

THE ANATOMY OF PERSISTENT SENTIMENT-DRIVEN FLUCTUATIONS*

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Abstract

We characterize the entire set of linear equilibria of beauty contest games under general information structures. In particular, we focus on equilibria in which *sentiments* - self-fulfilling changes in beliefs that are orthogonal to fundamentals and exogenous noise - can drive aggregate fluctuations. We show that, under rational expectations, there exists a continuum of sentiment-driven equilibria that generate aggregate fluctuations. Without having to take a stance on the private information agents might possess, we provide a general characterization of necessary and sufficient conditions under which a change in sentiments can have prolonged effects on aggregate outcomes and when it can only have short lived effects. In addition, we also provide a practical way to characterize these equilibria.

Keywords: Sentiments, Dispersed Information, Dynamic Correlated Equilibria, Dynamic Signal Extraction

JEL classifications: E20, E32, F44

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1 Introduction

Can a change in sentiments induce persistent macroeconomic fluctuations? Even though this is a very attractive proposition and has captured the minds of economists at least since Keynes and Pigou, this idea has been very hard to formalize under rational expectations. We revisit this question in this paper.

We explore this question in a set of stylized beauty contest models that have unique Rational Expectations Equilibria (REE) under full information. A key feature of the information structure is that agents in the economy have dispersed information and receive noisy *endogenous* signals about the aggregate action in the economy.¹ In particular, these noisy signals confound the information about current aggregate actions with payoff relevant fundamentals and agents must filter out the average action in order to make inferences about their appropriate action. This results in each agent's action to depend on the aggregate action even if the primitives of the model do not feature any coordination motive. This induced strategic complementarities allows for persistent fluctuations driven by self-fulfilling changes in beliefs. We refer to these self-confirming changes in beliefs as *sentiments* and aggregate fluctuations driven by these changes as *sentiment-driven fluctuations*. Importantly, these sentiment-driven fluctuations are independent of changes in fundamentals such as technologies, preferences, or government policies. In fact, they can even exist in an economy without any change in these aggregate fundamentals.

It is crucial to note at this point that our definition of sentiments is fundamentally different than the way the term is used in the fast growing theoretical and empirical literature which studies expectations-driven fluctuations.² This literature has largely modeled sentiments as an *exogenous* stochastic process which can affect the real economy. In contrast, in the context of our model, changing sentiments are self-fulfilling changes in beliefs which arise *endogenously* in our model in the sense that their evolution is disciplined by rational expectations. However, in a spirit similar to the large literature mentioned above, sentiments in our model are also orthogonal to any changes in fundamentals. Thus, our notion of sentiments is the same as in [Benhabib et al. \(2015\)](#), and can be interpreted as a correlated equilibrium.³ It is important to point out that while [Benhabib et al. \(2015\)](#) show how sentiments can generate stochastic self-fulfilling rational expectations equilibria, they do not explore whether such sentiments can

¹The availability of such information to agents in an economy is not hard to motivate. For example, firms, in finalizing their production decisions use information about expected *aggregate demand*, which is readily available from surveys of consumer expectations. Alternatively, one could think of such signals as market research by each firm regarding the demand for its own product.

²See for example [Angeletos and La'O \(2010, 2013\)](#), [Lorenzoni \(2009\)](#), [Barsky and Sims \(2012\)](#), [Acharya \(2013\)](#), [Nimark \(2014\)](#), [Rondina and Walker \(2014\)](#), [Angeletos et al. \(2014\)](#), [Huo and Takayama \(2015\)](#) among many others.

³This equilibrium concept is also explored in [Bergemann et al. \(2015\)](#).

generate persistent fluctuations.

In this paper, we characterize the entire set of linear-stationary-rational-expectations equilibria in such economies with particular focus on equilibria in which sentiments have aggregate consequences. In the spirit of the recent work by [Bergemann and Morris \(2013\)](#), our characterization of equilibria does not depend on the private information agents might possess. Our characterization highlights that there is generic multiplicity of equilibria in which aggregate outcomes are driven by sentiments. We then provide necessary and sufficient conditions for which we can partition this set into two parts: those equilibria in which sentiments result in persistent aggregate fluctuations and those in which they don't. Importantly, this characterization of necessary and sufficient conditions does not depend on the structure of private information agents may possess

If we restrict attention to the class of equilibria in which the dynamics of aggregate actions can be described by an invertible linear-stationary stochastic processes,⁴ we find that sentiments can only generate i.i.d aggregate fluctuations in any such REE. The consistency between perceived and actual laws of motion that REE demands rules out any possibility of the persistent effects of sentiments.⁵ However, this is not the full set of equilibria. Importantly, if aggregate actions are driven by non-invertible stochastic process, there exist a continuum of equilibria in which the response of aggregate outcomes to a change in sentiments can be described by non-invertible ARMA processes, and exhibit non-trivial impulse responses such as hump-shapes. However, as we show in [Section 4](#), even though such equilibria may exhibit a hump-shaped impulse response, as long as agents observe past actions and fundamentals, these fluctuations are observationally-indistinguishable from i.i.d.fluctuations. In other words, these persistent fluctuations are not predictable.

In fact, we show that a necessary condition for sentiment-driven fluctuations to be observationally -distinguishable from i.i.d. is that either: (1) agents do not observe past fundamentals without any noise or (2) they do not observe past aggregate outcomes. Furthermore, this characterization is robust to any specification of private information and the structure of fundamental shocks. If one of these conditions is met, we show that there exist another continuum of equilibria in which sentiments drive persistent fluctuations which are also predictable. These findings are encouraging since they suggest that sentiments might be able to account for persistent fluctuations which is a robust feature of macroeconomic time series. Additionally, in all equilibria which feature sentiment-driven fluctuations, sentiments can also alter the way in which aggregate fundamentals affect aggregate outcomes. Thus, sentiments might also be able to act as amplification mechanisms with regard to fundamental shocks.

⁴Consider a linear stochastic process $x_t = A(L)e_t$. Then this process is invertible if observing the history of realizations of x^t allows one to perfectly infer the sequence of realizations of the history e^t . A non-invertible process is one in which observing x^t does not allow one to perfectly infer the sequence e^t .

⁵The equilibrium in [Benhabib et al. \(2015\)](#) lies in this class of equilibria.

We also show that the existence of the continuum of sentiment equilibria is accompanied by the implication that agents can disagree forever.⁶ Agents can differ in their beliefs about which equilibrium in the continuum is being realized: some or even all the agents may entertain different models of how sentiments evolve over time. Interestingly, equilibrium does not disallow the possibility that all agents can be wrong about the actual evolution of sentiments and hence aggregate actions. Furthermore, observing the realizations of aggregate outcomes never contradicts the potentially incorrect beliefs of these agents. Thus, the usual tight link between objective and allowable subjective beliefs under rational expectations is not very strong.

The rest of the paper is structured as follows. In the next section, we present the economic environment. In Section 3 we present simple examples which can be solved analytically in order to explore the forces which generate sentiment-driven fluctuations. These examples also demonstrate a practical way to characterize sentiment equilibria. Section 4 then presents the general result which characterizes the set of sentiment equilibria for general information structures. Section 5 contains a discussion regarding the assumptions under which sentiments drive persistent fluctuations. It also relates our results to the large literature on sunspots and correlated equilibria after which we conclude.

2 Environment

We consider a standard beauty-contest game such as in [Morris and Shin \(2002\)](#). Our economy consists of a continuum of agents indexed by $i \in [0, 1]$. Agent i wants to choose an action $a_{i,t}$ every period which depends on their idiosyncratic fundamental shock $z_{i,t}$, an aggregate fundamental shock θ_t and the economy wide aggregate action a_t . Assume that the optimal action by agent i is given by:

$$a_{i,t} = \alpha \mathbb{E}[z_{i,t} \mid \mathcal{I}_{i,t}] + \varphi \mathbb{E}[\theta_t \mid \mathcal{I}_{i,t}] + \gamma \mathbb{E}[a_t \mid \mathcal{I}_{i,t}], \quad (1)$$

where

$$a_t = \int a_{i,t} \quad (2)$$

is defined as the aggregate action and $\mathcal{I}_{i,t}$ denotes the information set of agent i at date t . α and φ can take any value on the real line but we impose that the $\gamma < 1$. This assumption ensures that there is a unique full-information fundamental equilibrium.⁷ The processes for

⁶Our context here, with asymmetric information and non-invertible sentiment dynamics, of course differs from the context of the seminal work of [Geanakoplos and Polemarchakis \(1982\)](#).

⁷ γ is a measure of the strength of strategic complementarities. If $\gamma \geq 1$, this complementarity is strong enough to generate multiple equilibria. See for example [Cooper and John \(1988\)](#). Since we restrict $\gamma < 1$, our results do not depend on the strength of the strategic complementarity.

idiosyncratic and aggregate fundamental are given by:

$$z_{i,t} = \mathbf{h}(L)\boldsymbol{\zeta}_{i,t} = \sum_{k=0}^{\infty} \mathbf{h}_k \boldsymbol{\zeta}_{i,t-k} \quad (3)$$

$$\theta_t = \mathbf{g}(L)\boldsymbol{\nu}_t = \sum_{k=0}^{\infty} \mathbf{g}_k \boldsymbol{\nu}_{t-k}, \quad (4)$$

where $\boldsymbol{\zeta}_{i,t}$ and $\boldsymbol{\nu}_t$ are sequences of Gaussian white noise innovations to the idiosyncratic and aggregate fundamental respectively.⁸ $\boldsymbol{\zeta}_{i,t}$ is a vector of idiosyncratic shocks to agents' fundamental and satisfies an adding-up constraint $\int_i \boldsymbol{\zeta}_{i,t} = \mathbf{0}$ at each date t . In contrast, $\boldsymbol{\nu}_t$ is common across all agents. Furthermore, we assume that $\mathbf{h}(L)$ and $\mathbf{g}(L)$ are potentially infinite-order one-sided polynomials in positive powers of the lag operator L .⁹ We do not impose any restrictions on $\mathbf{h}(L)$ and $\mathbf{g}(L)$ except square-summability which implies that $z_{i,t}$ and θ_t are linear stationary processes. Also, note that for the rest of the paper, bold-face letters indicate vectors and matrices while non bold variables indicate scalars.

We can partition the vector of exogenous common shocks as $\boldsymbol{\nu}_t = \begin{bmatrix} \mathbf{v}_t & \boldsymbol{\eta}_t \end{bmatrix}'$, where $\boldsymbol{\eta}_t$ represents the vector of noise which is common across agents and their loading in $\mathbf{g}(L)$ is zero. In the literature, $\boldsymbol{\eta}_t$ is often interpreted as noise shocks, animal spirits or confidence shocks.¹⁰ $\boldsymbol{\zeta}_{i,t} = \begin{bmatrix} \mathbf{u}_{i,t} & \boldsymbol{\varsigma}_{i,t} \end{bmatrix}'$ denotes the vector of exogenous idiosyncratic shocks: both idiosyncratic fundamentals $\mathbf{u}_{i,t}$ and of idiosyncratic noise $\boldsymbol{\varsigma}_{i,t}$ (whose loading in $\mathbf{h}(L)$ may be zero).

Information Set of Agents We impose very little structure on the information sets that each agent possesses. Agents in the model have access to both *exogenous* and *endogenous* sources of information. Exogenous sources of information are those that are not affected by interactions among agents. These are modeled as a set of exogenous signals $\mathbf{y}_{i,t}$ which take the form:

$$\mathbf{y}_{i,t} = \mathbf{P}(L)\boldsymbol{\nu}_t + \mathbf{Q}(L)\boldsymbol{\zeta}_{i,t} \quad (5)$$

where $\mathbf{P}(L)$ and $\mathbf{Q}(L)$ can be any square summable, one sided polynomials in the lag operator L .¹¹ This structure is very general and encompasses commonly used assumptions in models with information frictions. For example, consider a situation in which each agent observes a

⁸Even though the idiosyncratic and aggregate fundamentals are univariate stochastic processes, we allow them to be driven by a vector of innovations.

⁹As is convention we define the lag operator L as $Lx_t := x_{t-1}$, $L^{-1}x_t := x_{t+1}$ and $L^n x_t = x_{t-n}$.

¹⁰See for example [Lorenzoni \(2009\)](#), [Angeletos and La'O \(2013\)](#), among many others.

¹¹In other words, the signals can only depend on past and current changes (not future) in the fundamental shocks.

public and a private signal about the aggregate fundamental,

$$\begin{aligned} y_{i,t}^1 &= v_t + \eta_t, \\ y_{i,t}^2 &= v_t + \varsigma_{i,t}, \end{aligned}$$

which is similar to the specification in [Morris and Shin \(2002\)](#). In terms of equation (5), this information structure can be represented as:

$$\mathbf{y}_{i,t} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{\mathbf{P}(L)} \begin{bmatrix} v_t \\ \eta_t \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\mathbf{Q}(L)} \varsigma_{i,t}$$

In contrast to *exogenous* sources of information, *endogenous* sources are affected by interactions among agents. In other words, the informativeness of such signals is determined in equilibrium. We model such sources of information as the set of signals $\mathbf{x}_{i,t}$:

$$\mathbf{x}_{i,t} = \mathbf{A}(L)a_t + \mathbf{B}(L)\boldsymbol{\nu}_t + \mathbf{C}(L)\boldsymbol{\zeta}_{i,t}, \quad (6)$$

The key distinction between $\mathbf{x}_{i,t}$ and $\mathbf{y}_{i,t}$ is that while $\mathbf{y}_{i,t}$ provides information about objects which aren't determined as part of equilibrium, $\mathbf{x}_{i,t}$ provides an agent with information about objects which are shaped by equilibrium.

