

UNDERSTANDING HANK: INSIGHTS FROM A PRANK*

Sushant Acharya[†]

Keshav Dogra[‡]

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Abstract

Using an analytically tractable heterogeneous agent New Keynesian model, we show that whether incomplete markets resolve New Keynesian ‘paradoxes’ depends on the cyclical risk. Incomplete markets reduce the effectiveness of forward guidance and multipliers in a liquidity trap only with procyclical risk. Countercyclical risk amplifies these ‘puzzles’. Procyclical risk permits determinacy under a peg; countercyclical risk may generate indeterminacy even under the Taylor principle. By affecting the cyclical risk, even ‘passive’ fiscal policy influences the effects of monetary policy.

Keywords: New Keynesian, incomplete markets, monetary and fiscal policy, determinacy, forward guidance, fiscal multipliers

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[†]Federal Reserve Bank of New York **Email:** sushant.acharya@ny.frb.org

[‡]Federal Reserve Bank of New York **Email:** keshav.dogra@ny.frb.org

Heterogeneity and market incompleteness have been proposed as a means to understand the monetary transmission mechanism (Kaplan et al., 2018), the forward guidance puzzle (McKay et al., 2016), the distributional effects of monetary policy (Gornemann et al., 2016), the efficacy of targeted transfers (Oh and Reis, 2012), automatic stabilizers (McKay and Reis, 2016b), among many other topics. These explorations have revealed that the introduction of market incompleteness into models with nominal rigidities can affect not just their substantive predictions, but also the determinacy properties of equilibrium (Ravn and Sterk, 2018; Auclert et al., 2018; Bilbiie, 2019a), which are central to fundamental questions in monetary economics - how is the price level determined, and what kind of policy regime ensures price stability? But the source of the differences between heterogeneous agent New Keynesian (HANK) and representative agent New Keynesian (RANK) economies, and the extent to which these differences are a general result rather than a consequence of particular modeling assumptions, remain obscure. This is because incomplete market models are generally analytically intractable since the wealth distribution is an infinite dimensional state variable; thus, most of these studies make use of computational methods. While these papers have highlighted striking differences in the behavior of HANK and RANK economies, the lack of analytical tractability makes it hard to pinpoint exactly which features are responsible for these differences.

In particular, the HANK literature emphasizes two broad differences relative to RANK: (i) precautionary savings and cyclical uninsurable risk; (ii) marginal propensity to consume (MPC) heterogeneity and the sensitivity of high-MPC households' income to the business cycle. But the precise role of each set of factors remains unclear. For example, Werning (2015) argues that the cyclical risk is the main factor affecting whether and how HANK differs from RANK, while Bilbiie (2019a) argues the cyclical risk of income for high MPC individuals is key to understanding these differences. Our goal is to isolate and understand the distinct effects of precautionary savings and the cyclical risk. To do so, we study an analytical HANK economy in which MPC heterogeneity is wholly absent - thus *any* differences relative to RANK are unambiguously due to precautionary savings and cyclical uninsurable risk.

Our environment is a standard NK economy, with the exception that individuals face idiosyncratic, uninsurable shocks to their endowment of labor. In our environment, idiosyncratic income risk varies endogenously with aggregate economic activity, and may be either procyclical or countercyclical. The economy permits closed-form solutions because household utility has constant absolute risk aversion (CARA) and income shocks are normally distributed. All households have the same (time-varying) MPC, which permits linear aggregation and gives us an exact closed-form aggregate Euler equation, without having to carry around an infinite-dimensional state variable or impose a degenerate wealth distribution. Thus, one can think of it as a Pseudo-Representative Agent New-Keynesian model - in short, a PRANK. One might worry that abstracting from MPC heterogeneity to obtain an analytical HANK throws the baby out with the bathwater. Again though, since our goal is to understand the role of precautionary savings and risk, it is an advantage of our approach, not a weakness, that we can study these latter features in isolation.

Even absent MPC heterogeneity, uninsurable income risk can alter the properties of a NK economy, in a way which depends on the cyclical risk of income. Firstly, market incompleteness can alter the determinacy properties of equilibrium. RANK models feature indeterminacy under an interest rate peg, or more generally, under interest rate rules which fail to satisfy the Taylor principle. HANK models can feature determinacy under a peg - but only when income risk is procyclical. When risk is procyclical, higher future output and inflation cannot be self-fulfilling - even under a peg - because it would also imply

higher income risk, reducing demand via the precautionary savings channel. Whereas procyclical income risk makes indeterminacy less likely, countercyclical risk makes it more likely - if risk is countercyclical, the standard Taylor principle may not even be sufficient to ensure determinacy. In this case, fear of lower output in the future implies higher risk, depressing demand via the precautionary savings channel and generating a self-fulfilling recession. This clarifies that [Ravn and Sterk \(2018\)](#)’s finding that incomplete markets make determinacy less likely is driven by the fact that risk is countercyclical in their model. We also derive an income-risk augmented Taylor principle which depends explicitly on the cyclicity of risk.

This cyclicity of income risk is endogenous. In particular, it depends on the cyclicity of fiscal policy, and on whether redistribution increases or decreases when output is low. This highlights a new dimension of monetary-fiscal interaction, distinct from (but related to) the traditional question concerning whether the fiscal authority adjusts surpluses in order to repay government debt along any hypothetical price path ([Leeper, 1991](#)). In HANK economies, what matters is not just the expected path of surpluses, but whether those surpluses are raised in ways that increase or decrease the variance of households’ after-tax income, and whether this depends on the overall level of economic activity.

Next we discuss how the cyclicity of risk affects whether incomplete markets can solve the “puzzles” arising in RANK models which have been the subject of much consternation in the recent literature. For example, in RANK, announcements of future interest rate cuts are equally, or more, effective than current policy changes in stimulating output and inflation. Market incompleteness can reverse this prediction, but only if income risk is strongly procyclical, so the expansionary effect of a promised future boom is offset by an increase in desired precautionary savings in response to the increased risk generated by the boom. If risk is countercyclical, this prediction is reversed, and incomplete markets *worsen* the ‘forward guidance puzzle’ (FGP). Interestingly, HANK models may feature a stronger forward guidance puzzle than RANK even if income risk is acyclical or weakly procyclical. Looser monetary policy effectively provides more consumption insurance against income shocks, reducing consumption risk (which is ultimately what matters for precautionary savings) for a given level of income risk, and boosting demand.

Next, we turn to another controversial prediction of RANK. RANK predicts that in a liquidity trap, the government spending multiplier is greater than 1 and increasing in the duration of the trap.¹ This is due to the expected inflation channel: when nominal interest rates are constrained due to the zero bound, higher future spending increases expected inflation, lowers real rates, and stimulates current spending. If income risk is procyclical, the precautionary savings channel can potentially outweigh the effect of expected inflation in our HANK economy. While future spending lowers real rates, it also increases risk, encourages households to save, and moderates the increase in current spending. Consequently, the multiplier can be *less* than 1 and *decreasing* in the duration of the liquidity trap. In contrast, if risk is countercyclical, precautionary savings and expected inflation both work in the same direction, increasing the multiplier.

These results reveal that, even absent MPC heterogeneity, the precautionary savings motive can change outcomes in a NK economy - albeit in a way which depends on the cyclicity of income risk. In [Section 5](#) and [Appendix F](#), we introduce MPC heterogeneity by adding hand-to-mouth (HTM) agents à la [Bilbiie \(2008\)](#), and show that this only affects the magnitude of the deviation from RANK, not the sign: whether

¹This prediction depends on the assumption that the monetary authority targets the zero-output, zero-inflation steady state (using an appropriately specified active Taylor rule) as soon as the underlying shocks abate and the ZLB is no longer binding. We maintain this assumption throughout our analysis. [Cochrane \(2017\)](#) discusses how fiscal multipliers change when this assumption is dropped and alternative criteria are used to select among the many bounded rational expectations equilibria consistent with a given path of nominal interest rates.

determinacy is more or less likely than in RANK, and whether puzzles are amplified or resolved, depends only on the cyclical risk.² This by no means implies that MPC heterogeneity is irrelevant; as a large literature has emphasized, it can change the contemporaneous sensitivity of output to interest rates (Bilbiie, 2008), the effectiveness of deficit financed fiscal expansions (Bilbiie et al., 2013), the transmission mechanism of monetary policy (Auclert, 2019; Kaplan et al., 2018) and so forth. Further, the fact that MPC heterogeneity and precautionary savings have different effects does not mean that they are unrelated - in a quantitative model, both features will be generated by the same structural features of the environment. For example, higher uninsurable risk or tighter liquidity constraints can increase both desired precautionary savings and MPCs (Carroll et al., 2019). But isolating precautionary savings and the cyclical risk is still useful to uncover the distinct *effects* of these factors on aggregate outcomes. By studying precautionary savings and the cyclical risk in isolation, our analysis clarifies that these features are sufficient to change determinacy and whether puzzles will emerge.

Related Literature The HANK literature has taken two broad approaches to understand how heterogeneity and market incompleteness change aggregate outcomes in monetary economies. The *quantitative* HANK literature answers this question by building rich models disciplined by micro and macro data. The *analytical* HANK literature, to which our paper contributes, makes simplifying assumptions to cleanly identify the forces at work in richer quantitative models. In particular, as emphasized above, this literature focuses on (i) precautionary savings and the cyclical risk of income and (ii) MPC heterogeneity, neither of which is present in RANK models. The two agent New Keynesian (TANK) models (Bilbiie, 2008; Debortoli and Galí, 2018) focus on MPC heterogeneity by assuming away uninsurable income risk. We do the opposite, abstracting from MPC heterogeneity to focus on precautionary savings.

Another branch of the analytical HANK literature (Ravn and Sterk, 2018; Bilbiie, 2019a,b; Werning, 2015; Challe, forthcoming) allows for a precautionary saving motive by using the simplifying assumption that agents are unable to borrow and the government issues no debt - the so called *zero liquidity limit*.³ This makes the wealth distribution degenerate, affording analytical tractability. While a useful simplifying assumption, this has the strong implication that income risk passes through one-for-one to consumption risk. In reality, households can partially insure consumption against income shocks through various mechanisms (Blundell et al., 2008; Heathcote et al., 2014); thus the pass-through from income to consumption risk is less than one, and may vary over time. Our approach does not impose zero liquidity, and allows for endogenous, time-varying pass-through of income to consumption risk. This component of the precautionary savings channel is already implicitly present in quantitative HANK models; the advantage of our approach is that we can observe it analytically.

Another difference, relative to Werning (2015), is that we discuss how the cyclical risk of income

²This characterization is true provided that there are not too many HTM agents or their income is not too procyclical. Introducing many HTM agents with very procyclical income can make the consumption of unconstrained agents co-move negatively with GDP. In this case, a fall in real interest rates, by increasing unconstrained agents' consumption, would decrease GDP. Naturally, determinacy and other outcomes are radically different in such an economy. While we rule out this case in our analysis, Bilbiie (2008) discusses it at length; see also Galí et al. (2004); Broer et al. (2019).

³Werning (2015) and Bilbiie (2019a) also consider models which relax the zero liquidity limit. In older work, Challe and Ragot (2011) and Challe et al. (2017) make assumptions on preferences, technology and market structure in order to construct analytically tractable *limited heterogeneity equilibria* in which the wealth distribution has finite support. These papers primarily study how the precautionary savings channel can amplify aggregate shocks, which is related to, but distinct from, the themes we address in this paper.

affects determinacy, not just the equilibrium response of consumption to interest rates. In prior work, [Bilbiie \(2019a\)](#) also discusses determinacy and presents a modified Taylor principle which, like ours, depends on the presence of compounding/discounting in the Euler equation.⁴ In his environment, compounding or discounting can arise provided that agents face some risk of becoming liquidity constrained and the sensitivity of their income to aggregate income while they are constrained differs from 1. Our model makes it clear that the cyclical risk can generate discounting or compounding in the Euler equation and affect determinacy even without HTM agents, liquidity constraints and so forth. Our determinacy results are also related to [Auclert et al. \(2018\)](#), who also analyze how incomplete markets affects determinacy and the economy’s response to increases in government spending, monetary policy shocks and forward guidance.⁵ Their analytical results are framed in terms of an infinite dimensional \mathbf{M} matrix which describes the response of consumption at any date to aggregate output at any other date; for example, they show that determinacy depends on the asymptotic properties of the far-out columns of this matrix. They also present numerical results which generally confirm the results in our closed-form solutions (procyclical risk permits determinacy under a peg; countercyclical risk makes determinacy less likely). The assumption of CARA utility allows us to analyze determinacy and the economy’s response to shocks in a transparent model permitting closed-form solutions.

[McKay et al. \(2016\)](#) argued that incomplete markets solve the “forward guidance puzzle” ([Del Negro et al., 2015](#)), i.e. the fact that in NK models, announcements of interest rate cuts far in the future are more effective at stimulating output and inflation than contemporaneous cuts. [McKay et al. \(2017\)](#) present a stylized incomplete markets model, again with zero liquidity, in which household consumption is described by a ‘discounted Euler equation’. We also derive a modified Euler equation in our framework (which does not rely on zero liquidity) and describe conditions under which forward guidance is less effective than in RANK. The model only generates a *discounted* Euler equation and weakens the power of forward guidance if income risk is sufficiently procyclical (as in [McKay et al. \(2017\)](#)). If instead risk is countercyclical, the model generates an *explosive* Euler equation, exacerbating the forward guidance puzzle.

The rest of the paper is structured as follows. Section 1 presents the model economy. Section 2 characterizes equilibrium. Section 3 shows how the cyclical risk affects determinacy in our HANK economy and derives a risk-adjusted Taylor principle. Section 4 discusses conditions under which market incompleteness and in particular the cyclical risk solves, or amplifies, two perceived ‘puzzles’ present in RANK: the power of forward guidance, and explosive government spending multipliers in a liquidity trap. Section 5 briefly discusses how MPC heterogeneity affects our results and Section 6 concludes.

1 Model

We introduce uninsurable income risk into an otherwise standard NK model. Households face idiosyncratic income risk and can only save in a nominally riskless bond. For simplicity, we consider an economy with idiosyncratic risk but no aggregate uncertainty.

⁴Bilbiie’s modified Taylor principle first appeared in the June 2017 version of [Bilbiie \(2019b\)](#) and subsequently in the December 2017 version of [Bilbiie \(2019a\)](#), then titled “*A Catch-22 for HANK Models: No Puzzles, No Amplification*”.

⁵Indeed, this paper began life as a discussion of the 2017 version of [Auclert et al. \(2018\)](#).

1.1 Households

There is a continuum of households in the economy indexed by $i \in [0, 1]$ who solve:

$$\begin{aligned} \max_{\{A_{t+1}^i, c_t^i\}_{t=0}^{\infty}} \quad & -\frac{1}{\gamma} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t e^{-\gamma c_t^i} \\ \text{subject to} \quad & P_t c_t^i + \frac{1}{1+i_t} A_{t+1}^i = A_t^i + P_t y_t^i \end{aligned}$$

Each household can save only in a risk free nominal bond A_{t+1}^i which has a price of $(1+i_t)^{-1}$ at date t and pays off 1 in nominal terms at $t+1$. c_t^i denotes consumption of the the final good. y_t^i denotes the income of household i in period t and can be written as:

$$y_t^i = (1 - \tau_t) \omega_t \ell_t^i + d_t^i + \frac{T_t}{P_t} \quad (1)$$

Each household's income is made up of three components: real labor income net of taxes $(1 - \tau_t) \omega_t \ell_t^i$, real dividends from the production sector, d_t^i and real transfers from the government, T_t/P_t , as we discuss next.

Following [Aiyagari \(1994\)](#), households have a stochastic endowment of labor ℓ_t^i each period which they supply inelastically at the prevailing real wage ω_t . Each period, household i 's endowment of labor is given by $\ell_t^i \sim N(\ell, \sigma_{\ell,t}^2)$ where ℓ is the aggregate endowment of labor in this economy. We normalize $\ell = 1$. τ_t denotes the linear tax on labor income. In particular we assume that $\sigma_{\ell,t}^2 = \sigma_{\ell}^2(y_t)$ where y_t denotes aggregate output. As in [McKay and Reis \(2016a\)](#), this specification allows for cyclical changes in the distribution of earnings risks in line with the empirical evidence documented by [Storesletten et al. \(2004\)](#).⁶

Each household also receives dividends $d_t^i = d_t + \delta_t (\ell_t^i - 1)$ from the productive sector, which may vary across households. The distribution of dividends is an important determinant of how an incomplete markets economy responds to various shocks ([Bilbiie, 2008](#); [Broer et al., 2019](#); [Werning, 2015](#)). This convenient specification nests many commonly used cases. For example, $\delta = 0$ implies that dividends are distributed equally across all households; $\delta > 0$ implies that households with larger labor income receive more dividends. We also allow δ_t to vary with economic activity: $\delta_t = \delta(y_t)$.

The last source of income is lump sum transfers net of taxes. The government makes a lump sum transfer T_t/P_t which is the same across all households in each period, and taxes labor income at the rate τ_t . For future reference, $\bar{y}_t = \int_0^1 y_t^i di$ denotes average household income. Given our assumptions, individual income y_t^i is normally distributed with mean \bar{y}_t and variance $\sigma_{y,t}^2 = [(1 - \tau_t)\omega_t + \delta_t]^2 \sigma_{\ell,t}^2$. The assumption that risk is i.i.d. is not essential and is made only for tractability. Appendix [E](#) shows that all our results are qualitatively unchanged if we assume that individual income is autocorrelated. The assumption of normality, however, is crucial in permitting aggregation.

It is worth noting that while [Guvenen et al. \(2014\)](#) find that the variance of household income is acyclical, they find that the left-skewness is countercyclical. Given our assumption of normally distributed income, skewness is constant in our model. If we relaxed the assumption of normality and permitted time varying left-skewness, our model would not permit linear aggregation. However, the economics in the model would be broadly similar: countercyclical left-skewness would have very similar effects to countercyclical

⁶None of our results depend on the assumption that σ_y^2 depends on y . Even if σ_{ℓ}^2 is constant, the variance of household income will generally still vary with economic activity, as Section [2.3](#) shows.

variance, since both tend to increase households' precautionary saving motive in recessions.

1.2 Final goods producers

Final goods are consumed by households and the government and also used by intermediate goods producers as inputs into production. A representative competitive final goods firm transforms the differentiated intermediate goods $x_t(j)$, $j \in [0, 1]$ into the final good x according to the CES aggregator $x_t = \left[\int_0^1 x_t(j)^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}}$. As is standard, the final good producer's demand for variety j is $x_t(j) = \left(\frac{P_t(j)}{P_t} \right)^{-\theta} x_t$.

1.3 Intermediate goods producers

There is a continuum of monopolistically competitive intermediate goods firms indexed by $j \in [0, 1]$. Following Basu (1995) and Nakamura and Steinsson (2010), each firm combines labor and the final good to produce a differentiated good $x(j)$ using a constant returns to scale technology:⁷ $x_t(j) = z m_t(j)^\alpha n_t(j)^{1-\alpha}$ where $m_t(j)$ is the quantity of the final good used by the firm producing variety j , $n_t(j)$ is the labor input used, and z is the constant level of productivity. Each firm faces a quadratic cost of changing the price of the variety it produces following Rotemberg (1982). Firm j solves:

$$\max_{\{P_t(j), n_t(j), m_t(j)\}_{t=0}^\infty} \sum_{t=0}^\infty Q_{t|0} \left\{ \left(\frac{P_t(j)}{P_t} \right)^{1-\theta} x_t - \frac{W_t}{P_t} n_t(j) - m_t(j) - \frac{\Psi}{2} \left(\frac{P_t(j)}{P_{t-1}(j)} - 1 \right)^2 x_t \right\}$$

subject to:

$$z m_t(j)^\alpha n_t(j)^{1-\alpha} \geq \left(\frac{P_t(j)}{P_t} \right)^{-\theta} x_t$$

where $Q_{t+s|t} = \prod_{k=0}^{s-1} \frac{1}{1+r_{t+k}}$, $1+r_t = \frac{1+i_t}{1+\Pi_{t+1}}$ denotes the real interest rate and $\frac{\Psi}{2} \left(\frac{P_t(j)}{P_{t-1}(j)} - 1 \right)^2 x_t$ denotes the cost of changing prices from last period's level. For simplicity, we assume that this cost is rebated lump sum to households along with dividends.⁸ $\Psi \geq 0$ is a constant which scales the cost.

