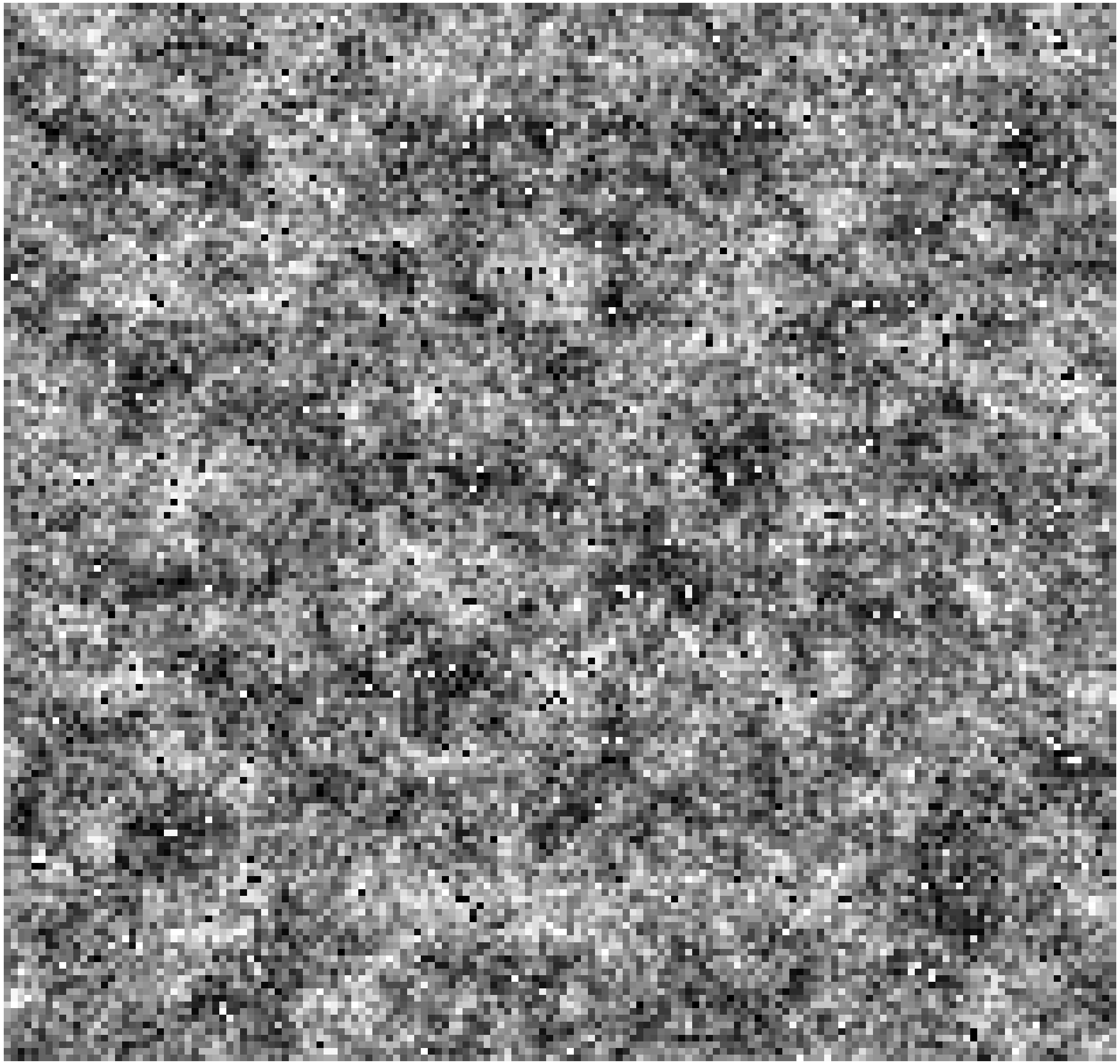
(Strgonly / i.i.d) White Noise Gaussian (1D image)



