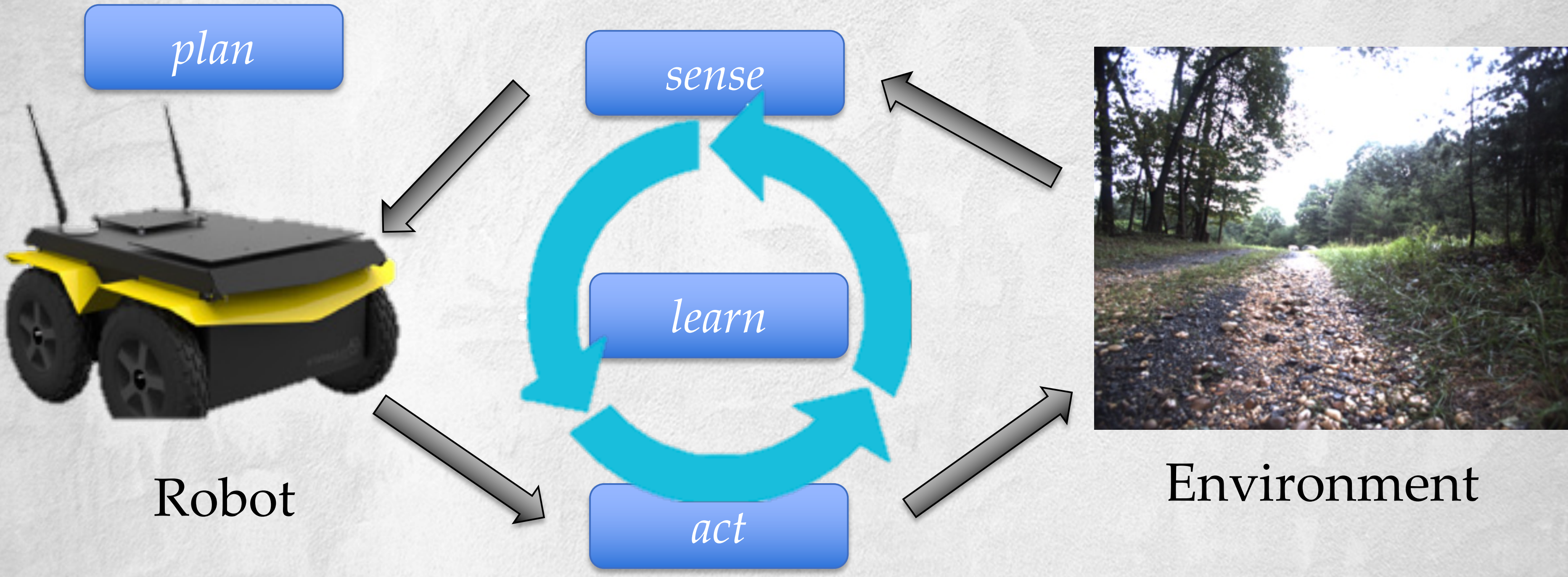
AI Robotics

Probability and Statistics

Overview

Robot Model



Robot Model

Sensor Measurements:
$Z_{0:t} = \{z_0, \ldots, z_t\}$

Robot State (or pose):
$X_{0:t} = \{X_0, \ldots, X_t\}$

sense

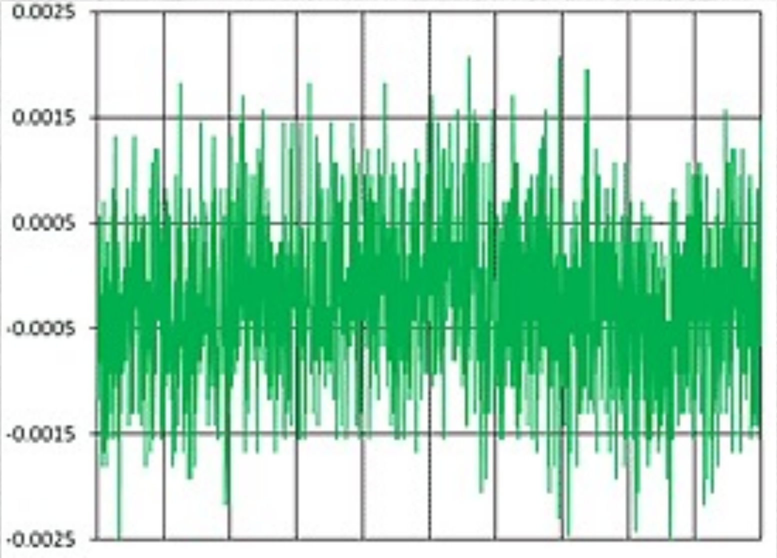
act

Landmark or Map:
$m = \{m_1, \ldots, m_n\}$

Robot Controls:
$u_{0:t} = \{u_0, u_1, \ldots, u_t\}$

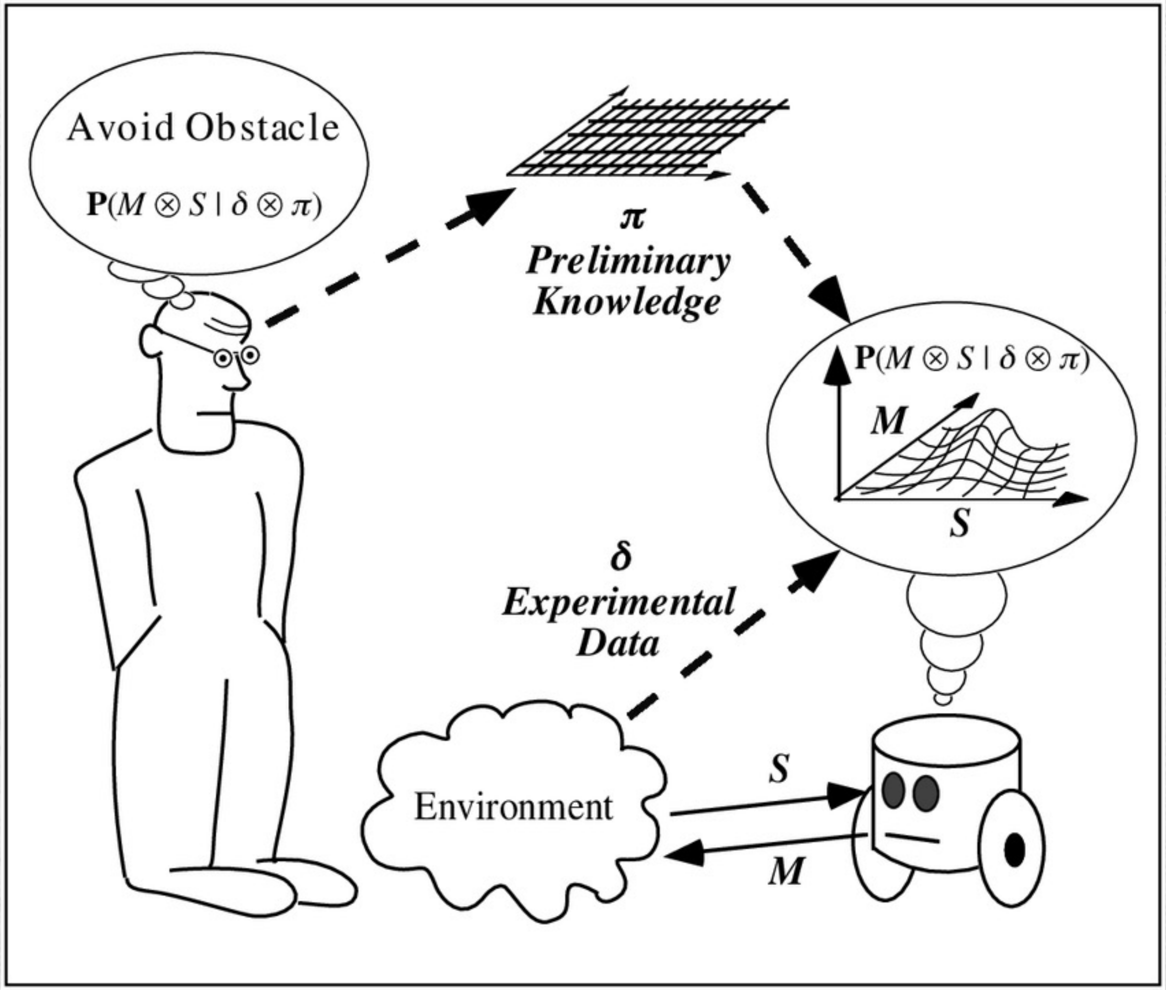
Uncertainty

Sensor uncertainty: sensor measurements are noisy

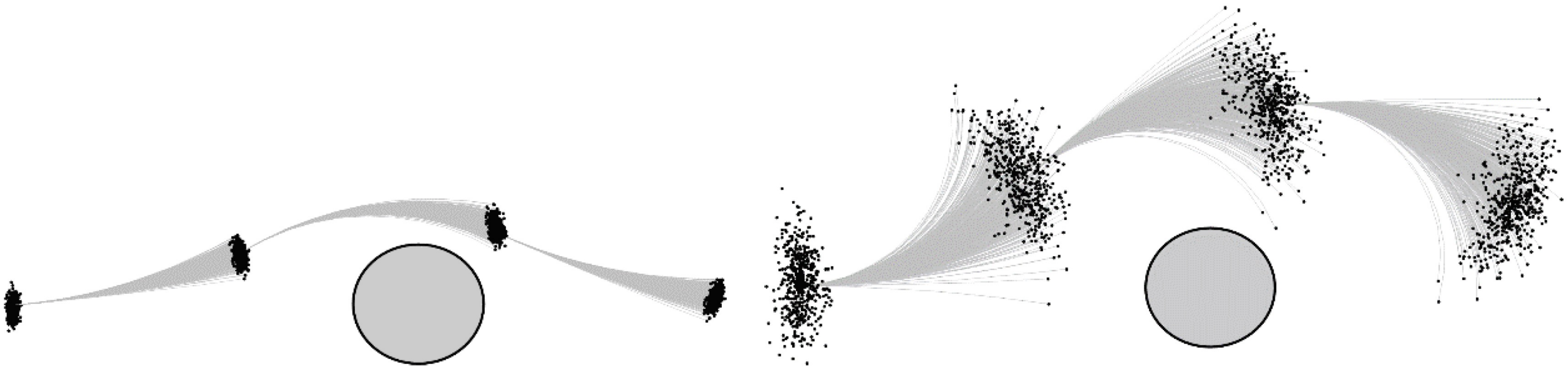


Motion uncertainty: robot control is noisy due to vibration, external forces

Probabilistic Robotics



Probabilistic Robotics