1.4 Policy

Monetary Policy We assume that the monetary authority sets nominal rates according to some rule:

$$1 + i_t = (1 + r) \Pi_t^{\Phi_\pi} \geq 1 \quad (2)$$

where $(1 + r)$ is the steady state real rate and Φ_π measures how monetary policy responds to inflation.

Fiscal Policy The budget constraint of the fiscal authority can be written as:

$$B_t + P_t g_t + T_t = P_t \tau_t \omega_t + \frac{1}{1 + i_t} B_{t+1} \quad (3)$$

⁷Since households supply their stochastic endowment of labor inelastically, the intermediate firms' utilization of the final good M_t is the factor which adjusts in response to demand. This specification of the supply side is just one possibility. Our results are general enough to apply to any specification of the supply side which yields a NK Phillips curve - with the understanding that different specifications of the supply side may affect whether income risk is procyclical or countercyclical.

⁸This assumption is made to simplify exposition. Even if we did not rebate this cost to households, it would be zero in a linear approximation of the economy around the zero inflation steady state.

where g_t denotes government purchases of the final goods and $s_t = \tau_t \omega_t - T_t/P_t - g_t$ denotes real primary surpluses.⁹ We allow the rate of labor income taxation τ_t to depend in a continuous but otherwise arbitrary fashion on aggregate output: $\tau_t = \tau(y_t)$. We assume throughout that lump-sum transfers T_t/P_t adjust as needed to ensure fiscal solvency: fiscal policy is “passive” in the sense of [Leeper \(1991\)](#).¹⁰

1.5 Market Clearing

The aggregate resource constraint implies $c_t + g_t = y_t$ where $c_t = \int_0^1 c_t^i di$ denotes aggregate consumption, $y_t = x_t - m_t$ denotes net output (GDP) and $m_t = \int_0^1 m_t(j) dj$ denotes firms’ total utilization of the final good as an intermediate input.

2 Characterizing Equilibrium

In this section we characterize equilibrium in our HANK economy. We start by solving the decision problem of each household.

2.1 Household decisions

The virtue of assuming CARA utility and normally distributed income shocks is that it allows us to characterize households’ optimal decisions in closed-form.

Proposition 1 (Individual decision problem). *Given a sequence of real interest rates, aggregate output and idiosyncratic risk $\{r_t, y_t, \sigma_{y,t}\}$,¹¹ household i ’s consumption decision can be expressed as:*

$$c_t^i = C_t + \mu_t (a_t^i + y_t^i) \quad (4)$$

where $a_t^i = A_t^i/P_t$ is real net worth at the start of date t and C_t and μ_t solve the following recursions:

$$C_t [1 + \mu_{t+1} (1 + r_t)] = -\frac{1}{\gamma} \ln \beta (1 + r_t) + C_{t+1} + \mu_{t+1} \bar{y}_{t+1} - \frac{\gamma \mu_{t+1}^2 \sigma_{y,t+1}^2}{2} \quad (5)$$

$$\mu_t = \frac{\mu_{t+1} (1 + r_t)}{1 + \mu_{t+1} (1 + r_t)} \quad (6)$$

Proof. See Appendix [A.1](#). □

Equation (4) shows that individual consumption can be decomposed into an aggregate component C_t and an idiosyncratic component $\mu_t (a_t^i + y_t^i)$. The idiosyncratic component states that at any date t all households have the same marginal propensity to consume (MPC) μ_t out of cash-on-hand.¹² This is a deliberate choice: to highlight the distinct roles of precautionary savings and MPC heterogeneity, we abstract altogether from the latter in order to focus on the former.

⁹Mean household income is equal to GDP minus fiscal surplus minus government expenditures $\bar{y}_t = y_t - s_t - g_t$.

¹⁰This is not an environment where the *fiscal theory of the price level* (FTPL) is at play ([Woodford, 1998](#); [Cochrane, 2017](#)). We make this assumption to highlight that in the presence of incomplete markets, fiscal policy affects the effects of monetary policy even when it is ‘passive’.

¹¹We restrict attention to sequences of interest rates for which there exists a terminal date $T < \infty$ after which $r_t > 0$.

¹²Note that since household income is independent across time, μ_t is also the MPC out of household income. This is not true more generally - in the extension with persistent individual income discussed in Appendix [E](#), the MPC out of household income differs out the MPC out of wealth.

The aggregate component \mathcal{C}_t can be decomposed into 3 terms. To see this, solve (5) forwards to get:

$$\mathcal{C}_t = \underbrace{\frac{1}{\gamma} \sum_{s=1}^{\infty} Q_{t+s|t} \frac{\mu_t}{\mu_{t+s}} \ln \left[\frac{1}{\beta(1+r_{t+s-1})} \right]}_{\text{impatience}} + \underbrace{\mu_t \sum_{s=1}^{\infty} Q_{t+s|t} \bar{y}_{t+s}}_{\text{PIH}} - \underbrace{\frac{\gamma \mu_t}{2} \sum_{s=1}^{\infty} Q_{t+s|t} \mu_{t+s} \sigma_{y,t+s}^2}_{\text{precautionary savings}} \quad (7)$$

The first term in (7) reflects the effect of impatience and interest rates on savings: if the interest rate or the discount factor β is high, current consumption is lower as households wish to save more. The second term reflects the permanent income hypothesis: higher expected discounted lifetime income increases current consumption. The final term reflects the precautionary savings motive. To the extent that households are prudent, $\gamma > 0$, higher future income risk lowers current consumption by increasing the desire to save.¹³ This third term indicates the effect of uninsurable income risk on aggregate consumption - if households face no idiosyncratic income risk, this term is zero and households are *permanent income consumers*. Note that what matters for the precautionary savings channel is the variance of *consumption*, or the extent to which the variance of income transmits to consumption. A given level of income risk σ_y^2 depresses current consumption more when the sensitivity of consumption to income, i.e. the MPC μ_t , is higher. In our framework, this sensitivity varies over time depending on the path of real interest rates, as we now explain.

Suppose a household i receives a dollar at date t . Since μ_t is the MPC, the increase in consumption today is $dc_t^i = \mu_t$ and the household saves $1 - \mu_t$, yielding $da_{t+1}^i = (1+r_t)(1-\mu_t)$ tomorrow out of which they consume $dc_{t+1}^i = \mu_{t+1} da_{t+1}^i$. The household would optimally smooth consumption by increasing date t and $t+1$ spending by the same amount $dc_t^i = dc_{t+1}^i$ which implies (6). When interest rates are temporarily high, households can afford to spend more out of an extra dollar today while still smoothing consumption, since a smaller amount saved earns a higher rate of return. Conversely, households whose income falls must cut consumption by a larger amount when interest rates are high. If instead they attempt to cushion the fall by borrowing as much as they would if interest rates were lower, this debt will accumulate at the high interest rate, forcing them to cut consumption even more in the future. More generally, iterating (6) forwards shows that MPC today depends positively on the future path of real interest rates $\mu_t = (\sum_{s=0}^{\infty} Q_{t+s|t})^{-1}$. Consumption responds to the present value of lifetime income, not current income per se; when interest rates are high, current income is a larger fraction of this present value, so consumption responds more to current income.¹⁴ This can be seen clearly in the case of constant real rates where $\mu_t = \frac{r}{1+r}$ is constant

¹³With CARA preferences, the coefficient of absolute prudence $-u'''(c)/u''(c) = \gamma$. While γ is also the coefficient of absolute risk aversion, it appears here in its role as the coefficient of absolute prudence.

¹⁴ If individual income was persistent instead of i.i.d., the MPC would be less sensitive to future interest rates. The MPC - the change in consumption following a unit income shock - equals the change in permanent income. Here permanent income is defined as the annuity value of the present discounted value of lifetime income:

$$y_{p,t}^i = \frac{1}{\sum_{s=0}^{\infty} Q_{t+s|t}} y_t^i + \sum_{s=1}^{\infty} \left(\frac{Q_{t+s|t}}{\sum_{s=0}^{\infty} Q_{t+s|t}} \right) \mathbb{E}_t y_{t+s}^i$$

Higher future interest rates (lower $\sum_{s=0}^{\infty} Q_{t+s|t}$) increase the weight of current income in permanent income. Since i.i.d. shocks only affect current income, they have a larger effect on permanent income when this weight is high, i.e. when interest rates are low. Persistent income shocks affect both current and future income, and thus have a large effect on permanent income whatever the relative weights on the current and future income. Thus, the MPC is less sensitive to interest rates in this case.

However, persistent income may actually make household's precautionary savings motive *more* sensitive to interest rates overall, because it increases the average MPC. When households are already exposed to substantial consumption risk, even a small increase in the sensitivity of consumption to income shocks has a large effect on desired precautionary savings. Appendix E.5 discusses this in detail.

across time and is simply the annuity value of a additional dollar of income today.¹⁵

This effect of interest rates on MPCs is absent in tractable HANK models which impose the zero liquidity limit (Ravn and Sterk, 2018; Werning, 2015; McKay et al., 2017). In these models, unconstrained households anticipate that their consumption will equal their income in all future periods; the precautionary savings channel is present, but its strength is not affected by variations in the sensitivity of consumption to income. While our framework shines a light on variations in the precautionary savings motive, they exist more generally in HANK models with CRRA preferences if the zero liquidity limit is not imposed.

2.2 Aggregate equilibrium conditions

The lack of MPC heterogeneity across households permits aggregation despite a non-degenerate distribution of wealth and allows us to focus on precautionary savings. Aggregating (4) across, aggregate consumption is $c_t = \mathcal{C}_t + \mu_t(a_t + \bar{y}_t)$ where $a_t = \int_0^1 a_t^i di$. Imposing asset and goods markets clearing, i.e. $c_t + g_t = y_t$ and $a_t = B_t/P_t$, we can derive the *aggregate IS equation*:¹⁶

$$y_t = y_{t+1} - \frac{\ln \beta (1 + r_t)}{\gamma} - \frac{\gamma \mu_{t+1}^2 \sigma_{y,t+1}^2}{2} + g_t - g_{t+1} \quad (8)$$

Absent risk ($\sigma_y = 0$) this *exact aggregate Euler equation* looks very much like the linearized IS equation from the 3 equation RANK model (except that this is not linearized). When $\sigma_y^2 > 0$, (8) also features a precautionary savings term which depends on 3 factors: the coefficient of absolute prudence γ , the sensitivity of individual consumption to individual income μ_{t+1} , and idiosyncratic income risk, $\sigma_{y,t+1}^2$. This last term is endogenous; we discuss its determinants in Section 2.3.

Appendix A.3 shows that firms' profit maximization described in section 1.3 yields the Phillips curve:

$$\Psi \Pi_t (\Pi_t - 1) = 1 - \theta \left(1 - \alpha^{-1} z^{-\frac{1}{\alpha}} x_t^{\frac{1-\alpha}{\alpha}} \right) + \Psi (\Pi_{t+1} - 1) \Pi_{t+1} \left[\frac{1}{1 + r_t} \frac{x_{t+1}}{x_t} \right] \quad (9)$$

Finally, the relation between net and gross output is given by $y_t = x_t - (x_t/z)^\alpha$. In summary, despite being an incomplete markets model, the entire model can be summarized by the following equations which describe the dynamics of only aggregate variables: the IS equation (8), the MPC recursion (6), the Phillips curve (9), the relation between net and gross output, the monetary policy rule (2) and finally, the Fisher equation $1 + i_t = (1 + r_t)\Pi_{t+1}$.

2.3 What determines the cyclicalty of income risk?

Household consumption, and the aggregate IS equation (8), depend on the path of idiosyncratic income risk $\sigma_{y,t}^2$. In equilibrium, this variance depends on aggregate output y_t :

$$\sigma_{y,t}^2 = \sigma^2(y_t) = \left[\left(1 - \tau(y_t) \right) \omega(y_t) + \delta(y_t) \right]^2 \sigma_\ell^2(y_t) \quad (10)$$

¹⁵Caballero (1990) and Wang (2003) solved a CARA decision problem with a constant real interest rate.

¹⁶See Appendix A.2 for details.

where $\omega(y_t)$ denotes real wages,¹⁷ $\delta(y_t)$ and $\tau(y_t)$ are defined above. This relation depends endogenously on many factors - wage cyclicality, time varying dividend policies and fiscal policy - as we now explain.

Define the **cyclicality of income risk** as $d\sigma^2(y)/dy$. This answers the question: If all exogenous variables were held fixed, and aggregate income was higher than its steady state level, would the variance of idiosyncratic income be higher or lower?¹⁸ Equation (11) shows that $d\sigma^2(y)/dy$ depends on 4 factors: (i) the cyclicality of wages $\omega'(y)$; (ii) the cyclicality of labor taxes, $\tau'(y)$; (iii) firms' dividend policy, i.e. whether dividends are more or less unequally distributed in good times compared to bad, $\delta'(y)$; (iv) the cyclicality of labor endowment risk $d\sigma_\ell^2(y)/dy$. This last factor can be thought of as unemployment risk which is not explicitly modeled here: if the probability of becoming unemployed is greater in recessions, i.e. the probability of drawing a low labor endowment is higher when y is low, then $d\sigma_\ell^2(y)/dy < 0$.

$$\frac{d\sigma^2(y)}{dy} = 2\sigma(y)\sigma_\ell(y) \left\{ \underbrace{(1 - \tau(y))\omega'(y)}_{\text{cyclicality of real wages}} - \underbrace{\tau'(y)\omega(y)}_{\text{cyclicality of taxes}} + \underbrace{\delta'(y)}_{\text{cyclicality of dividend risk}} \right\} + \underbrace{\frac{\sigma^2(y)}{\sigma_\ell^2(y)} \frac{d\sigma_\ell^2(y)}{dy}}_{\text{cyclicality of employment risk}} \quad (11)$$

Fiscal policy affects both the *level* and *cyclicality* of income risk. More redistribution (higher $\tau(y)$) reduces after-tax income risk; trivially, setting $\tau = 1$ would totally eliminate labor income risk. Similarly, the cyclicality of fiscal policy affects the cyclicality of risk. For example, if labor income tax rates are lower in recessions and higher in booms ($\tau'(y) > 0$), this tends to make income risk countercyclical. Conversely, if the government paid lump sum transfers in recessions financed by proportional taxes, i.e. $\tau'(y) < 0$, this would tend to make income risk more procyclical.

3 Determinacy of equilibrium in HANK economies

Armed with these results, we start by studying how market incompleteness affects determinacy of equilibria in our HANK economy. Recent work by [Ravn and Sterk \(2018\)](#) argues that the Taylor principle - nominal rates should respond more than one-for one to inflation - is not sufficient to ensure determinacy in an economy with incomplete markets. Conversely, [Auclert et al. \(2018\)](#) numerically find that in HANK models, the Taylor Principle may not even be necessary for determinacy, provided that income risk is not too countercyclical. Our framework sheds light on both results by deriving a new Taylor Principle which depends on the cyclicality of income risk. We show that procyclical income risk makes indeterminacy less likely, so a smaller Taylor rule coefficient below 1 suffices for determinacy, while countercyclical risk makes indeterminacy more likely, so a larger coefficient above 1 is required.

Linearizing (6),(8),(9) and (2) around the flexible price level of output y^* (normalized to 1), $\mu = \frac{r}{1+r}$

¹⁷Appendix A.3 shows that $\omega(y_t)$ is an increasing function of y_t .

¹⁸In a model with aggregate shocks, this notion of cyclicality - the one relevant for determinacy and policy puzzles - need not coincide with the definition used in the empirical literature, i.e., the correlation between income risk and aggregate economic activity ([Storesletten et al., 2004](#); [Guvenen et al., 2014](#)). Without aggregate shocks, the definitions are essentially equivalent.

and $\Pi = 1$, we have:¹⁹

$$\hat{y}_t = \Theta \hat{y}_{t+1} - \frac{1}{\gamma} (i_t - \pi_{t+1}) - \Lambda \hat{\mu}_{t+1} \quad (12)$$

$$\hat{\mu}_t = \tilde{\beta} \hat{\mu}_{t+1} + \tilde{\beta} (i_t - \pi_{t+1}) \quad (13)$$

$$\pi_t = \tilde{\beta} \pi_{t+1} + \kappa \hat{y}_t \quad (14)$$

$$i_t = \Phi_\pi \pi_t \quad (15)$$

where

$$\Theta = 1 - \frac{\gamma \mu^2}{2} \frac{d\sigma^2(y^*)}{dy}, \quad \Lambda = \gamma \mu^2 \sigma_y^2(y^*), \quad \tilde{\beta} = \frac{1}{1+r}$$

κ denotes the slope of the linearized Phillips curve, and $\hat{y}_t, \hat{\mu}_t, i_t$ and π_t denote the log-deviation of $y_t, \mu_t, 1 + i_t$ and Π_t from their steady state values.

Equation (14) and (15) are the standard linearized Phillips curve and interest rate rule. The difference between HANK and RANK is concentrated in the aggregate demand block, represented by the first two equations. The IS equation (12) relates aggregate output today to output tomorrow. In the standard RANK model, since there is no idiosyncratic risk, $\sigma = \frac{d\sigma^2(y)}{dy} = 0$, we have $\Theta = 1$ and $\Lambda = 0$, rendering (12) identical to the standard linearized IS equation. However, the presence of idiosyncratic risk can change this. If risk is procyclical $\frac{d\sigma^2(y)}{dy} > 0$, then $\Theta < 1$ and if risk is countercyclical $\frac{d\sigma^2(y)}{dy} < 0$, then $\Theta > 1$. If risk is acyclical then $\frac{d\sigma^2(y)}{dy} = 0$ and $\Theta = 1$ as in RANK.

To understand why the cyclicity of risk affects Θ , consider a scenario in which the real interest rates are fixed at their steady state value and (12)-(13) can be written as:

$$\hat{y}_t = \Theta \hat{y}_{t+1} \quad (16)$$

and $\hat{\mu}_t = 0, \forall t$. Θ measures the sensitivity of output today to output tomorrow. **Procyclical risk** implies that current spending moves less than one-for-one with future spending $\Theta \in (0, 1)$. Suppose that at date 0, households conjecture that output at date 1 will be higher than steady state, $\hat{y}_1 > 0$. This belief about date 1 output affects demand at date 0 in two ways. First, via the permanent income channel, agents anticipating higher income at date 1 demand more consumption at date 0. Higher demand raises income, further raising consumption, and ultimately an increase in \hat{y}_1 would increase \hat{y}_0 one-for-one. However, there is also a second effect. Since income risk is procyclical, higher output at date 1 also increases the idiosyncratic risk agents face at date 1. This tends to reduce date 0 consumption, and thus output, via the precautionary savings channel. Overall, an increase in \hat{y}_1 increases \hat{y}_0 less than one-for-one.

Equation (16) shows that procyclical risk delivers a “discounted” Euler equation as in McKay et al. (2016, 2017). Our derivation indicates that procyclical risk, rather than market incompleteness per se, is responsible for the discounting. Indeed, McKay et al. (2017) features strongly procyclical risk: “low productivity households receive a constant transfer from the government while high productivity households receive all cyclical wages and dividends, minus the acyclical transfers.” Thus for them, risk - the income gap between high and low productivity households - is highest in booms and lowest in recessions.

Countercyclical risk instead implies that that current spending moves more than one-for-one with

¹⁹Appendix B derives the linearized model and provides expressions describing y^* and r . We choose z to normalize $y^* = 1$.

future spending $\Theta > 1$ (generating an “*explosive*” Euler equation). Suppose that at date 0, households contemplate a lower output than steady state at date 1, $\hat{y}_1 < 0$. This directly depresses consumption via the permanent income channel as agents expect to be poorer in the future; on its own, this would make \hat{y}_0 fall one-for-one with \hat{y}_1 . In addition, however, agents understand that lower date 1 output implies higher idiosyncratic risk at date 1. This further lowers demand at date 0 via the precautionary savings channel. Overall, \hat{y}_0 falls *more* than one-for-one with \hat{y}_1 . Countercyclical unemployment can naturally generate countercyclical income risk, as discussed by Challe and Ragot (2016), Challe et al. (2017) and Ravn and Sterk (2018) among others.

Finally, **acyclical risk** implies that current spending moves one-for-one with future spending ($\Theta = 1$) as in RANK. While agents face risk in the acyclical case, changes in future GDP do not affect risk, and thus do not affect current spending via the precautionary savings channel.