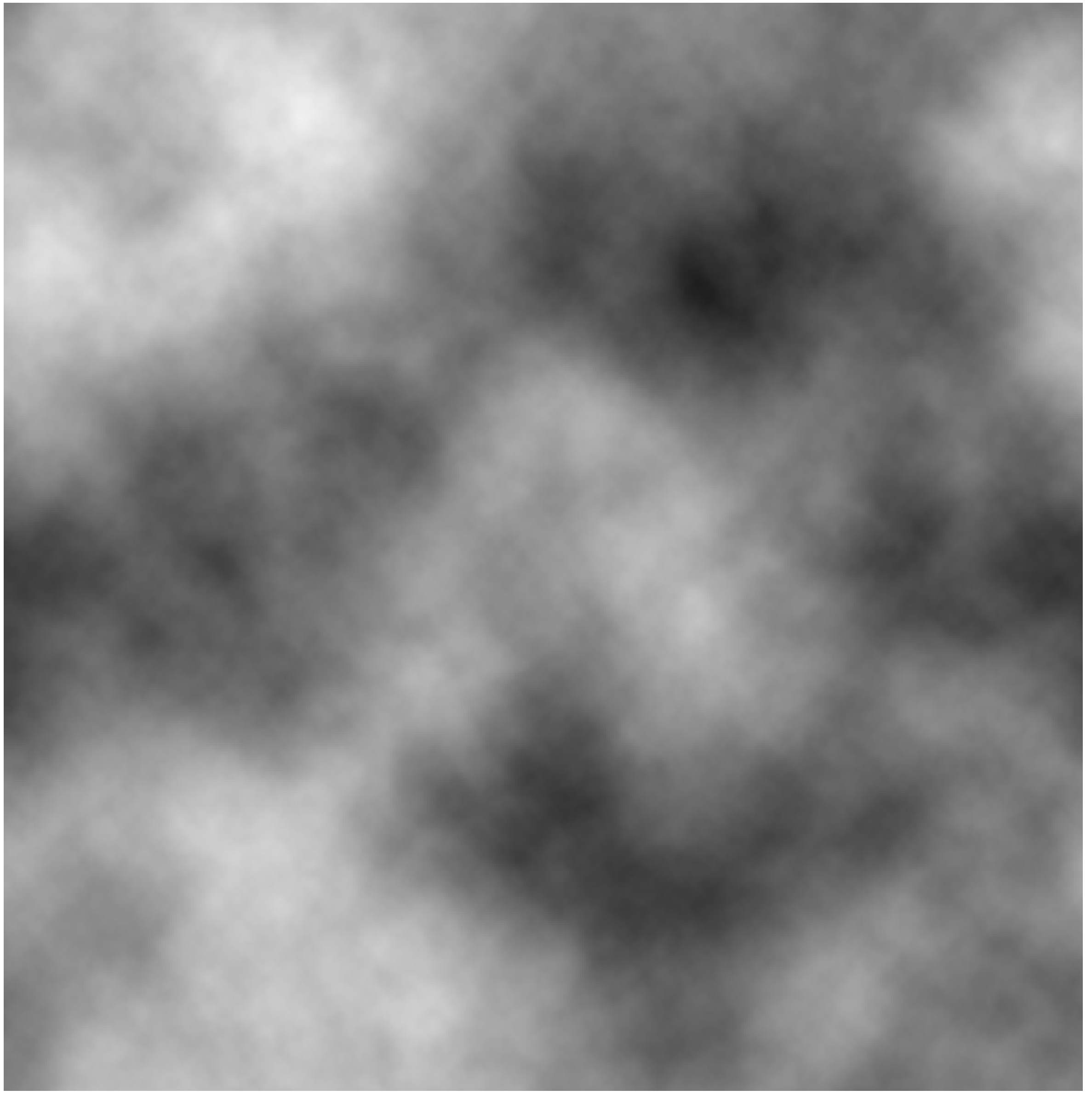
(Strongly / i.i.d.) White Noise
Gaussian (2D image)



