Manipulation Estimation and Controls: Assignment 1

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Q1.

Linear system is defined with in the form x = Ax + Bu

Where A and B are defined as

```
A = [[0,1,0];[0,0,1];[1,5,7]];
B = transpose([1,0,0]);
C = [0,1,3];
```

1 a. Stability Criterion for the above system is defined by the eigenvalues of A

```
eigen_A = eig(A)
```

```
eigen_A = 3x1 complex
   7.6690 + 0.0000i
  -0.3345 + 0.1361i
  -0.3345 - 0.1361i
```

Since one of the eigen values is positive, the linear system A is unstable

1 b. Controllability of the system

```
Q = ctrb(A,B)
```

```
Q = 3x3
     1     0     0
     0     0     1
     0     1     7
```

```
clear rank
rank_of_Q = rank(Q)
```

```
rank_of_Q = 3
```

Matrix Q is of rank n (same rank as of A). Therefore the system is controllable

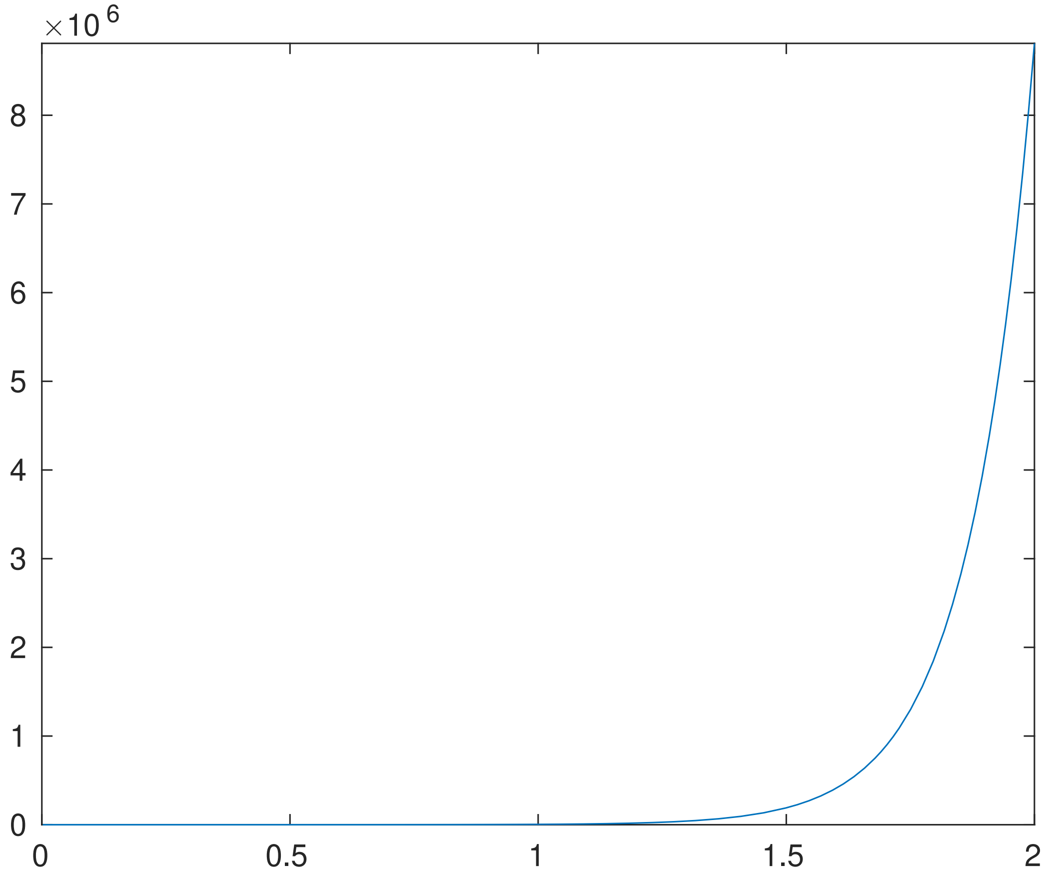
1 c. Initial State Vector is given as:

The output of the unforced system plotted below:

```
x_0 = [0;1;0]
```

```
x_0 = 3x1
     0
     1
     0
```

```
syms t
fplot(C*expm(A*t)*x_0, [0,2])
```



1 d. Find the feedback gain K to make the system stable

```
eig_values = [-1 + 1i, -1 - 1i, -2]
```

```
eig_values = 1x3 complex
  -1.0000 + 1.0000i  -1.0000 - 1.0000i  -2.0000 + 0.0000i
```

```
K = place(A,B,eig_values)
```