More generally, even though acyclical risk implies $\Theta = 1$, HANK still differs from RANK in this case - when real interest rates can vary - due to the third term on the RHS of (12), which represents variations in the precautionary savings motive. Recall that $\hat{\mu}_t$ denotes the log-deviation of households’ MPC. When the MPC is high, individual consumption responds more to individual income, and so a given level of income risk implies more volatile consumption, and thus a stronger precautionary savings motive. Thus when $\hat{\mu}_{t+1}$ is high, households seek to reduce consumption today relative to tomorrow. $\hat{\mu}_{t+1}$ in turn depends on the whole future path of interest rates as shown in (13). As we discussed in section 2.1, when interest rates are temporarily higher, households receiving a adverse income shock cut consumption more than they would have done had interest rates been lower. A transitory period of high interest rates thus exposes households to more consumption risk, increasing desired precautionary savings and depressing demand. In this sense, the $-\Lambda\hat{\mu}_{t+1}$ term in (12) and (13) represents a novel channel of monetary policy in addition to the standard *intertemporal substitution* channel of monetary policy, represented by the second term in (12).²⁰

The discussion above analyzed determinacy in a special case with a nominal interest rate peg and rigid prices. The following Proposition describes local determinacy in the more general case with sticky prices and an interest rate rule.

Proposition 2 (An income-risk augmented Taylor Principle). *Suppose $\Theta \in (0, \bar{\Theta})$ where $\bar{\Theta}$ is greater than 1 for σ_y^2 sufficiently small and is defined in Appendix C. Then, the following condition is necessary and sufficient for equilibrium to be locally determinate:*

$$\Phi_\pi > 1 + \frac{\gamma}{\kappa} \left[\frac{(1 - \tilde{\beta})^2}{(1 - \tilde{\beta}) + \gamma\tilde{\beta}\Lambda} \right] \underbrace{(\Theta - 1)}_{\text{cyclical risk}} \quad (17)$$

Proof. See Appendix C. □

Incomplete markets do not necessarily affect determinacy: with acyclical risk ($\Theta = 1$), determinacy requires $\Phi_\pi > 1$, as in the RANK model. Away from this benchmark, procyclical income risk makes determinacy more likely, while countercyclical risk makes it less likely, as in the fixed price limit. More precisely, if risk is procyclical ($\Theta < 1$), determinacy obtains even if the standard Taylor principle is violated

²⁰More generally, the strength of this channel depends on whether individual income is i.i.d or persistent. See footnote 14 for a brief discussion and Appendix E.5 for a detailed discussion.

and $\Phi_\pi < 1$ - unlike in the RANK model, raising nominal rates more than one-for-one with inflation is not *necessary* to ensure determinacy. Indeed, if risk is sufficiently procyclical, determinacy obtains even under a nominal interest rate peg, contrary to the classic result of [Sargent and Wallace \(1975\)](#). A HANK economy with procyclical risk contains a powerful stabilizing force: higher output implies higher risk, which reduces demand and prevents the rise in output from being self-fulfilling.

Conversely, with countercyclical risk ($\Theta > 1$), the standard Taylor principle $\Phi_\pi > 1$ is not even *sufficient* for determinacy, unlike in RANK. Countercyclical risk is *destabilizing*: lower output implies higher risk, reducing demand and allowing the fall in output to become self-fulfilling. Monetary policy must respond more aggressively to prevent such self-fulfilling fluctuations. This result resonates with the findings of [Ravn and Sterk \(2018\)](#) who argue that even if monetary policy satisfies the Taylor principle, equilibrium in a HANK model may be indeterminate. But this result is not a property of HANK models in general - indeed we have just seen that procyclical risk makes indeterminacy less likely in such models. Indeterminacy is more likely in HANK models only when income risk is countercyclical, as in [Ravn and Sterk \(2018\)](#).

Finally, all else equal, a higher Λ weakens the extent to which pro- or counter-cyclical income risk warrants a deviation from the classic Taylor principle. A higher Λ makes monetary policy more powerful: smaller changes in interest rates have a larger effect on aggregate output, operating not just through the intertemporal channel but also by changing the pass-through from income to consumption shocks and affecting desired precautionary savings.²¹

The result in Proposition 2 is similar to two other modified Taylor principles which were previously presented in the HANK literature. [Bilbiie \(2019a\)](#) derives a modified Taylor principle which, like ours, depends on a compounding/discounting factor “ δ ” (the analog of our Θ) in the aggregate IS equation. In his environment, δ can differ from 1 provided that (i) unconstrained households face some risk of becoming liquidity constrained and (ii) the sensitivity of constrained households income to aggregate income differs from 1. In our environment, there are no HTM agents and Θ depends solely on the cyclicity of risk, making it clear that this factor alters determinacy even absent HTM agents, liquidity constraints and so forth. [Auclert et al. \(2018\)](#) also derive a modified Taylor principle in terms of two endogenous objects: μ^Y (which they argue measures how much demand is shifted to the future) and μ^R (which they argue measures how responsive the economy is to interest rates). They show via simulations that μ^Y is related to the cyclicity of risk, consistent with our findings - determinacy obtains with a lower Φ_π , or under a real interest rate peg, if income risk is sufficiently procyclical.

4 Some RANK policy puzzles

Recent work argues that RANK models make unrealistic predictions about the effect of forward guidance and the size of government spending multipliers at the ZLB. These perceived shortcomings have often been explained by the notion that the intertemporal substitution channel is ‘too strong’ in the RANK model, in which the representative agent is essentially a permanent income consumer, and households are always on their Euler equation. This suggests that a HANK model might reverse these ‘unrealistic’ predictions. Our analytical model sheds light on how market incompleteness affects these predictions.

²¹As mentioned elsewhere, this channel would be absent if we had considered an economy with zero liquidity, in which case the pass-through from income to consumption shocks is always equal to 1 and $\hat{\mu}_t = 0$.

4.1 Calibration

While our results are primarily analytical, when presenting numerical examples we parameterize the model as follows. We set $g = 0$ and normalize aggregate steady state net output y^* to 1, implying that consumption and income of the median household is 1 in steady state. We calibrate the model to an annual frequency and set the standard deviation of income in steady state σ_y - which is also the standard deviation of the growth rate of income of the median household - to 0.5. This is in line with [Guvenen et al. \(2014\)](#) who using administrative data finds the standard deviation of 1 year log earnings growth rate to be slightly above 0.5.²² We set the slope of the Phillips curve $\kappa = 0.1$ following [Schorfheide \(2008\)](#). We set the coefficient of relative prudence for the median household, $-\frac{cu'''(c)}{u''(c)} = \gamma$, to be 3, within the range of estimates in the literature (see e.g. [Cagetti \(2003\)](#); [Fagereng et al. \(2017\)](#); [Christelis et al. \(forthcoming\)](#)). We set $r = 4\%$.

Finally, we consider a range of values for the cyclicity of income risk $\frac{d\sigma^2}{dy}$ in order to illustrate how this affects outcomes qualitatively. While there is no consensus regarding empirical estimates of $\frac{d\sigma^2}{dy}$, [Storesletten et al. \(2004\)](#) find that the standard deviation of the persistent shock to (log) household income increases from 0.12 to 0.21 as the aggregate economy moves from peak to trough. If the difference between growth in expansions and recessions is roughly 0.03, this implies $\frac{d\sigma^2(y^*)}{dy} = \frac{0.12^2 - 0.21^2}{0.03} \approx -1$, which we use as our baseline value (implying $\Theta = 1.002$).²³ An important caveat is that this estimate of the cyclicity of risk is unconditional; in a richer micro-founded model, risk may respond to different shocks (monetary, fiscal, productivity etc.) differently. In addition, to illustrate the qualitative effects of the cyclicity of risk, we also plot dynamics for the cases of acyclical risk ($\frac{d\sigma^2(y^*)}{dy} = 0$, implying $\Theta = 1$), strongly countercyclical risk ($\frac{d\sigma^2(y^*)}{dy} = -5$, $\Theta = 1.011$), and strongly procyclical risk ($\frac{d\sigma^2(y^*)}{dy} = 5$, $\Theta = 0.989$).²⁴

4.2 Forward guidance

The effect of forward guidance in RANK is best illustrated with a simple experiment ([Del Negro et al., 2015](#)). The monetary authority announces at date t a temporary interest rate cut at date $t + k$: $i_{t+k} = -\varepsilon < 0$, $i_{t+s} = 0$ for all $s \neq k$. How does the effect on date t output depend on the horizon of forward guidance k ? In RANK $\Theta = 1$, $\Lambda = 0$, so iterating the IS equation forward yields

$$\hat{y}_t = -\gamma^{-1} \sum_{k=0}^{\infty} (i_{t+k} - \pi_{t+k+1})$$

Under rigid prices ($\kappa = 0$), $\pi_{t+k} = 0$, and nominal and real rates move by the same amount. In this case, whatever the horizon of forward guidance k , output increases by ε at date t and remains at this level until $t + k$ - announcements about far future rate cuts are *equally* as effective as contemporaneous rate cuts. Under sticky (not rigid) prices ($\kappa > 0$), announcements about far future interest rates are *more* effective

²²This is also consistent with [Gorbachev \(2011\)](#) who using the Panel Survey of Income Dynamics finds that the variance of unpredictable annual changes in household income was between 0.2 and 0.25 in the mid 1990s (which would correspond to a standard deviation of slightly below 0.5).

²³ We thank the editor and an anonymous referee for suggesting that we use this as a baseline.

²⁴We are not arguing that there is empirical evidence suggesting that $\frac{d\sigma^2(y^*)}{dy}$ is as large as ± 5 ; our goal is just to illustrate the qualitative effects of pro- and countercyclical risk clearly.

than contemporaneous changes. Inflation can be written as the present discounted value of future output:

$$\pi_t = \kappa \sum_{k=0}^{\infty} \tilde{\beta}^k \hat{y}_{t+k}$$

A larger horizon k implies output will be high for longer, creating a larger increase in inflation, which in turn reduces real rates and stimulates output further. All these effects operate through the *intertemporal* channel of monetary policy. Lower real interest rates - caused both by lower nominal rates and higher expected inflation - induce a declining path of consumption and output. With date $t+k$ output fixed at Y^* (implicitly by an active Taylor rule after date $t+k$), lower consumption *growth* implies a higher *level* of consumption, and output, today.

The dotted black line in Figure 1a shows the response of output in RANK to the announcement of a unit cut in nominal interest rates at date $t+5$. The response of output remains positive until the announced change in policy is enacted at date 5, with the largest effect on the day of announcement - date 0. The black dotted curve in Figure 1b shows how the impact effect of a future one-time cut in interest rates depends on the horizon of the policy change. Announced future policy changes are more effective than contemporaneous policy changes. Del Negro et al. (2015) dub this the *forward guidance puzzle* (FGP).

Consider the same experiment in our HANK economy. With rigid prices, iterating the IS equation and μ_t recursion forward now yields

$$\hat{y}_t = -\gamma^{-1} \sum_{k=0}^{\infty} \Theta^k i_{t+k} - \Lambda \sum_{k=0}^{\infty} \Theta^k \sum_{s=1}^{\infty} \tilde{\beta}^s i_{t+k+s} \quad (18)$$

We can express the sensitivity of output at date t to interest rate changes at date $t+k$ as:

$$\frac{d\hat{y}_t}{di_{t+k}} = -\gamma^{-1} \Theta^k - \Lambda \sum_{s=1}^k \tilde{\beta}^s \Theta^{k-s} \quad (19)$$

With countercyclical income risk ($\Theta > 1$), Θ^k and $\sum_{s=1}^k \tilde{\beta}^s \Theta^{k-s}$ are increasing in k . Even with fixed prices, far future announcements are *more* effective in stimulating demand than contemporaneous policy:

$$\left| \frac{d\hat{y}_t}{di_{t+k+1}} \right| > \left| \frac{d\hat{y}_t}{di_{t+k}} \right|, \forall k \geq 0$$

In both HANK and RANK, lower future interest rates induce lower consumption growth and higher consumption (and output) on impact. But the FGP is more severe in this HANK economy for two reasons associated with precautionary savings. First, higher future output reduces income risk. Anticipating higher income and less risk at date $t+k$, households at date $t+k-1$ increase spending more than one-for-one, leading to a larger boom at date $t+k-1$, which in turn feeds a larger boom at date $t+k-2$, and so forth. Second, lower interest rates make consumption less responsive to current income ($\hat{\mu} < 0$). Thus, given income risk, consumption risk - which ultimately affects precautionary savings - is lower, further boosting spending today. Overall, incomplete markets *worsen* the FGP if income risk is countercyclical.

If income risk was acyclical $\Theta = 1$, the first effect just described would be absent, but the second effect would still be operative. Indeed with $\Theta = 1$, we have $\left| \frac{d\hat{y}_t}{di_{t+k}} \right| = \gamma^{-1} + \beta \Lambda \frac{1-\tilde{\beta}^k}{1-\tilde{\beta}}$ which is strictly increasing

in k . While forward guidance does not affect income risk in this case, it does reduce the sensitivity of consumption to income μ_t , and thus boosts consumption by lowering desired precautionary savings.

Thus procyclical risk ($\Theta < 1$) is essential (within our framework) if incomplete markets are to resolve the FGP. With $\Theta < 1$, the response of date t output to a cut in interest rates at $t+k$ has two components, as can be seen from (19). The first term $\gamma^{-1}\Theta^k$ is decreasing in k , making future interest rate cuts *less* effective than contemporaneous cuts. This is not because households are not forward-looking, or because they anticipate being borrowing constrained in the future; our households are infinitely lived and unconstrained. Instead, it is because risk is procyclical. At date $t+k-1$, households anticipate that the cut in rates at date $t+k$ will increase output and average income, but they also expect this to increase idiosyncratic risk. Thus while higher average income would induce them to increase consumption one-for-one, higher risk dampens this increase in consumption, so date $t+k-1$ spending increases less than one-for-one. Similarly, at date $t+k-2$, households increase spending less than one-for-one in response to a smaller increase in date $t+k-1$ income, and so forth.²⁵ McKay et al. (2016) argue that market incompleteness moderates the FGP, generating behavior which can be approximated in terms of a ‘discounted Euler equation’. Our framework clarifies that this is only the case when risk is sufficiently procyclical. These results resonate with Bilbiie (2019a), who argues that incomplete markets need not necessarily solve the FGP; they may make it worse. In Bilbiie (2019a), the FGP is mitigated by a different channel, namely that the income of HTM households responds less than one for one to aggregate income.

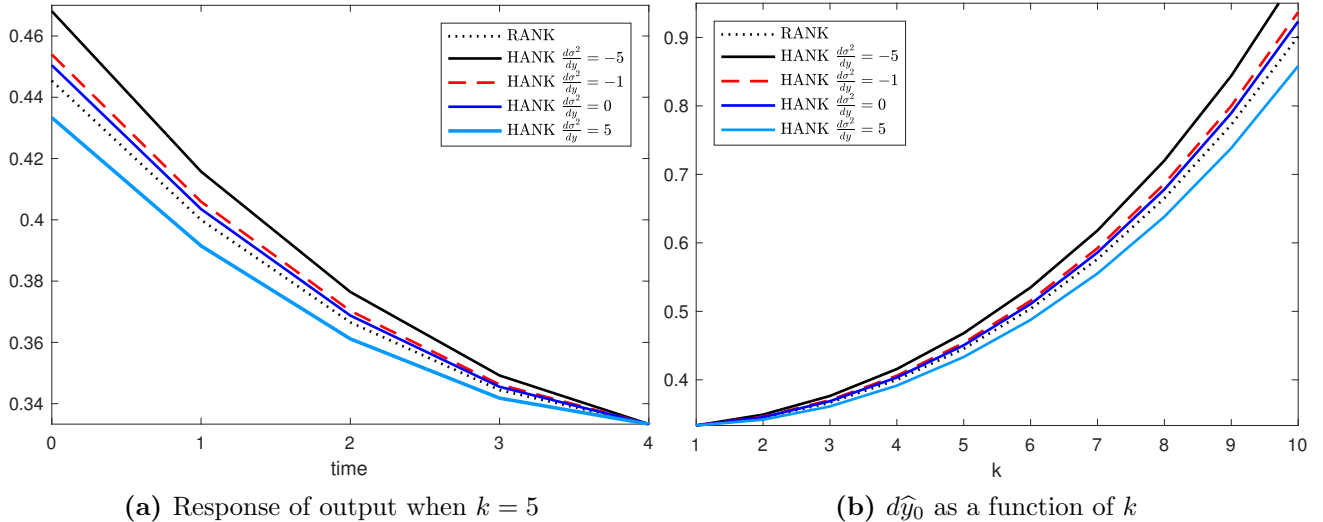


Figure 1. Response of output to a unit drop in nominal interest rates k periods in the future.

The cyclicity of income risk remains key even when prices are not perfectly rigid ($\kappa > 0$). In this case, the FGP is more pronounced even in RANK due to the expected inflation channel; this carries over to HANK economies. Thus, even with moderately procyclical income risk, the puzzle may persist in the sense that announced changes have a larger effect on output than contemporaneous changes. Nonetheless, more procyclical income risk (lower Θ) still reduces both the size of the output response to announced policy changes, and the gains (if any) from future announcements. Figure 1a illustrates the paths of output in response to a unit cut in nominal interest rates at date 5, for various values of $\frac{d\sigma^2}{dy}$. As can be seen,

²⁵For mildly procyclical risk (Θ close to 1), the fall in μ may still reduce desired precautionary savings on net, increasing the impact of far future interest rate cuts, and leaving the FGP unresolved. However, for sufficiently procyclical risk ($\Theta \ll 1$) desired precautionary savings are increased on net, reducing the effectiveness of far future interest rate cuts.

procyclical risk ($\frac{d\sigma^2}{dy} > 0$ or lower Θ) reduces the effect of forward guidance on output at all horizons while countercyclical risk ($\frac{d\sigma^2}{dy} < 0$ or higher Θ) increases the effect.

Given our calibration, the differences in the cyclicity of income risk have quantitatively small effects on the strength of forward guidance. In particular, while qualitatively forward guidance is more effective in the acyclical HANK economy (solid dark blue line) than in RANK (dotted black line), consistent with our analytical results for the fixed price case, this difference is very small quantitatively. This is because our CARA-normal economy generates a low MPC for reasonable values of r , since $\mu = r/(1+r) = 0.0385$ - well below empirical estimates of the MPC out of transitory income shocks. This low MPC reduces consumption risk for a given level of income risk, and thus reduces the strength of the precautionary savings channel given by $\Lambda = \gamma\mu^2\sigma^2$, which is the only difference between RANK and acyclical HANK. In a quantitative HANK model with a higher MPC, the precautionary saving channel would be stronger and the difference between acyclical HANK and RANK economies would be more pronounced. Even within our CARA-normal framework, relaxing the assumption of i.i.d. income risk increases the MPC out of income, increases consumption risk, and amplifies the difference between these economies, as we show in Appendix E. By the same token, the degree of countercyclicity of risk in our baseline economy with $\frac{d\sigma^2}{dy} = -1$ (dashed red line) implies a Θ of only 1.002 and generates only a small difference relative to the acyclical case, owing to the low MPC. In a quantitative HANK model, these differences would be magnified.²⁶

4.3 Fiscal multipliers

The textbook 3-equation RANK model predicts large declines in output and inflation during a liquidity trap, when the natural rate of interest is negative and the nominal rate is constrained by the zero lower bound (ZLB). Temporary increases in government spending during the liquidity trap have unusually large multipliers (substantially greater than 1) which grow with the duration of the liquidity trap. We now explore whether, and how, these predictions are modified in an incomplete markets model.