Correlated Noise



Correlated Noise



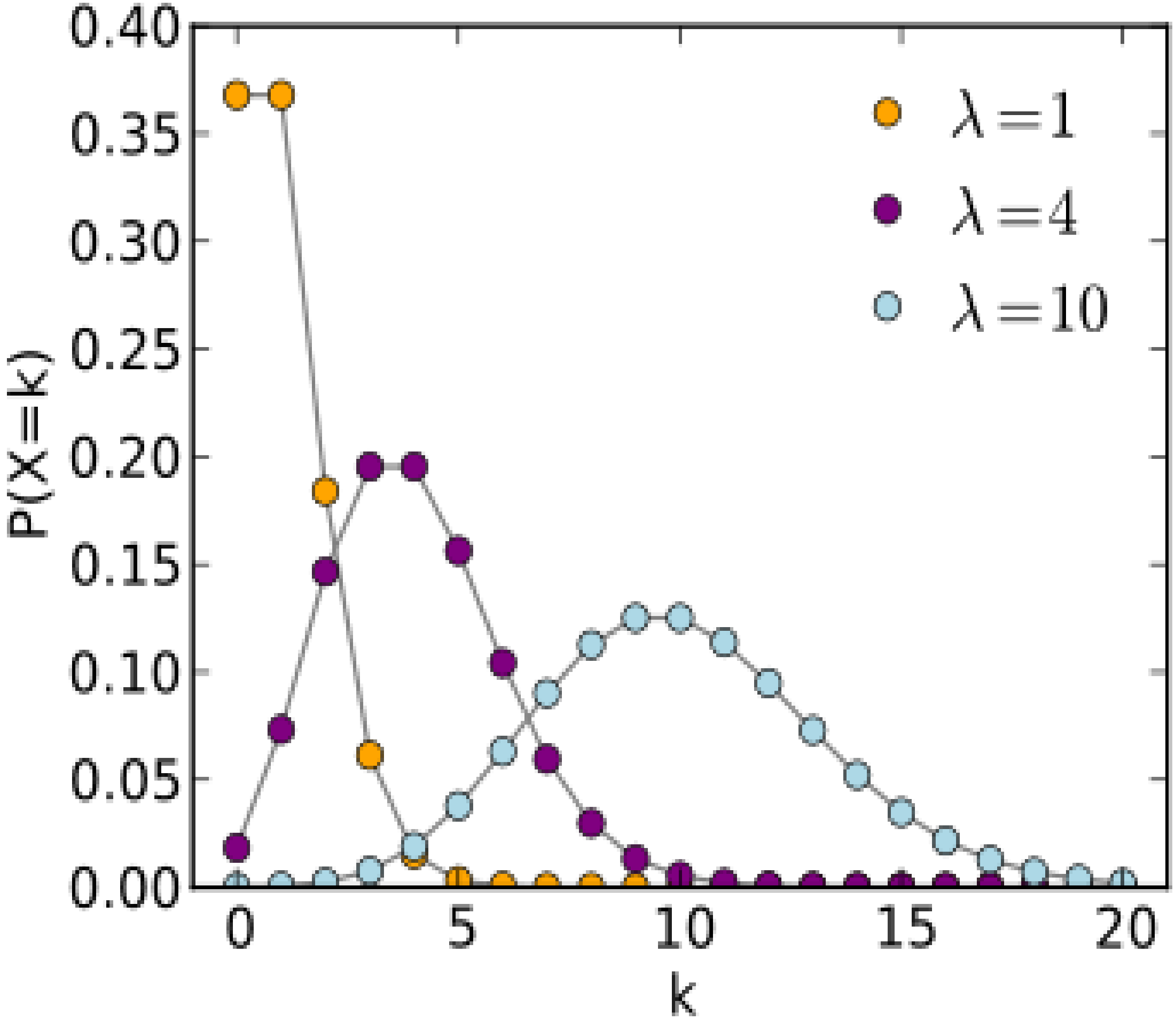
Noise Models in Medical Imaging

- Gaussian PDF
  - Most common
  - Thermal noise in electronic systems
  - Signal-to-noise ratio (SNR)
    - SNR (X) = E [X] / SD (X) = mean / sigma
    - Noise level : sigma