The set of signals $\mathbf{x}_{i,t}$ are linear combinations of current and past innovations and aggregate action. As before, the only restriction we impose is that $\mathbf{A}(L)$, $\mathbf{B}(L)$ and $\mathbf{C}(L)$ be square-summable and one sided polynomials in the lag operator L . If $\mathbf{A}(L) \neq \mathbf{0}$, then agents observe signals which provide information directly about equilibrium actions and not just about changes in exogenous fundamentals. The amount of information $\mathbf{x}_{i,t}$ provides to the agent depends on the equilibrium. To see this clearly, consider the case in which $\mathbf{B}(L) = \mathbf{C}(L) = \mathbf{0}$ and $\mathbf{A}(L) = 1$, i.e. $x_{i,t} = a_t$. Suppose that in equilibrium a_t responds one-for one to changes in the aggregate fundamental θ_t , then observing $x_{i,t}$ provides agent i enough information to infer the realization of θ_t perfectly. In contrast, if in equilibrium a_t does not respond to changes in the aggregate fundamental θ_t , then observing $x_{i,t}$ does not provide the agent with any information about the realization of θ_t . Thus, the informativeness of signals $\mathbf{x}_{i,t}$ is determined as part of equilibrium rather than being exogenously specified.¹²

This specification of endogenous information is general enough to encompass assumptions that are made commonly in the information frictions literature. For example, suppose agents

¹²Notice that setting $\mathbf{A}(L) = \mathbf{0}$ shuts off this property of endogenous informativeness of the signal.

observe a public and a private signal about current and past aggregate action a_t and a_{t-1} :

$$\begin{aligned} x_{i,t}^1 &= a_t + \eta_t, \\ x_{i,t}^2 &= a_{t-1} + u_{i,t}. \end{aligned}$$

These signals can be written compactly in terms of equation (6) as:

$$\mathbf{x}_{i,t} = \underbrace{\begin{bmatrix} 1 \\ L \end{bmatrix}}_{\mathbf{A}(L)} a_t + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\mathbf{B}(L)} \eta_t + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\mathbf{C}(L)} u_{i,t}$$

It is important to notice that if $\mathbf{A}(L) = 0$, equation 6 encompasses the noisy public signal in equation (5). In principle, the vector of signals $\mathbf{x}_{i,t}$ can contain both endogenous and exogenous signals. Thus, even though it is not necessary, we define equation (5) separately from (6) because of the notational convenience it provides in Section 4.

In summary, we can express the information set of any agent i at date t as:¹³

$$\mathcal{I}_{i,t} = \mathbb{V}(\mathbf{y}_i^t, \mathbf{x}_i^t) \vee \mathbb{M} \quad (7)$$

where $\mathbb{V}(\mathbf{y}_i^t, \mathbf{x}_i^t)$ denotes the smallest sub-space spanned (at date t) by the past and current realizations of \mathbf{y}_i^t and \mathbf{x}_i^t . Finally, since we are going to concentrate only on rational expectations equilibria, all agents have knowledge of the so called *cross-equation restrictions* imposed by a rational expectations equilibrium. This is denoted by \mathbb{M} and simply means that the agent knows that the dynamics of the economy are determined by equations (1) - (4).¹⁴ It is useful to define two more assumptions before we proceed:

Assumption 1. *The past aggregate action is observable, i.e, $\mathcal{I}_{i,t} \supseteq \mathbb{V}(a^{t-1})$.*

Assumption 2. *The past exogenous aggregate shocks can be perfectly inferred, i.e, $\mathcal{I}_{i,t} \supseteq \mathbb{V}(\boldsymbol{\nu}^{t-1})$.*

If the condition in Assumption 1 is satisfied, then one of the signals in $\mathbf{x}_{i,t}$ will be a_{t-1} . Under Assumption 2 we will assume that the information structure is such that at any date t each agent can infer the innovations to the aggregate fundamental at date $t - 1$. However, notice that for Assumption 2 to be true, it is not necessary that agents to observe $\boldsymbol{\nu}_{t-1}$ directly. For example, if the aggregate fundamental follows an AR(1) process, $\theta_t = \rho\theta_{t-1} + v_t$, observing past fundamentals $\{\theta^{t-1}\}$ allows agents to infer past shocks $\{v^{t-1}\}$ perfectly.

¹³ $\mathbf{X} \vee \mathbf{Y}$ denotes the smallest closed subspace which contains the subspaces \mathbf{X} and \mathbf{Y} .

¹⁴In the paper we assume that agents cannot observe the current fundamentals perfectly when making their decisions. Although this is not key, we will generally assume that the information available to agents will not be sufficient to infer the aggregate and idiosyncratic fundamentals perfectly.

These assumptions are not essential in the definition of equilibrium, but later it will become clear that they play an important role in determining the properties of the equilibria which can exist. We organize our investigation in Sections 3 and 4 around Assumptions 1-2 as these allow us to neatly partition the complete set of equilibria and uncover the properties of the different kind of equilibria which can arise.

Equilibria In this paper, we focus on linear rational expectations equilibrium (REE). We classify these REE into two broad classes which are labeled as *fundamental* equilibrium and *sentiment* equilibrium. By fundamental equilibrium, we refer to those equilibria in which the aggregate action a_t is driven solely by exogenous aggregate shocks and is formalized in the definition below.

Definition 1 (Fundamental Equilibrium). *In any fundamental equilibrium, the aggregate action is driven purely by changes in the aggregate fundamental innovations \mathbf{v}_t and common noise $\boldsymbol{\eta}_t$ in the exogenous information:*

$$a_t = \boldsymbol{\psi}(L)\boldsymbol{\nu}_t \quad (8)$$

where $\boldsymbol{\psi}(L)$ is a vector of square-summable rational polynomial in positive powers of the lag operator L . Furthermore, a_t is consistent with the agents' optimal choice given the information set $\mathcal{I}_{i,t}$ in (7)

$$a_t = \boldsymbol{\psi}(L)\boldsymbol{\nu}_t = \int \left\{ \alpha \mathbb{E}[z_{i,t} \mid \mathcal{I}_{i,t}] + \varphi \mathbb{E}[\theta_t \mid \mathcal{I}_{i,t}] + \gamma \mathbb{E}[a_t \mid \mathcal{I}_{i,t}] \right\} di \quad (9)$$

In the fundamental equilibrium aggregate fluctuations are driven solely by changes in exogenous fundamentals of the economy.¹⁵ For example, these exogenous shocks can be aggregate TFP or preference shocks. Furthermore, we allow fundamental equilibria to include those in which agents may not directly observe the fundamentals θ_t . In such a setting aggregate noise in signals can also result in aggregate fluctuations. Thus, this class of equilibria encompasses the standard full-information equilibrium as well as those in economies with information frictions. In the latter, the definition is general enough to include both equilibria with exogenous information and endogenous information.

Definition 2 (Sentiment Equilibria). *Consider any payoff irrelevant white noise process $\{\epsilon_t\}$ where $\{\epsilon_t\} \perp \{\boldsymbol{\nu}_t\}, \{\boldsymbol{\zeta}_{i,t}\}$. A sentiments equilibrium is one in which the aggregate action is driven by changes in fundamental innovations \mathbf{v}_t , exogenous noise $\boldsymbol{\eta}_t$ and also by changes in payoff irrelevant sentiments ϵ_t :*

$$a_t = \boldsymbol{\psi}(L)\boldsymbol{\nu}_t + \phi(L)\epsilon_t \quad (10)$$

¹⁵All fundamental equilibria lie in the Hilbert space $\mathcal{H}(\mathbf{v}, \boldsymbol{\eta})$ (the space spanned by square-summable linear combinations of \mathbf{v}^t and $\boldsymbol{\eta}^t$).

where $\epsilon_t \sim N(0, 1)$. $\psi(L)$ and $\phi(L)$ are square-summable rational polynomials in positive powers of the lag operator L . Moreover, a_t is consistent with the agents' optimal choice

$$a_t = \psi(L)\boldsymbol{\nu}_t + \phi(L)\epsilon_t = \int \left\{ \alpha \mathbb{E}[z_{i,t} \mid \mathcal{I}_{i,t}] + \varphi \mathbb{E}[\theta_t \mid \mathcal{I}_{i,t}] + \gamma \mathbb{E}[a_t \mid \mathcal{I}_{i,t}] \right\} di \quad (11)$$

The key difference between the two classes is that in the sentiment equilibria, in addition to fluctuations driven by forces in a fundamental equilibrium, aggregate fluctuations can also arise due to changes in a payoff irrelevant factor ϵ_t .¹⁶ Notice that ϵ_t is completely unrelated to changes in fundamentals $\boldsymbol{\nu}_t$ and exogenous noise $\boldsymbol{\eta}_t$. Strictly speaking, sentiment equilibria are correlated equilibria.

In order to explain what sentiments are, it is useful to explain what they are not. As explained above, it may be the case that agents only observe aggregate fundamentals v_t with measurement error or noise η_t and this noise itself may drive aggregate fluctuations. According to our definition, this is *not* an example of sentiments driven fluctuations. Sentiments ϵ_t , as we have defined them, must be orthogonal to the vector $\boldsymbol{\nu}_t$, which includes η_t - they are not part of an agent's exogenous sources of information $\mathbf{y}_{i,t}$. As another example, one could consider an environment outside of our framework in which all agents observe a public randomization device (Aumann, 1974).¹⁷ If agents have a motive for coordinating their actions, such an environment might permit an equilibrium in which agents use this device to coordinate, and thus the aggregate outcome responds to the coordination device. Again according to our definition this is not a sentiments driven equilibrium. Unlike the realization of such a public coordination device, ϵ_t is not part of the exogenous information set: therefore it is not observed directly, nor is it an exogenously given feature of the environment.

Given that exogenous sources of information $\mathbf{y}_{i,t}$ provide no information about ϵ_t , it follows that agents can only get information about ϵ_t through the endogenous sources of information $\mathbf{x}_{i,t}$. Moreover, even within $\mathbf{x}_{i,t}$, the only way ϵ_t affects an agent's information set is through the aggregate action a_t . This is in contrast to η_t which can appear independently of a_t in $\mathbf{x}_{i,t}$. Thus, unlike η_t , ϵ_t is an *endogenous* source of aggregate fluctuations. The next section explains why such sentiment-driven fluctuations are possible even under rational expectations. In particular, our main focus in this paper is to study whether sentiments can drive persistent fluctuations.

¹⁶All sentiment equilibria lies in the Hilbert space $\mathcal{H}(\mathbf{v}, \boldsymbol{\eta}, \epsilon)$ (the space spanned by square-summable linear combinations of \mathbf{v}^t , $\boldsymbol{\eta}^t$ and ϵ^t).

¹⁷See a more detailed discussion in Section 5.2.

3 Understanding the forces

In this section we present some simple illustrations of the forces which allow sentiments to endogenously generate aggregate fluctuations in a rational expectations equilibrium. In order to provide a clean characterization and some intuition, we make some simplifying assumptions on the structure of the economy and the information structure of agents. We relax these assumptions and present our main result in Section 4. Also, for the entirety of this section, we will assume that Assumptions 1 and 2 hold. Then, in Section 4, we show the role these assumptions play in shaping the nature of permissible sentiments equilibria.

Simplifying Assumptions Relative to the general setup in the previous section we assume that the idiosyncratic fundamental for each individual is i.i.d, i.e. $z_{i,t} = u_{i,t}$. Also, we impose that the aggregate fundamental is described by a linear stationary stochastic process by $\theta_t = g(L)v_t$ where $v_t \sim N(0, \sigma_v^2)$.

Information Set For the rest of section 3, we will assume that the information set of each agent can be described as follows. As part of the exogenous information, we assume that each agent observes all past realizations of the aggregate fundamentals. Thus, at any date t , each agent perfectly observes the realization of the sequence θ^{t-1} . In addition, each agent has access to some endogenous information. In particular, at date t , each agent can perfectly observe all the past realizations of the aggregate action, i.e., the sequence a^{t-1} . Therefore, Assumption 1 and 2 are satisfied. In addition, each agent also observes a private signal given by:¹⁸

$$x_{i,t} = \beta a_t + (1 - \beta) u_{i,t}, \beta \in (0, 1)$$

$x_{i,t}$ is a signal of the current aggregate action which is convoluted by the idiosyncratic fundamental for each agent. This endogenous infiltration is what will turn out to be crucial for the existence of fluctuations driven by sentiments. In summary, the information set of agent i at time t can be written as $\mathbb{I}_{i,t} = \{a^{t-1}, x_i^t\} \equiv \mathbb{V}(\theta^{t-1}, a^{t-1}, x_i^t) \vee \mathbb{M}$.

3.1 An economy driven purely by sentiments

To isolate the forces at play, for the rest of subsection 3.1, we assume that aggregate fundamentals do not move. In other words, $\theta_t = 0$ for all t . Furthermore, we assume that this information is common knowledge. By shutting off any changes in aggregate fundamentals, the idea is to concentrate on an environment in which aggregate fluctuations are driven purely by changes in

¹⁸Chahrour and Gaballo (2016) show that such signals can arise endogenously in a islands economy.

sentiments. Recall that with the optimal action of each agent can be described as:

$$a_{i,t} = \alpha \mathbb{E}_{i,t}[u_{i,t}] + \gamma \mathbb{E}_{i,t}[a_t]. \quad (12)$$

where we have already imposed that $\mathbb{E}_{i,t}\theta_t = \theta_t = 0$ for all agents i and all time t .

Remark 1. *The unique fundamental equilibrium in this economy is one in which the aggregate action is constant over time at $a_t = 0$. In the fundamental equilibrium, each agent takes the action $a_{i,t} = \alpha x_{i,t} = \alpha u_{i,t}$ and thus, $\int_i a_{i,t} = 0$.*

In the fundamental equilibrium, the aggregate action is constant and this is common knowledge. Thus, even though each agent cannot directly observe the idiosyncratic fundamental $u_{i,t}$, observing the signal $x_{i,t}$ allows them to infer it perfectly. The fundamental equilibrium in this case is the same as the full information equilibrium in which each agent can observe $u_{i,t}$.

The next question we ask is whether there can be aggregate fluctuations in such an economy? The answer is clearly no if we restrict fluctuations to be driven by fundamentals. However, as we show next, the answer is yes (even under rational expectations) if we allow for sentiment-driven fluctuations. Recall that we defined a sentiment equilibrium in which the aggregate action a_t takes the following form

$$a_t = \phi(L)\epsilon_t. \quad (13)$$

where ϵ_t is some white noise process. Notice that ϵ_t does not appear directly in any of the formulation of the private signal $x_{i,t} = \beta a_t + (1 - \beta)u_{i,t}$. The only potential way it can affect actions is indirectly through the endogenous signal $x_{i,t}$. $x_{i,t}$ depends on the aggregate action a_t which in turn may depend on ϵ_t through the conjecture (13). Notice that the characterization of sentiment-driven equilibria in equation (13) is general enough to account for equilibria in which a shock to sentiments ϵ at date t can have persistent effects on aggregate outcomes.