With government purchases and time varying β_t , our linearized Euler equation becomes:

$$\hat{y}_t = \Theta \hat{y}_{t+1} - \gamma^{-1} (i_t - \pi_{t+1} - \rho_t) - \Lambda \hat{\mu}_{t+1} + \hat{g}_t - \hat{g}_{t+1} \quad (20)$$

where $-\rho_t$ denotes the log-deviation of β_t . We consider a scenario in which $\rho_t = -\bar{\rho} < 0$ for $t < T$ and $\rho_t = 0$ for $t > T$. Monetary policy is assumed to be constrained by the ZLB until date T : in log-deviations, $i_t = -\iota < 0$ for $t < T$.²⁷ Starting at date T , we assume that monetary policy implements the zero-output gap, zero inflation equilibrium (for example, with an appropriately specified active Taylor rule) so that $i_t = 0$. Furthermore, consider a fiscal policy that sets $\hat{g}_t = g > 0$ for the duration of the liquidity trap ($0 \leq t < T$) and zero thereafter. Again, it is instructive to start with the fixed price limit. Then, the fiscal multiplier is $\frac{\partial \hat{y}_t}{\partial g} = \Theta^{T-t-1}$ for $0 \leq t < T$, 0 otherwise. In RANK, $\Theta = 1$, and the multiplier at each date during the liquidity trap is 1, independent of the duration of the trap T . Incomplete markets, per se, need not change this: acyclical risk also implies $\Theta = 1$ and so the multiplier is 1 at all horizons.

Pro-cyclical income risk ($\Theta < 1$) delivers a multiplier below 1 and *decreasing* in the duration of the trap $T - t$; equivalently, the multiplier becomes larger as the end of the trap approaches. Intuitively, when households anticipate higher government spending throughout the duration of the trap, they also expect

²⁶Again, this is also true in our extension with persistent idiosyncratic income shocks, as shown in Appendix E.

²⁷More generally, our analysis applies to the case of a temporary interest rate peg.

higher aggregate income; but because idiosyncratic risk is procyclical, this carries with it a higher level of risk faced by each household, inducing them to spend less when real interest rates are fixed.

Countercyclical risk ($\Theta > 1$), however, delivers a multiplier greater than 1 and increasing in the duration of the trap. These large and increasing multipliers are not due to the expected inflation channel (Woodford, 2011; Eggertsson, 2011) which is absent since prices are fixed. Instead, they are due to the precautionary savings channel: higher future government spending increases aggregate output which reduces idiosyncratic risk. Anticipating this, households consume more given a fixed real interest rate.

The intuition broadly carries over to an environment with sticky rather than fixed prices where the expected inflation channel is also at work. This tends to raise fiscal multipliers, especially in more protracted liquidity traps. However, it remains true that procyclical risk tends to dampen the fiscal multiplier and its dependence on the duration of the liquidity trap episode, while countercyclical risk amplifies the fiscal multipliers further relative to RANK. Figure 2a plots the multiplier $\frac{d\hat{y}}{dg}$ for a liquidity trap that lasts 5 periods (with sticky rather than rigid prices).

With fixed prices, multipliers were identical in RANK and the acyclical risk HANK economy. This is not literally true when prices are somewhat flexible. In RANK, higher future government spending raises inflation and reduces real rates, stimulating output. This expected inflation channel is *strengthened* in HANK. Lower real rates lower μ_t , the sensitivity of household consumption to income shocks. Even with income volatility fixed (acyclical risk), this reduces the volatility of future household consumption, further stimulating demand via the precautionary savings channel. Again, in Figure 2a, this second effect is quantitatively small due to the low MPC in our economy, with the black dotted line (RANK) lying on top of the dark blue line (HANK economy with $\frac{d\sigma^2}{dy} = 0$ or $\Theta = 1$). Instead, as in the fixed price case, the most important way in which incomplete markets affect the multiplier is again, via the cyclicity of income risk. As can be seen in Figure 2a, more procyclical risk (higher $\frac{d\sigma^2}{dy}$ and lower Θ) lowers the multiplier while countercyclical risk amplifies the multiplier relative to RANK.

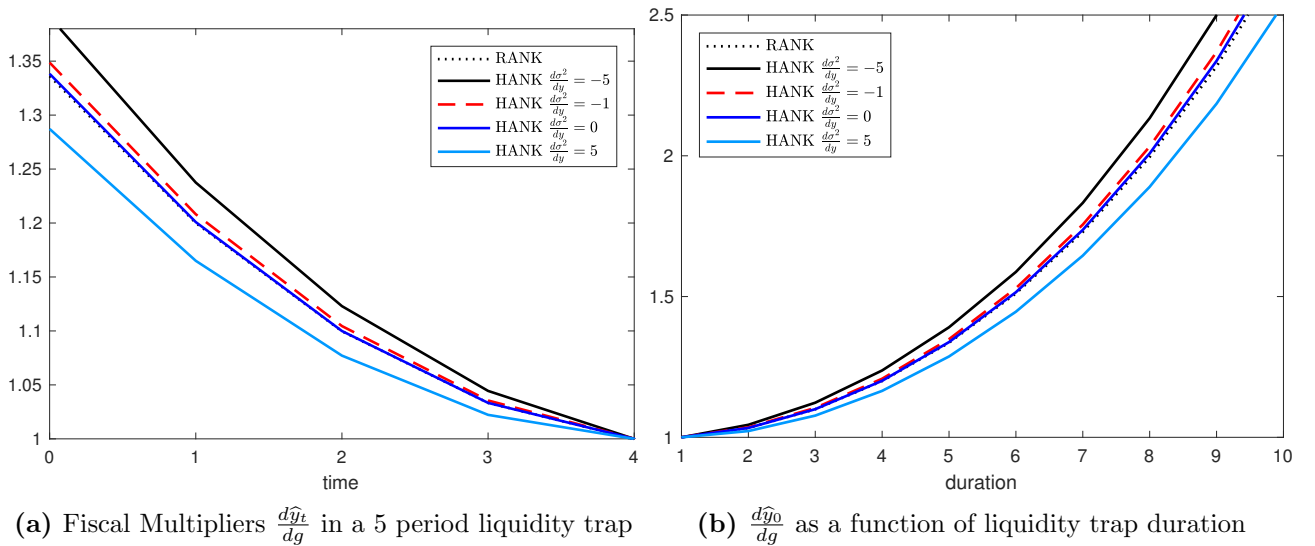


Figure 2. Fiscal Multipliers

Figure 2b illustrates how the duration of the liquidity trap affects the impact fiscal multiplier $d\hat{y}_0/dg$. In RANK, the multiplier is greater than 1 and increasing in the duration of the trap (again, due to the expected inflation channel). Procyclical risk ($\Theta < 1$) reduces the impact multiplier. The expected inflation

channel is counteracted by procyclical risk: while a longer period of higher output increases inflation more, it also raises risk more and thus depresses private spending more. If instead risk is countercyclical, expected inflation and precautionary savings work in the same direction, making the multiplier larger and strongly increasing in the duration of the trap. Again, while these effects are quantitatively small in our model with low MPC, they would be larger in a quantitative HANK with a higher MPC. The differences are magnified even within our CARA-normal framework when we allow for persistent idiosyncratic income shocks in Appendix E.

5 The role of MPC heterogeneity

The nascent HANK literature has sometimes suggested that incomplete markets matter to the extent that some households are or expect to be ‘off their Euler equation’, because this makes households less forward-looking, increases the MPC and weakens the strength of the intertemporal channel. In our model, households are never off their Euler equation and there is no MPC heterogeneity. Yet the precautionary savings motive alone can change the model’s predictions substantially, relative to RANK.²⁸

Of course, MPC heterogeneity can *also* change the predictions of a HANK model, relative to RANK, but it does so in a different way to the precautionary savings motive. Bilbiie (2008) studied a model with some hand-to-mouth (HTM) agents, which features MPC heterogeneity and no idiosyncratic risk. While this does not create discounting or compounding in his Euler equation, it can change the contemporaneous sensitivity of output to interest rates, which in RANK - and in our economy without MPC heterogeneity - simply equals the IES. Whether and how introducing HTM agents changes this sensitivity depends on the cyclicity of HTM agents’ income to aggregate income.

Appendix F replicates Bilbiie (2008)’s result in our economy. Suppose only fraction $1 - \eta$ households are unconstrained and η are HTM households who consume their entire after-tax income, which equals $\chi \bar{y}_t$ where \bar{y}_t denotes after-tax *aggregate* income.²⁹ χ denotes the cyclical sensitivity of HTM agents’ income - for example, if $\chi > 1$, HTM agents income falls more in recessions than the income of unconstrained agents. The linearized aggregate Euler equation becomes:

$$\hat{y}_t = -\frac{\Xi}{\gamma} (i_t - \pi_{t+1}) + (\Xi\Theta + 1 - \Xi) \hat{y}_{t+1} - \Xi\Lambda \hat{\mu}_{t+1} \quad (21)$$

where $\Xi = \frac{1-\eta}{1-\eta\chi} > 0$. Whether the Euler equation is “discounted” or “explosive” depends solely on whether income risk is procyclical or countercyclical - whether $\Theta \leq 1$ - and is unaffected by MPC heterogeneity. MPC heterogeneity can affect the *magnitude* of discounting or compounding, but not the direction of the deviation from the acyclical risk benchmark. We further show in the Appendix that our modified Taylor principle is unaffected by the introduction of MPC heterogeneity.³⁰

This also indicates that the cyclicity of income risk affects outcomes in a HANK economy in a different way than heterogeneous sensitivities to the business cycle, $\chi \leq 1$ - whose income falls more in recessions. The cyclicity of risk affects the coefficient on \hat{y}_{t+1} in the aggregate Euler equation - whether output is

²⁸ As is well known, in partial equilibrium, precautionary savings arises even for an unconstrained individual if the third derivative of the period utility function is positive, as it is in our CARA economy (Leland, 1968; Sandmo, 1970).

²⁹ For unconstrained households, the income process is now given by $y_t^i \sim N\left(\frac{1-\eta\chi}{1-\eta} \bar{y}_t, \sigma_{y,t}^2\right)$.

³⁰ This holds given our assumption that $\Xi > 0$. See Bilbiie (2008) for an analysis of the case in which $\Xi < 0$.

more sensitive to current or future interest rates - while χ primarily affects how sensitive output is to interest rates at any horizon. Our approach allows us to study cyclical income risk absent heterogeneous sensitivities. As we discuss in Appendix F.1, other analytical HANK models which exploit the zero liquidity limit have not made this distinction.

MPC heterogeneity and the cyclicalities of HTM income, on the one hand, and precautionary savings and the cyclicalities of idiosyncratic risk, on the other hand, are conceptually distinct, and change the NK model in different ways. While distinct, they are not unrelated: any structural model will make a prediction about the relation between these two features. The merit of our modeling approach is that we can shut down MPC heterogeneity altogether to study the effects of cyclical risk and precautionary savings in isolation.

6 Conclusion

We have presented an analytical HANK model which abstracts from MPC heterogeneity in order to study the effects of precautionary savings and cyclicalities of risk on aggregate outcomes. Our framework nests several different specifications of HANK economies in the theoretical and computational literature so far, and explains how differences in their conclusions can be explained by the differences in the cyclicalities of income risk in these economies. Even absent MPC heterogeneity, the cyclicalities of risk determines whether incomplete markets resolve or worsen the FGP, make determinacy more or less likely, and so forth.

Our analysis also uncovers a new dimension of monetary-fiscal interaction. Since fiscal policy affects the cyclicalities of risk, it plays a vital role in mediating the effect of monetary policy in a HANK economy. This fiscal-monetary interaction is logically distinct from the issues discussed in the FTPL literature. In our economy fiscal policy always raises surpluses to ensure solvency. However, *how* these surpluses are raised, and how this varies with economic activity, affects the cyclicalities of risk and thus the effects of monetary policy.

The importance of the cyclicalities of risk by no means implies that MPC heterogeneity is irrelevant. Indeed, as mentioned above, it can change the contemporaneous sensitivity of aggregate output to interest rates, which is unaffected by the cyclicalities of income risk. More generally, MPC heterogeneity would also affect the aggregate demand effects of redistributive policies, the effects of the supply of government debt, and so forth. Indeed, there may be interesting interactions between MPC heterogeneity and risk. Since a given change in income risk has a larger effect on consumption risk and precautionary savings when the MPC is high, countercyclical risk could be more powerful when it is felt primarily by high MPC agents.

Finally, while we have relied on an analytically tractable model to derive closed-form results, our qualitative results would be stronger in more quantitatively realistic settings. Our CARA-normal economy with i.i.d risk features a low MPC. Even within our environment, persistent rather than i.i.d. individual income increases the MPC (the pass-through from income to consumption risk), increases desired precautionary savings, and amplifies deviations from RANK (see Appendix E).³¹

³¹For example, we find that the difference between the effectiveness of forward guidance and the size of fiscal multipliers in an acyclical risk HANK and RANK is more pronounced when individual income is persistent (see Figures 4 and 5). This reflects that the precautionary savings channel of monetary policy can be stronger when individual income is persistent.

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Appendix

A The non-linear model

A.1 Deriving the consumption decision rule

Given a path of real interest rates $\{r_t\}$ and aggregate output $\{y_t\}$, household i ’s problem is:

$$\begin{aligned} \max_{\{c_t^i, a_{t+1}^i\}} \quad & -\gamma^{-1} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t e^{-\gamma c_t^i} \\ \text{s.t.} \quad & c_t^i + \frac{1}{1+r_t} a_{t+1}^i = a_t^i + y_{i,t} \end{aligned} \quad (22)$$

$$y_t^i \sim N(\bar{y}_t, \sigma_{y,t}^2) \quad (23)$$

where $a_t^i = A_t^i/P_t$ denotes the real value of household i ’s wealth at the beginning of date t . The optimal choices of the household can be summarized by the standard Euler equation:

$$e^{-\gamma c_t^i} = \beta (1+r_t) \mathbb{E}_t e^{-\gamma c_{t+1}^i} \quad (24)$$

Taking logs on both sides, the equation above can be written as:

$$-\gamma c_t^i = \ln \beta (1+r_t) + \ln \mathbb{E}_t e^{-\gamma c_{t+1}^i} \quad (25)$$

Next, we guess that the consumption decision rule of household i takes the form:

$$c_t^i = \mathcal{C}_t + \mu_t (a_t^i + y_t^i) \quad (26)$$

where \mathcal{C}_t and μ_t are deterministic processes that are common across all households. Given this guess, we can use the budget constraint (22) to write:

$$a_{t+1}^i = (1+r_t) (1-\mu_t) (a_t^i + y_t^i) - (1+r_t) \mathcal{C}_t \quad (27)$$

Using equation (27), one can express consumption at date $t + 1$ as:

$$\begin{aligned} c_{t+1}^i &= \mathcal{C}_{t+1} + \mu_{t+1} (a_{t+1}^i + y_{t+1}^i) \\ &= \mathcal{C}_{t+1} + \mu_{t+1} [(1 + r_t)(1 - \mu_t)(a_t^i + y_t^i) - (1 + r_t)\mathcal{C}_t + y_{t+1}^i] \end{aligned} \quad (28)$$

Then it is straightforward to see that:

$$\mathbb{E}_t [-\gamma c_{t+1}^i] = -\gamma \mathcal{C}_{t+1} - \gamma \mu_{t+1} [(1 + r_t)(1 - \mu_t)(a_{i,t} + y_{i,t}) - (1 + r_t)\mathcal{C}_t + \bar{y}_{t+1}] \quad (29)$$

and

$$\mathbb{E}_t (-\gamma c_{t+1}^i - E_t [-\gamma c_{t+1}^i])^2 = \gamma^2 \mu_{t+1}^2 \sigma_{y,t+1}^2 \quad (30)$$

and using the property of log-normals:

$$\ln \mathbb{E}_t e^{-\gamma c_{t+1}^i} = -\gamma \mathcal{C}_{t+1} - \gamma \mu_{t+1} [(1 + r_t)(1 - \mu_t)(a_{i,t} + y_{i,t}) - (1 + r_t)\mathcal{C}_t + \bar{y}_{t+1}] + \frac{\gamma^2 \mu_{t+1}^2 \sigma_{y,t+1}^2}{2} \quad (31)$$

Using this in the Euler equation (64) and matching coefficients:

$$\mu_t = \frac{\mu_{t+1}(1 + r_t)}{1 + \mu_{t+1}(1 + r_t)} \quad (32)$$

$$\mathcal{C}_t [1 + \mu_{t+1}(1 + r_t)] = -\frac{1}{\gamma} \ln \beta (1 + r_t) + \mathcal{C}_{t+1} + \mu_{t+1} \bar{y}_{t+1} - \frac{\gamma \mu_{t+1}^2 \sigma_{y,t+1}^2}{2} \quad (33)$$

which verifies the guess. Next, solving (33) forwards and using (32) yields equation (7) in the main text:

$$\mathcal{C}_t = \sum_{s=1}^{\infty} Q_{t+s|t} \frac{\mu_t}{\gamma \mu_{t+s}} \ln \left[\frac{1}{\beta (1 + r_{t+s-1})} \right] + \mu_t \sum_{s=1}^{\infty} Q_{t+s|t} \bar{y}_{t+s} - \frac{\gamma \mu_t}{2} \sum_{s=1}^{\infty} Q_{t+s|t} \mu_{t+s} \sigma_{y,t+s}^2 \quad (34)$$

where $Q_{t+s|t} = \prod_{k=0}^{s-1} \left(\frac{1}{1 + r_{t+k}} \right)$. If real interest rates are constant at a level $r > 0$, then (32) implies that

$$\mu_t = \mu = \frac{r}{1 + r} > 0, \forall t \quad (35)$$

A.2 Deriving the Aggregate Euler Equation

In order to derive the aggregate Euler equation, we start with the individual consumption decision rules. Since μ_t and \mathcal{C}_t do not have i superscripts, i.e. they are the same across all households, independent of wealth of income. Thus, we can linearly aggregate this economy to get an *aggregate consumption function*:

$$c_t = \int c_t^i di = \mathcal{C}_t + \mu_t \int (a_t^i + y_t^i) di = \mathcal{C}_t + \mu_t \left(\frac{B_t}{P_t} + \bar{y}_t \right) \quad (36)$$

where we have used asset market clearing $\frac{B_t}{P_t} = \int a_t^i di$ and the fact that $\bar{y}_t = \int y_t^i di$. Using (36) in (33):

$$\begin{aligned} \left[c_t - \mu_t \left(\frac{B_t}{P_t} + \bar{y}_t \right) \right] [1 + \mu_{t+1} (1 + r_t)] &= -\frac{1}{\gamma} \ln \beta (1 + r_t) + c_{t+1} - \mu_{t+1} \left(\frac{B_{t+1}}{P_{t+1}} + \bar{y}_{t+1} \right) + \mu_{t+1} \bar{y}_{t+1} \\ &\quad - \frac{\gamma \mu_{t+1}^2 \sigma_{y,t+1}^2}{2} \end{aligned}$$

Next, using (3) and (32), we can rewrite the equation above as:

$$\begin{aligned} \left[c_t - \mu_t \left(s_t + \frac{B_{t+1}}{P_{t+1}} \frac{1}{1 + r_t} + \bar{y}_t \right) \right] \left[\frac{1}{1 - \mu_t} \right] &= -\frac{1}{\gamma} \ln \beta (1 + r_t) + c_{t+1} - \mu_{t+1} \left(\frac{B_{t+1}}{P_{t+1}} + \bar{y}_{t+1} \right) \\ &\quad + \mu_{t+1} \bar{y}_{t+1} - \frac{\gamma \mu_{t+1}^2 \sigma_{y,t+1}^2}{2} \end{aligned}$$

Recall that $y_t = s_t + g_t + \bar{y}_t$ and in equilibrium, $c_t + g_t = y_t$. Combining this with the information above:

$$y_t = -\frac{1}{\gamma} \ln \beta (1 + r_t) + y_{t+1} - \frac{\gamma \mu_{t+1}^2 \sigma_{y,t+1}^2}{2} + g_t - g_{t+1} \quad (37)$$

This is the same as equation (8) in the main text and (20) is the linearized version of this equation.