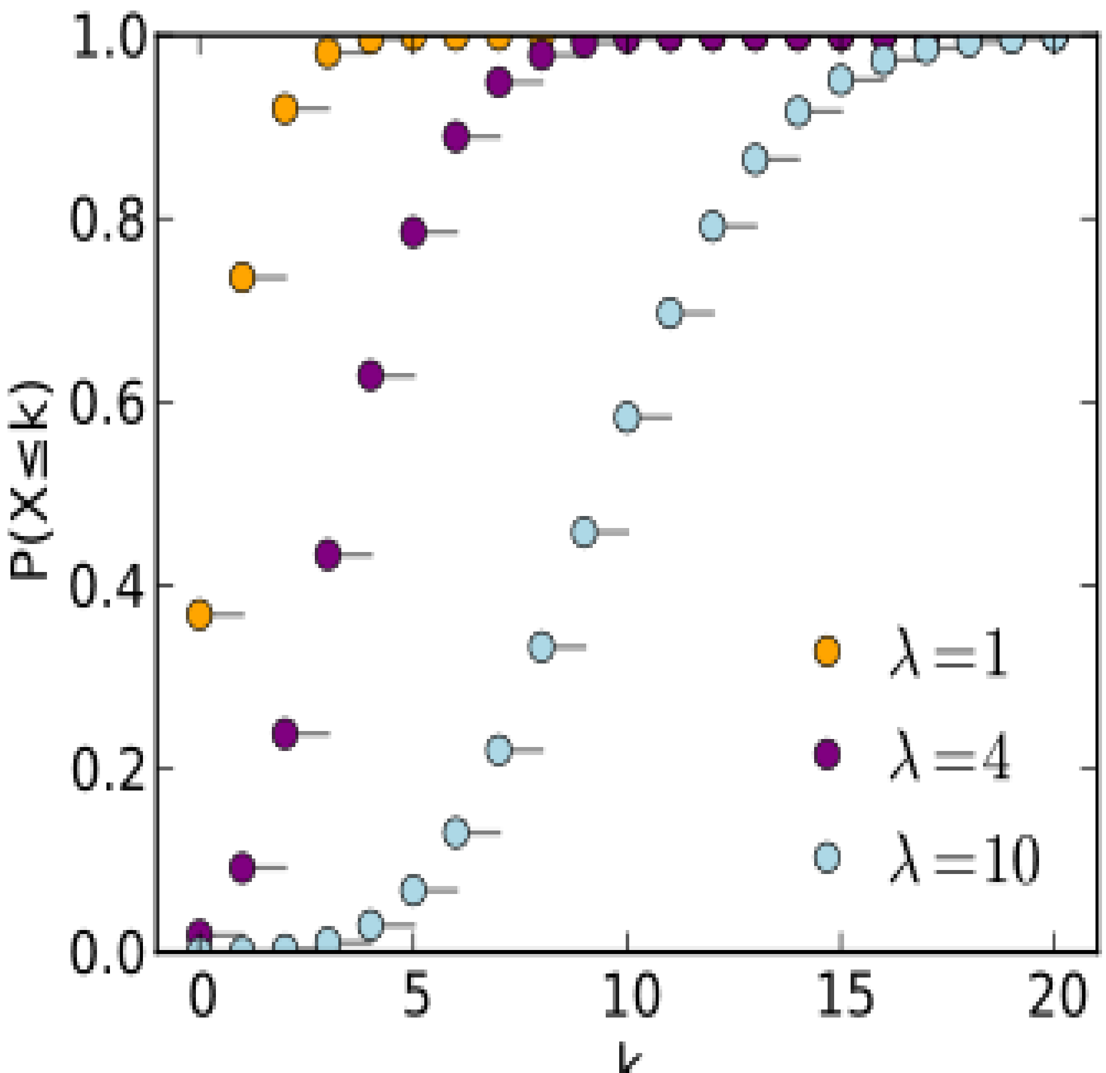
Noise Models in Medical Imaging

- Poisson PMF
  - Image created by accumulation of photons over a detector
    - X-ray, X-ray CT, CCD cameras
  - Given : Average number of captured photons (parameter) $\lambda \in R$ during a certain duration of imaging
    - λ can be non-integer
  - Probability of $k \in Z+$ events occur: $P(X=k;\lambda) = \frac{\lambda^k \exp(-\lambda)}{k!}$

Poisson Distribution (PMF)



Poisson Distribution (CDF)



Noise Models in Medical Imaging

- Poisson PMF
  - E [X] = Var [X] = lambda
  - SNR (X) = sqrt (lambda)

Poisson Noise (Simulation)

- Poisson PMF : SNR increases with lambda

Noise Models in Medical Imaging

- SNR increases with addition of independent RVs
  - Let X1, X2 (independent) have same PDF as X
  - E [X1 + X2] = 2 E [X]
  - Var (X1 + X2) = Var (X1) + Var (X2) = 2 Var (X)
  - SD (X1 + X2) = sqrt (2) SD (X)
  - SNR (X1 + X2) = sqrt (2) SNR (X)