(**a**)Avoiding obstacle with low uncertainty (**b**)Avoiding obstacle with high uncertainty

Basic Probability Concepts

Mathematical Foundations

1. Random variables
   a. Discrete variables
   b. Continuous variables
2. Expectation and variance
3. Sum rule
4. Gaussian distributions
5. Joint distributions
   a. Multivariate Gaussian
6. Independence and Conditional independence

Random Variables

- Discrete variables: has a countable number of possible values
  - Flipping a coin
  - Rolling dice
  - Classifying the type of an obstacle
- Continuous variables: can take on any of the values in some interval
  - Position of a robot
  - Position of an obstacle
- Random numbers can be generated in Python using *numpy.random.random* or *numpy.random.randint*

Probability Function

- Expresses the probability that a random variable takes on a certain value

Discrete variables

$P(\text{coin} = \text{heads}) = 0.5$

$P(\text{dice} = 1) = 0.167$

$P(\text{obstacle} = \text{car}) = 0.3$

Continuous variables

$P(x_{robot} > 1) = 0.2$

$P(y_{obstacle} < 10) = 0.8$

Sum Rule

- The probability of <u>mutually–exclusive</u> events is the sum of probabilities of those events
  - P(dice = 1 or dice = 2) =

    = P(dice = 1) + P(dice = 2)

    = 0.167 + 0.167 = 0.333

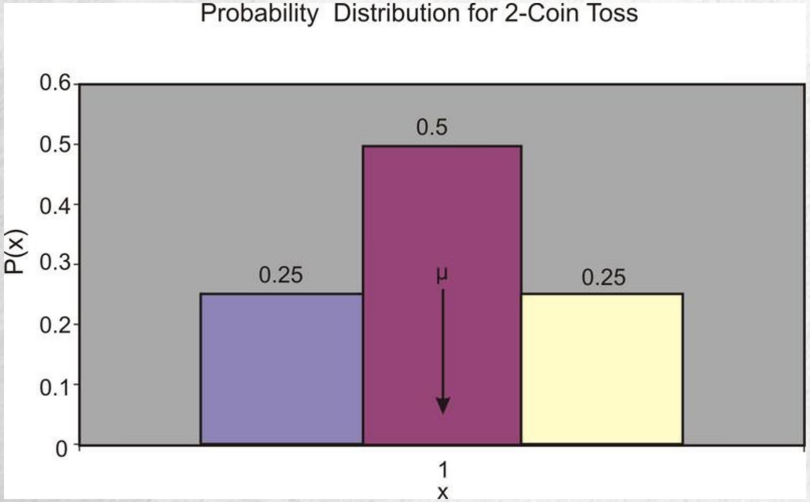
P (A or B) = P(A) + P(B)