A.3 Supply side

The firm's problem can be written as the Lagrangian:

$$\begin{aligned} \mathcal{L} &= \sum_{t=0}^{\infty} Q_{t|0} \left\{ \left(\frac{P_t(j)}{P_t} \right)^{1-\theta} x_t - \frac{W_t}{P_t} n_t(j) - m_t(j) - \frac{\Psi}{2} \left(\frac{P_t(j)}{P_{t-1}(j)} - 1 \right)^2 x_t \right\} \\ &\quad + \sum_{t=0}^{\infty} \xi_t Q_{t|0} \left\{ z m_t(j)^\alpha n_t(j)^{1-\alpha} - \left(\frac{P_t(j)}{P_t} \right)^{-\theta} x_t \right\} \end{aligned}$$

The first order conditions (imposing symmetric equilibrium $P_t(j) = P_t, m_t(j) = m_t$ and $n_t(j) = n_t$) are:

$$-x_t \{ (\theta - 1 - \theta \xi_t) + \Psi (\Pi_t - 1) \Pi_t \} + Q_{t+1|t} x_{t+1} \Psi (\Pi_{t+1} - 1) \Pi_{t+1} = 0 \quad (38)$$

$$\xi_t \alpha z m_t^{\alpha-1} n_t^{1-\alpha} = 1 \quad (39)$$

$$\xi_t (1 - \alpha) z m_t^\alpha n_t^{-\alpha} = \frac{W_t}{P_t} \quad (40)$$

In equilibrium, the labor market clears ($n_t = 1$) and so gross output $x_t = z m_t^\alpha$. Using this result and (39) in (38):

$$\Psi (\Pi_t - 1) \Pi_t = 1 - \theta \left(1 - \alpha^{-1} z^{-\frac{1}{\alpha}} x_t^{\frac{1-\alpha}{\alpha}} \right) + \frac{1}{1 + r_t} \frac{x_{t+1}}{x_t} \Psi (\Pi_{t+1} - 1) \Pi_{t+1}$$

Divide (40) by (39) and using this information:

$$\frac{W_t}{P_t} = \frac{1 - \alpha}{\alpha} \left(\frac{x_t}{z} \right)^{\frac{1}{\alpha}}$$

Finally, net output $y_t = x_t - m_t = x_t - (x_t/Z)^{\frac{1}{\alpha}}$ and aggregate real dividends can then be written as $d_t = x_t - \frac{1}{\alpha} \left(\frac{x_t}{z}\right)^{\frac{1}{\alpha}}$. Real wages are given by the function $\omega(y)$ which solves $y = z \left(\frac{\alpha}{1-\alpha}\right)^{\alpha} \omega^{\alpha} - \left(\frac{\alpha}{1-\alpha}\right) \omega$ with the understanding that we only consider the smaller of the two solutions for ω .

B The 4 Equation HANK Model

In this section we present the linearized model. Recall that the aggregate dynamics of our HANK economy can be fully described by:

$$y_t = -\frac{1}{\gamma} \ln \beta (1 + r_t) + y_{t+1} - \frac{\gamma \mu_{t+1}^2 \sigma^2(y_t)}{2} + g_t - g_{t+1} \quad (41)$$

$$\mu_t = \frac{\mu_{t+1} (1 + r_t)}{1 + \mu_{t+1} (1 + r_t)} \quad (42)$$

$$\Psi (\Pi_t - 1) \Pi_t = 1 - \theta \left(1 - \alpha^{-1} z^{-\frac{1}{\alpha}} x_t^{\frac{1-\alpha}{\alpha}}\right) + \frac{1}{1 + r_t} \frac{x_{t+1}}{x_t} \Psi (\Pi_{t+1} - 1) \Pi_{t+1} \quad (43)$$

$$1 + i_t = (1 + r) \Pi_t^{\Phi_{\pi}} \quad (44)$$

where $1 + r_t = \frac{1+i_t}{\Pi_{t+1}}$ and $y_t = x_t - \left(\frac{x_t}{z}\right)^{\frac{1}{\alpha}}$. The zero inflation steady state is given by:

$$\ln \beta (1 + r) = -\frac{\gamma^2 \mu^2 \sigma^2(y)}{2} \quad (45)$$

$$\mu = \frac{r}{1 + r} \quad (46)$$

$$x = z^{\frac{1}{\alpha}} \left[\frac{\alpha(\theta - 1)}{\theta} \right]^{\frac{\alpha}{1-\alpha}} \quad (47)$$

which implies that the flexible price level of output is:

$$y^* = z^{\frac{1}{\alpha}} \left[\frac{\alpha(\theta - 1)}{\theta} \right]^{\frac{\alpha}{1-\alpha}} \left[1 - \frac{\alpha(\theta - 1)}{\theta} z^{\frac{1-\alpha}{\alpha}} \right] \quad (48)$$

We choose z to normalize $y^* = 1$. It is immediate from equations (45)-(46) that the steady state of this economy must satisfy:

$$\frac{(1 + r)^2 \ln \left[\frac{1}{\beta(1+r)} \right]}{\gamma^2 r^2} = \frac{1}{2} \sigma^2(y)$$

The LHS is a decreasing function of r . In other words, a high level of idiosyncratic risk must be accompanied by a lower level of real interest rates in order to clear the savings market. It is straightforward to see that this equation has a unique solution for r which is positive since the LHS is monotonically decreasing in r and asymptotes to infinity as $r \rightarrow 0$. Another advantage of our CARA economy is that we can establish uniqueness of steady states. [Toda \(2017\)](#) shows that in a similar economy with CARA utility, there may exist multiple steady state interest rates if the labor endowment process is not i.i.d.

Next, we linearize the model. In what follows, $\hat{y}_t, \hat{i}_t, \pi_t$ and $\hat{\mu}_t$ denote the log-deviations of $y_t, 1 + i_t, \Pi_t$ and μ_t from their steady state values $y^* = 1, 1 + i = 1 + r, \Pi = 1$ and $\mu = \frac{r}{1+r}$ respectively. \hat{g}_t denotes the

deviations in levels g_t from the steady state level of $g = 0$. The linearized model is given by:

$$\hat{y}_t = \Theta \hat{y}_{t+1} - \gamma^{-1} (i_t - \pi_{t+1} - \rho_t) - \Lambda \hat{\mu}_{t+1} + \hat{g}_t - \hat{g}_{t+1} \quad (49)$$

$$\hat{\mu}_t = \tilde{\beta} (\hat{\mu}_{t+1} + i_t - \pi_{t+1}) \quad (50)$$

$$\pi_t = \tilde{\beta} \pi_{t+1} + \kappa \hat{y}_t \quad (51)$$

$$i_t = \Phi_\pi \pi_t \quad (52)$$

where $-\rho_t$ is the log-deviation of β from steady state, $\Lambda = \gamma \mu^2 \sigma^2(y^*)$, $\Theta = 1 - \frac{\gamma \mu^2}{2} \frac{d\sigma^2(y^*)}{dy_t}$, $\tilde{\beta} = \frac{1}{1+r}$ is the inverse of the steady state real interest rate. As $\Psi \rightarrow \infty$ (prices become perfectly rigid) $\kappa \rightarrow 0$.

B.1 The 3 equation RANK model

The standard 3 equation RANK model is a special case of our 4 equation HANK model. In the case of a representative agent model, $\sigma^2(y) = \frac{d\sigma^2(y)}{dy} = 0$ and $\tilde{\beta} = \beta$. Thus, in the RANK model, $\Lambda = 0$ and $\Theta = 1$. Thus, we can write the system as:

$$\hat{y}_t = \hat{y}_{t+1} - \gamma^{-1} (i_t - \pi_{t+1} - \rho_t) + \hat{g}_t - \hat{g}_{t+1} \quad (53)$$

$$\pi_t = \tilde{\beta} \pi_{t+1} + \kappa \hat{y}_t \quad (54)$$

$$i_t = \Phi_\pi \pi_t \quad (55)$$

Notice that the dynamics of $\hat{\mu}_t$ given by $\hat{\mu}_t = \tilde{\beta} (\hat{\mu}_{t+1} + i_t - \pi_{t+1})$ no longer affect the dynamics of \hat{y}_t and π_t . Thus, we can ignore that equation in the RANK model.

B.2 Determinacy properties of the RANK model under a peg

It is commonly known that if the monetary authority follows a nominal interest rate peg, $\Phi_\pi = 0$ then the standard RANK model features local indeterminacy. In other words, with $\Phi_\pi = 0$ there are multiple bounded paths of \hat{y}_t and π_t which satisfy equations (53)-(55). More generally, as long as $|\Phi_\pi| < 1$, the standard RANK model features local indeterminacy. See [Sargent and Wallace \(1975\)](#), [Bullard and Mitra \(2002\)](#) and [Galí \(2015\)](#) for a detailed exposition. This indeterminacy is generally associated with unanchored inflation. If prices are sticky $\kappa > 0$, the indeterminacy in prices also manifests itself in output. However, if prices are perfectly rigid, indeterminacy under a nominal peg manifests only in output since prices cannot move. To see this, notice that with perfectly rigid prices, the RANK model can be written as (with $\hat{g}_t = 0$ without loss of generality):

$$\hat{y}_t = \hat{y}_{t+1} \quad (56)$$

$$\pi_t = 0 \quad (57)$$

Any constant level of output is consistent with a bounded equilibrium in this case. With fixed prices, a fixed nominal interest rate translates into a fixed real interest rate. In this extreme case, output is demand determined and expectations of higher income in the future are self-fulfilling, raising income today by the same amount.

C Determinacy in HANK

Setting $\hat{g}_t = \rho_t = 0$, equations (49)-(52) can be written in matrix form as

$$\begin{bmatrix} \hat{y}_{t+1} \\ \hat{\pi}_{t+1} \\ \hat{\mu}_{t+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{\Theta} + \frac{\kappa(\gamma^{-1} - \Lambda)}{\tilde{\beta}\Theta} & \frac{(\gamma^{-1} - \Lambda)(\Phi_\pi - \tilde{\beta}^{-1})}{\Theta} & -\frac{\Lambda}{\tilde{\beta}\Theta} \\ -\frac{\kappa}{\tilde{\beta}} & \frac{1}{\tilde{\beta}} & 0 \\ -\frac{\kappa}{\tilde{\beta}} & \frac{1}{\tilde{\beta}} - \Phi_\pi & \frac{1}{\tilde{\beta}} \end{bmatrix} \begin{bmatrix} \hat{y}_t \\ \hat{\pi}_t \\ \hat{\mu}_t \end{bmatrix}$$

The characteristic polynomial of this system is:

$$\mathcal{P}(z) = \frac{1 + \gamma^{-1}\kappa\Phi_\pi}{\tilde{\beta}^2\Theta} - z^3 + \frac{\tilde{\beta}^2 + 2\tilde{\beta}\Theta + \tilde{\beta}\kappa(\gamma^{-1} - \Lambda)}{\tilde{\beta}^2\Theta} z^2 - \frac{2\tilde{\beta} + \tilde{\beta}\kappa\Phi_\pi(\gamma^{-1} - \Lambda) + \Theta + \kappa\gamma^{-1}}{\tilde{\beta}^2\Theta} z \quad (58)$$

Determinacy requires that all the roots of \mathcal{P} lie outside the unit circle. Clearly we have $\lim_{z \rightarrow -\infty} \mathcal{P}(z) = +\infty$, $\lim_{z \rightarrow +\infty} \mathcal{P}(z) = -\infty$. Under the assumption that $\gamma^{-1} > \Lambda$, $\mathcal{P}(z) > 0 \forall z \leq 0$. Thus, determinacy requires that $\mathcal{P}(1) > 0$. Otherwise, the polynomial would have at least one real root within the unit circle. Thus we need

$$\mathcal{P}(1) = \frac{(1 - \Theta)(1 - \tilde{\beta})^2}{\tilde{\beta}^2\Theta} + \frac{\kappa\gamma^{-1}(1 - \tilde{\beta}) + \tilde{\beta}\kappa\Lambda}{\tilde{\beta}^2\Theta} (\Phi_\pi - 1) > 0$$

which implies

$$\Phi_\pi > 1 + \frac{1}{\kappa} \left[\frac{\gamma(1 - \tilde{\beta})^2}{(1 - \tilde{\beta}) + \gamma\tilde{\beta}\Lambda} \right] (\Theta - 1) \quad (59)$$

Under our maintained assumptions, this condition is always necessary for determinacy. Next, we show that it is sufficient for determinacy under the following assumptions.

Assumption 1. $2\tilde{\beta} - 1 - \tilde{\beta}^3 > 0$, $\Lambda < \gamma^{-1}$, and income risk is not too countercyclical, i.e.

$$\Theta - 1 < \min \left\{ \frac{(1 - \tilde{\beta})(1 - \tilde{\beta}^2 + \gamma^{-1}\kappa) + \tilde{\beta}(\gamma^{-1}\kappa - \tilde{\beta}^2)\gamma\Lambda}{\tilde{\beta} - \frac{(1 - \tilde{\beta})^2}{(1 - \tilde{\beta}) + \beta\gamma\Lambda} - \tilde{\beta}^3(1 - \gamma\Lambda)}, \frac{1}{\tilde{\beta}\Lambda} - 1 + \frac{\gamma^{-1}\kappa}{\tilde{\beta}^2}, \frac{1 - \tilde{\beta} + \gamma^{-1}\kappa}{\tilde{\beta} - \frac{(1 - \tilde{\beta})^2}{1 - \tilde{\beta} + \beta\gamma\Lambda}} \right\}$$

Note that when $\Lambda = 0$, this last assumption reduces to

$$\Theta - 1 < \min \left\{ \frac{(1 - \tilde{\beta})(1 - \tilde{\beta}^2 + \gamma^{-1}\kappa)}{2\tilde{\beta} - 1 - \tilde{\beta}^3}, \frac{1 - \tilde{\beta} + \gamma^{-1}\kappa}{2\tilde{\beta} - 1} \right\}$$

which is strictly positive, under our other assumptions. Thus for Λ sufficiently close to zero, our risk-adjusted Taylor principle is sufficient for determinacy even under moderately countercyclical income risk. The other assumptions in 1 are satisfied for reasonable parameter values of the discount factor β as we now show.

Lemma 1. *If $\beta > \frac{\sqrt{5}}{2} - \frac{1}{2}$, then $2\tilde{\beta} - 1 - \tilde{\beta}^3 > 0$ and $\Lambda < \gamma^{-1}$.*

Proof. Recall that we define $\tilde{\beta} = \frac{1}{1+r}$, $\Lambda = \gamma \left(\frac{r}{1+r} \right)^2 \sigma_y^2$, and r solves

$$\frac{(1+r)^2 \ln \left[\frac{1}{\beta(1+r)} \right]}{\gamma^2 r^2} = \frac{1}{2} \sigma_y^2$$

It is immediate that $\tilde{\beta} \in (\beta, 1)$, so $\tilde{\beta} > \beta > \frac{\sqrt{5}}{2} - \frac{1}{2}$ and $2\tilde{\beta} - 1 - \tilde{\beta}^3 > 0$. Using the definition of Λ and the fact that $r > 0$, and rearranging,

$$\gamma \Lambda < 2 \ln \left(\frac{1}{\beta} \right) < 2 \ln \left(\frac{2}{\sqrt{5} - 1} \right) < 1$$

So we are done. \square

We can now provide sufficient conditions for determinacy. Lemma 2 provides sufficient conditions for a general cubic polynomial to have all its roots outside the unit circle, as required for determinacy. Lemma 3 then concludes by showing that these conditions are satisfied if Assumption 1 holds and (59) is satisfied.

Lemma 2. *Consider the characteristic polynomial*

$$\mathcal{P}(z) = -z^3 + A_2 z^2 + A_1 z + A_0 = (z_1 - z)(z_2 - z)(z_3 - z)$$

Suppose $A_2 > 0, A_1 < 0, A_0 > 0$. Then the following two conditions are sufficient³² for \mathcal{P} to have three roots outside the unit circle:

$$A_0^2 - A_0 A_2 - A_1 - 1 > 0 \tag{60}$$

$$-1 + A_2 + A_1 + A_0 > 0 \tag{61}$$

$$A_1 > 1 \tag{62}$$

Proof. Assume the conditions hold. Since $\mathcal{P}(1) = -1 + A_2 + A_1 + A_0 > 0$ and $\lim_{z \rightarrow +\infty} \mathcal{P}(z) = -\infty$, there is at least one real root above 1; let this be z_3 . Either z_1, z_2 are complex conjugates, or they are both real. Suppose they are complex conjugates. Note that

$$(z_1 z_2 - 1)(z_2 z_3 - 1)(z_3 z_1 - 1) = A_0^2 - A_0 A_2 - A_1 - 1 > 0$$

$z_2 z_3 - 1$ and $z_3 z_1 - 1$ are complex conjugates, so their product is a positive real number. So we must have $z_1 z_2 = |z_1| = |z_2| > 1$, i.e. all eigenvalues lie outside the unit circle in this case. Suppose then that z_1, z_2 are both real. $\mathcal{P}(0) = A_0 > 0$, so \mathcal{P} has either two real roots in $(0, 1)$ or none (in which case we are done, since it has no negative real roots). Suppose $z_1, z_2 \in (0, 1)$. By (61), we have

$$z_3^2(z_1 z_2 - 1)(z_1 - z_3^{-1})(z_2 - z_3^{-1}) > 0$$

$z_3^2(z_1 z_2 - 1) < 0$ by assumption, so we must have (letting $z_1 < z_2$ without loss of generality)

$$0 < z_1 < z_3^{-1} < z_2 < 1$$

³²The first two conditions are also necessary.

So $z_1 z_2 z_3 < z_3^{-1} z_2 z_3 = z_2 < 1$. Since we have assumed $A_0 = z_1 z_2 z_3 < 1$, this case is ruled out. Then it must be that all eigenvalues lie outside the unit circle. \square

Under Assumption 1, (59) is sufficient to ensure that the conditions in Lemma 2 obtain:

Lemma 3. *Suppose Assumption 1 holds and Φ_π satisfies (59). Then $A^{-1}B$ has three eigenvalues outside the unit circle.*

Proof. Suppose (59) holds: then (61) holds. Given our assumptions, we have $A_0, -A_1, A_2 > 0$. It only remains to show (60) and (62). Using the definition of the characteristic polynomial in (58), some algebra yields

$$\begin{aligned} A_0^2 - A_0 A_2 - A_1 - 1 &= \underbrace{\beta^{-1} \left(\frac{1 + \gamma^{-1} \kappa \Phi_\pi}{\tilde{\beta} \Theta} - 1 \right)}_{B_1} \underbrace{\left(\frac{1 + \gamma^{-1} \kappa \Phi_\pi}{\tilde{\beta}^2 \Theta} - \frac{\tilde{\beta} + \Theta + \kappa(\gamma^{-1} - \Lambda)}{\tilde{\beta} \Theta} + \beta \gamma (\gamma^{-1} - \Lambda) \right)}_{B_2} \\ &\quad + \underbrace{\frac{\gamma}{\beta \Theta} + \gamma \left(\frac{\gamma^{-1} \kappa}{\tilde{\beta}^2 \Theta} - 1 \right) \Lambda}_{B_3} > 0 \end{aligned}$$

We will show $B_1, B_2, B_3 > 0$. First take B_2 . Multiplying through by the positive number $\tilde{\beta}^2 \Theta$ and using the lower bound on Φ_π given by (59), we have

$$\tilde{\beta}^2 \Theta B_2 = (1 - \tilde{\beta})[\kappa \gamma^{-1} + 1 - \tilde{\beta}^2] + \tilde{\beta}(\gamma^{-1} \kappa - \tilde{\beta}^2) \gamma \Lambda - (\Theta - 1) \left[\tilde{\beta} - \frac{(1 - \tilde{\beta})^2}{(1 - \tilde{\beta}) + \tilde{\beta} \gamma \Lambda} - \tilde{\beta}^3 (1 - \gamma \Lambda) \right]$$

The term in square brackets is minimized when $\Lambda = 0$, in which case it equals $2\tilde{\beta} - 1 - \tilde{\beta}^3 > 0$. So it is positive, and we will have $B_2 > 0$ provided that

$$\Theta - 1 < \frac{(1 - \tilde{\beta})[\kappa \gamma^{-1} + 1 - \tilde{\beta}^2] + \tilde{\beta}(\gamma^{-1} \kappa - \tilde{\beta}^2) \gamma \Lambda}{\tilde{\beta} - \frac{(1 - \tilde{\beta})^2}{(1 - \tilde{\beta}) + \tilde{\beta} \gamma \Lambda} - \tilde{\beta}^3 (1 - \gamma \Lambda)}$$

which is guaranteed by Assumption 1. Next we show $B_3 > 0$. We have

$$B_3 = \frac{\gamma}{\beta} + \frac{\kappa \Lambda}{\beta^2} - \gamma \Lambda \Theta$$

which will be positive provided that

$$\Theta < \frac{1}{\tilde{\beta} \Lambda} + \frac{\gamma^{-1} \kappa}{\tilde{\beta}^2}$$

as ensured by Assumption 1. Next we show $B_1 > 0$. Given (59),

$$B_1 = \frac{1 + \gamma^{-1} \kappa \Phi_\pi}{\tilde{\beta} \Theta} - 1 > \frac{1}{\tilde{\beta} \Theta} \left[1 + \gamma^{-1} \kappa + \left(\frac{(1 - \tilde{\beta})^2}{(1 - \tilde{\beta}) + \gamma \tilde{\beta} \Lambda} \right) (\Theta - 1) - \beta \Theta \right]$$

which is positive provided that

$$\Theta - 1 < \frac{1 - \tilde{\beta} + \gamma^{-1}\kappa}{\tilde{\beta} - \frac{(1-\tilde{\beta})^2}{1-\tilde{\beta}+\beta\gamma\Lambda}}$$

as ensured by Assumption 1. This establishes that (60) is satisfied. To apply Lemma 2 we only need to check condition (62), i.e.

$$A_0 = \frac{1 + \gamma^{-1}\kappa\Phi_\pi}{\tilde{\beta}^2\Theta} > 1$$

Since $B_1 = \frac{1+\gamma^{-1}\kappa\Phi_\pi}{\tilde{\beta}\Theta} - 1 > 0$ and $\tilde{\beta} \in (0, 1)$, this is immediate. So we are done. \square

D Deriving the stationary distribution of cash-on-hand

The baseline model in the main text does not admit a stationary distribution of cash on hand as individual wealth follows a random walk under the optimal savings decisions. In this appendix, we augment the basic model to generate a stationary distribution so that we can derive an expression for the fraction of households who have negative consumption. We do so by introducing a perpetual youth structure. As we show next, while this change allows us to derive a stationary cash-on-hand distribution, it does not alter the aggregate dynamics of the economy up to the first order.