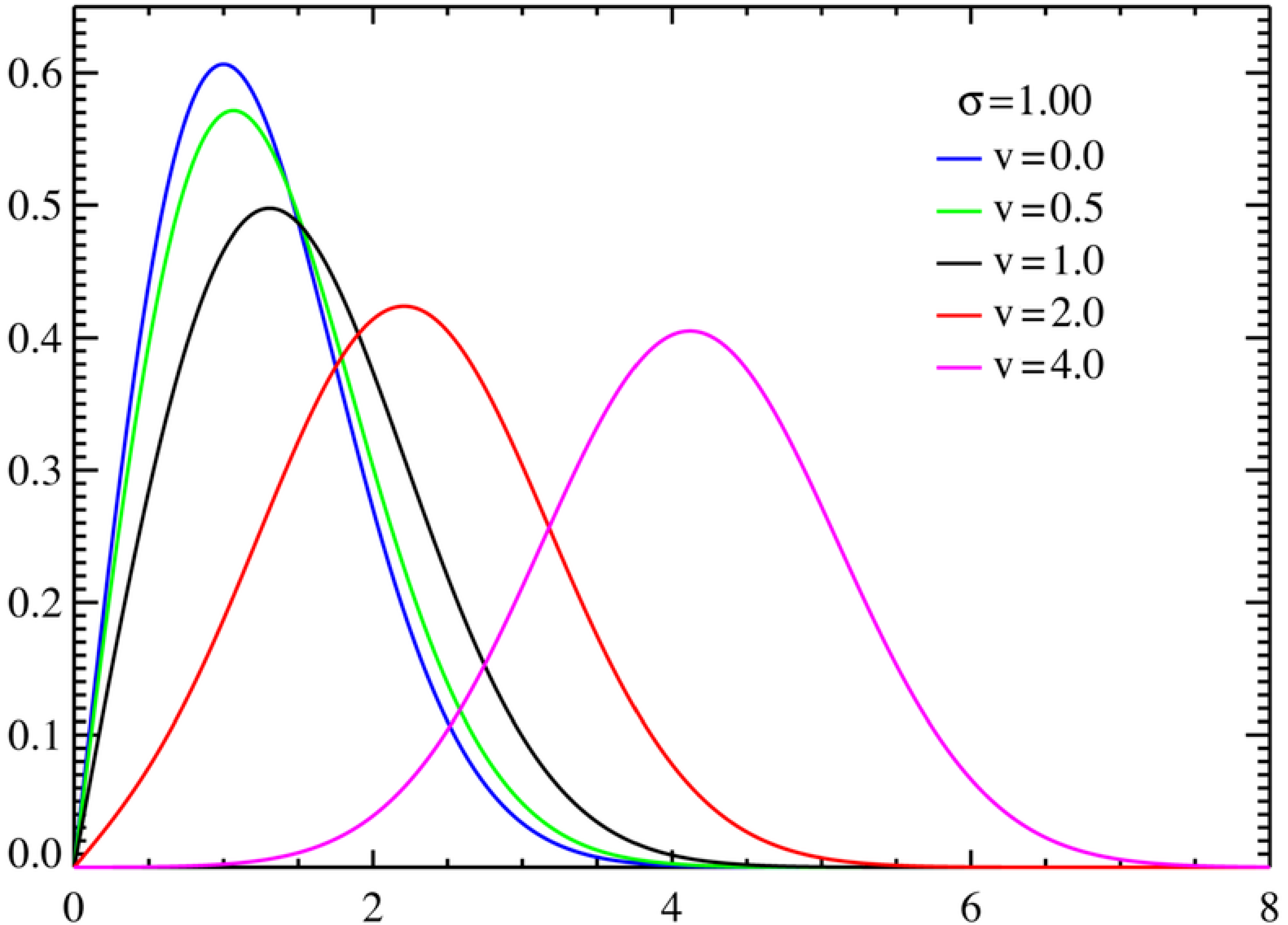
Noise Models in Medical Imaging

- Compound Poisson PMF
  - Practical X-ray CT uses polychromatic (broad-beam) X rays
  - PMF = sum of Poisson PMFs
    - One poisson PMF for each frequency
  - May be approximated by a Gaussian