- The probabilities of all possible values of a random variable should sum to one.
  - P(coin = heads) + P(coin = tails) = 1

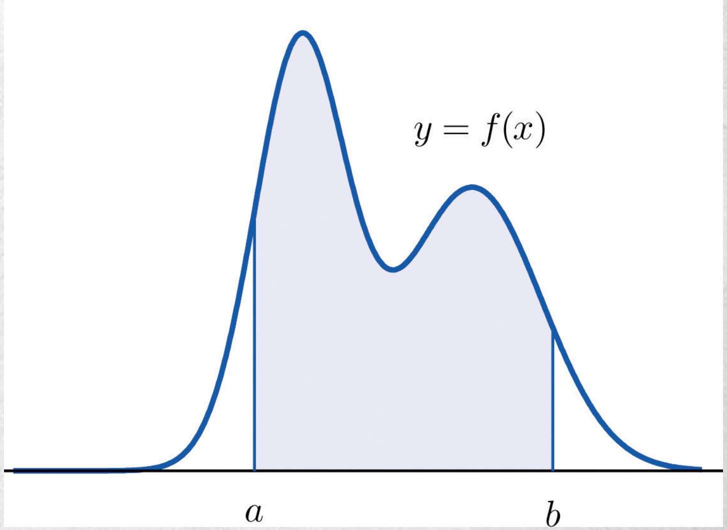
Probability Distribution

- Expresses the different probabilities of different possible values of a random variable

Discrete variables

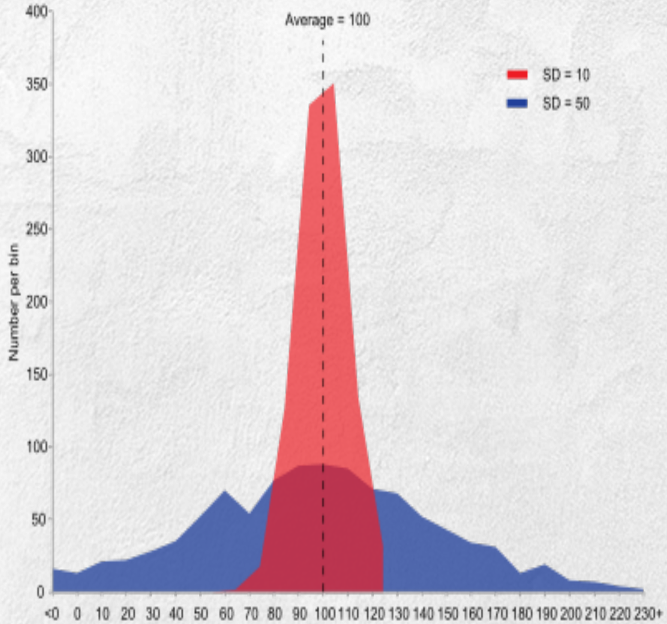


Continuous variables



Expectation & Variance

- **Expectation**: The mean / average / expected value of a random variable
- **Variance**: The dispersion / how far the distribution of values are spread out from the average
- **Standard deviation**: The square-root of the variance



Expectation & Variance

	Discrete Random Variable	Continuous Random Variable
Mean (Expected Value)	$\mu = E(X) = \sum_{i=1}^{n} xf(x)$	$\mu = E(X) = \int_{-\infty}^{\infty} xf(x)dx$
Variance	$\sigma^2 = V(X) = \sum_{i=1}^{n} (x-\mu)^2 f(x)$	$\sigma^2 = V(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$

Expectation & Variance

Expected value of rolling a 6-sided dice

$$E(\text{dice}) = 1(0.167) + 2(0.167) + 3(0.167) + 4(0.167) + 5(0.167) + 6(0.167) = 3.5$$

Expected #heads when flipping 3 coins

$$E(\text{heads}) = 0(0.125) + 1(0.375) + 2(0.375) + 3(0.125) = 1.5$$

Expectation & Variance

Variance of rolling a 6-sided dice

$$V(\text{dice}) = (1-3.5)^2(0.167) + (2-3.5)^2(0.167) + (3-3.5)^2(0.167)$$
$$+ (4-3.5)^2(0.167) + (5-3.5)^2(0.167) + (6-3.5)^2(0.167)$$
$$= 2.92$$