There is a continuum of households in the economy indexed by $i \in [0, 1]$. A household alive at date t survives until date $t + 1$ with probability $\vartheta \in (0, 1)$. Each period a measure $1 - \vartheta$ of households are born with zero wealth. Households can borrow and save in one period, nominal actuarial bonds. Buying one actuarial bond at date t entitles the bearer to \$1 if she survives until date $t + 1$; selling one bond obliges the bearer to pay \$1 if she survives until date $t + 1$. Intermediaries can trade both actuarial bonds and nominal one-period government bonds, which have price $\frac{1}{1+i_t}$. No arbitrage implies that the price of an actuarial bond is $\frac{\vartheta}{1+i_t}$. Household i solves

$$\begin{aligned} \max \quad & -\frac{1}{\gamma} \mathbb{E}_0 \sum_{t=0}^{\infty} (\beta\vartheta)^t e^{-\gamma c_t^i} \\ \text{subject to} \quad & P_t c_t^i + \frac{\vartheta}{1+i_t} A_{t+1}^i = A_t^i + P_t y_t^i \end{aligned}$$

Otherwise, the model is as in the main text. Letting $1 + r_t$ denote the real interest rate on government debt and a_t^i real holdings of actuarial bonds, the budget constraint is

$$a_{t+1}^i = \frac{1+r_t}{\vartheta} [a_t^i + y_t^i - c_t^i]$$

and the household Euler equation is

$$e^{-\gamma c_t^i} = \beta\vartheta \frac{1+r_t}{\vartheta} \mathbb{E}_t e^{-\gamma c_{t+1}^i} \quad (63)$$

where \mathbb{E}_t denotes the expectation conditional on survival. Taking logs on both sides, the equation above

can be written as:

$$-\gamma c_t^i = \ln \beta (1 + r_t) + \ln \mathbb{E}_t e^{-\gamma c_{t+1}^i} \quad (64)$$

which is the same as in the baseline model. Following the same steps as we did in Appendix A.1, one can show that the consumption decision of a household can be written as:

$$c_t^i = C_t + \mu_t(a_t^i + y_t^i)$$

where

$$C_t [1 + \mu_{t+1} (1 + r_t)] = -\frac{1}{\gamma} \ln \beta (1 + r_t) + C_{t+1} + \mu_{t+1} y_{t+1} - \frac{\gamma \mu_{t+1}^2 \sigma_{y,t+1}^2}{2} \quad (65)$$

$$\mu_t = \frac{\mu_{t+1} (1 + r_t)}{\vartheta + \mu_{t+1} (1 + r_t)} \quad (66)$$

As before, we can impose market clearing and (65) can be written as:

$$Y_t = -\frac{1}{\gamma} \ln [\beta(1 + r)] + Y_{t+1} - \frac{\gamma}{2} \mu_{t+1}^2 \sigma_{y,t+1}^2 \quad (67)$$

Linearizing (66) and (67) around steady state yields:

$$\begin{aligned} \hat{y}_t &= \Theta \hat{y}_{t+1} - \frac{1}{\gamma} (i_t - \pi_{t+1}) - \Lambda \hat{\mu}_{t+1} \\ \hat{\mu}_t &= \tilde{\beta} (i_t - \pi_{t+1} + \hat{\mu}_{t+1}) \end{aligned}$$

where $\tilde{\beta}$ is now equal to $\vartheta/(1 + r)$. In a similar fashion to the baseline model, one can linearize the firm's first order condition to get the Phillips curve:³³

$$\pi_t = \tilde{\beta} \pi_{t+1} + \kappa \hat{y}_t$$

Thus, the dynamics of the aggregate variables are identical to the model in the main text. However, unlike the baseline model, this version allows for a stationary distribution of cash-on-hand. To derive this, notice that individual wealth is a random walk (for survivors) where innovations have variance σ_y^2 . Define cash on hand $x_t^i = a_t^i + y_t^i$.

We assume that population is constant and normalized to 1. In steady state, at each date t we have $1 - \vartheta$ households entering the economy with 0 assets. Thus, cash-on-hand for this group is a random variable $x_{t|t}^i \sim N(y, \sigma_y^2)$ and the mass of such households with cash-on-hand less than x in the economy is $(1 - \vartheta) \Phi\left(\frac{x-y}{\sigma_y}\right)$ where Φ denotes the standard normal cdf. Among the households that entered at $t - 1$, only a fraction ϑ survive till date t and their distribution of cash on hand is given by $x_{t|t-1}^i \sim N(y, 2\sigma_y^2)$. The mass of such households with cash-on-hand below x is $(1 - \vartheta) \vartheta \Phi\left(\frac{x-y}{\sqrt{2}\sigma_y}\right)$. In a similar fashion, one can go back further in time and derive the distribution of cash-on-hand today of survivors from cohorts born at any date in the past. As a result, the cdf of the cash-on-hand distribution in steady state can be

³³Here we are assuming that the firm discounts its profits at the same rate as is paid by the actuarial bond.

written as:

$$F(x) = (1 - \vartheta) \sum_{k=1}^{\infty} \vartheta^{k-1} \Phi \left(\frac{x - y}{\sigma_y \sqrt{k}} \right)$$

and since consumption is linear in x : $c^i = (1 - \mu)y + \mu c^i$, the steady state distribution of consumption is:

$$G(c) = \Pr[(1 - \mu)y + \mu c^i < c] = F \left(\frac{c - (1 - \mu)y}{\mu} \right) = (1 - \vartheta) \sum_{k=1}^{\infty} \vartheta^{k-1} \Phi \left(\frac{c - y}{\mu \sigma_y \sqrt{k}} \right)$$

Thus, the fraction of households with negative consumption in steady state is given by:

$$G(0) = (1 - \vartheta) \sum_{k=1}^{\infty} \vartheta^{k-1} \Phi \left(\frac{-y}{\mu \sigma_y \sqrt{k}} \right)$$

Following [Benhabib and Bisin \(2006\)](#), we set the survival probability $\vartheta = 1 - 62^{-1}$ consistent with a average working life of 62 years. The rest of the parameters follow the same the same calibration we employ to make plots in the main paper. This yields $G(0) = 0.0006$.

E Model with persistent individual income

The model in the main paper assumes that idiosyncratic shocks to labor endowment are i.i.d. and hence so is income. The setup can be easily extended to a case in which the idiosyncratic income is persistent. This appendix shows that all our results remain unchanged in this case.

E.1 Household Problem

The stochastic process governing each household's labor endowment is:

$$\ell_t^i = (1 - \lambda) \bar{\ell} + \lambda \ell_{t-1}^i + \sigma_{\ell,t} e_t^i \quad (68)$$

where $e_t^i \sim N(0, 1)$. $\lambda \in [0, 1]$ denotes the autocorrelation in ℓ_t .³⁴ Also, the budget constraint (in real terms) of each household can still be written as:

$$a_{t+1}^i = (1 + r_t) (a_t^i + y_t^i - c_t^i)$$

where $1 + r_t$ denotes the real interest rate. As in the main text, the income of household i is:

$$\begin{aligned} y_t^i &= (1 - \tau_t) \omega_t \ell_t^i + d_{i,t} + \frac{T_t}{P_t} \\ &= \underbrace{d_t + \frac{T_t}{P_t}}_{\bar{y}_t} + [(1 - \tau_t) \omega_t + \delta_t] \bar{\ell} + \underbrace{[(1 - \tau_t) \omega_t + \delta_t]}_{\varpi_t} (\ell_t^i - \bar{\ell}) \end{aligned} \quad (69)$$

where we have used the definition of $d_{i,t}$ from the main text. As in our baseline case, \bar{y}_t and ϖ_t are deterministic processes since our environment does not feature aggregate risk. Notice that λ also denotes

³⁴With $\lambda = 0$, this nests our baseline model.

the auto-correlation of income y_t^i . As before, the Euler equation of household i can be written as:

$$-\gamma c_t^i = \ln \beta (1 + r_t) + \ln \mathbb{E}_t e^{-\gamma c_{t+1}^i} \quad (70)$$

Next, guess that the consumption function can be written as:

$$c_t^i = C_t + \mu_t (a_t^i + \bar{y}_t) + \zeta_t (\ell_t^i - \bar{\ell})$$

Plugging this into the household budget constraint yields:

$$a_{t+1}^i = (1 + r_t) [(1 - \mu_t) (a_t^i + \bar{y}_t) + (\varpi_t - \zeta_t) (\ell_t^i - \bar{\ell}) - C_t]$$

which can be used to derive an expression for consumption at date $t + 1$:

$$\begin{aligned} c_{t+1}^i &= C_t + \mu_{t+1} (a_{t+1}^i + \bar{y}_{t+1}) + \zeta_{t+1} (\ell_{t+1}^i - \bar{\ell}) \\ &= C_{t+1} + \mu_{t+1} \bar{y}_{t+1} - \mu_{t+1} (1 + r_t) C_t + \mu_{t+1} (1 + r_t) (1 - \mu_t) (a_t^i + \bar{y}_t) \\ &\quad + [\lambda \zeta_{t+1} + \mu_{t+1} (1 + r_t) (\varpi_t - \zeta_t)] (\ell_t^i - \bar{\ell}) + \zeta_{t+1} \sigma_{\ell,t+1} e_{t+1}^i \end{aligned}$$

where we have also used (68). Thus, we have:

$$\begin{aligned} \mathbb{E}_t c_{t+1}^i &= C_{t+1} + \mu_{t+1} \bar{y}_{t+1} - \mu_{t+1} (1 + r_t) C_t + \mu_{t+1} (1 + r_t) (1 - \mu_t) (a_t^i + \bar{y}_t) \\ &\quad + [\lambda \zeta_{t+1} + \mu_{t+1} (1 + r_t) (\varpi_t - \zeta_t)] (\ell_t^i - \bar{\ell}) \end{aligned}$$

and

$$\mathbb{V}_t c_{t+1}^i = \zeta_{t+1}^2 \sigma_{\ell,t+1}^2$$

and since c_{t+1}^i is a normal random variable, we can rewrite the euler equation as:

$$\begin{aligned} c_t^i &= -\frac{1}{\gamma} \ln [\beta(1 + r_t)] - \frac{1}{\gamma} \ln \mathbb{E}_t e^{-\gamma c_t^i} \\ &= -\frac{1}{\gamma} \ln [\beta(1 + r_t)] - \frac{1}{\gamma} \ln e^{-\gamma \mathbb{E}_t c_t^i + \frac{\gamma^2}{2} \mathbb{V}_t(c_t^i)} \\ &= -\frac{1}{\gamma} \ln [\beta(1 + r_t)] C_{t+1} + \mu_{t+1} \bar{y}_{t+1} - \mu_{t+1} (1 + r_t) C_t + \mu_{t+1} (1 + r_t) (1 - \mu_t) (a_t^i + \bar{y}_t) \\ &\quad + [\lambda \zeta_{t+1} + \mu_{t+1} (1 + r_t) (\varpi_t - \zeta_t)] (\ell_t^i - \bar{\ell}) - \frac{1}{2} \gamma \zeta_{t+1}^2 \sigma_{\ell,t+1}^2 \end{aligned} \quad (71)$$

Matching coefficients and rearranging, we have:

$$C_t = -\frac{1}{\gamma} \ln [\beta(1 + r_t)] + C_{t+1} + \mu_{t+1} \bar{y}_{t+1} - \mu_{t+1} (1 + r_t) C_t - \frac{1}{2} \gamma \zeta_{t+1}^2 \sigma_{\ell,t+1}^2 \quad (72)$$

$$\mu_t = \frac{\mu_{t+1} (1 + r_t)}{1 + \mu_{t+1} (1 + r_t)} \quad (73)$$

$$\frac{\zeta_t}{\mu_t} = \lambda (1 + r_t)^{-1} \frac{\zeta_{t+1}}{\mu_{t+1}} + \varpi_t \quad (74)$$

Solving (74) forwards reveals that:

$$\zeta_t = \mu_t \sum_{s=0}^{\infty} \lambda^s Q_{t+s|t} \varpi_{t+s} = \frac{\sum_{s=0}^{\infty} \lambda^s Q_{t+s|t} \varpi_{t+s}}{\sum_{s=0}^{\infty} Q_{t+s|t}}$$

Because individual income is now persistent, ζ_t denotes the MPC out of a positive shock to ℓ_t^i at date t . Such a shock increases the present discounted value of lifetime income of the household by $\sum_{s=0}^{\infty} \lambda^s Q_{t+s|t} \varpi_{t+s}$ and agents consume a fraction μ_t out of this increase in lifetime income. Using this expression, one can write the consumption function as:

$$\begin{aligned} c_t^i &= C_t + \mu_t (a_t^i + y_t^i) + \mu_t \sum_{s=1}^{\infty} \lambda^s Q_{t+s|t} \varpi_{t+s} (\ell_t^i - \bar{\ell}) \\ &= C_t + \mu_t \left(a_t^i + y_t^i + \mathbb{E}_t \sum_{s=1}^{\infty} Q_{t+s|t} \varpi_{t+s} (\ell_{t+s}^i - \bar{\ell}) \right) \end{aligned} \quad (75)$$

which nests the case in the main text when $\lambda = 0$.

E.2 General Equilibrium

In equilibrium, goods market and asset markets must clear. This requires $c_t = \bar{y}_t = y_t = \int y_t^i di$ and $\int a_t^i di = 0$.³⁵ To get an expression for aggregate consumption, we can aggregate (75):

$$c_t = C_t + \mu_t y_t$$

Using goods market clearing this implies that $C_t = (1 - \mu_t) y_t$. Using this in (72) along with (73):

$$y_t = -\frac{1}{\gamma} \ln [\beta(1 + r_t)] + y_{t+1} - \frac{1}{2} \gamma \zeta_{t+1}^2 \sigma_{\ell, t+1}^2 \quad (76)$$

Finally, $\varpi_t = \varpi(y_t)$ where y_t denotes aggregate output and $\varpi'(y)$ denotes the cyclicity of risk.

E.3 The linearized economy

Log-linearizing (73), (74) and (76) around the deterministic steady state, the linearized demand block is:

$$\hat{y}_t = -\gamma^{-1} (i_t - \pi_{t+1}) + \hat{y}_{t+1} - \Lambda \hat{\zeta}_{t+1} \quad (77)$$

$$\hat{\mu}_t = \tilde{\beta} (i_t - \pi_{t+1} + \hat{\mu}_{t+1}) \quad (78)$$

$$\hat{\zeta}_t = \tilde{\beta} \lambda \hat{\zeta}_{t+1} + (1 - \lambda) \hat{\mu}_t - \frac{\Theta - 1}{\Lambda} \hat{y}_t \quad (79)$$

where $\Lambda = \gamma \left(\frac{r}{1+r-\lambda} \right)^2 \sigma_y^2$ and $\Theta = 1 - \gamma \zeta^2 \left(1 - \tilde{\beta} \lambda \right) \frac{\sigma_{\ell}^2 \varpi'(y^*)}{\varpi(y^*)} = 1 - \frac{1}{2} \frac{\gamma(1-\tilde{\beta})^2}{1-\tilde{\beta}\lambda} \frac{d\sigma_y^2}{dy}$ denotes the cyclicity of risk as before.³⁶ Here $\sigma_y^2 = \varpi(y^*)^2 \sigma_{\ell}^2$ is the variance of innovations to income in steady state. Notice that while Θ does not appear in front of \hat{y}_{t+1} in (77), it does appear in (79) which describes the MPC out of labor endowment shocks. In the limit as $\lambda \rightarrow 0$, (79) can be substituted into (77), yielding the same Euler

³⁵We have assumed $g_t = T_t = B_t = 0 \forall t$ without loss of generality. This can be relaxed without affecting any of the results.

³⁶In deriving the expressions above we have used the fact that in steady state $\mu = 1 - \tilde{\beta}$ and $\zeta = (1 - \tilde{\beta}) / (1 - \tilde{\beta}\lambda) \varpi(y^*)$.

equation as in the baseline. The rest of the model consists of the Phillips curve and the interest rate rule which are the same as in the baseline model:

$$\pi_t = \tilde{\beta}\pi_{t+1} + \kappa\hat{y}_t \quad (80)$$

$$i_t = \Phi_\pi\pi_t \quad (81)$$

Next we establish the determinacy properties of this model and show that they depend crucially on the cyclicity of income risk. As in the main text, we start with the special case in which prices are fixed $\pi_t = 0$ for all t and nominal interest rates are pegged, i.e. $i_t = 0$.

E.4 Determinacy of Equilibrium

E.4.1 Special case: rigid prices + nominal rate peg

In this case, the system of five equations above can be reduced to two equations:

$$\begin{aligned} \hat{y}_t &= \hat{y}_{t+1} - \Lambda\hat{\zeta}_{t+1} \\ \hat{\zeta}_t &= \tilde{\beta}\lambda\hat{\zeta}_{t+1} - \frac{\Theta - 1}{\Lambda}\hat{y}_t \end{aligned}$$

This system can be expressed in matrix form as:

$$\begin{bmatrix} \hat{y}_t \\ \hat{\zeta}_t \end{bmatrix} = \begin{bmatrix} 1 & -\Lambda \\ -\frac{\Theta-1}{\Lambda} & \Theta - 1 + \tilde{\beta}\lambda \end{bmatrix} \begin{bmatrix} \hat{y}_{t+1} \\ \hat{\zeta}_{t+1} \end{bmatrix}$$

The characteristic polynomial of the system above can be written as:

$$\mathcal{P}(\varphi) = \varphi^2 - (\tilde{\beta}\lambda + \Theta)\varphi + \tilde{\beta}\lambda$$

Case 1: Acyclical Risk $\Theta = 1$ In this case, the characteristic polynomial simplifies to:

$$\mathcal{P}(\varphi) = \varphi^2 - (\tilde{\beta}\lambda + 1)\varphi + \tilde{\beta}\lambda$$

which implies that the two eigenvalues are $\varphi = \tilde{\beta}\lambda$ and $\varphi = 1$. Since only one root is inside the unit circle, this system features indeterminacy.

Case 2: Countercyclical Risk $\Theta > 1$ Notice that evaluating at $\varphi = 1$ we have $\mathcal{P}(1) = 1 - \Theta < 0$ and $\varphi \rightarrow \infty$ we have $\mathcal{P}(\varphi) \rightarrow \infty$. This means that there is one root between 1 and ∞ (outside the unit circle). Consequently, countercyclical income risk results in indeterminacy under a peg.