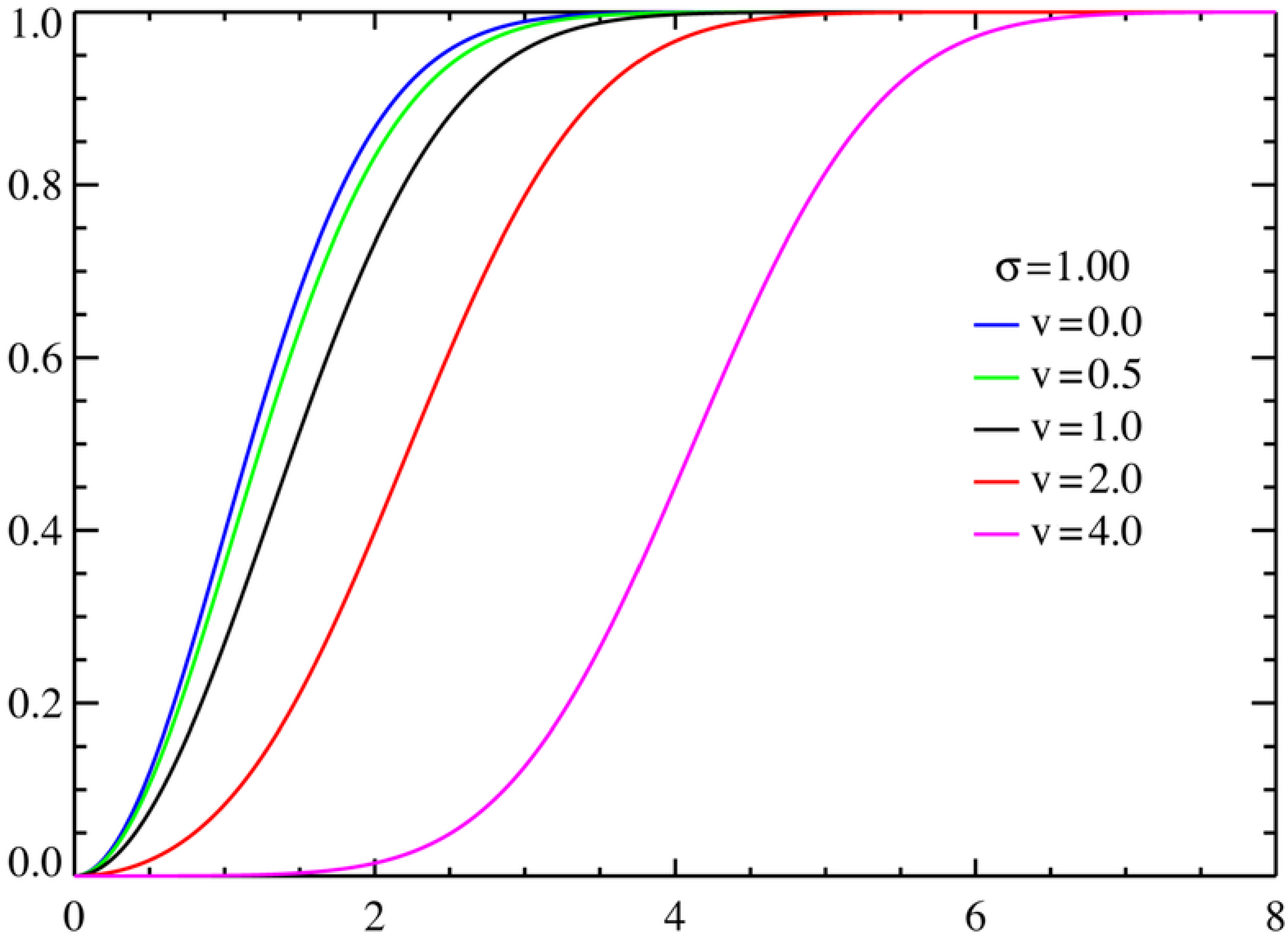
Noise Models in Medical Imaging

- Rician PDF
  - Magnitude images obtained from MRI
  - In MRI data (complex), noise in each component is i.i.d. zero-mean Gaussian
  - $p(x|\nu,\sigma) = \frac{x}{\sigma^2}\exp(-\frac{x^2+\nu^2}{2\sigma^2})I_0(\frac{x\nu}{\sigma^2})$,
  - nu = magnitude of complex signal (uncorrupted)
    - Not E [X]
  - sigma = std. dev. of Gaussian in Re/Im part
    - Not SD (X)
  - p(x) defined only for non-negative x
  - PDF mass shifted to right of nu

Rician Distribution (PDF)



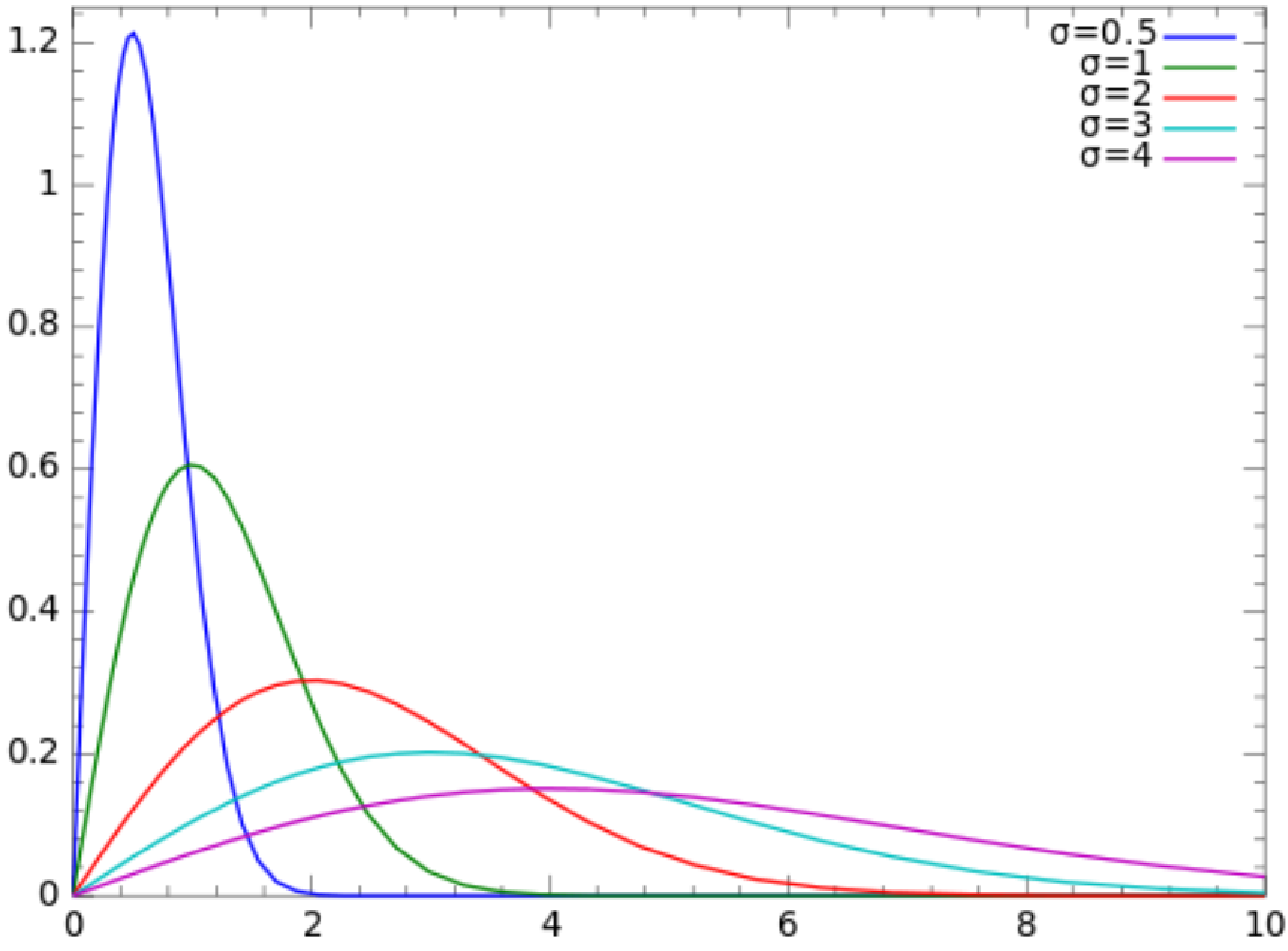
Rician Distribution (CDF)