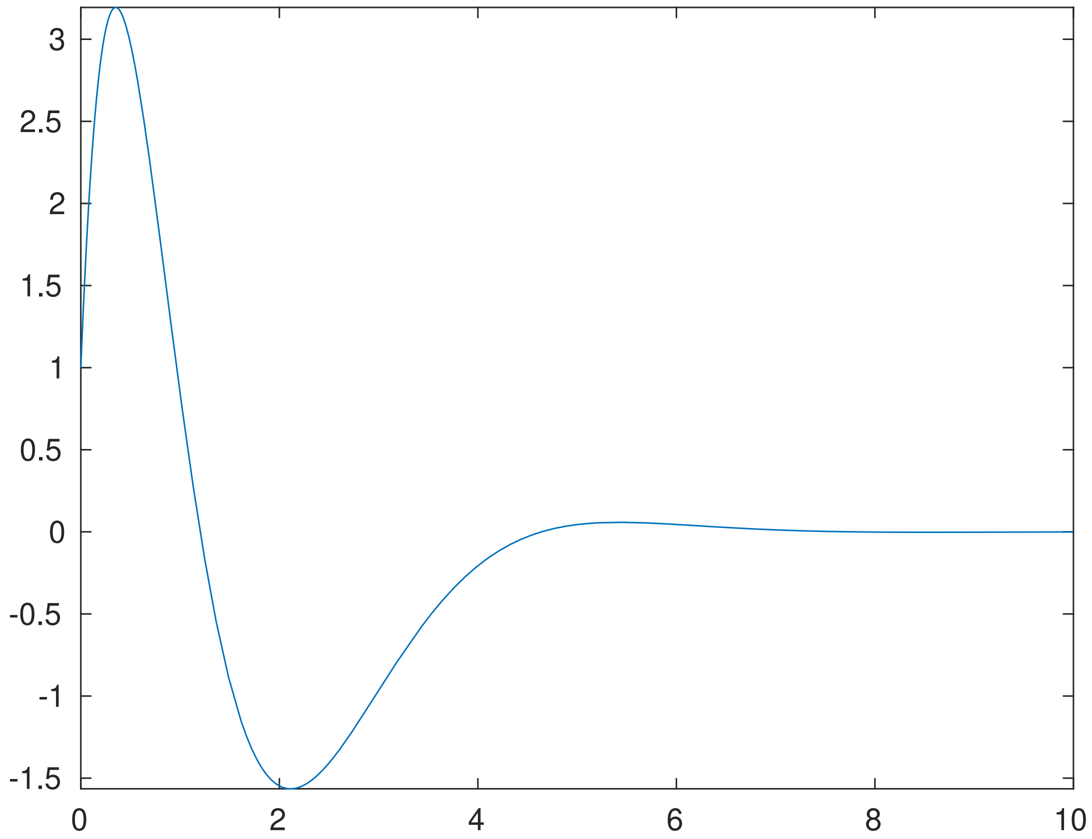
```
K = 1x3
   11.0000   60.0000   88.0000
```

1 e. Plot output of the system after control input

```
new_mat = A-B*K
```

```
new_mat = 3x3
  -11.0000  -59.0000  -88.0000
         0         0    1.0000
    1.0000    5.0000    7.0000
```

```
fplot(C*expm(new_mat*t)*x_0, [0,10])
```



Q 2.

"Pendulum on a cart" system

2 a. Cart pendulum equations as non-linear state-space equations

Cart pendulum equations are written as shown below:

```
syms F
syms z
F = gamma*xc_ddot - beta*phi_ddot*cos(phi) + beta*phi_dot*phi_dot*sin(phi) + mu*xc_dot % = F
```

F =

$$\beta \sin(\phi)\ \dot{\phi}^2 + \gamma\,\ddot{\mathrm{xc}} + \mu\,\dot{\mathrm{xc}} - \beta\,\ddot{\phi}\,\cos(\phi)$$

```
z = 0
```

```
z = 0
```

```
z = alpha*phi_ddot - beta*xc_ddot*cos(phi) - D*sin(phi)
```

z =

$$\alpha\,\ddot{\phi} - \mathrm{D}\,\sin(\phi) - \beta\,\ddot{\mathrm{xc}}\,\cos(\phi)$$

The state vector x is given as:

```
syms xc_ddot xc_dot xc gamma beta phi phi_dot phi_ddot mu alpha D u
x = [xc; phi; xc_dot; phi_dot]
```

x =

$$\begin{pmatrix} \mathrm{xc} \\ \phi \\ \dot{\mathrm{xc}} \\ \dot{\phi} \end{pmatrix}$$

The standard mechanical form is given as:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau.$$

The state-space representation of the standard mechanical form is given as:

<u>The state variable x in state space form is shown below:</u>

$$x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix} \quad \text{where} \quad q = \begin{bmatrix} x_c \\ \phi \end{bmatrix} ; \quad \dot{q} = \begin{bmatrix} \dot{x}_c \\ \dot{\phi} \end{bmatrix}$$

Using the standard mechanical form, the non-linear state space equation is:

$$\dot{x} = F(x,u)$$

$$\dot{x} = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ M^{-1}(u-G-C) \end{bmatrix}$$

The derivation of the above equation is shown below:

```
syms xc_ddot xc_dot xc gamma beta phi phi_dot phi_ddot mu alpha D u

M = [[gamma, -beta*cos(phi)];[-beta*cos(phi),alpha]]
```

M =

$$\begin{pmatrix} \gamma & -\beta\,\cos\left(\phi\right) \\ -\beta\,\cos\left(\phi\right) & \alpha \end{pmatrix}$$

```
C = [beta*phi_dot*phi_dot*sin(phi) + mu*xc_dot; 0]
```

C =

$$\begin{pmatrix} \beta\,\sin\left(\phi\right)\,\dot{\phi}^2 + \mu\,\dot{\mathrm{xc}} \\ 0 \end{pmatrix}$$

```
G = [0; -D*sin(phi)]
```

G =

$$\begin{pmatrix} 0 \\ -\mathrm{D}\,\sin\left(\phi\right) \end{pmatrix}$$

```
U = [u; 0]
```

U =

$$\begin{pmatrix} u \\ 0 \end{pmatrix}$$

```
q_dd = inv(M)*(u - G - C)
```

q_dd =

$$\begin{pmatrix} \frac{\beta\,\cos(\phi)\,(u+\mathrm{D}\,\sin(\phi))}{\sigma_1} - \frac{\alpha\,\left(\beta\,\sin(\phi)\,\dot{\phi}^2-u+\mu\,\dot{\mathrm{xc}}\right)}{\sigma_1} \\ \frac{\gamma\,(u+\mathrm{D}\,\sin(\phi))}{\sigma_1} - \frac{\beta\,\cos(\phi)\,\left(\beta\,\sin(\phi)\,\dot{\phi}^2-u+\mu\,\dot{\mathrm{xc}}\right)}{\sigma_1} \end{pmatrix}$$

where

$$\sigma_1 = \alpha\,\gamma - \beta^2\cos\left(\phi\right)^2$$

```
state_space_form = [xc_dot; phi_dot; q_dd]
```

state_space_form =

$$\begin{pmatrix} \dot{\mathrm{xc}} \\ \dot{\phi} \\ \frac{\beta\,\cos(\phi)\,(u+\mathrm{D}\,\sin(\phi))}{\sigma_1} - \frac{\alpha\,\left(\beta\,\sin(\phi)\,\dot{\phi}^2-u+\mu\,\dot{\mathrm{xc}}\right)}{\sigma_1} \\ \frac{\gamma\,(u+\mathrm{D}\,\sin(\phi))}{\sigma_1} - \frac{\beta\,\cos(\phi)\,\left(\beta\,\sin(\phi)\,\dot{\phi}^2-u+\mu\,\dot{\mathrm{xc}}\right)}{\sigma_1} \end{pmatrix}$$

where

$$\sigma_1 = \alpha\,\gamma - \beta^2\cos\left(\phi\right)^2$$

2 b. Describe the set of equilibrium points

2b.) Describe equilibrium equations

Ans.) The condition for equilibrium is given by:

$$\dot{x} = f(x_e, u) = 0 \quad (\text{when i/p } u = 0)$$

$$\dot{x} = \begin{bmatrix} \dot{x}_c \\ \dot{\phi} \\ \dfrac{\beta\cos\phi(u + D\sin\phi)}{\alpha\gamma - \beta^2\cos^2(\phi)} - \dfrac{\alpha\left(\beta\sin(\phi)\dot{\phi}^2 - u + \mu\dot{x}_c\right)}{\alpha\gamma - \beta^2\cos^2(\phi)} \\ \dfrac{\gamma\left(u + D\sin(\phi)\right)}{\alpha\gamma - \beta^2\cos^2(\phi)} - \dfrac{\beta\cos(\phi)\left[\beta\sin(\phi)\dot{\phi}^2 - u + \mu\dot{x}\right]}{\alpha\gamma - \beta^2\cos^2(\phi)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

from above relation:

$$\dot{x}_c = 0 \quad ; \quad \dot{\phi} = 0 \quad ; \quad u = 0$$

also, $\ddot{x}_c = 0 \quad ; \quad \ddot{\phi} = 0$ at equilibrium

Solving for the third row of above matrix $(\ddot{x}_c = 0)$

$$\frac{\beta\cos\phi(\cancel{u}^{0} + D\sin\phi)}{\alpha\gamma - \beta^2\cos^2(\phi)} - \frac{\alpha\left(\cancel{\beta\sin(\phi)\dot{\phi}^2}^{0} - \cancel{u}^{0} + \cancel{\mu\dot{x}_c}^{0}\right)}{\alpha\gamma - \beta^2\cos^2(\phi)} = 0$$

$$\Rightarrow D\beta\cos\phi\sin\phi = 0$$

$$\therefore \boxed{\cos\phi\sin\phi = 0}$$

Solving for the fourth row of above matrix $(\ddot{\phi} = 0)$

$$\frac{\gamma\left(\cancel{u}^{0} + D\sin(\phi)\right)}{\alpha\gamma - \beta^2\cos^2(\phi)} - \frac{\beta\cos(\phi)\left[\cancel{\beta\sin(\phi)\dot{\phi}^2}^{0} - \cancel{u}^{0} + \cancel{\mu\dot{x}}^{0}\right]}{\alpha\gamma - \beta^2\cos^2(\phi)} = 0$$

$$\Rightarrow \gamma D\sin\phi = 0$$

$$\boxed{\sin\phi = 0}$$

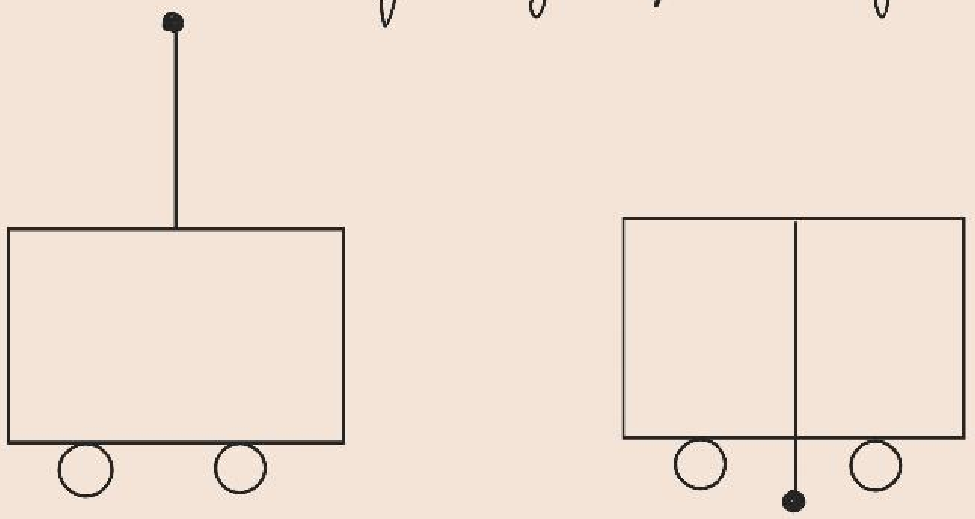
$\therefore$ equilibrium states are when $x_2 = 0, \pi, 2\pi \ldots$

i.e $\boxed{x_2 = n\pi}$

Ans 2b.) The equilibrium of $x_2 = \phi = 0$

or $\phi = 180^\circ$

would mean that the pendulum is vertically straight up or straight down.



2 c. Compute the eigen values of matrix A which represents the linearized system about the equilibrium point at x = 0

```
A = [[0,0,1,0];[0,0,0,1];[0,1,-3,0];[0,2,-3,0]];

B = [0;0;1;1];

eig_A = eig(A)
```

```
eig_A = 4x1
         0
   -3.3301
    1.1284
   -0.7984
```

Since one the eigenvalues of A have a real positive value the system is unstable about the state x = 0