Case 3: Procyclical Risk $0 < \Theta < 1$ With $\Theta \in (0, 1)$, we have:

$$\begin{aligned} \mathcal{P}(0) &= \tilde{\beta}\lambda \in (0, 1) \\ \mathcal{P}(1) &= 1 - \Theta \in (0, 1) \\ \mathcal{P}(\Theta) &= \tilde{\beta}\lambda(1 - \Theta) = \mathcal{P}(0)\mathcal{P}(1) \end{aligned}$$

So $\mathcal{P}()$ is convex. If there are real roots they must be between 0 and 1 which implies determinacy. If the roots are complex, then the roots are complex conjugates i.e., $|\varphi_1| = |\varphi_2|$. Consequently, $\varphi_1\varphi_2 = |\varphi_1|^2 = \tilde{\beta}\lambda \in (0, 1)$. So both complex roots lie in the unit circle and hence there is determinacy.

E.4.2 Determinacy: General Case

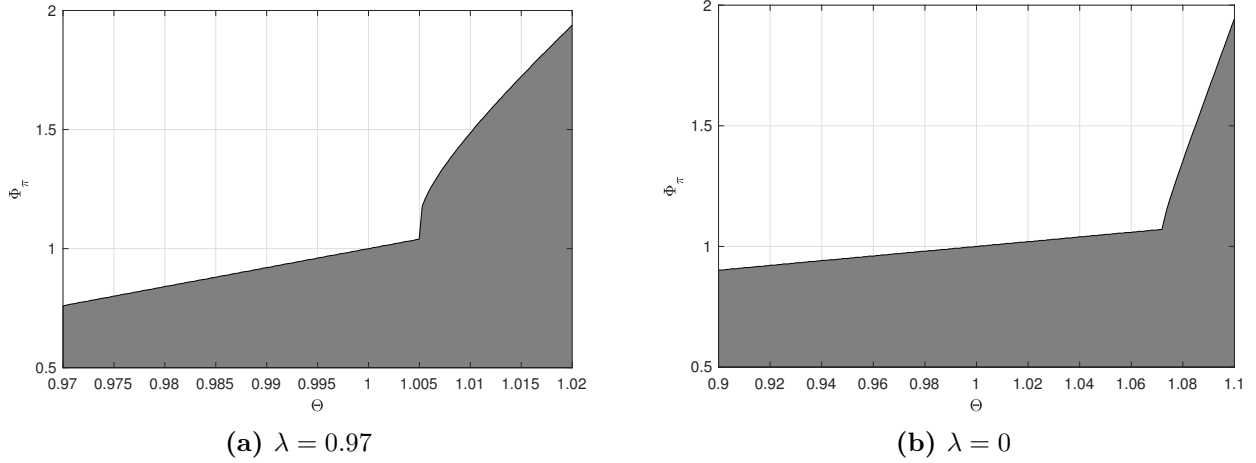


Figure 3. Combinations of Θ and Φ_π which result in local determinacy.

In the general case without rigid prices, while it is hard to derive an analytical condition for determinacy we numerically show that numerically that the cyclical risk Θ is still the key. In this case, the dynamic 4×4 system can be written as:

$$\begin{bmatrix} 1 + \frac{\kappa}{\beta\gamma} + \frac{\Theta-1}{\beta\lambda} & -\frac{1}{\gamma} \left(\frac{1}{\beta} - \Phi_\pi \right) & -\frac{(1-\lambda)\Lambda}{\beta\lambda} & \frac{\Lambda}{\beta\lambda} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} & 0 & 0 \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} - \Phi_\pi & \frac{1}{\beta} & 0 \\ \frac{\Theta-1}{\beta\Lambda} & 0 & -\frac{1-\lambda}{\beta\lambda} & \frac{1}{\beta\lambda} \end{bmatrix} \begin{bmatrix} \hat{y}_t \\ \pi_t \\ \hat{\mu}_t \\ \hat{\zeta}_t \end{bmatrix} = \begin{bmatrix} \hat{y}_{t+1} \\ \pi_{t+1} \\ \hat{\mu}_{t+1} \\ \hat{\zeta}_{t+1} \end{bmatrix}$$

Determinacy requires that all four eigenvalues of the system lie outside the unit circle. Figure 3 plots Θ on the x-axis and the Taylor rule coefficient Φ_π on the y-axis. The left panel is the case in which income is highly autocorrelated, $\lambda = 0.97$ based on [Heathcote et al. \(2010\)](#) while the right panel corresponds to the i.i.d. case $\lambda = 0$ described in the main text. Here we set the steady state variance of income shocks to 0.02 following [Heathcote et al. \(2010\)](#); the other parameters are the same as in the baseline calibration. The dark shaded region depicts the combinations of (Θ, Φ_π) for which the economy features local indeterminacy. As can be seen in both panels, for procyclical risk, $\Theta \in (0, 1)$, the economy can display determinacy even when the standard Taylor principle fails ($\Phi_\pi < 1$). More generally, the two panels show that regardless of λ , a higher cyclical risk Θ requires that the monetary authority respond in a stronger fashion to inflation in order to maintain determinacy.

E.4.3 Forward Guidance Puzzle and Fiscal Multipliers

Figures 4 and 5 show that our characterization in Sections 4.2 and 4.3 continue to hold in the case with persistent rather than i.i.d income. As in the main paper, the figures show that the strength of forward

guidance is weakest when risk is procyclical $\Theta < 1$. The same is true for the magnitude of fiscal multipliers in a liquidity trap. Notice that the difference between outcomes in RANK and acyclical HANK is, if anything, larger than in the baseline model, suggesting that the strength of the precautionary savings channel is amplified with persistent individual income (of course this depends on calibration). We discuss this next.

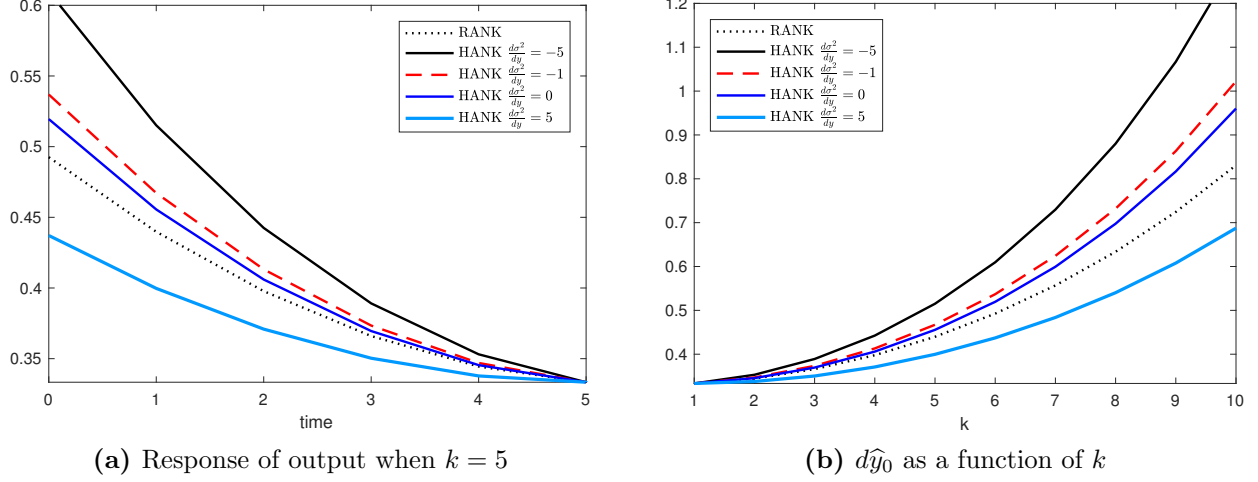


Figure 4. Response of output to a unit drop in nominal interest rates k periods in the future.

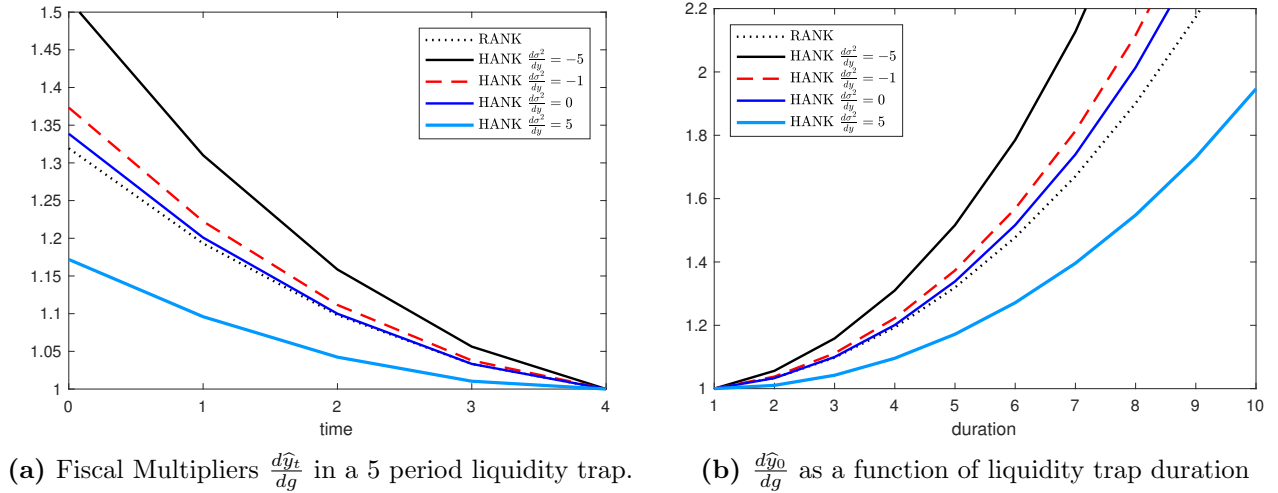


Figure 5. Fiscal Multipliers

E.5 Strength of the precautionary savings channel of monetary policy

Recall that in the model with i.i.d shocks we identified $-\Lambda\hat{\mu}_{t+1}$ and (78) as the precautionary savings channel of monetary policy. When policy raises interest rates at date $t + 1$, the MPC rises, consumption becomes more dependent on current income and therefore more risky, and the precautionary savings motive increases, depressing demand at date t .

Moving from i.i.d. income $\lambda = 0$ to $\lambda > 0$ has ambiguous effects on the strength of this precautionary savings channel. With persistent income, this channel can no longer be identified with the term $-\Lambda\hat{\mu}_{t+1}$, since the MPC out of the current labor-endowment shock is now ζ_t , which differs from the MPC out of

wealth μ_t (see equation (77)). In general, (79) shows that $\hat{\zeta}_t$ in turn depends on the future path of both real interest rates and aggregate output. For the purposes of studying the strength of the precautionary savings channel, however, it suffices to consider acyclical risk ($\Theta = 1$) in which case $\hat{\zeta}_t$ depends only on the path of real interest rates:

$$\hat{\zeta}_t = (1 - \lambda) \sum_{s=0}^{\infty} \sum_{k=0}^{\infty} \lambda^s \tilde{\beta}^{s+k} r_{t+k+s}$$

Overall, the effect of a change in λ on the strength of the precautionary savings channel $-\Lambda_{\hat{\zeta}_{t+1}}$ is non-monotonic. More persistent income (higher λ) strictly reduces the sensitivity of $\hat{\zeta}_t$ to changes in interest rates at any future horizon. At the same time however, higher λ increases the steady state value of ζ making consumption risk higher and increasing $\Lambda = \gamma \zeta^2 \sigma_\ell^2$. To see this, suppose monetary policy announces at date t a rate hike at $t + 1$. The effect on the precautionary savings motive is:

$$\frac{d}{dr_{t+1}} \Lambda_{\hat{\zeta}_{t+1}} = \gamma \sigma_\ell^2 \varpi^2 (1 - \lambda) \left(\frac{r}{1 + r - \lambda} \right)^2$$

The size of this effect is increasing in λ for $\lambda < 1 - r$, decreasing when $\lambda > 1 - r$. In this sense, even a substantial increase in the persistence of individual income can actually increase the strength of the precautionary savings channel of monetary policy: e.g. if $r = 4$ percent in steady state, this would be true as long as $\lambda \in [0, 0.96)$. However, as λ goes to 1, this channel becomes weaker than in the i.i.d. case. In fact when $\lambda = 1$, the precautionary savings channel of monetary policy vanishes.

Intuitively, an increase in the persistence of individual income has two effects on the sensitivity of desired precautionary saving to changes in interest rates. On the one hand, persistence reduces the sensitivity of the MPC to interest rates. When income is persistent, they affect earnings in the future as well as today. Thus, a change in interest rates - the relative price of current and future earnings - has less effect on the innovation to permanent income and consumption. In fact in the limit as $\lambda \rightarrow 1$, any shock increases permanent income and consumption one-for-one, whatever the time path of interest rates.

On the other hand, persistent income increases the average level of MPC, for the same reason - when income is persistent, shocks have a larger effect on permanent income. Thus, with a higher λ , households have a higher MPC but one which is less sensitive to changes in interest rates. The precautionary savings motive depends not on the level of the MPC but on consumption risk, i.e. and therefore on MPC squared. Even a small increase in the sensitivity of consumption to income shocks can have a large effect on desired precautionary savings when households are already exposed to substantial consumption risk. This is why the effect of λ on the sensitivity of precautionary savings to interest rates is non-monotonic. For $\lambda < 1 - r$, the second effect (the higher average MPC) dominates. For $\lambda > 1 - r$, the first effect (less sensitive MPC) dominates.

E.6 Interpreting Holm (2017) through the lens of our model

Holm (2018) argues that an increase in income risk weakens monetary transmission. This might seem inconsistent with our argument that uninsurable income risk introduces a new channel of monetary policy transmission, because changes in interest rates change the sensitivity of consumption to income and therefore the income risk faced by households. On closer inspection however, our results are consistent with Holm - in fact, our framework can be used to explain his results in a model permitting closed-form

solutions. [Holm \(2018\)](#) performs two quite different exercises to support the claim that an increase in income risk weakens monetary transmission:

1. [Holm \(2018\)](#) uses a continuous time partial equilibrium model with a constant real interest rate and constant level of income risk and argues that consumption is less sensitive to interest rate changes if risk is higher. He shows this by computing the cross-derivative of the steady state consumption function with respect to the steady state real interest rate and variance of innovations to income, for a household with zero wealth. He finds that in the neighborhood of zero risk, this cross-derivative is positive (higher r reduces consumption less when risk is high) if the mean reversion of income is high enough (specifically, if the rate of mean reversion of the income process is less than the interest rate).
2. In a general equilibrium HANK model, [Holm \(2018\)](#) compares two economies with a different standard deviation of idiosyncratic risk. He varies the discount rate in the two economies to ensure that the steady state real interest rate is the same in both. Next, he subjects both economies to a persistent monetary policy shock which raises real interest rate by 1 basis point on impact. He finds that the impact effect on output of the monetary policy shock is smaller when risk is higher.

In this Appendix we evaluate these claims through the lens of our model and provide a caveat to the conclusion of [Holm \(2018\)](#). As we show next, while the first claim is also true in the context of our model, the second one is true *only* if risk is procyclical. Thus, a higher level of steady state risk lowers the effect of monetary policy only if risk is procyclical. In order to demonstrate this, we use the model described in Appendix [E](#) in which households face a persistent income process.

Claim 1 In our model, following from [\(72\)](#), the steady state consumption of a household with zero wealth and labor endowment ℓ^i can be written as:

$$\begin{aligned} c(a=0) &= \mathcal{C} + \mu\bar{y} + \xi(\ell^i - \bar{\ell}) \\ &= -\frac{1}{\gamma r} \ln[\beta(1+r)] + \bar{y} - \frac{\gamma}{2} \frac{r}{(1+r-\lambda)^2} \sigma_y^2 + \xi(\ell^i - \bar{\ell}) \end{aligned}$$

Thus,

$$\frac{\partial^2 c(a=0)}{\partial r \partial \sigma_y^2} = \frac{\gamma}{2} \frac{r - (1-\lambda)}{(1+r-\lambda)^3}$$

which is positive if the mean reversion of income $1-\lambda$ is less than the real interest rate r . This confirms the first claim of [Holm \(2018\)](#).

Claim 2 Recall that the linearized model in the persistent income case can be written as:

$$\begin{aligned} \hat{y}_t &= -\frac{1}{\gamma}(i_t - \pi_{t+1}) + \hat{y}_{t+1} - \Lambda \hat{\zeta}_{t+1} \\ \hat{\mu}_t &= \tilde{\beta}(i_t - \pi_{t+1} + \hat{\mu}_{t+1}) \\ \hat{\zeta}_t &= (1-\lambda)\hat{\mu}_t + \tilde{\beta}\lambda\hat{\zeta}_{t+1} + \frac{1-\Theta}{\Lambda}\hat{y}_t \\ \pi_t &= \tilde{\beta}\pi_{t+1} + \kappa\hat{y}_t \\ i_t &= \Phi_\pi\pi_t + \varepsilon_t \end{aligned}$$

where we have appended a monetary policy ‘shock’ ε_t to the interest rate rule. Based on [Holm \(2018\)](#), we assume that there is a one time unanticipated increase in ε at date 0 after which it dissipates slowly as $\varepsilon_t = \rho^t \varepsilon_0$. All agents know the path of ε_t at date 0. The object of interest in this experiment is how the initial impact on output \hat{y}_0 depends on the level of risk. In what follows, we consider the minimum state variable equilibrium which will be unique for large enough Φ_π .

Since ε_t is the only state variable, it is clear that every other variable will be a linear function of ε_t : $i_t = i_\varepsilon \varepsilon_t$, $\pi_t = \pi_\varepsilon \varepsilon_t$, etc. Thus real interest rates will be $r_t = i_t - \pi_{t+1} = (i_\varepsilon - \rho \pi_\varepsilon) \rho^t \varepsilon_0$. As in [Holm \(2018\)](#), we require that the impact effect of the shock on interest rates has a unit size, i.e. $(i_\varepsilon - \rho \pi_\varepsilon) \varepsilon_0 = 1$. This is equivalent to simply dropping the Taylor rule and Phillips curve and letting real interest rates follow the path $r_t = \rho^t$. Thus, the system above can be simplified to:

$$\hat{y}_t = -\frac{1}{\gamma} r_t + \hat{y}_{t+1} - \Lambda \hat{\zeta}_{t+1} \quad (82)$$

$$\hat{\mu}_t = \tilde{\beta}(r_t + \hat{\mu}_{t+1}) \quad (83)$$

$$\hat{\zeta}_t = (1 - \lambda) \hat{\mu}_t + \tilde{\beta} \lambda \hat{\zeta}_{t+1} + \frac{1 - \Theta}{\Lambda} \hat{y}_t \quad (84)$$

Since we can write all variables as linear functions of the state variable r_t : $\hat{y}_t = y_r r_t$, $\hat{\mu}_t = \mu_r r_t$, $\hat{\zeta}_t = \zeta_r r_t$, we can rewrite (82)-(84) as:

$$\begin{aligned} y_r r_t &= \left[-\frac{1}{\gamma} + y_r \rho - \Lambda \rho \zeta_r \right] r_t \\ \mu_r r_t &= \tilde{\beta}(1 + \rho \mu_r) r_t \\ \zeta_r r_t &= \left[(1 - \lambda) \mu_r + \tilde{\beta} \lambda \rho \zeta_r + \frac{1 - \Theta}{\Lambda} y_r \right] r_t \end{aligned}$$

which can be simplified to yield:

$$y_r = -\frac{\gamma^{-1} + \frac{(1-\lambda)\Lambda\rho\tilde{\beta}}{(1-\tilde{\beta}\rho)(1-\tilde{\beta}\lambda\rho)}}{1 - \rho + \frac{\rho(1-\Theta)}{1-\tilde{\beta}\lambda\rho}}$$

All else equal, the effect of a unit shock to real interest rates on output (in absolute value) is increasing in Λ and Θ . [Holm \(2018\)](#) considers two economies with a different standard deviation of idiosyncratic risk, varying the discount rate to ensure that the steady state real interest rate is the same in both cases. In our economy, this would correspond to keeping $\tilde{\beta} := (1 + r)^{-1}$ fixed. As a result, steady state $\mu = 1 - \tilde{\beta}$ and $\zeta = (1 - \tilde{\beta})(1 - \tilde{\beta}\lambda)^{-1} \varpi(y^*)$ are also the same across the two economies. The only parameters that change are Λ and Θ . Recall that

$$1 - \Theta = \Lambda \left(1 - \tilde{\beta}\lambda \right) \frac{\varpi'(y^*)}{\varpi(y^*)} \quad \text{and} \quad \Lambda = \gamma \zeta^2 \sigma_\ell^2 = \gamma \left(\frac{1 - \tilde{\beta}}{1 - \tilde{\beta}\lambda} \right)^2 \sigma_y^2$$

Using these in the expression for y_r :

$$y_r = - \frac{\gamma^{-1} + \frac{(1-\lambda)\tilde{\beta}\Omega}{1-\tilde{\beta}\rho}}{1 - \rho + (1 - \tilde{\beta}\lambda)\frac{\varpi'(y^*)}{\varpi(y^*)}\Omega}$$

where $\Omega := \frac{\rho\Lambda}{1-\tilde{\beta}\lambda\rho}$ is monotonically *increasing* in risk σ_y^2 , holding steady state real interest rates fixed.