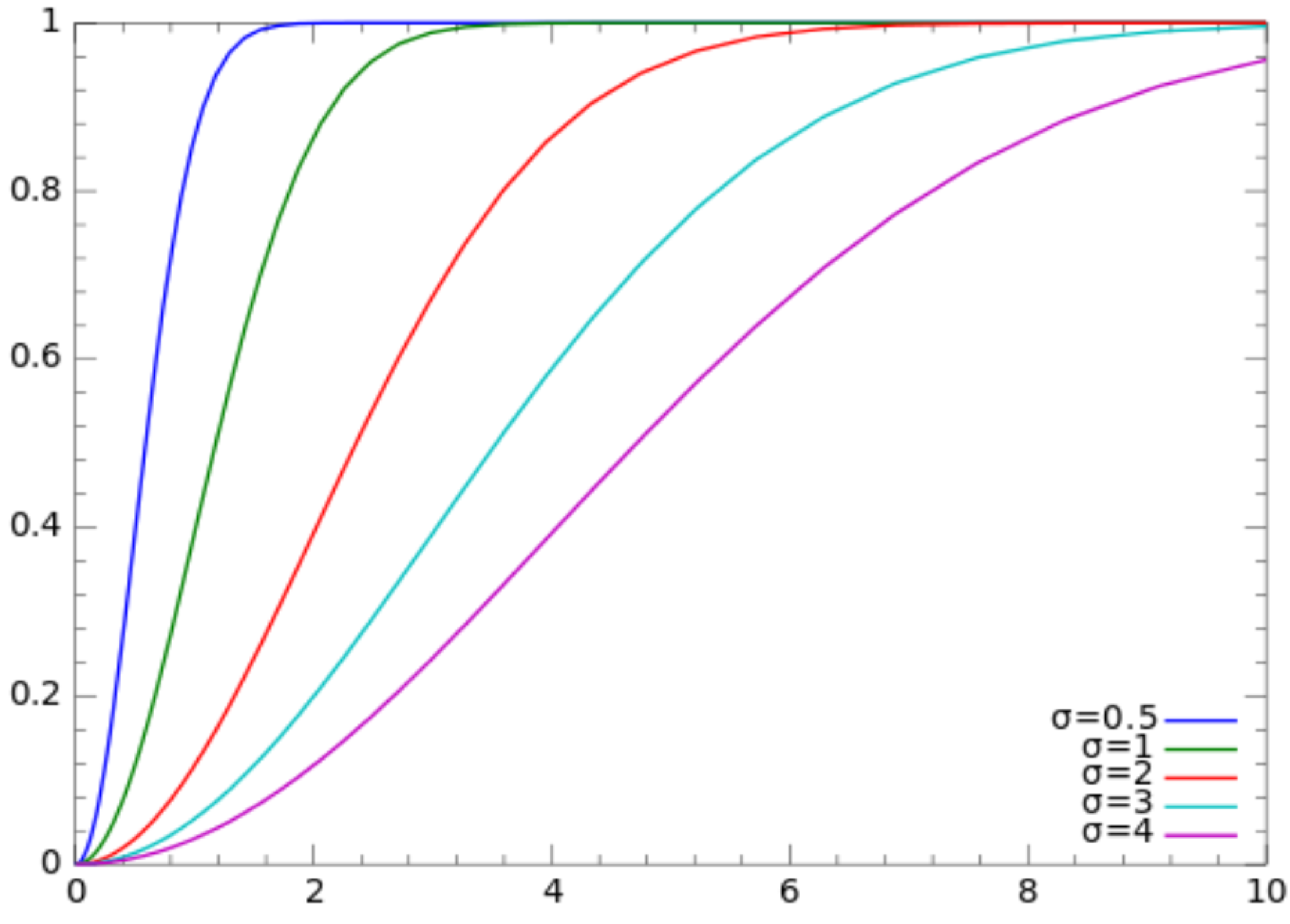
Noise Models in Medical Imaging

- Rician PDF
  - Special cases :
    - nu = 0
      - Rician PDF reduces to the Rayleigh PDF $p(x;\sigma) = \frac{x}{\sigma^2}\exp(-\frac{x^2}{2\sigma^2})$
    - nu >> sigma
      - Rician PDF tends to a Gaussian PDF $G(x;\nu,\sigma^2)$