There are two possible forms that $\phi(L)$ can take. First, $\phi(L)$ is *invertible*, i.e., there is no root of $\phi(z)$ ¹⁹ that lies inside the unit circle. Intuitively, this means that by observing the sequence a^t is equivalent to observing ϵ^t . In other words, since each agent can observe the past realizations of the aggregate action, each of them can perfectly infer all the past realizations of the white noise process ϵ_t . The other possibility is that $\phi(L)$ is *non-invertible*, i.e., there exists at least one root of $\phi(z)$ that lies inside the unit circle. In this case, by observing all past realizations of the aggregate action a^{t-1} , agents cannot recover the sequence ϵ^{t-1} and agents will never be able to infer innovations $\{\epsilon_\tau\}_{\tau=-\infty}^{t-1}$ perfectly. We start the analysis by first considering the *invertible* case.

¹⁹ $\phi(z)$ is the z -transform of $\phi(L)$.

3.1.1 Invertible process for sentiments

Suppose that $\phi(L)$ is invertible. The following proposition shows that the only possible sentiment equilibrium is one in which $\phi(L)$ is a constant. Thus, under the restriction that $\phi(L) = \phi_0 + \phi_1 L + \phi_2 L^2 \dots$ is invertible, even though we did not assume that $\phi_1, \phi_2 \dots = 0$. In other words, in a REE with invertible ϕ sentiments can only cause i.i.d aggregate fluctuations.

Proposition 1. *For α large enough, there exists sentiment equilibria with $\phi(L)$ being invertible. Furthermore, in any such sentiment equilibrium the aggregate action $a_t = \phi(L)\epsilon_t$ follows the following i.i.d process*

$$\phi(L) = \phi_0 = \pm \frac{\sigma_u}{\beta} \sqrt{\frac{(1-\beta)(\alpha\beta + \beta - 1)}{1-\gamma}}. \quad (14)$$

where $|\phi_0|$ is the standard deviation of aggregate fluctuations driven by sentiments in any REE. In other words, sentiment shocks can only affect aggregate outcomes contemporaneously.

Proof. See Appendix A. □

Note that Benhabib et al. (2015) obtain a similar result in a static setting, where by construction, sentiments can only affect aggregate outcomes in an i.i.d fashion. While Proposition 1 is fundamentally different from Benhabib et al. (2015): $\phi(L)$ is allowed to be any invertible process, but the equilibrium consistency requirements imposed by rational expectations restricts the effects of sentiments on aggregate outcomes to be i.i.d. in any REE.

Furthermore, notice that in Proposition 1, REE restricts the variance of sentiment driven fluctuations (and also allowable fluctuations). But this is not the only restriction that REE imposes. The REE not only pins down the variance but also the entire autocorrelation structure. Here REE ensures that all autocorrelations $\mathbb{E}[a_t a_{t-k}] = 0$ for $k \neq 0$.

To gain some intuition regarding the content of Proposition 1, notice that since aggregate outcomes are driven solely by contemporaneous changes in sentiments, the evolution of aggregate outcomes can be written as $a_t = \omega \epsilon_t$. Consequently, all past signals are irrelevant to the agent in trying to forecast the aggregate action today. Thus, agents optimal forecasts only depend on current signals and can be written as:²⁰

$$\mathbb{E}_{i,t}[a_t] = \chi x_{i,t} \quad (15)$$

$$\mathbb{E}_{i,t}[u_{i,t}] = \frac{1-\beta\chi}{1-\beta} x_{i,t} \quad (16)$$

where

$$\chi = \frac{\beta\omega^2}{\beta^2\omega^2 + (1-\beta)^2\sigma_u^2} \quad (17)$$

²⁰See Appendix A for a full proof

for some $\omega \in \mathbb{R}$. This implies that the action of each agent in period t is only driven by information relating to the current period:

$$a_{i,t} = \gamma \mathbb{E}_{i,t} a_t + \alpha \mathbb{E}_{i,t} u_{i,t} = \left[\gamma \chi + \alpha \frac{1 - \beta \chi}{1 - \beta} \right] x_{i,t}$$

and so does the aggregate action:

$$a_t = \int a_{i,t} = \left[\gamma \chi + \alpha \frac{1 - \beta \chi}{1 - \beta} \right] \int x_{i,t} = \beta \left[\gamma \chi + \alpha \frac{1 - \beta \chi}{1 - \beta} \right] a_t$$

As can be seen, consistency requires that $\beta \left[\gamma \chi + \alpha \frac{1 - \beta \chi}{1 - \beta} \right] = 1$ which can only hold only if the standard deviation of fluctuations is the same as specified in Proposition 1. In other words ω must equal ϕ_0 . Thus, the consistency requirements of a REE prevent sentiments from having any persistent effect on aggregate outcomes.

Example: Why can't sentiments drive persistent fluctuations? To further appreciate why sentiments cannot have persistent effects in this case, for a moment suppose there is an equilibrium in which agents perceive that sentiments have persistent effects on aggregate outcomes. In particular, assume that this persistent effect is described by an AR(1) process:

$$a_t = \rho a_{t-1} + \omega \epsilon_t$$

where both ρ and ω are determined as part of equilibrium.²¹ Given the perceived law of motion of aggregate action, agent i can transform her private signal $x_{i,t}$ into:²²

$$\hat{x}_{i,t} = x_{i,t} - \beta \rho a_{t-1} = \beta \omega \epsilon_t + (1 - \beta) u_{i,t} \quad (18)$$

and the optimal forecast of a_t and $u_{i,t}$ are given by

$$\mathbb{E}_{i,t}[a_t] = \rho a_{t-1} + \chi \hat{x}_{i,t} \quad \text{and} \quad \mathbb{E}_{i,t}[u_{i,t}] = \frac{1 - \beta \chi}{1 - \beta} \hat{x}_{i,t} \quad (19)$$

where χ is defined in equation (17). Notice that in this case, since sentiments have persistent effects, past aggregate actions also give agents information about sentiments in the past and hence helps them forecast current actions. Given these optimal forecasts, agent i 's optimal

²¹Note that the AR(1) $a_t = \frac{\omega}{1 - \rho L} \epsilon_t$ satisfies the property that it is invertible since $\frac{1}{1 - \rho L}$ has no zeros inside the unit circle.

²²Note that these two forms are informationally equivalent since each agent observes a_{t-1} perfectly at date t .

action can be written as:

$$a_{i,t} = \gamma \rho a_{t-1} + \gamma \chi \hat{x}_{i,t} + \alpha \frac{1 - \beta \chi}{1 - \beta} \hat{x}_{i,t} \quad (20)$$

and thus, the actual law of motion of the aggregate action can be written as:

$$a_t = \gamma \rho a_{t-1} + \beta \omega \left[\gamma \chi + \alpha \frac{1 - \beta \chi}{1 - \beta} \right] \epsilon_t \quad (21)$$

Rational Expectations requires that the actual and perceived laws of motion be consistent. Consequently, it must be the case that in any REE $\rho = \gamma \rho$. Given our assumption that $\gamma < 1$, the only way this can be satisfied if $\rho = 0$ or in other words, sentiments cannot have a persistent effect on aggregate outcomes.²³ This example shows that rational expectations puts strong restrictions on how sentiments can evolve and affect the aggregate action. In this sense, rational expectations determines the allowable set of processes for sentiments in equilibrium and hence, sentiments can be thought of as arising endogenously in our setting.

3.1.2 Non-invertible process for sentiments

The previous section imposed the restriction that $\phi(L)$ be invertible. However, there is no economic reason to a-priori exclude the case in which $\phi(L)$ is non-invertible. In fact many standard economic models feature non-invertible processes for equilibrium quantities without any pathologies. For example, the simple life-cycle-consumption hypothesis yields a process for savings which is non-invertible:²⁴

$$\Delta s_t = \left(L - 1 + \frac{1}{R} \right) e_t, \text{ where } R > 1 \text{ is the interest rate.}$$

This process is non-invertible since $z - 1 + R^{-1}$ has a zero inside the unit circle at $z = 1 - \beta < 1$.

Thus, we turn our attention to equilibria in which the aggregate action $a_t = \phi(L)\epsilon_t$ is non-invertible. If this is the case, then agents cannot recover $\{\epsilon_\tau\}_{\tau=-\infty}^t$ by observing $\{a_\tau\}_{\tau=-\infty}^t$.

²³Even if we relax that assumption on γ and allow $\gamma = 1$, then for such an equilibrium to exist, a knife edge condition that $\alpha = (1 - \beta)/\beta$ must be satisfied. Suppose we have the knife edge condition holding $\alpha = \frac{1-\beta}{\beta}$ and $\gamma = 1$. Notice that then the optimal response of any agent can be written as:

$$a_{i,t} = \frac{1}{\beta} \mathbb{E}_{i,t} [\beta a_t + (1 - \beta) u_{i,t}] \equiv \frac{1}{\beta} x_{i,t}, \quad (22)$$

In this knife edge case, agents do not need to solve any signal extraction problem any more since they miraculously see a perfect signal describing their optimal action.

²⁴Assume that the consumer has utility function $u(c) = ac - \frac{b}{2}c^2$ and that she discounts the future at rate δ . For simplicity assume that the return on savings $R = \delta^{-1} > 1$. Also, assume that income y_t is i.i.d. over time. In this very standard setting, it is straightforward to show that savings out of current income follows the non-invertible process mentioned in the main text.

Technically, if $\phi(L)$ is non-invertible, then there exists at least one $|\lambda| < 1$ such that $\phi(\lambda) = 0$. In this section, for ease of exposition we restrict attention to the case in which $\phi(L)$ has exactly one root inside the unit circle, and leave the general case to Section 4. Before we characterize equilibria in this class, we first define the Blaschke factor as it will be the key to the non-invertible sentiment equilibrium.

Definition 3. A Blaschke Factor with parameter λ is defined as

$$\mathcal{B}(L; \lambda) = \frac{L - \lambda}{1 - \lambda L}, \quad (23)$$

where $\lambda \in (-1, 1)$.

The following proposition characterizes the set of sentiment equilibria if $\phi(L)$ has one inside root. Notably, all the equilibria are proportional to a Blaschke factor.

Proposition 2. There exists a continuum of sentiment equilibria with one inside root. All these sentiment equilibria $a_t = \phi(L)\epsilon_t$ take the following form

$$\phi(L) = \pm \frac{\sigma_u}{\beta} \sqrt{\frac{(1 - \beta)(\alpha\beta + \beta - 1)}{1 - \gamma}} \mathcal{B}(L; \lambda), \quad (24)$$

with $\lambda \in (-1, 1)$

Proposition 2 shows that in order to understand the property of non-invertible sentiment equilibrium, it is sufficient to understand the properties of the Blaschke factor.

First, as can be seen in Figure 1, the impulse response of a stochastic process which can be described by a Blaschke factor displays persistence. Let $\{\vartheta_0, \vartheta_1, \dots\}$ denote the coefficients of the infinite moving-average representation of the Blaschke factor:

$$\mathcal{B}(L; \lambda)\epsilon_t = \sum_{\tau=0}^{\infty} \vartheta_{\tau} \epsilon_{t-\tau} \quad (25)$$

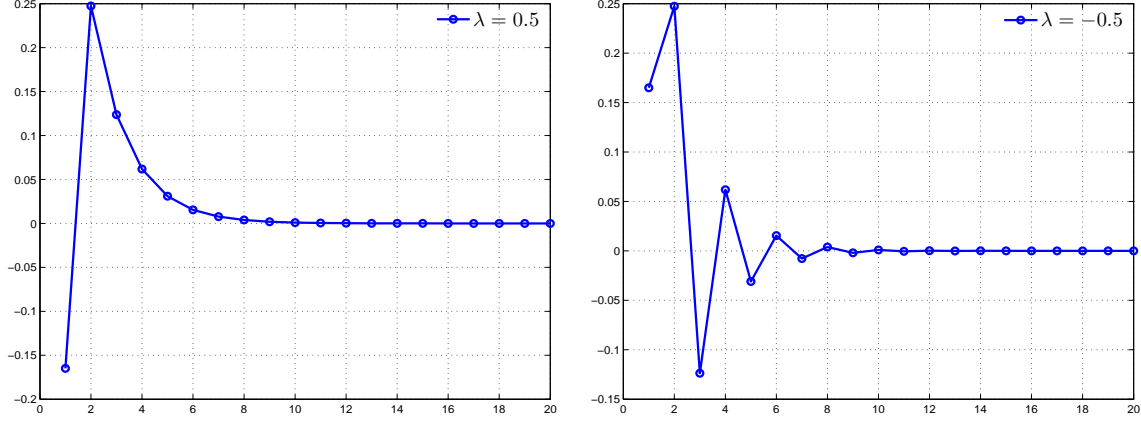
and the coefficients are given by

$$\vartheta_0 = \lambda \quad \vartheta_1 = (\lambda^2 - 1) \quad \vartheta_2 = \lambda\vartheta_1 \quad \dots \quad \vartheta_{\tau} = \lambda\vartheta_{\tau-1}$$

Second, the impulse response of a Blaschke factor is non-monotonic:

1. If $\lambda > 0$, then $\text{sign } \vartheta_0 \neq \text{sign } \vartheta_1$ and $\text{sign } \vartheta_{\tau} = \text{sign } \vartheta_{\tau+1}$ for $\tau > 0$.
2. If $\lambda < 0$, then $\text{sign } \vartheta_0 = \text{sign } \vartheta_1$ and $\text{sign } \vartheta_{\tau} \neq \text{sign } \vartheta_{\tau+1}$ for $\tau > 0$.

Figure 1: Impulse Response of Blaschke Factor



To give a numerical example, we choose $\lambda = \pm 0.5$, and Figure 1 shows a typical impulse response graph of the Blaschke factor.