This means that the vertical upright pendulum is not in stable equilibrium

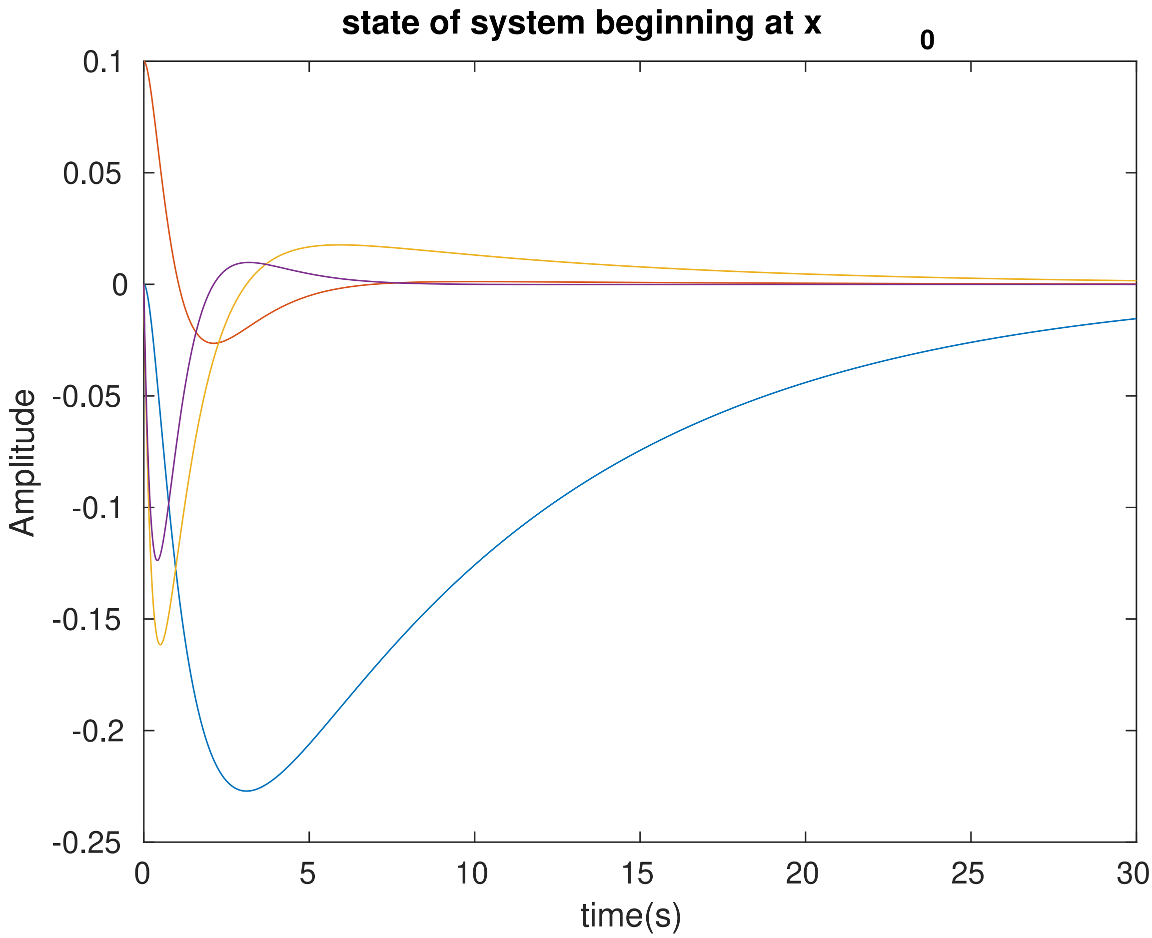
2 d. Finding the optimal feedback control gain

```
Q = [[1,0,0,0];[0,5,0,0];[0,0,1,0];[0,0,0,5]];
R = 10;

K = lqr(A, B, Q, R)
```