Rayleigh Distribution (PDF)



Rayleigh Distribution (CDF)



Noise Models in Medical Imaging

- Ultrasound
  - Complicated to characterize because depends on
    - Device type, e.g., coherent or incoherent waves
    - Scattering density of tissue
  - Variety of distributions have been used for magnitude images
    - Rician, Raylegh, K

Noise Models in Medical Imaging

- Ultrasound
  - One of the currently used models
    - $X = \mu + \sqrt{\mu} Y$
    - X = observed intensity
    - mu = true intensity
    - Y = Gaussian with mean 0, std. dev. Sigma
    - P (X) = Gaussian with
      - Mean : E [X] = mu
      - Variance : Var [X] = mu * Var (Y)

Denoising Scanner Data

- Denoising by averaging multiple acquisitions
  - Assumption
    - E [Data | TrueSignal] = TrueSignal
    - Valid for Gaussian, Poisson, ultrasound model
    - Invalid for Rician, Rayleigh
  - Motivation, Strategy
    - Interpreting expectation as the average value obtained after performing a large number of experiments
  - Strategy
    - Image multiple times
    - Average scanner data

Denoising Scanner Data

- Practical Analysis
  - If we repeat the imaging process N times, then how close is the average to the (noiseless) signal value ?
  - Let X1 , · · · , XN be N RVs
    - i.i.d., mean “m”, std. dev. “s”
  - Estimator = sample mean = $X := (1/N) \sum_{n=1}^{N} X_n$

Denoising Scanner Data

- What is expectation of sample mean X ?
  - E [X] = m
- What is the variance of sample mean X ?
  - SD (X) = s / sqrt (N)
- If we perform an infinite number of experiments, then SD (X) = 0 and E [X] = m
  - Sample mean X converges to noiseless signal value !

Denoising Scanner Data

- Limitations of averaging method
  - Works only when noise PDF is centered at true signal value
    - What happens in case of Rician noise ? Nu = 0 ?
  - More general methods are required ...

Denoising Scanner Data

- Maximum-Likelihood (ML) estimation
  - Likelihood function
    - Function of model parameters and data
    - Gives the likelihood that the data was generated from a specific model (parameter values)
    - Consider model P (**x** | theta)
      - **x** = vector of observed data values
      - theta = parameter values
    - Likelihood function = L (theta | **x**) = P (**x** | theta)
    - Example
      - Assume, N repeated independent measurements from MRI
      - $L(\theta|x_1, \cdots, x_N) = P(x_1, \cdots, x_N|\mu, \sigma) = \Pi_n G(x_n; \mu, \sigma)$

Denoising Scanner Data

- Maximum-Likelihood (ML) estimation
  - Example: Gaussian-distributed data
    - Estimate parameters theta by maximizing likelihood function
      - Or maximizing logarithm of likelihood function
    - What is ML estimate for mu ?
      - Sample mean
    - What is ML estimate for sigma ?
      - Sample variance

Denoising Scanner Data

- Maximum-Likelihood (ML) estimation
  - Example : Poisson-distributed data
    - Poisson PMF: $P(x;\lambda) = \frac{\lambda^x \exp(-\lambda)}{x!}$
      $L(\lambda|x_1,\cdots,x_N) = \Pi_n P(x;\lambda)$
      ML estimate for λ equals the sample mean = $(1/N)\sum_n x_n$

Denoising Scanner Data

- Maximum-Likelihood (ML) estimation
  - Example: Rayleigh-distributed data
    - Consider MR image (intensity = magnitude)
    - Consider pixels with background / air
    - We can use ML estimation to estimate noise level
    - Rayleigh PDF: $p(x;\sigma) = \frac{x}{\sigma^2} \exp(-\frac{x^2}{2\sigma^2})$
      $L(\sigma | x_1, \cdots, x_N) = \Pi_n P(x;\sigma)$
      ML estimate for σ^2 equals $(1/2) \sum_n x_n^2 / N$

Denoising Scanner Data

- Maximum-Likelihood (ML) estimation
  - Example : Rician-distributed data
    - Consider MR image (intensity = magnitude)
    - Consider noise level = sigma is known
    - Rician PDF: $p(x|\nu,\sigma) = \frac{x}{\sigma^2}\exp(-\frac{x^2+\nu^2}{2\sigma^2})I_0(\frac{x\nu}{\sigma^2})$
    - $L(\nu,\sigma|x_1,\cdots,x_N) = \Pi_n P(x|\nu,\sigma)$
    - ML estimation for nu requires iterative optimization
      - e.g., gradient-descent optimization

Denoising Scanner Data

- Limitations of denoising by ML estimation
  - What if we can’t make repeated measurements ?
    - Expensive ?
    - Time consuming ?
  - We can compensate for lack of data by using “prior” information about what we want to estimate
    - In our case, we want to estimate the noiseless image
    - We know that intensities vary smoothly over space
  - How to use prior information for denoising ?