Third, the Blaschke factor is a unitary operator,²⁵ or in other words has the same auto-correlation generating function as an i.i.d process:

$$g(z) = \mathcal{B}(z^{-1}; \lambda) \mathcal{B}(z; \lambda) = \frac{z^{-1} - \lambda}{1 - \lambda z^{-1}} \frac{z - \lambda}{1 - \lambda z} = 1. \quad (26)$$

As a result, any time series generated by a Blaschke factor is observationally equivalent to one generated by an i.i.d process given data up to any date t even if the impulse response to an ϵ shock potentially displays non-trivial dynamics.

These properties of the Blaschke factor imply that the non-invertible sentiment equilibria displays a persistent and non-monotonic impulse response to innovations in sentiments, but the resulting time series cannot be distinguished from an i.i.d process. A direct implication of this property is that it is very hard statistically to reject the hypothesis that sentiments do not have persistent effects even if an actual innovation to sentiments today can continue to affect aggregate outcomes in the future.

Another interesting thing to note is that Proposition 1 and 2 together imply that all the sentiment equilibria described so far, invertible or non-invertible, imply that aggregate outcomes have the same variance which can be expressed as:

$$\mathbb{V}[a_t] = \frac{\sigma_u^2 (1 - \beta)(\alpha\beta + \beta - 1)}{\beta^2 (1 - \gamma)}. \quad (27)$$

²⁵A unitary operator is a bounded linear operator U defined on a Hilbert space \mathcal{H} that satisfies $U^*U = UU^* = I$ where U^* is the adjoint of U and I is the identity operator.

However, REE no longer restricts sentiments to only have i.i.d effects on aggregate outcomes if $\phi(L)$ is allowed to be non-invertible. Even so, rational expectations still imposes strict restrictions on the processes of sentiments (and their effect on aggregate outcomes) allowable in equilibrium. As mentioned earlier, this is the sense in which our model features an *endogenous* notion of sentiments. By restricting the way in which sentiments can affect aggregate actions dynamically, the model features endogenous sentiment-driven dynamics.

Importantly, Proposition 2 implies that no other non-invertible process is possible in equilibrium,²⁶ and the reason is similar to the invertible case. With persistent process, REE imposes an infinite number of consistency requirements, and the only way to satisfy all of them is to restrict the process to be the form of Blaschke factor. The following example, using the simplest persistent process which is non-invertible and different from the Blaschke factor, shows why this has to be the case.

Example: ARMA(1,1) process For a moment let us assume that in equilibrium, the sentiment equilibrium $a_t = \phi(L)\epsilon_t$ evolves according to the following stationary ARMA(1,1) process:

$$a_t = \omega \frac{L - \xi}{1 - \lambda L} \epsilon_t, \quad (28)$$

where ξ, ω and λ are determined in equilibrium. Nevertheless, let us start by assuming that $|\xi| < 1$, which makes the process in Equation (28) be noninvertible.²⁷ Given this perceived law of motion of the aggregate action, we next characterize each individual's forecasts about the aggregate action a_t and idiosyncratic fundamental $u_{i,t}$.²⁸

Given the information set of each agent, their optimal forecast of a_t and $u_{i,t}$ can be written as:

$$\mathbb{E}_{i,t}[a_t] = \chi x_{i,t} + (1 - \beta\chi) (\lambda - \xi) \sum_{j=0}^{\infty} \xi^j a_{t-1-j}, \quad (29)$$

$$\mathbb{E}_{i,t}[u_{i,t}] = \frac{1 - \beta\chi}{1 - \beta} x_{i,t} - \frac{\beta(1 - \beta\chi)}{1 - \beta} (\lambda - \xi) \sum_{j=0}^{\infty} \xi^j a_{t-1-j} \quad (30)$$

where χ is the same as in equation (17). Using these, the optimal action of agent i can be

²⁶This is true as long as we restrict there to be only one zero inside the unit circle at $\phi(\lambda) = 0$ for $\lambda \in (-1, 1)$. For the more general case, see Section 4.

²⁷This is without loss of generality since $|\zeta| > 1$ would imply that the process described in equation (28) is invertible which we already considered in Proposition 1. Of course, since we are looking for stationary equilibria, this requires that $|\lambda| < 1$.

²⁸See Appendix B for more details.

written as:

$$\begin{aligned} a_{i,t} &= \gamma \mathbb{E}_{i,t} a_t + \alpha \mathbb{E}_{i,t} u_{i,t}, \\ &= \left(\gamma \chi + \alpha \frac{1 - \beta \chi}{1 - \beta} \right) x_{i,t} + (1 - \beta \chi) (\lambda - \xi) \left(\gamma - \frac{\alpha \beta}{1 - \beta} \right) \sum_{j=0}^{\infty} \xi^j a_{t-1-j}. \end{aligned} \quad (31)$$

Aggregating the individual actions, the aggregate action can be expressed as:

$$a_t = \beta \left(\gamma \chi + \alpha \frac{1 - \beta \chi}{1 - \beta} \right) [\lambda a_{t-1} - \omega \xi \epsilon_t + \omega \epsilon_{t-1}] + (1 - \beta \chi) (\lambda - \xi) \left(\gamma - \frac{\alpha \beta}{1 - \beta} \right) \sum_{j=0}^{\infty} \xi^j a_{t-1-j}. \quad (32)$$

Note that equation (32) is the actual law of motion of the aggregate action. In order for this constitute a REE, this must be identical to the perceived law of motion (28). Thus, REE imposes that the following conditions:

$$\begin{aligned} \beta \left(\gamma \chi + \alpha \frac{1 - \beta \chi}{1 - \beta} \right) &= 1, \\ \lambda &= \xi, \end{aligned}$$

which can only be satisfied if $\omega = \pm \frac{\sigma_u}{\beta} \sqrt{\frac{(1-\beta)(\alpha\beta+\beta-1)}{1-\gamma}}$. Thus, REE restricts $\xi = \lambda$. Notice that with $\xi = \lambda$, in their optimal forecasts of a_t and $u_{i,t}$ given by equations (29)-(30), each agent puts no weight on the past realizations of the aggregate action. This reveals that in such an equilibrium, even though shocks to sentiments can affect aggregate outcomes over time, looking at past realizations of aggregate actions does not provide any useful information to each agent about the average action today. Consequently, these fluctuations are observationally equivalent to an i.i.d process.

In other words, the consistency requirements imposed by REE ensure that any fluctuations driven by sentiments cannot be predictable based on past realizations of aggregate outcomes. This restriction is satisfied if sentiments actually only had i.i.d. effects on aggregate actions. However, Proposition 2 showed that even if sentiments did have persistent effects on aggregate outcomes, REE puts very strong restrictions on the structure of how such sentiments can affect aggregates. We show in Section 4 that this restriction on sentiment-driven fluctuations is robust and holds for very general information structures. In particular, Section 4 shows that the broader restriction that REE imposes is that as long as Assumptions 1-2 hold, in any sentiments equilibria, sentiment -driven aggregate fluctuations can only be described by the following:

$$a_t = \omega \prod_{i=1}^n \mathcal{B}(L, \lambda_i)$$

for any n -sequence $\{\lambda_i\}_{i=1}^n \in (0, 1)^n$ for any $n \in \mathbb{Z}_+$.²⁹

These restrictions, imposed by REE, on how sentiments can affect aggregate actions have strong implications for what an econometrician can learn about the role of sentiments in explaining aggregate fluctuations. Consider an econometrician who can observe all past realizations of the aggregate outcomes. On the basis of this data, she will not be able to reject the hypothesis that sentiments only affect aggregate outcomes in an i.i.d fashion.

3.2 Can agents disagree forever in a REE?

Proposition 2 stated that under rational expectations, sentiments can generate persistent aggregate fluctuations. Moreover, the Proposition also stated that there is a continuum of such non-invertible sentiment equilibria indexed by $\lambda \in (-1, 1)$. Usually in games with multiple equilibria, a particular equilibrium arises when all agents choose to play a particular equilibrium and all agents are in agreement about which equilibrium is being played. Thus, even though each agent may have dispersed information, each agent typically shares the same subjective beliefs about the model (or the distribution of outcomes) as all other agents, which are also the same the objective beliefs imposed by the equilibrium. This is indicative of the tight link between objective and subjective beliefs that rational expectations demands in equilibrium.

This link, however, is much weaker in our sentiments equilibria even though these are still rational expectations equilibria. The following Proposition formalizes this notion by showing that in any one of the continuum of sentiment equilibria described in Propositions 1 and 2, agents do not necessarily have to agree on which equilibrium is being played. Yet still, their behavior will be consistent with the conditions that need to be satisfied in a rational expectations equilibrium.

Proposition 3. *For α large enough, given any $\lambda \in (-1, 1)$, there exists a sentiments equilibrium in which aggregate action is given by:*

$$a_t = \omega \mathcal{B}(L; \lambda) \text{ , where } \omega = \pm \frac{\sigma_u}{\beta} \sqrt{\frac{(1 - \beta)(\alpha\beta + \beta - 1)}{1 - \gamma}}$$

In such an equilibrium, all agents agree on ω but each agent i can entertain arbitrary beliefs about the equilibrium λ as long as this belief lies in $(-1, 1)$.

Proof. Pick an equilibrium indexed by a particular $\lambda = \bar{\lambda}$. Then, aggregate dynamics can be described as:

$$a_t = \omega \mathcal{B}(L; \bar{\lambda}) \epsilon_t, \tag{33}$$

²⁹ n is the number of roots of the polynomial $\phi(z)$ which lie inside the unit circle. In this section we had restricted our assumption to $n = 1$. Notice that this product of Blaschke factors are also observationally equivalent to an i.i.d. process given the information set since Blaschke factors are unitary.

Suppose that agent i believes that the law of motion of aggregate action is instead given by

$$a_t = \omega_i \frac{L - \lambda_i}{1 - \lambda_i L} \epsilon_t \quad (34)$$

where $\lambda_i \in (-1, 1)$ and $\lambda_i \neq \bar{\lambda}$. The information set of this agent i is the same as specified at the beginning of Section 3.1. Thus, at date t , the agent observes the signal: $x_{i,t} = \beta a_t + (1 - \beta)u_{i,t}$ and all past realizations of the aggregate actions a^{t-1} . Given the information set and this perceived law of motion, the optimal action of agent i can be written as:³⁰

$$a_{i,t} = \alpha \mathbb{E}_{i,t}[u_{i,t}] + \gamma \mathbb{E}_{i,t}[a_t] = \chi_i x_{i,t}$$

with

$$\chi_i = \frac{\beta \omega_i^2}{\beta^2 \omega_i^2 + (1 - \beta)^2 \sigma_u^2}$$

Note that if agent i has the correct beliefs about ω , i.e., $|\omega_i| = |\omega|$, then $\chi_i \equiv \chi$ (defined in equation (17)) is independent of λ_i . As a result, the implied aggregate action is

$$a_t = \int \chi_i x_{i,t} = \chi \beta a_t$$

The only consistency requirement imposed by rational expectations is, $\beta \chi = 1$, which requires that

$$\omega_i = \omega = \pm \frac{\sigma_u}{\beta} \sqrt{\frac{(1 - \beta)(\alpha \beta - 1 + \beta)}{1 - \gamma}}.$$

□

Thus, rational expectations does not restrict any agents belief about λ . In other words, agents can disagree forever about which equilibrium is being played. Furthermore, these beliefs will not get contradicted in equilibrium even though dynamics are different in equilibria with different λ ! To understand why this is possible, recall that an important restriction that REE imposed in Proposition 2 was that sentiment driven fluctuations are unpredictable given the past realizations and in equilibrium, i.e., agents could not learn anything about the current and future aggregate actions from observing past actions. Crucially, when agents believe the aggregate law of motion is proportional to a Blaschke factor $\mathcal{B}(L; \lambda)$, then their best forecast also does not depend on the past signals, and thus is unaffected by the actual λ even though it governs the relationship between the aggregate action over time.

Corollary 1. *Consider an equilibrium in which the evolution of aggregate outcomes can be*

³⁰The mechanics are very similar to those for the ARMA(1,1) example in Appendix B.

expressed as:

$$a_t = \omega \frac{L - \lambda}{1 - \lambda L}$$

The following statements hold in any such equilibrium:

1. Consider any sequence of $\{\lambda_i\}_{i \in [0,1]}$ where $\lambda_i \in (-1, 1)$ denotes agent i 's beliefs about the true λ . Then it can be the case that such that $\lambda_i \neq \lambda$ for all $i \in [0, 1]$. In other words, all agents can be wrong about which equilibrium is being played.
2. The actual aggregate dynamics which depend on λ is independent of the distribution of beliefs λ_i that agents might have about the aggregate λ . Consequently, knowing the distribution of beliefs about λ_i in the population gives no information about the actual λ .
3. Agents cannot learn which equilibrium is being played based on past realizations.

The above corollary which simply follows from Proposition 3 and states that in equilibrium, all agents can entertain the incorrect model indexed by λ without ever learning the true λ . Importantly, this wedge between subjective and objective beliefs is still consistent with rational expectations. Given the information set of agents, they cannot identify any differences between the actual and their perceived laws of motion even though they observe all past realizations of aggregate outcomes.

3.3 Do aggregate fundamentals limit the role of sentiments?

In the previous subsection, we had assumed that there was no change in aggregate fundamentals. Thus, by construction, all fluctuations were driven purely by changes in sentiments. This stark setting allowed us to highlight in a transparent fashion the fact that REE can still feature a role for sentiment driven fluctuations, but that the way in which sentiments can affect aggregate outcomes are strongly restricted by rational expectations. This brings us to the question of whether stochastic aggregate fundamentals weaken or tighten these restrictions. Can sentiments still separately drive aggregate fluctuations in the presence of aggregate fundamentals? In this section we introduce a stochastic aggregate fundamental θ_t into the economy. In particular, we assume that aggregate fundamentals are governed by an exogenous linear stationary stochastic process:

$$\theta_t = g(L)v_t = g_0v_t + g_1v_{t-1} + g_2v_{t-2} + \dots, \quad (35)$$

where v_t denotes the innovations to aggregate fundamentals. Further, as is common in the literature, we assume that $g(L)$ is invertible and $v_t \sim \mathcal{N}(0, \sigma_v^2)$.³¹ Similar to Section 3.1, we still maintain the assumption that the idiosyncratic shock follows an i.i.d process, i.e., $z_{i,t} = u_{i,t}$

³¹The setting easily generalizes if one wishes to accommodate non-invertible processes for $g(L)$.

and that Assumptions 1-2 still hold. In such a hybrid equilibrium, the evolution of the aggregate action can be expressed in general as:

$$a_t = \psi(L) v_t + \phi(L) \epsilon_t.$$

where $\psi(L)$ and $\phi(L)$ are potentially infinite order polynomials in positive powers of L .³² The following Proposition describes the entire set of linear equilibria in this setting.