Variance of rolling a 20-sided dice

$$V(\text{dice}) = (1-10.5)^2(0.05) + (2-10.5)^2(0.05) + (3-10.5)^2(0.05)$$
$$+ \ldots \ldots$$
$$= 33.25$$

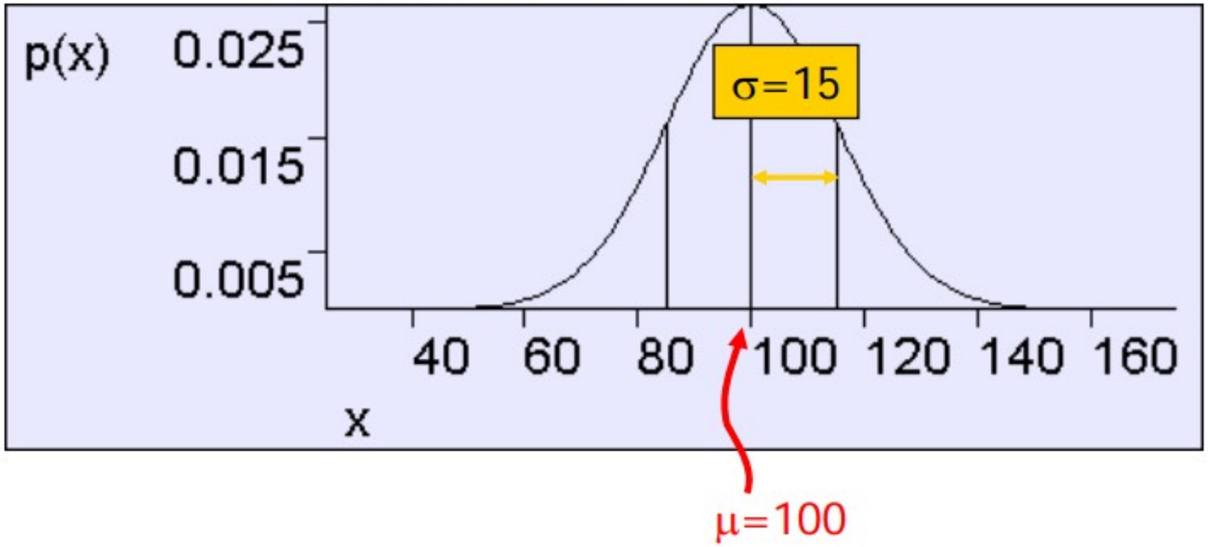
Gaussian Distribution

- Also known as the normal distribution / bell curve

Example: *x* position of a robot

General Gaussian

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2} \right)$$



$$E[X] = \mu$$

$$\mathrm{Var}[X] = \sigma^2$$

Joint Distributions

- Probabilities of combinations of <u>multiple</u> random variables
- **Expectation / mean vector**: The mean / average / expected values of the random variables

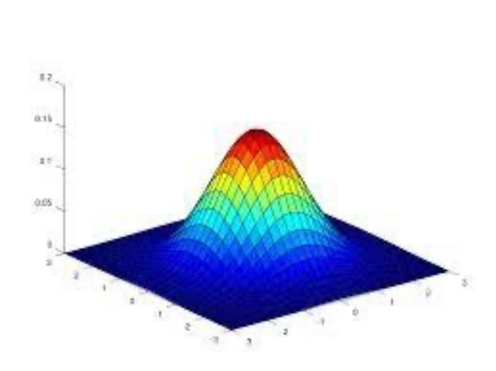
$$E[\boldsymbol{X}] = \sum \boldsymbol{X}\, f(\boldsymbol{X})$$

- **Covariance matrix:** Measure of the joint variability between multiple random variables

$$\begin{aligned}\operatorname{cov}(\mathbf{X},\mathbf{X}) &= \operatorname{E}\left[(\mathbf{X}-\operatorname{E}[\mathbf{X}])(\mathbf{X}-\operatorname{E}[\mathbf{X}])^{\mathrm{T}}\right] \\ &= \operatorname{E}\left[\mathbf{X}\mathbf{X}^{\mathrm{T}}\right] - \operatorname{E}[\mathbf{X}]\operatorname{E}[\mathbf{X}]^{\mathrm{T}}.\end{aligned}$$

Multivariate Gaussian

Example: (x,y,z) position of a robot



Write r.v. $\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_m \end{pmatrix}$ Then define $X \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ to mean

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{m/2} \| \boldsymbol{\Sigma} \|^{1/2}} \exp\left(-\tfrac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right)$$

Where the Gaussian's parameters have...