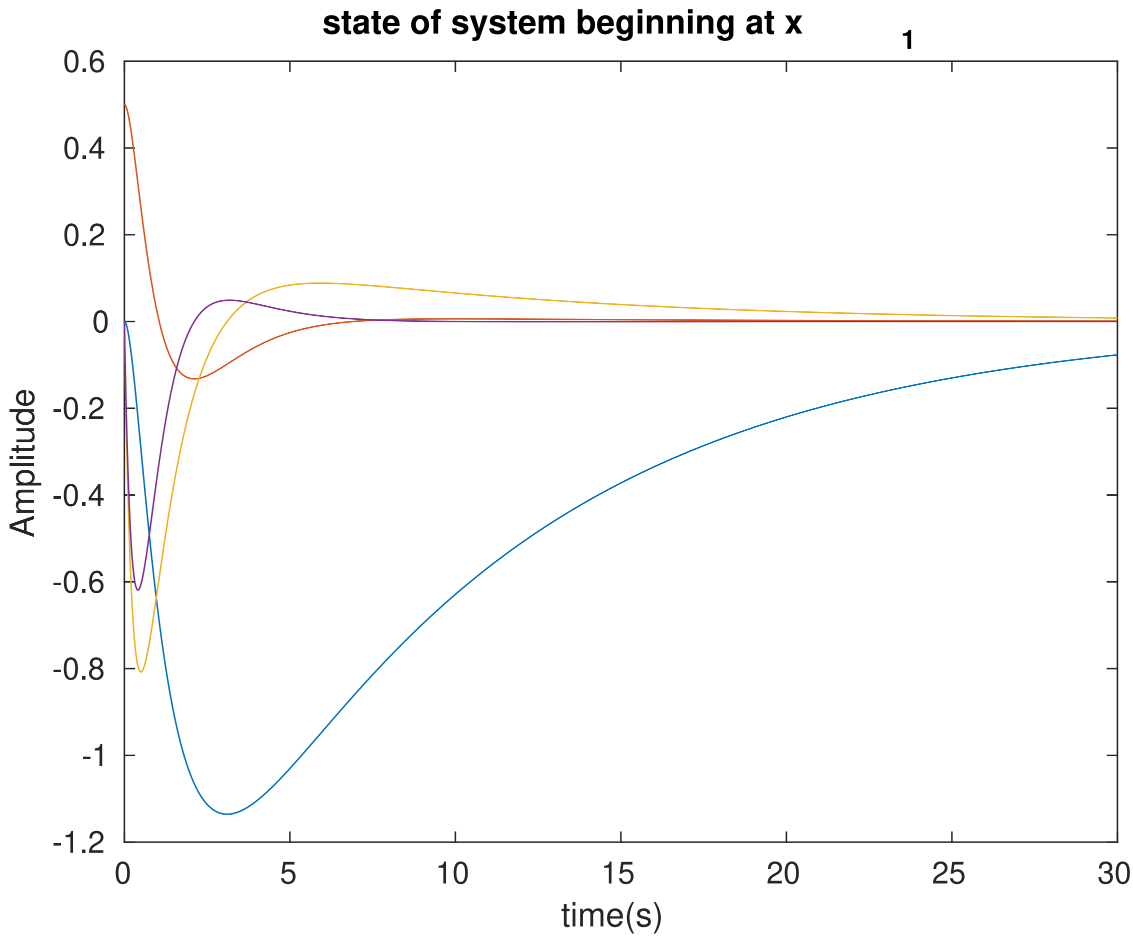
```
K = 1x4
   -0.3162   10.2723   -6.7857    9.2183
```

Using the above feedback control to plot the state of the linearized system

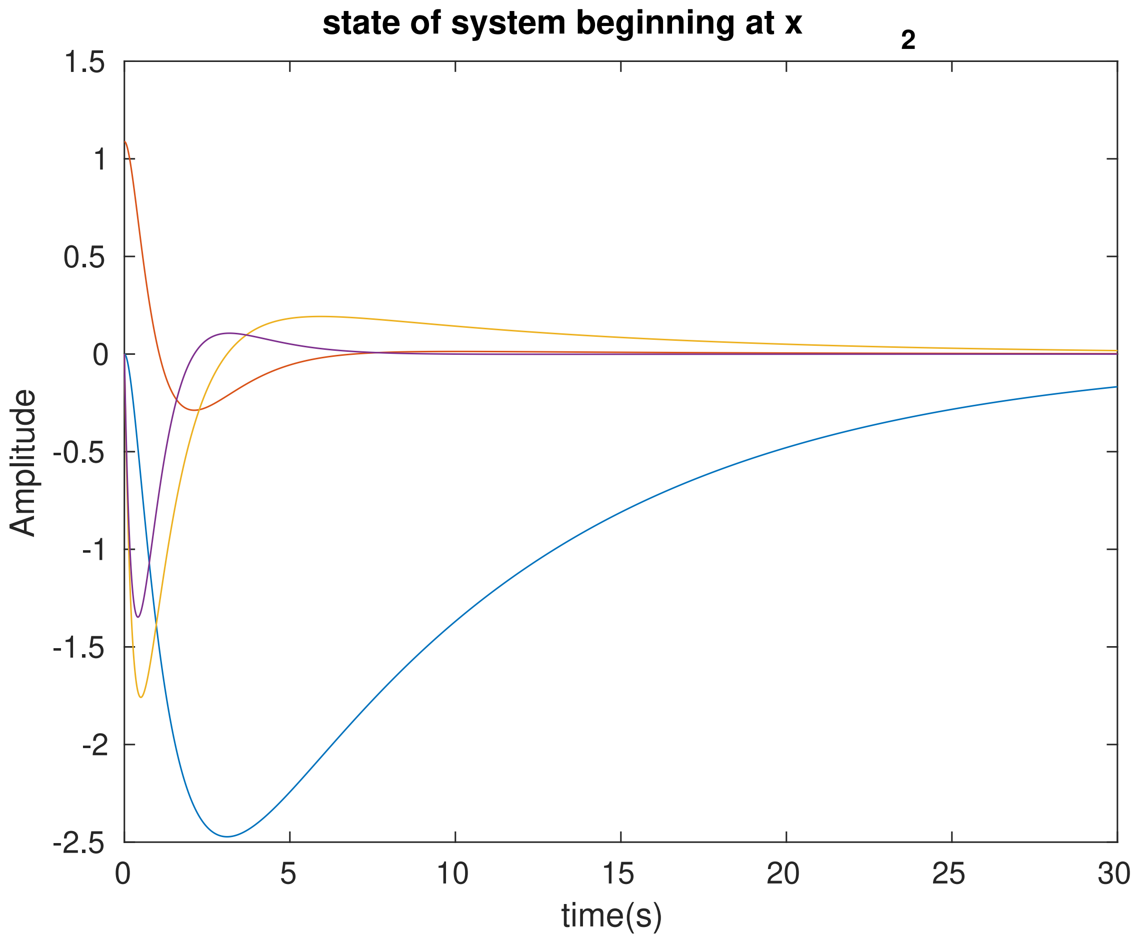
```
t = 0 : 0.01 : 30 ;

% define three initial state vectors x_0, x_1, and x_2
x_0 = [0; 0.1; 0; 0];
x_1 = [0; 0.5; 0; 0];
x_2 = [0; 1.0886; 0; 0];
x_3 = [0; 1.1; 0; 0];

% plotting the state of system beginning at x_0
[t,x0] = ode45(@(t,x)(A-B*K)*x, t, x_0);
plot(t0,x0)
title('state of system beginning at x_0')
ylabel('')
xlabel('time(s)')
```