Proposition 4. *For α large enough, there exists a continuum of equilibria in the following form:*

$$\psi(L) = \frac{\varphi}{1-\gamma} g(L) + \frac{\sigma_u^2 (1-\beta) \psi_0 [\alpha\beta - (1-\beta)\gamma] - \varphi g_0 [\sigma_u^2 (1-\beta)^2 + \beta^2 \phi_0^2]}{(1-\gamma) [\beta^2 (\sigma_v^2 \psi_0^2 + \phi_0^2) + \sigma_u^2 (1-\beta)^2]} \quad (36)$$

$$\phi(L) = \begin{cases} \pm \phi_0, & \text{if } \phi(L) \text{ invertible,} \\ \pm \phi_0 \prod_{i=1}^n \mathcal{B}(L; \lambda_i), & \text{if } \phi(L) \text{ noninvertible} \end{cases} \quad (37)$$

for any $n \in \mathbb{Z}_+$ and any n -sequence $\{\lambda_i\} \in (-1, 1)^n$ where (ψ_0, ϕ_0) lie on the real valued circle with center at $(-\varphi\sigma_v g_0/2, 0)$ and radius \bar{r} which is given by:

$$\bar{r} = \sqrt{\frac{\sigma_u^2 (1-\beta) [\alpha\beta + \beta - 1]}{\beta^2 (1-\gamma)} + \frac{\varphi^2 g_0^2 \sigma_v^2}{4 (1-\gamma)^2}} \quad (38)$$

Proof. See Appendix A and the general proof in section 4. □

The first thing to note about equilibria in this setting is that even with the presence of fundamentals, sentiments can only affect aggregate outcomes in the same way as in the case with only sentiments. In particular, if we restrict $\phi(L)$ to only be invertible, then much like the earlier case, rational expectations does not permit sentiments to affect aggregate outcomes for more than one period. However, if one allows for non-invertability then even under rational expectations sentiments can drive persistent fluctuations. Much like the case with only sentiments, REE restricts the way in which sentiments can affect aggregate outcomes. Any fluctuations in aggregate outcomes due to sentiments must be proportional to the Blaschke factor.

Second, fluctuations based on aggregate fundamentals are also affected by the presence of sentiments. In the case where aggregate fundamentals also move, there is an additional source of multiplicity. This can be seen in terms of the values of ϕ_0 . As is stated in Proposition 4, there is a continuum of equilibrium indexed by $\phi_0 \in [0, r]$, that is, the standard deviation of aggregate fluctuations that is explained by sentiments. Thus, there is one equilibrium in which

³²As before, we further impose the restriction that $\psi(L)$ and $\phi(L)$ be square-summable which ensures that a_t is stationary. As in the case before, we allow for $\phi(L)$ to be both invertible and non-invertible.

all fluctuations are explained by fundamentals and none of the fluctuations are explained by sentiments, $\phi_0 = 0$. The other extreme equilibrium is one in which aggregate fundamentals do not have any impact on contemporaneous aggregate actions. In this other extreme all contemporaneous aggregate fluctuations are explained by sentiments.³³

Notice that in the example above, agents did not have any direct information about the aggregate fundamental. Theorem 2 in the next section shows that providing the agents additional information about aggregate fundamentals may affect the equilibrium $\psi(L)$ but cannot affect the form of $\phi(L)$ as long as Assumptions 1-2 hold. In other words, even if agents had more information about aggregate fundamentals, as long as they observe lagged innovations to fundamentals and lagged aggregate outcomes, the form of the sentiment equilibria is not affected even though $\psi(L)$ may change. To appreciate this, consider the extreme case in which all agents perfectly observe the contemporaneous aggregate fundamental θ_t . This case is very similar to the example in section 3.1 as the aggregate fundamental is common knowledge and thus the characterization of the sentiment equilibria is the same as in Proposition 4. Agents can perfectly track aggregate fundamentals and $\psi(L) = \frac{\varphi}{1-\gamma}g(z)$. Even in this case there is still a continuum of sentiment equilibria.³⁴

Even though the examples above featured stylized assumptions about the information set of agents, the broad underlying message from the examples above is that the characterization of sentiment equilibria in Propositions 2 and 4 is robust. In fact, Theorem 2 below shows that this characterization continues to hold for very general information structures as long as agents observe past aggregate actions and aggregate fundamentals regardless of the structure of private information agents might possess.

4 The general characterization

Theorem 1. *If $\phi(L)$ is a sentiment equilibrium, then*

$$\tilde{\phi}(L) = \phi(L) \prod_{k=0}^K \mathcal{B}(L; \lambda_k), \quad (40)$$

³³The total variance of aggregate fluctuations explained by sentiments is maximized in this equilibrium and is given by \bar{r}^2 . There is also a continuum of intermediate equilibria where aggregate fluctuations respond both to sentiments and fundamentals on impact and can be indexed by (ψ_0, ϕ_0) which lie on the circle below:

$$\left(\psi_0 - \frac{\varphi\sigma_v g_0}{2}\right)^2 + \phi_0^2 = \bar{r}^2 \quad (39)$$

The above equation clearly shows that a higher value of $|\phi_0|$ implies a lower value of $|\psi_0|$.

³⁴However, the sentiments do not affect the response of aggregate fluctuations to fundamentals.

is also a sentiment equilibrium, where K can be any nonnegative integer and λ_k can be any real number that lies in $(-1, 1)$.

Proof. See Appendix D for a proof. \square

Theorem 1 formalizes the intuition developed in Section 3. Once we find a sentiment equilibrium, we can construct a continuum of equilibria around the original one. These equilibria display different impulse responses, but they share the same auto-covariance generating functions. Agents can disagree with each other on which one they choose from this class of equilibria, but their actions are rational and consistent with the requirements in a rational expectations equilibrium. Note that this result is driven by the fact that the sentiment shock ϵ_t enters agents' information set only through the aggregate action a_t contained in the endogenous signals. In contrast, exogenous common noise η_t do not share the same properties.

One can probably interpret Theorem 1 as a way to enlarge the set of equilibria. The following theorem reduces the set of sentiment equilibrium once additional assumptions are made in terms of the information structure.

Theorem 2. *Under Assumption 1 and 2, sentiment equilibria $\phi(L)$ can only take the following form*

$$\phi(L) = \omega \prod_{k=0}^K \mathcal{B}(L; \lambda_k), \quad (41)$$

where K can be any nonnegative integer and $\lambda_k \in (-1, 1)$.

Proof. We prove this theorem in two steps. First, we prove that if the sentiment process $\phi(L)$ is invertible, then the sentiment process can only be i.i.d, $\phi(L) = \omega$. If Assumption 1 and 2 are satisfied, $\mathbb{E}[\boldsymbol{\nu}_{t-1} \mid \mathbf{y}_i^t] = \boldsymbol{\nu}_{t-1}$ and agents also observe $a_{t-1} = \boldsymbol{\psi}(L)\boldsymbol{\nu}_{t-1} + \phi(L)\epsilon_{t-1}$. As a result, agents observe $\phi(L)\epsilon_{t-1}$ perfectly. If $\phi(L)$ is invertible, then past sentiment shocks $\{\epsilon_{t-k}\}_{k=1}^\infty$ can be inferred perfectly.

Now consider an impulse response of the signals to an ϵ_t shock, where $\epsilon_0 = 1$, and $\epsilon_t = 0$ for $t \neq 0$. Note that for the sentiment process $\phi(L) = \sum_{t=0}^\infty \phi_t L^t$, ϕ_t is the same as the response of a_t at time t . Therefore, to show that $\phi(L) = \omega$, it is sufficient to show that the impulse response of a_t is zero after the initial period.

Before agents observe $\mathbf{y}_{i,0}, \mathbf{x}_{i,0}$, and a_{-1} , the past realizations of all signals are zero, and it follows $\mathbb{E}_{i,t}[\boldsymbol{\zeta}_{i,t-k}] = \mathbf{0}$, $\mathbb{E}_{i,t}[\boldsymbol{\nu}_{t-k}] = \mathbf{0}$, and $\mathbb{E}_{i,t}[\epsilon_{t-k}] = 0$ for $t < 0$ and $k \geq 0$.

When $t = 0$, agents observe

$$\begin{aligned} \mathbf{y}_{i,0} &= \mathbf{0}, \\ \mathbf{x}_{i,0} &= \mathbf{A}(0)\phi_0, \\ a_{-1} &= 0 \end{aligned}$$

In this initial period, agents may not be able to distinguish all the shocks, and the response of a_t is given by

$$a_0 = \phi_0 = \int \alpha \mathbb{E}_{i,0}[\mathbf{h}(L)\boldsymbol{\zeta}_{i,0}] + \varphi \mathbb{E}_{i,0}[\mathbf{d}(L)\boldsymbol{\nu}_0] + \gamma \mathbb{E}_{i,0}[\phi(L)\epsilon_t] \quad (42)$$

$$= \int \alpha \mathbb{E}_{i,0} \left[\sum_{k=0}^{\infty} \mathbf{h}_k \boldsymbol{\zeta}_{i,0-k} \right] + \varphi \mathbb{E}_{i,0}[\mathbf{d}(0)\boldsymbol{\nu}_0] + \gamma \mathbb{E}_{i,0}[\phi(0)\epsilon_t], \quad (43)$$

which may be different from zero.³⁵

When $t > 0$, all shocks convert to zeros

$$\begin{aligned} \mathbf{y}_{i,t} &= \mathbf{0}, \\ \mathbf{x}_{i,t} &= \mathbf{0}, \\ a_{t-1} &= 0. \end{aligned}$$

Particularly, when $t = 1$, agents observe $\boldsymbol{\nu} = 0$ and can infer $\epsilon_0 = 1$ perfectly through $\phi(L)\epsilon_0 = a_0 - \boldsymbol{\psi}_0\boldsymbol{\nu}_0$. Comparing with the information at time $t = -1$, the signals at $t = 0$ offer no additional information about idiosyncratic shocks. It follows that $\mathbb{E}_{i,t}[\boldsymbol{\zeta}_{i,t-k}] = 0$, $\mathbb{E}_{i,t}[\boldsymbol{\nu}_{t-k}] = 0$, and $\mathbb{E}_{i,t}[\epsilon_{t-k}] = 0$ for $t > 0$ and $k \geq 0$. Therefore, the impulse response with $t > 0$ is given by

$$a_t = \phi_t = \int \alpha \mathbb{E}_{i,t}[\mathbf{h}(L)\boldsymbol{\zeta}_{i,t}] + \varphi \mathbb{E}_{i,t}[\mathbf{d}(L)\boldsymbol{\nu}_t] + \gamma \mathbb{E}_{i,t}[\phi(L)\epsilon_t] = \gamma \phi_t \epsilon_0. \quad (44)$$

Given that $\gamma \neq 1$ and $\epsilon_0 = 1$, it has to be that $\phi_t = 0$ for $t > 0$ and $\phi(L) = \phi_0$.

Now we proceed to the second step and suppose that $\phi(L)$ is non-invertible. Let $\tilde{\phi}(L)$ denote the fundamental representation of $\phi(L)$, which is an invertible process. By Theorem 1, $\phi(L)$ is an equilibrium if and only if $\tilde{\phi}(L)$ is an equilibrium. In the first step, we have shown that if $\tilde{\phi}(L)$ is invertible, it can only be an i.i.d process. As a result, if there exists a sentiment equilibrium $\phi(L)$, it has to be the case that the auto-covariance generating function of $\phi(L)$ is the same as an i.i.d process. \square

A couple of remarks should be made here. First, this result is quite general. We do not impose any restrictions on the number of shocks nor the number of signals, and we do not impose restrictions on whether the signal process is invertible or non-invertible. Second, the Theorem states that under Assumptions 1-2 if a sentiment equilibrium exists, the impulse response of the sentiment equilibrium can be persistent, but that the autocovariance generating function is observationally equivalent to an i.i.d process. In other words, sentiment driven fluctuations are indistinguishable from an i.i.d. process as long as Assumptions 1-2 hold. Finally, it is

³⁵Note that the inference about $\boldsymbol{\zeta}_{i,-k}$ could be different from zero.

important to notice that Theorem 2 is not about the existence of sentiment equilibrium, it only shows that if a sentiment equilibrium exists, it can only take a particular form. Whether a sentiments-equilibria exists or not depends on the exact information set. However, Corollary 2 shows that Theorem 2 is still useful in determining whether sentiment equilibria exist.

Corollary 2. *Under Assumption 1 and 2, there exists a sentiment equilibrium if and only if there exists an i.i.d sentiment equilibrium $\phi(L) = \omega$. Further more, if $\phi(L) = \omega$ is an equilibrium, then*

$$\phi(L) = \omega \prod_{k=0}^K \mathcal{B}(L; \lambda_k), \quad (45)$$

is also a sentiment equilibrium, where K can be any nonnegative integer and λ_k can be any real number that lies in $(-1, 1)$.

Proof. This follows directly from Theorem 1 and Theorem 2. □

Given a set of assumptions about the full information set of agents, Corollary 2 provides a practical way to check if a sentiment equilibria exists and to find all sentiment equilibria. One can first look for an i.i.d sentiment equilibrium, which is relatively easy to solve. If the i.i.d sentiment equilibria exists, then the rest of sentiment equilibria can be constructed by multiplying the i.i.d equilibrium by some Blaschke factors. This procedure allows us to characterize the entire set of sentiment equilibria as long as Assumptions 1-2 hold.