Then, $|y_r|$ will be *increasing* in Ω if

$$\frac{\varpi'(y^*)}{\varpi(y^*)} < \frac{1 - \lambda}{(1 - \tilde{\beta}\lambda)(1 - \tilde{\beta}\rho)|y_r|}$$

This condition is satisfied if income is not too procyclical. Conversely, if income is sufficiently procyclical, $|y_r|$ will be *decreasing* in Ω , and therefore decreasing in the level of idiosyncratic income risk σ_y^2 . In fact, when income follows a random walk ($\lambda = 1$), *any* level of procyclical income risk tends to reduce $|y_r|$.

[Holm \(2018\)](#)'s result - that higher risk reduces the effect of a shock to real interest rates - is consistent with the second of these two cases. This makes sense, as his model features procyclical income risk: the only source of idiosyncratic income risk comes from idiosyncratic shocks to labor productivity, which are multiplied by a procyclical wage. He also studies an economy with close to random walk income (the autocorrelation of income shocks is 0.97). Again, this is consistent with our model, which states that when $\lambda \approx 1$, even a small degree of procyclicality is sufficient to make $|y_r|$ decreasing in σ_y^2 .

The finding that persistent real interest rate shocks have less impact on output in the presence of procyclical income risk is intuitive. A persistent increase in interest rates reduces output both today and in the future. But in the presence of procyclical risk, households anticipate that lower *average* income in the future will also be accompanied by lower income *risk*, as labor income becomes less volatile. This tends to mitigate the decline in consumer demand on impact, as households reduce their precautionary saving. If steady state risk is higher, the magnitude of this offsetting effect becomes larger, and the effect of persistent monetary policy shocks on output is smaller in magnitude.

Note, however, that this result is special to the case of procyclical income risk. In the presence of countercyclical risk, higher steady state risk would increase the effect of persistent monetary policy shocks on output. In this case, a contractionary monetary policy shock still reduces both present and future demand, but now the fall in future output makes households anticipate *higher* risk, leading to more precautionary saving and a larger fall in output on impact. Thus, [Holm \(2018\)](#)'s finding that higher income risk weakens monetary policy transmission appears to be sensitive to the assumption of procyclical risk.

F MPC Heterogeneity

In this appendix, we augment our standard model to accommodate a mass of agents who are hand-to-mouth. Suppose now that out of the unit mass of households, only those indexed by $i \geq \eta$ are unconstrained and those with $i \in [0, \eta]$ are hand-to-mouth consumers who consume their entire after-tax incomes, i.e. for these agents $c_t^i = y_t^i$ for all t . With the introduction of these new agents, the economy now features MPC heterogeneity as a fraction η of agents have MPC 1 and a fraction $1 - \eta$ have MPC $\mu_t < 1$. Varying η

allows us to vary the average MPC in the economy which is given by $\eta + (1 - \eta)\mu_t$.

As Bilbiie (2008, 2019a,b) has shown, the effect of introducing hand-to-mouth (henceforth HTM) agents into the economy depends critically on the sensitivity of these agents' income to aggregate income. Following Bilbiie (2019a), we assume that the after-tax income of HTM agents is proportional to aggregate after tax income, y_t . In particular, we assume that the average income of HTM agents, $\frac{1}{\eta} \int_0^\eta y_t^i di = \chi \bar{y}_t$ where \bar{y}_t denotes after-tax aggregate income.³⁷ Meanwhile for unconstrained households, the income process is now given by $y_t^i \sim N\left(\frac{1-\eta\chi}{1-\eta}\bar{y}_t, \sigma_{y,t}^2\right)$. Constrained and unconstrained agents might have different cyclical sensitivities of income for many reasons. The simplest case in which $\chi > 1$ is that the government directly imposes lump sum taxes (proportional to GDP) on the unconstrained households and rebates the proceeds lump sum to constrained households. In other words, if the transfer to constrained agents is $(\chi - 1)y_t$ and the tax on unconstrained agents is $\eta(\chi - 1)y_t/(1 - \eta)$, the after-tax income of both agents is as described. Similarly, $\chi < 1$ if transfers are in the other direction.

We focus on the case in which $\eta\chi < 1$, i.e. HTM agents do not receive all of the after-tax aggregate income. This assumption also ensures that the income of the unconstrained is still increasing in GDP. Bilbiie (2008) allows for the case in which the income accruing to the unconstrained actually *decreases* when GDP increases holding everything else constant. The average after-tax income of unconstrained agents is $\frac{1-\eta\chi}{1-\eta}\bar{y}_t$. Proposition 1 is still valid for these unconstrained agents, except that their mean after-tax income is $\frac{1-\eta\chi}{1-\eta}\bar{y}_t$ instead of just \bar{y}_t as in the Proposition 1, where \bar{y}_t denotes aggregate after-tax income. Hence their consumption decision rules can be described by (4):

$$c_t^i = C_t + \mu_t(a_t^i + y_t^i) \text{ for } i \in (\eta, 1]$$

where μ_t is still defined as before by (32) but C_t is now defined as:

$$C_t [1 + \mu_{t+1} (1 + r_t)] = -\frac{1}{\gamma} \ln \beta (1 + r_t) + C_{t+1} + \frac{1 - \eta\chi}{1 - \eta} \mu_{t+1} \bar{y}_{t+1} - \frac{\gamma \mu_{t+1}^2 \sigma_{y,t+1}^2}{2} \quad (85)$$

Define c_t^u as the average consumption of the unconstrained agents, i.e., $c_t^u = \frac{1}{1-\eta} \int_\eta^1 c_t^i di$. As a result we can express c_t^u as:

$$c_t^u = C_t + \mu_t \left(a_t + \frac{1 - \eta\chi}{1 - \eta} \bar{y}_t \right)$$

Asset market clearing is now given by $(1 - \eta)a_t = \frac{B_t}{P_t}$ since only the unconstrained agents hold assets. Imposing asset market clearing, we can write the above as:

$$c_t^u = C_t + \frac{\mu_t}{1 - \eta} \left(\frac{B_t}{P_t} + (1 - \eta\chi) \bar{y}_t \right)$$

Using this expression in (85) and using (3) and (32):

$$y_t = -\frac{1 - \eta}{1 - \eta\chi} \frac{1}{\gamma} \ln \beta (1 + r_t) + y_{t+1} - \Delta g_{t+1} + \frac{\eta\chi}{1 - \eta\chi} \Delta s_{t+1} - \frac{\gamma}{2} \left(\frac{1 - \eta}{1 - \eta\chi} \right) \mu_{t+1}^2 \sigma_{y,t+1}^2$$

³⁷Because these households do not make savings decisions, the distribution of incomes among these households does not affect aggregate outcomes.

Setting $g_t = T_t = s_t = 0$, we have:

$$y_t = -\frac{\Xi}{\gamma} \ln \beta (1 + r_t) + y_{t+1} - \Xi \frac{\gamma \mu_{t+1}^2}{2} \sigma_{y,t+1}^2 \quad (86)$$

where $\Xi = \frac{1-\eta}{1-\eta\chi} > 0$ given our assumption that $\eta\chi < 1$. In order to understand the role played by χ , note that the resource constraint implies:³⁸

$$y_t = c_t = \eta\chi y_t + (1-\eta)c_t^u \quad \Rightarrow \quad y_t = \frac{1-\eta}{1-\eta\chi} c_t^u \equiv \Xi c_t^u$$

In other words, Ξ is the amount by which GDP goes up if the per capita consumption of unconstrained households goes up by 1 unit, which may be due to a cut in interest rates, lower risk which reduces precautionary savings etc. Suppose first that $\chi = 1$, i.e., constrained and unconstrained individuals have the same average income. A unit increase in the consumption of the unconstrained, directly increases GDP by $1-\eta$ units. This increase in aggregate income in turn increases the income of HTM agents by $1-\eta$; these individuals spend the whole increment, increasing aggregate consumption and further increasing aggregate GDP by $\eta(1-\eta)$. This in turn induces a third round effect and so on. The net effect is that a unit increase in c_t^u increases GDP by $(1-\eta)(1+\eta+\eta^2+\dots) = \frac{1-\eta}{1-\eta} = 1$, however small the fraction of unconstrained individuals $1-\eta$. In the language of [Kaplan et al. \(2018\)](#), a larger fraction of HTM agents, η reduces the *direct* effect of an increase in c_t^u on GDP but is exactly compensated by larger *indirect* effects from the second round effects of increased spending by HTM agents. Thus, with $\chi = 1$, c_t^u moves one-for-one with y_t , for any value of $\eta \in [0, 1]$ and the aggregate Euler equation is the same as if there were no HTM agents in the economy. Notice that if $\chi = 1$, then $\Xi = 1$ and (86) is the same as equation (8) (with G_t set to 0).

However, as [Bilbiie \(2008, 2019a,b\)](#) has forcefully argued, this irrelevance result is special to the case where $\chi = 1$. If instead, $\chi < 1$ (the income of HTM's is less cyclically sensitive than that of unconstrained agents), then $\Xi < 1$, and an increase in the consumption of the unconstrained increases GDP less than one-for-one, especially to the extent that the fraction of HTM agents η is large. Intuitively, if the income and spending of HTM agents increases less than one-for-one with GDP, the second round multiplier effects described in the previous paragraph are naturally dampened. If instead $\chi > 1$ and HTM agents have more cyclically sensitive incomes, then $\Xi > 1$, then an increase in the consumption of unconstrained raises GDP more than one-for-one. Finally if $\Xi < 0$ (which is ruled out by our assumption that $\eta\chi < 1$) then an increase in the consumption of the unconstrained would *decrease* GDP in equilibrium. This case was first considered by [Bilbiie \(2008\)](#).

It is straightforward to see how MPC heterogeneity can affect the interest rate sensitivity of aggregate spending. Since real interest rates do not directly affect the behavior of HTM agents, they affect GDP directly only via the behavior of unconstrained agents. The sensitivity of consumption growth to real interest rates for unconstrained agents is not affected by the introduction of HTM agents and equals γ^{-1} as before. But the sensitivity of GDP to the consumption of the unconstrained is given by Ξ as we described above. Thus, the overall sensitivity of GDP to real interest rates is given by $\frac{dy_t}{d \ln(1+r_t)} = \frac{dy_t}{dc_t^u} \frac{dc_t^u}{d \ln(1+r_t)} = \frac{\Xi}{\gamma} \equiv \tilde{\gamma}^{-1}$, as shown in (86). When $\chi = 1$, MPC heterogeneity does not affect aggregate interest rate sensitivity. When $\chi < 1$, HTM agents dampen the interest rate sensitivity, while when $\chi > 1$, they amplify it.³⁹

³⁸Setting $g_t = T_t = s_t = 0$ without loss of generality

³⁹Again we restrict attention to the case with $\chi \in (1, \eta^{-1})$.

MPC heterogeneity changes the predictions of the NK model in a qualitatively different way than uninsurable idiosyncratic risk. While MPC heterogeneity can change the contemporaneous response of GDP to interest rates, depending on the cyclicity of HTM income, precautionary savings and uninsurable risk do not affect the contemporaneous interest rate sensitivity, regardless of the cyclicity of income risk.

To proceed further it is instructive to linearize (86) around steady state to yield:

$$\hat{y}_t = -\tilde{\gamma}^{-1} (i_t - \pi_{t+1}) + \tilde{\Theta} \hat{y}_{t+1} - \tilde{\Lambda} \hat{\mu}_{t+1} \quad (87)$$

where $\tilde{\Theta} = \Xi\Theta + 1 - \Xi$ and $\tilde{\Lambda} = \Xi\Lambda$. As discussed above, when the incomes of constrained and unconstrained households are equally sensitive to aggregate income ($\chi = 1$), GDP moves one-for-one with unconstrained households' consumption ($\Xi = 1$), and the aggregate Euler equation is unaffected by the introduction of some HTM agents. In particular, $\tilde{\Theta} = \Theta$ and whether the Euler equation is “discounted” or “explosive” depends solely on whether income risk is procyclical, countercyclical or acyclical and is unaffected by MPC heterogeneity.

Suppose instead that HTM incomes are less cyclically sensitive ($\chi < 1$), so that GDP moves less than one-for-one with c_t^u ($\Xi < 1$). In this case $\tilde{\Theta}$ is just the weighted average of Θ and 1. Thus, (87) is similar to the Euler equation without MPC heterogeneity (12), except that the coefficient on future GDP is now closer to 1. Thus, in this case, MPC heterogeneity reduces the effect of idiosyncratic risk on aggregate outcomes, pulling the HANK economy back towards the RANK benchmark. If risk is procyclical, MPC heterogeneity makes the Euler equation “less discounted”, increasing the sensitivity of current spending to far future changes in interest rates, making determinacy under a peg less likely and so forth. Similarly, if risk is countercyclical, MPC heterogeneity makes the Euler equation less “explosive”. While MPC heterogeneity makes a discounted Euler equation “less discounted”, and a explosive Euler equation “less explosive”, it cannot turn a discounted Euler equation into an explosive one, or vice versa.

The effect of MPC heterogeneity is reversed if HTM incomes are more cyclically sensitive $\chi > 1$, so that GDP moves more than one-for-one with c_t^u , ($\Xi > 1$). In this case, $\Xi > 1$ drives $\tilde{\Theta}$ further away from 1, pushing the HANK economy further away from the RANK benchmark. If risk is procyclical, MPC heterogeneity now makes the Euler equation even more discounted; if risk is countercyclical, it makes the Euler equation even more explosive. Again, in both cases, MPC heterogeneity amplifies the deviation from the RANK benchmark but does not change the direction of this deviation.

MPC heterogeneity and determinacy of equilibrium Recall that Proposition 2 showed that determinacy in our HANK economy without MPC heterogeneity depended on whether monetary policy satisfied the risk augmented Taylor principle (17). Since the aggregate Euler equation with MPC heterogeneity (87) has the same form as (12), it is immediate that the augmented Taylor principle in this case takes the form:

$$\Phi_\pi > 1 + \frac{\tilde{\gamma}}{\kappa} \left[\frac{(1 - \tilde{\beta})^2}{(1 - \tilde{\beta}) + \tilde{\gamma}\tilde{\beta}\tilde{\Lambda}} \right] (\tilde{\Theta} - 1) = 1 + \frac{\gamma}{\kappa} \left[\frac{(1 - \tilde{\beta})^2}{(1 - \tilde{\beta}) + \gamma\tilde{\beta}\Lambda} \right] (\Theta - 1)$$

where the second equality holds because $\tilde{\gamma} = \gamma/\Xi$, $\tilde{\Theta} - 1 = \Xi(\Theta - 1)$ and $\tilde{\Lambda} = \Xi\Lambda$. That is, the introduction of MPC heterogeneity does not change the determinacy properties of equilibrium, even if $\chi \neq 1$. The cyclicity of income risk remains the most important feature affecting determinacy. Intuitively, MPC

heterogeneity alters the static relation between the spending of unconstrained households and GDP, but does not affect the dynamic relationship between the spending of unconstrained households today and in the future. Thus, while χ might be very important for other questions, it does not affect determinacy given the cyclical nature of income risk. It is important to know that this result depends on the maintained assumption that $\eta\chi < 1$ which restricts $\Xi > 0$, i.e. the consumption of unconstrained increases in GDP. Bilbiie (2008) shows that determinacy may require a violation of the Taylor principle when $\Xi < 0$.⁴⁰

Forward Guidance Puzzle and size of fiscal multipliers See the earlier working paper version of this paper⁴¹ for a detailed analysis of the FGP and size of fiscal multipliers in this model with MPC heterogeneity. The analysis shows that it is primarily the cyclical nature of risk which determines whether the FGP is muted or if fiscal spending multipliers are large or small, not MPC heterogeneity.

F.1 Difference from zero-liquidity models

MPC heterogeneity and uninsurable risk are two logically distinct (but not necessarily unrelated) features of an incomplete markets economy. Our framework allows us to make this distinction explicit: taking σ_y to zero in the limit while introducing an $\eta > 0$ fraction of HTM agents gives us MPC heterogeneity without risk, while $\sigma_y > 0$ and $\eta = 0$ gives us risk but no MPC heterogeneity. More subtly, the *cyclical nature* of risk is distinct from the cyclical nature of HTM agents' income. Cyclical nature of risk concerns the question "does risk go up or down in a recession?". A separate question is, "Who loses disproportionately from a recession?" - in particular, does income fall more or less for constrained (high MPC) agents or unconstrained agents?

Our framework clearly distinguishes between the cyclical nature of risk which is given by $\frac{d\sigma^2}{dy}$ and the cyclical nature of HTM income which is given by χ . Recent work by Bilbiie (2019a) and Werning (2015) have emphasized the importance of both these forces. However, the analytical framework used in these papers does not permit the two factors to be distinguished - the same parameters which generate countercyclical risk also generate also imply a higher sensitivity of HTM income. This leads Werning (2015) for example to conclude that countercyclical income risk increases the sensitivity of consumption to contemporaneous changes in interest rates.

To see this, consider the following simple example based on Werning (2015). There is a continuum of agents in the economy and two income states $y^i \in \{y_h(y_t), y_l(y_t)\}$ which potentially depend on aggregate income Y_t in different ways: $y'_h(y) \neq y'_l(y)$. The probability of switching from income state s to s' is denoted by $p_{s'|s}$. Agents can save in a zero net-supply risk free bond and cannot borrow. It is straightforward to see that in equilibrium all households consume their entire income and high income households are unconstrained, thus pricing the bond. The low income individuals are effectively HTM agents. The high income agents' Euler equation is:

$$u'(y_h(y_t)) = \beta(1 + r_t) [p_{l|h} u'(y_l(y_{t+1})) + p_{h|h} u'(y_h(y_{t+1}))]$$

⁴⁰See also Galí et al. (2004) who find that adding HTM agents can affect determinacy in a model with capital.

⁴¹Available at https://www.newyorkfed.org/medialibrary/media/research/staff_reports/sr835.pdf.

Taking logs on both sides and linearizing, one gets (specializing to the CARA utility case):

$$\hat{y}_t = -\frac{1}{\gamma y'_h(y)} r_t + \frac{p_{l|h} e^{\gamma(y_h - y_l)} \left(\frac{y'_l(y)}{y'_h(y)} \right) + p_{h|h}}{p_{l|h} e^{\gamma(y_h - y_l)} + p_{h|h}} \hat{y}_{t+1}$$

A natural way to generate countercyclical income risk in this setting would be to posit $y'_l(y) > 1 > y'_h(y)$. Since low income agents are constrained, this also has the effect of increasing the cyclical sensitivity of HTM income. A lower value of $y'_h(y)$ increases the sensitivity of GDP to real interest rates. But this sensitivity increases not because income risk is countercyclical, but because the income of HTMs is more cyclically sensitive (thus, the income of unconstrained agents is less sensitive). To see this, consider the limit where $p_{l|h} \rightarrow 0$ and the each agent's type is fixed (as in our model with $\eta > 0$ and $\sigma_y \rightarrow 0$). In the limit, idiosyncratic income risk is zero, and thus $y'_h(y) < 1$ clearly does not generate countercyclical idiosyncratic income risk. However, it still generates a higher interest rate elasticity of GDP as can be seen from the equation above. This example (which closely mimics the environment in [Bilbiie \(2019a\)](#) and [Werning \(2015\)](#)) suggests that the cyclical risk is not the primary force affecting the interest rate elasticity of aggregate spending and instead it is the differential sensitivity of constrained and unconstrained agents' income which is key. However, in their framework it is difficult to distinguish these two features since they are governed by the same parameters. In contrast, our framework allows us to independently vary the cyclical risk and HTM income. Doing so reveals that cyclical risk of HTM income, not cyclical risk of risk, affects the interest rate sensitivity of aggregate spending.