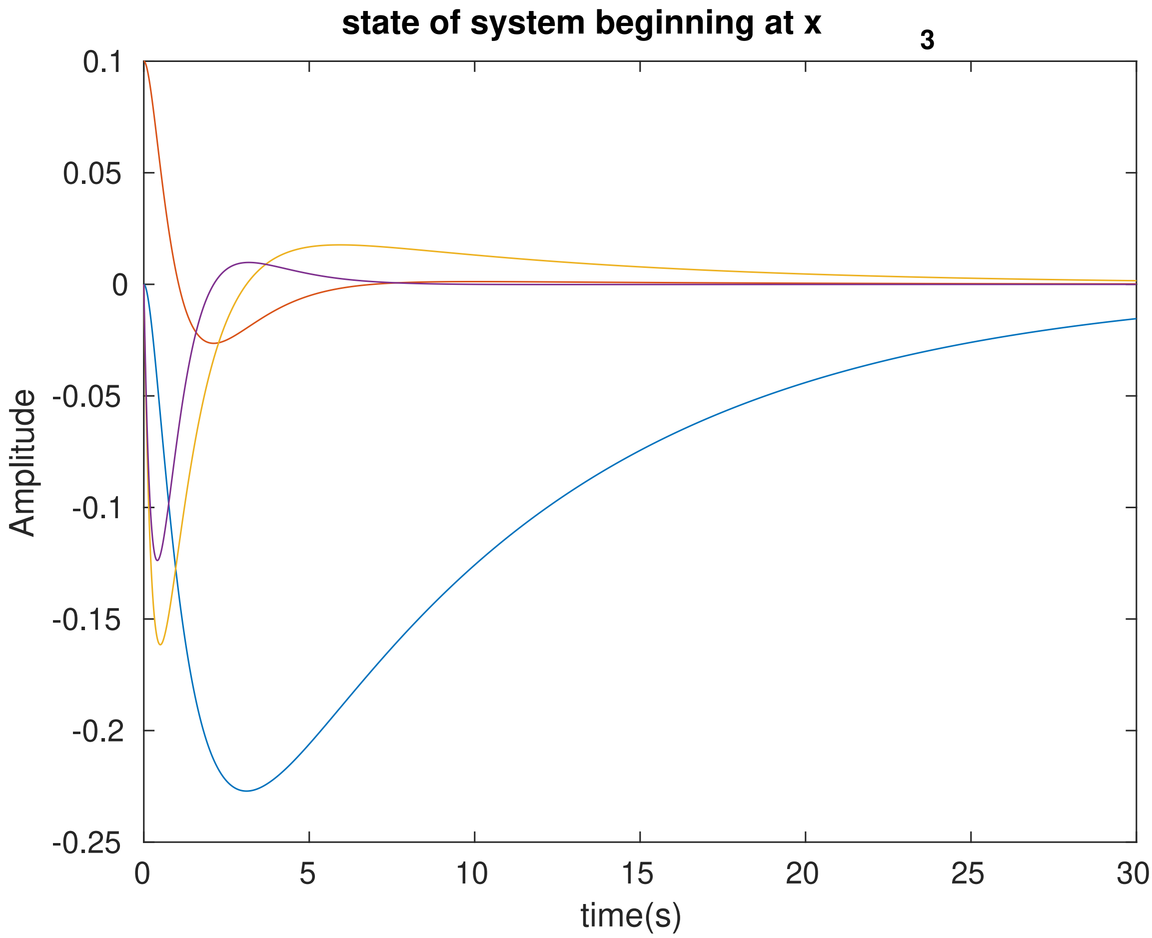

```
% plotting the state of system beginning at x_1
[t,x1] = ode45(@(t,x)(A-B*K)*x, t, x_1);
plot(t,x1)
title('state of system beginning at x_1')
ylabel('')
xlabel('time(s)')
```




```
% plotting the state of system beginning at x_2
[t,x2] = ode45(@(t,x)(A-B*K)*x, t, x_2);
plot(t,x2)
title('state of system beginning at x_2')
ylabel('')
xlabel('time(s)')
```