Another important implication of Theorem 2 is that in order for sentiment-driven fluctuations to display predictable persistence and be distinguishable from i.i.d., Assumption 1 and/or Assumption 2 must not hold. In other words, if there exist any equilibria in which sentiments can drive persistent and predictable fluctuations, it must be the case that the agents' information set does not contain the one period lagged innovations associated with aggregate shocks or past aggregate action. This is summarized in the corollary below and discussed in greater detail in Section 4.1.

Corollary 3. *If sentiment-driven fluctuations are persistent and predictable, then it must be the case that $\mathbb{V}(\nu^{t-1}, a^{t-1}) \not\subset \mathcal{I}_{i,t}$.*

This result provides a helpful insight to the large literature which studies sentiment driven equilibria such as Benhabib et al. (2013, 2015) and Chahrour and Gaballo (2016) among others. This literature has largely concentrated on studying i.i.d fluctuations driven by sentiments. The result above serves as a guide for the minimum ingredients required to study persistent sentiment driven fluctuations.³⁶

³⁶In addition to this growing literature which explores sentiment driven equilibria, there is a large literature which studies economic fluctuations arising from information frictions. This literature uses very similar models

4.1 Persistence and predictability of sentiment-driven fluctuations

Corollary 3 stated that if there is any possibility of sentiment equilibria in which sentiment-driven fluctuations are persistent and predictable, then it must be that Assumptions 1 and/or 2 not be satisfied. This raises the question whether there exist any equilibria in which sentiments can drive persistent and predictable aggregate fluctuations even if we relax these assumptions. In this section we show that this is in fact the case by showing two examples of such equilibria when we relax these Assumptions. These examples show that the set of such sentiment equilibria is not empty. In fact, this set features even more multiplicity.

The properties of such equilibria strongly depend on the assumptions about information sets that agents possess. In other words, the properties of such equilibria are no longer robust to the specification of private information. Since the main goal of this paper is to characterize robust properties of sentiment equilibria, we have made very few assumptions regarding the information structure and hence do not attempt to provide a general characterization of the properties of such equilibria. We begin by relaxing Assumption 1 and then consider the case of relaxing Assumption 2.

4.1.1 Agents do not observe past aggregate actions

We start by investigating whether persistent and predictable sentiment-driven fluctuations could arise if Assumption 1 was violated, i.e. if we restricted the agents to only observe aggregate actions with a lag of greater than one period. Suppose, at date t , agents only observe a_{t-k} for $k > 1$. Does this necessarily imply that sentiment-driven fluctuations must be distinguishable from i.i.d? The answer is no and is in fact not surprising if one recalls that consistency requirements imposed by rational expectations. Recall from the discussion in sections 3.1 and 3.3 that in any sentiment equilibrium, the agents' forecast of the current aggregate action optimally did not depend on lagged aggregate actions.³⁷ Thus, in any REE, each agent's forecast at date t of the current aggregate action a_t is not affected by not observing a_{t-1} . Consequently, neither is the characterization of equilibrium. Thus, even if agents did not observe past aggregate outcomes (or observed it with an extended lag) the i.i.d. like sentiments equilibria still exists.

However, removing past actions from the information set of agents also introduces additional equilibria in which sentiment-driven equilibria are not observationally equivalent to an i.i.d. process. To see this, consider the following example. For simplicity, assume that aggregate fundamentals are known and fixed permanently at 0 and assume that agent-specific idiosyncratic

but commonly makes the assumptions that the realizations of aggregate fundamentals and aggregate outcomes in the past are common knowledge in order to avoid the complexity of dealing with the problem referred to as *forecasting the forecasts of others*, Townsend (1983). Theorem 2 shows that these assumptions on the information set of agents rules out the possibility of persistent sentiment-driven fluctuations.

³⁷Recall the restrictions placed by REE on equations (19) and (29).

fundamentals $z_{i,t}$ follow an AR(1) process: $z_{i,t} = \rho z_{i,t-1} + u_{i,t}$.³⁸ As before, each agent receives a private endogenous signal of the form:

$$x_{i,t} = \beta a_t + (1 - \beta)u_{i,t}$$

However, unlike the previous examples we assume that at date t , each agent is unable to observe a_{t-1} and can only observe the realizations of the aggregate outcome up till date $t - 2$. Appendix C.1 shows that there exist multiple equilibria in which sentiment-driven fluctuations can be described by a MA(1):

$$a_t = \phi_0 \epsilon_t + \phi_1 \epsilon_{t-1}$$

and the expressions describing ϕ_0 and ϕ_1 are in Appendix C.1. This example shows that relaxing the conditions in Theorem 2 does in fact allow for sentiment equilibria in which sentiments can drive aggregate fluctuations which are persistent and predictable.

4.1.2 Agents do not observe past aggregate fundamentals

In a similar spirit as the exercise above, we relax Assumption 2. Again, the aim of this exercise is to provide a demonstration that relaxing this assumption can result in equilibria which feature non-i.i.d. looking sentiment driven fluctuations. To see this consider another simple example in which the aggregate fundamental $\theta_t = v_t$ is described by an i.i.d process. As before, each agent observes the noisy signal $x_{i,t} = \beta a_t + (1 - \beta)u_{i,t}$ where $u_{i,t}$ denotes the idiosyncratic fundamental. Each agent also observes the past aggregate outcomes, a^{t-1} . However, unlike in the previous subsections, we assume that at each date t agents are unable to observe $\theta_{t-1} = v_{t-1}$ and must wait two periods to observe it, i.e. at date t , they observe the sequence of realizations v^{t-2} but not the realizations v_{t-1} and v_t . Appendix C.2 shows that in this case, there exist a continuum of equilibria of the form

$$a_t = \phi(L)\epsilon_t + \psi(L)v_t,$$

such that

$$\phi(L) = \phi_0 + \phi_1 L,$$

$$\psi(L) = \psi_0 + \psi_1 L,$$

where $\phi_0^2 + \psi_0^2 + \phi_1^2 + \psi_1^2 = \frac{\alpha - 1 + \varphi \psi_0}{1 - \gamma}$, $\phi_0 \phi_1 = -\psi_0 \psi_1$ and $\left| \frac{\phi_1}{\phi_0} \right| < 1$. Thus, this example shows that a violation of Assumption 2 also adds another continuum of sentiment-equilibria in which sentiments can drive persistent and predictable aggregate fluctuations.

³⁸In the earlier examples in Section 3, we had set $\rho = 0$.

5 Some additional observations

5.1 Discussion of Assumptions 1 and 2

Our findings suggest that whether or not assumptions 1 and 2 hold when agents make their decision has strong implications for the nature of sentiment-driven fluctuations that can arise in equilibrium. In particular, sentiment-driven equilibria can only be persistent and distinguishable from i.i.d iff assumption 1 and/or 2 are not satisfied. As we argue below, it is not hard to believe that these assumptions may not be satisfied.

Past aggregate outcomes such as GDP, inflation, unemployment rate and other national statistics are released frequently by statistical agencies. Despite these releases often being widely available, there are at least two potential reasons why Assumption 1 may still not hold. First, the availability of information does not imply that agents can incorporate it immediately into their decision making process. This might be because economic agents have limited information processing capacity or find it too costly to acquire information (See for example, the literature on rational inattention and sticky-information such as [Sims \(2003\)](#), [Mankiw and Reis \(2002\)](#), [Maćkowiak and Wiederholt \(2009\)](#), [Maćkowiak and Wiederholt \(2015\)](#), [Acharya and Wee \(2016\)](#) among others). Second, public data releases about economic aggregates are subject to measurement errors and frequent revisions, which can be interpreted as past aggregate outcomes not being perfectly observable. For example, [Orphanides \(2001\)](#) shows that the use of the real-time data versus the eventual final release strongly matters for the conduct of monetary policy. Similarly, [Zucman \(2013\)](#) shows that official statistics can substantially underestimate the net foreign asset position of a country while [Aruoba \(2008\)](#) finds that initial announcements of statistical agencies may be biased.

In terms of Assumption 2, perfectly observing the past realizations of aggregate shocks is probably only a conceptual idealization. Even for total factor productivity or labor productivity, economists have not really reached an agreement about its measurement ([Fernald, 2012](#); [McGrattan and Prescott, 2014](#)).

Thus, the possible failure of Assumptions 1 and 2 when economic agents make their decisions is not very unlikely. In such a situation, sentiments can drive persistent and predictable aggregate fluctuations. In addition, the interaction between sentiment and fundamental shocks can induce potential interesting dynamics as noted in sub-sections 3.3 and 4.1.2.

5.2 Sunspots and correlated equilibria

Sentiment-driven fluctuations, in our paper, took the form of self confirming beliefs about aggregate outcomes. Thus, one could interpret the sentiments equilibria as sunspots. However, it is important to realize that the continuum of sentiment equilibria that we characterized are not

simple sunspot randomizations over multiple fundamental equilibria as in many macroeconomic models.

There exists however a significant literature showing that sunspot equilibria can exist in models where the fundamental equilibrium is unique. The seminal paper of [Cass and Shell \(1983\)](#) demonstrates this in a two period model with a unique fundamental equilibrium by introducing securities traded in the first period with returns that are sunspot contingent and can induce wealth effects. [Peck and Shell \(1991\)](#) obtain a similar result by postulating imperfect competition and non-Walrasian trades in the post-sunspot market that also gives rise to wealth effects.³⁹ By contrast [Mas-Colell \(1992\)](#) and [Gottardi and Kajii \(1999\)](#) explicitly rule out securities with payoffs contingent on sunspot realizations, again in a context where there is a unique fundamental equilibrium. Since agents can be heterogeneous in endowments and preferences, they can still trade in first period securities with appropriate payoffs, without being able to insure against sunspot fluctuations. Such trades however change second period endowments and attain endowments from which there exists multiple spot market equilibria, and sunspots can randomize over these. Thus according to [Gottardi and Kajii \(1999\)](#) what accounts for the existence of sunspot equilibria is “potential multiplicity” in future spot markets.⁴⁰ It is clear that these forces are not generating the multiple sunspot equilibria in our economy as agents do not trade assets and do not make any intertemporal decisions. Instead, the multiple sunspot equilibria in our model arise due to signal extraction problems in a setting with endogenous information sources.

The sentiment equilibria that we obtain are closely related to the correlated equilibria of [Aumann \(1974\)](#), as further developed by [Maskin and Tirole \(1987\)](#).^{41,42} [Maskin and Tirole \(1987\)](#) were the first to give an example of sunspot equilibria under asymmetric information in a competitive economy. In their model, information is asymmetric and signals to agents are imperfectly correlated because only some but not all agents observe a signal that coordinates their actions. The importance of this result comes from the fact that if signals are perfectly correlated across all agents, then equilibrium payoffs must lie in the convex hull of the ordinary Nash equilibria. However, if signals are imperfectly correlated equilibrium payoffs need not lie in the convex hull of the ordinary Nash equilibria. In such a setting, there can be a unique fundamental equilibrium, but also other correlated equilibria. [Maskin and Tirole \(1987\)](#) also

³⁹See also [Spear \(1989\)](#) for an overlapping generations model with two islands where prices in one island act as sunspots for the other.

⁴⁰[Mas-Colell \(1992\)](#) and [Gottardi and Kajii \(1999\)](#) give examples of such economies characterized by endowments, preferences and security payoffs. [Gottardi and Kajii \(1999\)](#) also provide a systematic method to generically construct such economies with sunspot equilibria.

⁴¹See also [Peck and Shell \(1991\)](#), example 5.7.

⁴²[Aumann et al. \(1988\)](#) provide an excellent overview of the relation between correlated and sunspot equilibria under asymmetric information with a set of examples in market games that in the limit converge to a competitive equilibrium, and also illustrate that under asymmetric information there can be correlated equilibria even though the fundamental equilibrium is unique.

show that under the signal structure of their simple competitive economy correlated equilibria exist only if there are Giffen goods. In our model, however, where the correlated equilibria are induced by noisy sentiments, agents face a signal extraction problem where optimal actions depend continuously on the variances of the sentiment and idiosyncratic shock distributions. All goods can be normal and demand functions downward sloping, as in [Benhabib et al. \(2013, 2015\)](#). The market clearing requirement for all realizations of sentiment shocks are achieved by restrictions imposed by equilibria on the allowable set of sentiment processes.

6 Conclusion

The objective of this paper was to establish whether endogenously arising sentiments could drive persistent aggregate fluctuations in the context of rational expectations equilibria. Within the class of the commonly used beauty contest, we provided a complete characterization of all stationary linear rational expectations equilibria and showed that there exist a multiplicity of equilibria in which sentiments can drive aggregate fluctuations. Furthermore, we identified necessary and sufficient conditions under which these sentiments equilibria can account for persistent aggregate fluctuations. We also showed that these predictions are robust to a very general specification of the information structure and thus do not strongly depend on the private information agents might possess.

More precisely we showed that sentiments can generate persistent aggregate fluctuations under very general conditions. To generate persistent fluctuations which are *observationally distinct* from i.i.d, one of two conditions must be satisfied: (i) agents do not observe aggregate fundamentals in the current or preceding period and/or (ii) agents do not observe lagged aggregate outcomes (or observe lagged actions with a delay). If agents can observe lagged aggregate fundamentals and past aggregate outcomes, then sentiments can never generate aggregate fluctuations which are observationally distinct from an i.i.d. fluctuations, whatever the structure of private information. However, even if sentiment-driven fluctuations are observationally equivalent to i.i.d., they may still display a hump shaped response to a change in sentiments. Thus, persistent sentiment driven aggregate fluctuations can exist even when agents can observe current fundamentals, but these fluctuations are not predictable. This characterization serves as an important guide for a growing literature in the field of macroeconomics which is trying to theoretically and quantitatively evaluate the importance of sentiments or correlated equilibria in trying to understand aggregate fluctuations. Additionally, we showed that sentiments equilibria also allow agents to potentially disagree forever. This could be useful for understanding persistent disagreements amongst agents in the economy or amongst professional forecasters.

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Appendix

A Equilibrium with invertible sentiments

In this Appendix we construct a rational expectations equilibrium in which sentiments are invertible and can only affect aggregate outcomes in an i.i.d fashion. Recall that the evolution of aggregate fundamentals is governed by an exogenous stochastic process and can be written as:

$$a_t = g(L)v_t = g_0v_t + g_1v_{t-1} + g_2v_{t-2} + \dots, \text{ where } v_t \sim N(0, \sigma_v^2)$$

For the purpose of this appendix we restrict ourselves to the same information sets as in Section 3 and leave the generalization to Section 4. We describe the information set below once more for convenience.