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_m \end{pmatrix} \qquad \boldsymbol{\Sigma} = \begin{pmatrix} \sigma^2{}_1 & \sigma_{12} & \cdots & \sigma_{1m} \\ \sigma_{12} & \sigma^2{}_2 & \cdots & \sigma_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1m} & \sigma_{2m} & \cdots & \sigma^2{}_m \end{pmatrix}$$

Independence

- If the occurrence of A has no effect on B, then A and B are independent
  - E.g. Rolling multiple dice, picking a random card from a deck multiple times after replacing it

- The probability of <u>independent</u> events is the product of probabilities of those events
  - $P(\text{coin}_1 = \text{heads and coin}_2 = \text{heads}) =$

    $= P(\text{coin}_1 = \text{heads}) + P(\text{coin}_2 = \text{heads})$

    $= 0.5\ (0.5) = 0.25$

$P(A, B) = P\ (A \text{ and } B) = P(A)\ P(B)$

Conditional Probability

- Probability of a random variable, A, given that another random variable, B, is known

$$P(A\,|B) = \frac{P(A,B)}{P(B)}$$

$P(Crash) = 10\%$ $\quad P(Speeding) = 13\%$

$P(Crash, Speeding) = 9\%$

$$P(Crash\,|\,Speeding) = \frac{P(Crash, Speeding)}{P(Speeding)}$$

$= 69\%$

Independence & Conditional Independence

- If A and B are independent, then

$$P(A \mid B) = P(A) \qquad P(B \mid A) = P(B)$$

- If A and B are conditionally-independent given C, then

$$P(A, B \mid C) = P(A \mid C)\, P(B \mid C)$$

$$P(A \mid B, C) = P(A \mid C)$$

$$P(B \mid A, C) = P(B \mid C)$$

Independence & Conditional Independence

- Independence relations are important in probabilistic robotics because it characterizes the effect of random variables on other variables
- If A and B are <u>conditionally-independent</u> given C, that means that B carries no further information about A if C is known

e.g. it can be assumed that the robot state x is sufficient to predict the (potentially noisy) measurement z. Knowledge of any other variable, such as past measurements, controls or even past states, is irrelevant if x is completely known

Bayes Rule

- Can be directly derived from the equations for joint and conditional probabilities
- Usually used to update the probability of a belief given prior knowledge and a related piece of evidence

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Law of Total Probability

$$P(A) = \sum_i P(A, B = b_i)$$

$$= \sum_i P(A \mid B = b_i)\, P(B = b_i)$$

- The individual probability of a random variable, A, can be obtained by summing up the probabilities of it occurring together with all possible values of another random variable, B
- P(A) is also known as the <u>marginal probability</u>

Law of Total Probability

Example:

- A robot has a 5% probability of being manufactured with a defect
- A non-defective robot has a 90% probability of successfully navigating to the goal
- A defective robot has a 50% probability of successfully navigating to the goal
- What is the overall / total probability of success?

$$P(Success) = P(Success \mid Defect)\, P(Defect) + P(Success \mid Nondefect)\, P(Nondefect)$$

$$= 0.5\,(0.05) + 0.9(0.95) = 0.88$$

Practice Questions

Problem 1

- Assume that we have 2 fair coins. What is the probability that flipping the second coin results in heads given that flipping the first coin results in heads?

Problem 2

- The probability that a robot is at position $x=0$ is 0.2
- The probability that a robot is at position $x=1$ is 0.3
- The probability that a robot is at position $x=2$ is 0.5
- What is the expected value of the robot's position?

Problem 3

- A robot navigates randomly between two locations, an office and a hallway
- The overall probability of the robot being located in the office is 50%
- When in the office, the robot has a 80% probability of detecting a person
- When in the hallway, the robot has a 20% probability of detecting a person
- Given that the robot detected a person, what is the probability that the robot is located in the office?

References

1. https://www.cc.gatech.edu/~bboots3/STR-Spring2018/readings/Gaussians.pdf
2. https://bayes.wustl.edu/etj/prob/book.pdf
3. https://dellaert.github.io/21S-3630/notes/N3_Probabilistic_Actions.pdf