Information Sets Every period, each agent can observe a private signal about the aggregate action and their idiosyncratic shock: $x_{i,t} = \beta a_t + (1 - \beta) u_{i,t}$. In addition, agents can observe the past realizations of both the aggregate action and aggregate fundamental, which implies the past realizations of idiosyncratic shocks are also observable. The information set of agent i at time t is then given by $\mathbb{I}_{i,t} = \{a^{t-1}, x_i^t\} \equiv \mathbb{U}(\theta^{t-1}) \vee \mathbb{V}(a^{t-1}, x_t^t) \vee \mathbb{M}$.

In general in any such linear rational expectations equilibrium, aggregate outcomes can be described as:

$$a_t = \psi(L)v_t + \phi(L)\epsilon_t$$

where ϵ_t is a white noise sequence which we refer to as innovations to sentiments. Further for this appendix we concentrate on the case where $\phi(z)$ has no zeros inside the unit circle. Then the mapping between signals and innovations from the perspective of agent i can be written as:

$$\begin{bmatrix} a_{t-1} \\ x_{i,t} \\ \theta_{t-1} \end{bmatrix} = \begin{bmatrix} L\phi(L) & 0 & L\psi(L) \\ \beta\phi(L) & 1 - \beta & \beta\psi(L) \\ 0 & 0 & Lg(L) \end{bmatrix} \begin{bmatrix} \varepsilon_t \\ u_{i,t} \\ v_t \end{bmatrix} \Leftrightarrow z_{i,t} = M(L) e_{i,t} \quad (46)$$

Note that the determinant of $M(z)$ is

$$\det[M(z)] = z^2 Q(z) d(z) \quad (47)$$

and there are two roots inside the unit circle. This mapping can also be represented by an observationally equivalent invertible representation of system (46) and can be written as:

$$z_{i,t} = \underbrace{M(L) \Sigma W_1 \mathbb{B}(L; 0) W_2 \mathbb{B}(L; 0)}_{\widetilde{M}(L)} \underbrace{\mathbb{B}(L^{-1}; 0)' W_2' \mathbb{B}(L^{-1}; 0)' W_1' \Sigma^{-1} e_{i,t}}_{\widetilde{e}_{i,t}}$$

where

$$\begin{aligned}\mathbb{B}(L; \lambda) &= \begin{bmatrix} \frac{L^{-1}-\lambda}{1-\lambda L^{-1}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ W_1 &= \begin{bmatrix} \frac{\sigma_v \psi_0}{\sqrt{\phi_0^2 + \sigma_v^2 \psi_0^2}} & \frac{\phi_0}{\sqrt{\phi_0^2 + \sigma_v^2 \psi_0^2}} & 0 \\ 0 & 0 & 1 \\ -\frac{\phi_0}{\sqrt{\phi_0^2 + \sigma_v^2 \psi_0^2}} & \frac{\sigma_v \psi_0}{\sqrt{\phi_0^2 + \sigma_v^2 \psi_0^2}} & 0 \end{bmatrix} \\ W_2 &= \begin{bmatrix} 0 & 1 & 0 \\ \frac{\sigma_u(1-\beta)}{\sqrt{\sigma_u^2(1-\beta)^2 + \beta^2(\phi_0^2 + \sigma_v^2 \psi_0^2)}} & 0 & \beta \frac{\sqrt{\phi_0^2 + \sigma_v^2 \psi_0^2}}{\sqrt{\sigma_u^2(1-\beta)^2 + \beta^2(\phi_0^2 + \sigma_v^2 \psi_0^2)}} \\ -\frac{\beta \sqrt{\phi_0^2 + \sigma_v^2 \psi_0^2}}{\sqrt{\sigma_u^2(1-\beta)^2 + \beta^2(\phi_0^2 + \sigma_v^2 \psi_0^2)}} & 0 & \frac{\sigma_u(1-\beta)}{\sqrt{\sigma_u^2(1-\beta)^2 + \beta^2(\phi_0^2 + \sigma_v^2 \psi_0^2)}} \end{bmatrix}\end{aligned}$$

Using the Kolomogrov-Weiner projection formulas:

$$\begin{aligned}\int \mathbb{E}_{i,t} a_t &= \left[\phi(L) - \frac{\sigma_u^2 (1-\beta)^2 \phi_0}{\beta^2 (\phi_0^2 + \sigma_v^2 \psi_0^2) + \sigma_u^2 (1-\beta)^2} \right] \epsilon_t + \left[\psi(L) - \frac{\sigma_u^2 (1-\beta)^2 \psi_0}{\sigma_u^2 (1-\beta)^2 + \beta^2 (\phi_0^2 + \sigma_v^2 \psi_0^2)} \right] v_t \\ \int \mathbb{E}_{i,t} [\theta_t] &= \frac{\beta^2 \sigma_v^2 g_0 \psi_0 \phi_0}{\beta^2 (\phi_0^2 + \sigma_v^2 \psi_0^2) + \sigma_u^2 (1-\beta)^2} \epsilon_t + \left[g(L) - \frac{\beta^2 \phi_0^2 + \sigma_u^2 (1-\beta)^2}{\beta^2 (\phi_0^2 + \sigma_v^2 \psi_0^2) + \sigma_u^2 (1-\beta)^2} g_0 \right] v_t \\ \int \mathbb{E}_{i,t} u_{i,t} &= \frac{1}{1-\beta} \left[\beta a_t - \beta \int \mathbb{E}_{i,t} a_t \right]\end{aligned}$$

Also, recall that equilibrium must satisfy:

$$a_t = \gamma \int \mathbb{E}_{i,t} a_t + \varphi \int \mathbb{E}_{i,t} \theta_t + \alpha \int \mathbb{E}_{i,t} u_{i,t}$$

As a result, it must be the case that:

$$\begin{aligned}\psi(z) &= \frac{\varphi}{1-\gamma} g(z) + \frac{\sigma_u^2 (1-\beta) \psi_0 [\alpha\beta - (1-\beta)\gamma] - \varphi g_0 [\sigma_u^2 (1-\beta)^2 + \beta^2 \phi_0^2]}{(1-\gamma) [\beta^2 (\sigma_v^2 \psi_0^2 + \phi_0^2) + \sigma_u^2 (1-\beta)^2]} \\ \phi(z) &= \frac{\sigma_u (1-\beta) [\alpha\beta - \gamma(1-\beta)] + \varphi \beta^2 \sigma_v^2 g_0 \psi_0}{(1-\gamma) [\beta^2 (\phi_0^2 + \sigma_v^2 \psi_0^2) + \sigma_u^2 (1-\beta)^2]} \phi_0\end{aligned}$$

Evaluating at $z = 0$:

$$\begin{aligned}\psi_0 &= \frac{\sigma_u (1-\beta) [\alpha\beta - \gamma(1-\beta)] + \varphi \beta^2 \sigma_v^2 g_0 \psi_0}{(1-\gamma) [\beta^2 (\phi_0^2 + \sigma_v^2 \psi_0^2) + \sigma_u^2 (1-\beta)^2]} \psi_0 \\ \phi_0 &= \frac{\sigma_u (1-\beta) [\alpha\beta - \gamma(1-\beta)] + \varphi \beta^2 \sigma_v^2 g_0 \psi_0}{(1-\gamma) [\beta^2 (\phi_0^2 + \sigma_v^2 \psi_0^2) + \sigma_u^2 (1-\beta)^2]} \phi_0\end{aligned}$$

For these to be true, it has to be that there exists a solution $\{\psi(0), \phi(0)\}$ to the following equation

$$\sigma_v^2 \left(\psi_0^2 - \frac{\varphi g_0 \psi_0}{1 - \gamma} \right) + \phi_0^2 = \frac{\sigma_u^2 (1 - \beta) [\alpha \beta + \beta - 1]}{\beta^2 (1 - \gamma)} \quad (48)$$

Completing the square:

$$\sigma_v^2 \left(\psi_0 - \frac{\varphi g_0}{2(1 - \gamma)} \right)^2 + \phi_0^2 = \frac{\sigma_u^2 (1 - \beta) [\alpha \beta + \beta - 1]}{\beta^2 (1 - \gamma)} + \frac{\varphi^2 g_0^2 \sigma_v^2}{4(1 - \gamma)^2} \quad (49)$$

As long as $\alpha > \frac{1 - \beta}{\beta}$ (which is the same condition as the one that is required for the existence of a fundamental equilibrium), the equation above describes a real valued circle with center at $\left(\frac{\varphi g_0}{2(1 - \gamma)}, 0 \right)$ in the (ψ_0, ϕ_0) plane with radius $\sqrt{\frac{\sigma_u^2 (1 - \beta) [\alpha \beta + \beta - 1]}{\beta^2 (1 - \gamma)} + \frac{\varphi^2 g_0^2 \sigma_v^2}{4(1 - \gamma)^2}}$. Thus, there exists a continuum of equilibrium which can be characterized by the circle above.

A.1 Case with only sentiment shocks

Notice that this Appendix encompasses the case with no fundamentals. To see that set $g(z) = 0$ for all z . We set $\sigma_v = 1$ without loss of generality in this case. This implies that the aggregate fundamental $\theta_t = 0$ for all t . In this case, equation (48) can be written as:

$$\psi(z) = \psi_0 \frac{\sigma_u^2 (1 - \beta) [\alpha \beta - (1 - \beta) \gamma]}{(1 - \gamma) [\beta^2 \phi_0^2 + \sigma_u^2 (1 - \beta)^2]}$$

which implies that $\psi(z) = \psi_0 = 0$ for all z . Then from equation (49) we can write:

$$\phi(0) = \pm \frac{\sigma_u}{\beta} \sqrt{\frac{(1 - \beta) (\alpha \beta + \beta - 1)}{1 - \gamma}}$$

which is the same condition in Proposition 1 for the economy with only invertible sentiments.

B Signal Extraction and Projections in the ARMA(1,1) example

In the ARMA(1,1) example, we assumed that perceived law of motion of the aggregate action is given by:

$$a_t = \omega \frac{L - \xi}{1 - \lambda L} \epsilon_t \quad \text{or} \quad a_t = \lambda a_{t-1} - \omega \xi \epsilon_t + \omega \epsilon_{t-1}$$

with $-1 < \xi < 1$. Recall that we endowed each agent with a private signal $x_{i,t}$:

$$x_{i,t} = \beta a_t + (1 - \beta) u_{i,t}$$

Since each agent also observes the past realizations of the aggregate actions, a^{t-1} , we can transform the signal $x_{i,t}$ into an informationally equivalent signal $\hat{x}_{i,t}$ which is defined as:

$$\hat{x}_{i,t} = x_{i,t} - \beta \lambda a_{t-1} = -\beta \omega (\xi \epsilon_t - \epsilon_{t-1}) + (1 - \beta) u_{i,t}$$

Then, for each agent the mapping between signals and innovations can be written as:

$$\begin{bmatrix} x_{i,t} \\ a_{t-1} \end{bmatrix} = \begin{bmatrix} \frac{\beta(L-\xi)\omega}{1-L\lambda} & (1-\beta)\sigma_u \\ \frac{L(L-\xi)\omega}{1-L\lambda} & 0 \end{bmatrix} \begin{bmatrix} \epsilon_t \\ \frac{u_{i,t}}{\sigma_u} \end{bmatrix}$$

or equivalently $s_{i,t} = M(L)e_{i,t}$. Notice that the determinant of $M(z)$ has 2 zeros inside the unit circle: at $z = 0$ and $z = \xi$. Consequently, $M(L)$ is not invertible in positive powers of L implying that observing all the past and current realizations of x_i^t and a^{t-1} is not sufficient to infer the sequences ϵ^t and u_i^t . In order to solve the agents signal extraction and forecasting problem, we can write the above system in a observationally equivalent form:

$$s_{i,t} = \underbrace{M(L)W\mathbb{B}(L;0)V\mathbb{B}(L;\xi)}_{M^*(L)} \underbrace{\mathbb{B}(L^{-1};\xi)'V'\mathbb{B}(L^{-1};0)'W'}_{e_{i,t}^*} e_{i,t}$$

where

$$W = \begin{bmatrix} \frac{-\sigma_u(1-\beta)}{\sqrt{\sigma_u^2(1-\beta)^2 + \beta^2\xi^2\omega^2}} & \frac{\beta\xi\omega}{\sqrt{\sigma_u^2(1-\beta)^2 + \beta^2\xi^2\omega^2}} \\ -\frac{\beta\xi\omega}{\sqrt{\sigma_u^2(1-\beta)^2 + \beta^2\xi^2\omega^2}} & \frac{-\sigma_u(1-\beta)}{\sqrt{\sigma_u^2(1-\beta)^2 + \beta^2\xi^2\omega^2}} \end{bmatrix} \quad V = \begin{bmatrix} \frac{\sigma_u(1-\beta)}{\sqrt{\sigma_u^2(1-\beta)^2 + \beta^2\omega^2}} & \frac{\beta\omega}{\sqrt{\sigma_u^2(1-\beta)^2 + \beta^2\omega^2}} \\ -\frac{\beta\omega}{\sqrt{\sigma_u^2(1-\beta)^2 + \beta^2\omega^2}} & \frac{\sigma_u(1-\beta)}{\sqrt{\sigma_u^2(1-\beta)^2 + \beta^2\omega^2}} \end{bmatrix}$$

and

$$\mathbb{B}(L;\lambda) = \begin{bmatrix} \mathcal{B}(L,\lambda) & 0 \\ 0 & 1 \end{bmatrix}$$

Then we can use the Kolmogrov-Weiner projection formulas to get:

$$\begin{aligned} \mathbb{E}_{i,t}a_t &= \chi[\hat{x}_{i,t} + \beta\lambda a_{t-1}] + (1-\beta\chi)(\lambda-\xi) \sum_{j=0}^{\infty} \xi^j a_{t-1-j} \\ \mathbb{E}_{i,t}u_{i,t} &= \frac{1-\beta\chi}{1-\beta}\hat{x}_{i,t} + \frac{\beta\lambda(1-\beta\chi)}{1-\beta}a_{t-1} - \frac{\beta(1-\beta\chi)}{1-\beta}(\lambda-\xi) \sum_{j=0}^{\infty} \xi^j a_{t-1-j} \end{aligned}$$

where $\chi = \frac{\beta\omega^2}{\beta^2\omega^2 + \sigma_u^2(1-\beta)^2}$. Then the optimal action of agent i can be written as:

$$\begin{aligned} a_{i,t} &= \gamma\mathbb{E}_{i,t}a_t + \alpha\mathbb{E}_{i,t}u_{i,t} \\ &= \left(\gamma\chi + \alpha\frac{1-\beta\chi}{1-\beta}\right)(\hat{x}_{i,t} + \beta\lambda a_{t-1}) + (1-\beta\chi)(\lambda-\xi) \left(\gamma - \frac{\alpha\beta}{1-\beta}\right) \sum_{j=0}^{\infty} \xi^j a_{t-1-j} \end{aligned}$$

Aggregating over all individuals, the actual law of motion can be written as:

$$a_t = \beta \left(\gamma\chi + \alpha\frac{1-\beta\chi}{1-\beta} \right) [\lambda a_{t-1} - \omega\xi\epsilon_t + \omega\epsilon_{t-1}] + (1-\beta\chi)(\lambda-\xi) \left(\gamma - \frac{\alpha\beta}{1-\beta} \right) \sum_{j=0}^{\infty} \xi^j a_{t-1-j}$$

For there to be consistency between actual and perceived law of motion, the following must be true:

$$\begin{aligned} \beta \left(\gamma\chi + \alpha\frac{1-\beta\chi}{1-\beta} \right) &= 1 \\ \lambda &= \xi \end{aligned}$$

Notice that the for the first condition to be true, it must be the case that:

$$\omega^2 = \frac{\sigma_u^2 (1 - \beta) (\alpha\beta - 1 + \beta)}{\beta^2 (1 - \gamma)}$$

which is the same as the variance defined in Proposition 2.

C Persistent Predictable Sentiments-Driven Fluctuations

C.1 Agents do not observe past aggregate actions

Consider the following environment: The best response is the same as in (1) where it is common knowledge that the aggregate fundamental is fixed at $\theta_t = 0$ for all t . The idiosyncratic fundamental $z_{i,t}$ is given by a AR(1):

$$z_{i,t} = \frac{1}{1 - \rho L} u_{i,t}$$

Agents receive two signals every period: (1) two periods before aggregate action a_{t-2} ; (2) noisy signal $x_{i,t}$ about current aggregate action:

$$x_{i,t} = \beta a_t + (1 - \beta) u_{i,t}$$

where

$$a_t = \phi(L) \epsilon_t$$

An educated guess for the equilibrium path of aggregate action is:

$$\phi(L) = \phi_0 + \phi_1 L$$

Given this guess, the problem can be transformed into a static problem with the relevant information encoded in the following modified signals:

$$\begin{aligned} w_{i,t}^1 &= \beta \phi_0 \epsilon_{t-1} + (1 - \beta) u_{i,t-1} \\ w_{i,t}^2 &= \beta (\phi_0 \epsilon_t + \phi_1 \epsilon_{t-1}) + (1 - \beta) u_{i,t} \end{aligned}$$

The covariance matrix of $w_{i,t} = [w_{i,t}^1, w_{i,t}^2]'$ can be written as:

$$\Omega = \begin{bmatrix} \beta^2 \phi_0^2 + (1 - \beta)^2 \sigma_u^2 & \beta^2 \psi_0 \psi_1 \\ \beta^2 \psi_0 \psi_1 & \beta^2 (\phi_0^2 + \phi_1^2) + (1 - \beta)^2 \sigma_u^2 \end{bmatrix}$$

Then using the Kalman filter, any equilibrium satisfies:

$$\begin{aligned} \phi_0 &= \pm \sigma_u \sqrt{\frac{(1 - \beta)(1 - \gamma)(\alpha\beta + \beta - 1) \pm \sqrt{(1 - \beta)^2(1 - \gamma)^2((\alpha\beta + \beta - 1)^2 - 4\alpha^2\beta^2\rho^2)}}{\beta^2(1 - \gamma)^2}} \\ \phi_1 &= \phi_0 \frac{(1 - \beta)(1 - \gamma)(\alpha\beta + \beta - 1) \pm \sqrt{(1 - \beta)^2(1 - \gamma)^2((\alpha\beta + \beta - 1)^2 - 4\alpha^2\beta^2\rho^2)}}{2\alpha\beta\rho(1 - \beta)(1 - \gamma)} \end{aligned}$$

C.2 Agents do not observe lagged innovations to aggregate fundamentals

Suppose that the information structure of agents is given by:

$$\begin{aligned} z_{i,t} &= u_{i,t} \\ \theta_t &= v_t \end{aligned}$$

Agents receive three signals every period: (1) exogenous signal about the past value of common aggregate fundamental $y_t = \nu_{t-2}$; (2) an endogenous signal about idiosyncratic fundamental

$$x_{i,t} = \beta a_t + (1 - \beta)u_{i,t},$$

(3) and last period aggregate action a_{t-1} . This information structure implies that last period aggregate fundamental ν_{t-1} is not directly observable, and an educated guess is that

$$\begin{aligned} \phi(L) &= \phi_0 + \phi_1 L \\ \psi(L) &= \psi_0 + \psi_1 L \end{aligned}$$

The equivalent signal process is

$$\begin{aligned} w_{i,t}^1 &= \phi_0 \epsilon_{t-1} + \psi_0 v_{t-1} \\ w_{i,t}^2 &= \beta(\phi_0 \epsilon_t + \phi_1 \epsilon_{t-1} + \psi_0 v_t + \psi_1 v_{t-1}) + (1 - \beta)u_{i,t} \end{aligned}$$

Indeed, we can verify that the set of persistent sentiment equilibria is

$$\begin{aligned} \phi_1 &= \pm \sigma_v^2 \psi_0 \sqrt{\frac{(1 - \beta)(1 - \alpha\beta - \beta)\sigma_u^2 - \beta^2 \varphi \sigma_v^2 \psi_0 + \beta^2(1 - \gamma)(\phi_0^2 + \sigma_v^2 \psi_0^2)}{\beta^2(\gamma - 1)\sigma_v^2(\phi_0^2 + \sigma_v^2 \psi_0^2)}} \\ \psi_1 &= \mp \phi_0 \sqrt{\frac{(1 - \beta)(1 - \alpha\beta - \beta)\sigma_u^2 - \beta^2 \varphi \sigma_v^2 \psi_0 + \beta^2(1 - \gamma)(\phi_0^2 + \sigma_v^2 \psi_0^2)}{\beta^2(\gamma - 1)\sigma_v^2(\phi_0^2 + \sigma_v^2 \psi_0^2)}} \end{aligned}$$

Note that ϕ_0 and ψ_0 have to satisfy

$$\frac{(1 - \beta)(1 - \alpha\beta - \beta)\sigma_u^2 - \beta^2 \varphi \sigma_v^2 \psi_0}{\beta^2(\gamma - 1)} > \phi_0^2 + \sigma_v^2 \psi_0^2$$

and

$$\left| \frac{\phi_1}{\phi_0} \right| < 1$$

In equilibrium

$$\begin{aligned} \phi_0^2 + \psi_0^2 \sigma_v^2 + \phi_1^2 + \psi_1^2 \sigma_v^2 &= \frac{(1 - \beta)(1 - \alpha\beta - \beta)\sigma_u^2 - \beta^2 \varphi \sigma_v^2 \psi_0}{\beta^2(\gamma - 1)} \\ \phi_0 \phi_1 &= -\sigma_v^2 \psi_0 \psi_1 \end{aligned}$$

D Proof of Theorem 1

Proof of Theorem 1. Assume that the aggregate action a_t is given by

$$a_t = \psi(L)\nu_t + \phi(L)\epsilon_t$$

Let m, n, r, ℓ denote the dimensions of exogenous signals, endogenous signals, exogenous aggregate shocks, and exogenous idiosyncratic shocks, respectively. Agents' information structure specified in Section 2 can be represented by the following matrix

$$\mathbf{s}_{i,t} \equiv \begin{bmatrix} y_{i,t}^1 \\ \vdots \\ y_{i,t}^m \\ x_{i,t}^1 \\ \vdots \\ x_{i,t}^n \end{bmatrix} = \begin{bmatrix} 0 & P_{11}(L) & \dots & P_{1r}(L) & Q_{11}(L) & \dots & Q_{1\ell}(L) \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & P_{m1}(L) & \dots & P_{mr}(L) & Q_{m1}(L) & \dots & Q_{m\ell}(L) \\ A_1(L)\phi(L) & A_1(L)\psi_1(L) + B_{11}(L) & \dots & A_1(L)\psi_r(L) + B_{1r}(L) & C_{11}(L) & \dots & C_{1\ell}(L) \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ A_n(L)\phi(L) & A_n(L)\psi_1(L) + B_{n1}(L) & \dots & A_n(L)\psi_r(L) + B_{nr}(L) & C_{n1}(L) & \dots & C_{n\ell}(L) \end{bmatrix} \begin{bmatrix} \epsilon_t \\ \nu_t^1 \\ \vdots \\ \nu_t^r \\ \zeta_{i,t}^1 \\ \vdots \\ \zeta_{i,t}^\ell \end{bmatrix}$$

More compactly, we can represent the information structure by

$$\mathbf{s}_{i,t} = \begin{bmatrix} \mathbf{y}_{i,t} \\ \mathbf{x}_{i,t} \end{bmatrix} = \mathbf{M}(L) \begin{bmatrix} \epsilon_t \\ \nu_t \\ \zeta_{i,t} \end{bmatrix}.$$

Define $\tilde{\phi}(L)$ as

$$\tilde{\phi}(L) = \phi(L) \prod_{k=1}^K \mathcal{B}(L; \lambda_k),$$

where K can be any non-negative integer, and $\lambda_k \in (-1, 1)$. Denote $\tilde{\mathbf{M}}(L)$ as the matrix where $\phi(z)$ is replaced by $\tilde{\phi}(z)$. This replacement implies that agent i believes that the process for sentiments follow $\tilde{\phi}(L)$ instead of $\phi(L)$. We will show that the forecast rules based on $\mathbf{M}(L)$ is the same as those based on $\tilde{\mathbf{M}}(L)$. Note that

$$\tilde{\phi}(L)\tilde{\phi}(L^{-1}) = \phi(L)\phi(L^{-1}) \prod_{k=1}^K \mathcal{B}(L; \lambda_k) \prod_{k=1}^K \mathcal{B}(L^{-1}; \lambda_k) = \phi(L)\phi(L^{-1}),$$

where the second equality is due to the property of Blaschke factors. Utilizing the structure of the matrices $\mathbf{M}(L)$ and $\tilde{\mathbf{M}}(L)$, it follows that

$$\tilde{\mathbf{M}}(L)\tilde{\mathbf{M}}'(L^{-1}) = \mathbf{M}(L)\mathbf{M}'(L^{-1}).$$

This equality implies that the fundamental representation of $\mathbf{M}(L)$ and $\tilde{\mathbf{M}}(L)$ is the same. Denoting the fundamental representation of $\mathbf{M}(L)$ as $\mathbf{B}(L)$, we have

$$\mathbf{M}(L)\mathbf{M}'(L^{-1}) = \tilde{\mathbf{M}}(L)\tilde{\mathbf{M}}'(L^{-1}) = \mathbf{B}(L)\mathbf{B}'(L^{-1}),$$

where $\mathbf{B}(L)$ is invertible. For any stochastic variable

$$f_{i,t} = \mathbf{F}(L) \begin{bmatrix} \epsilon_t \\ \boldsymbol{\nu}_t \\ \mathbf{u}_{i,t} \end{bmatrix},$$

the Wiener-Hopf prediction formula using $\mathbf{M}(L)$ is given by

$$\mathbb{E}_{i,t}[f_{i,t}] = [\mathbf{F}(L)\mathbf{M}'(L^{-1})\mathbf{B}'(L^{-1})]_+ \mathbf{B}(L)^{-1}\mathbf{s}_{i,t},$$

and the forecasting rule using $\widetilde{\mathbf{M}}(L)$ is given by

$$\widetilde{\mathbb{E}}_{i,t}[f_{i,t}] = [\mathbf{F}(L)\widetilde{\mathbf{M}}'(L^{-1})\mathbf{B}'(L^{-1})]_+ \mathbf{B}(L)^{-1}\mathbf{s}_{i,t}.$$

Supposing an agent wants to forecast a stochastic variable driven by exogenous aggregate or idiosyncratic fundamentals, we have

$$\mathbf{F}(L) = [0, F_2(L), F_3(L) \dots, F_{r+\ell+1}(L)].$$

It is straightforward to verify that

$$\mathbf{F}(L)\mathbf{M}'(L^{-1}) = \mathbf{F}(L)\widetilde{\mathbf{M}}'(L^{-1}).$$

As a result, $\mathbb{E}_{i,t}[f_{i,t}] = \widetilde{\mathbb{E}}_{i,t}[f_{i,t}]$.

Supposing an agent wants to forecast the aggregate action, i.e., $f_{i,t} = a_t$. If the agent believes that $a_t = \phi(L)\epsilon_t$, then

$$\mathbf{F}(L) = [\phi(L), 0, \dots, 0].$$

Similarly, if the agent believes $a_t = \widetilde{\phi}(L)\epsilon_t$, then

$$\mathbf{F}(L) = [\widetilde{\phi}(L), 0, \dots, 0].$$

Due to that $\widetilde{\phi}(L)\widetilde{\phi}(L^{-1}) = \phi(L)\phi(L^{-1})$, the following identity holds

$$[\phi(L), 0, \dots, 0] \mathbf{M}'(L^{-1}) = [\widetilde{\phi}(L), 0, \dots, 0] \widetilde{\mathbf{M}}'(L^{-1}),$$

which implies that $\mathbb{E}_{i,t}[a_t] = \widetilde{\mathbb{E}}_{i,t}[a_t]$. □