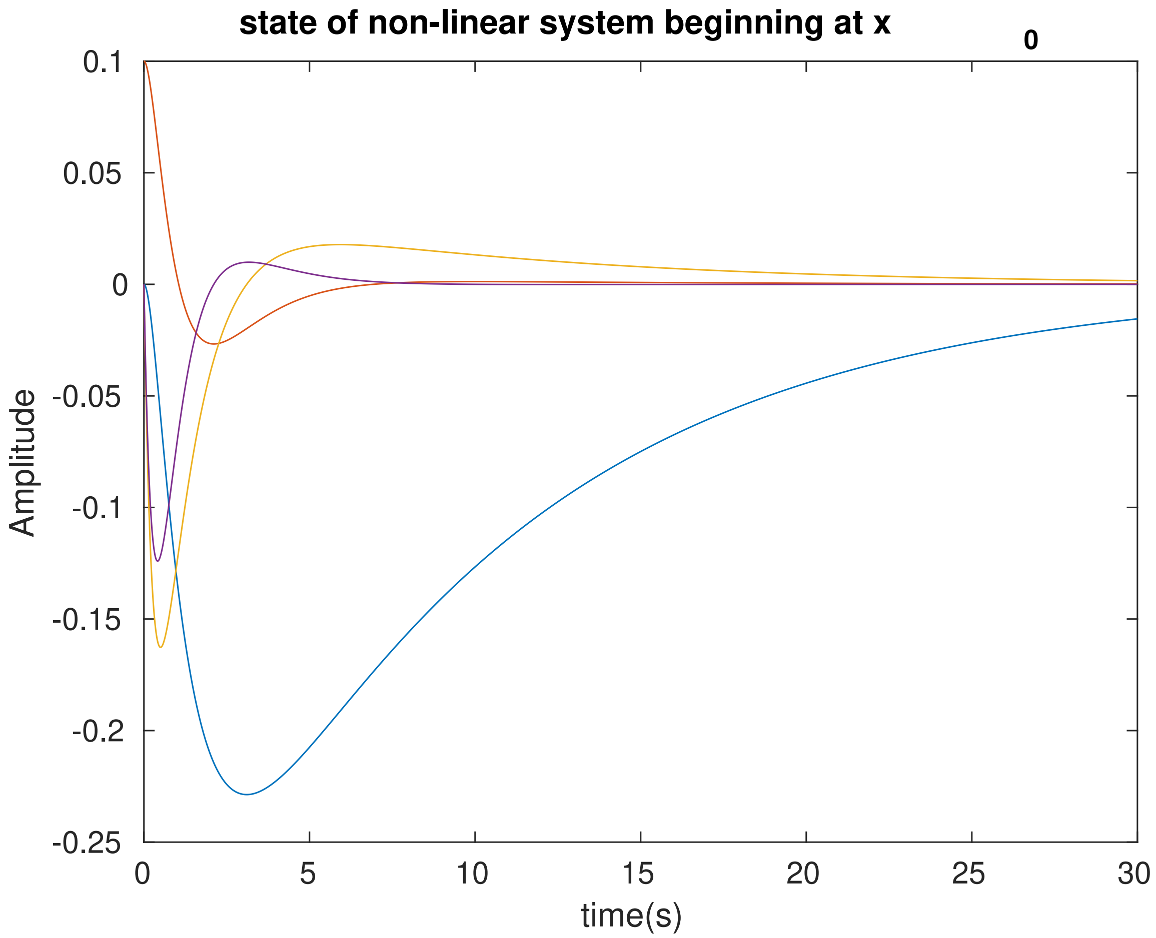

```
% plotting the state of system beginning at x_0
[t,x3] = ode45(@(t,x)(A-B*K)*x, t, x_3);
plot(t,x0)
title('state of system beginning at x_3')
ylabel('')
xlabel('time(s)')
```



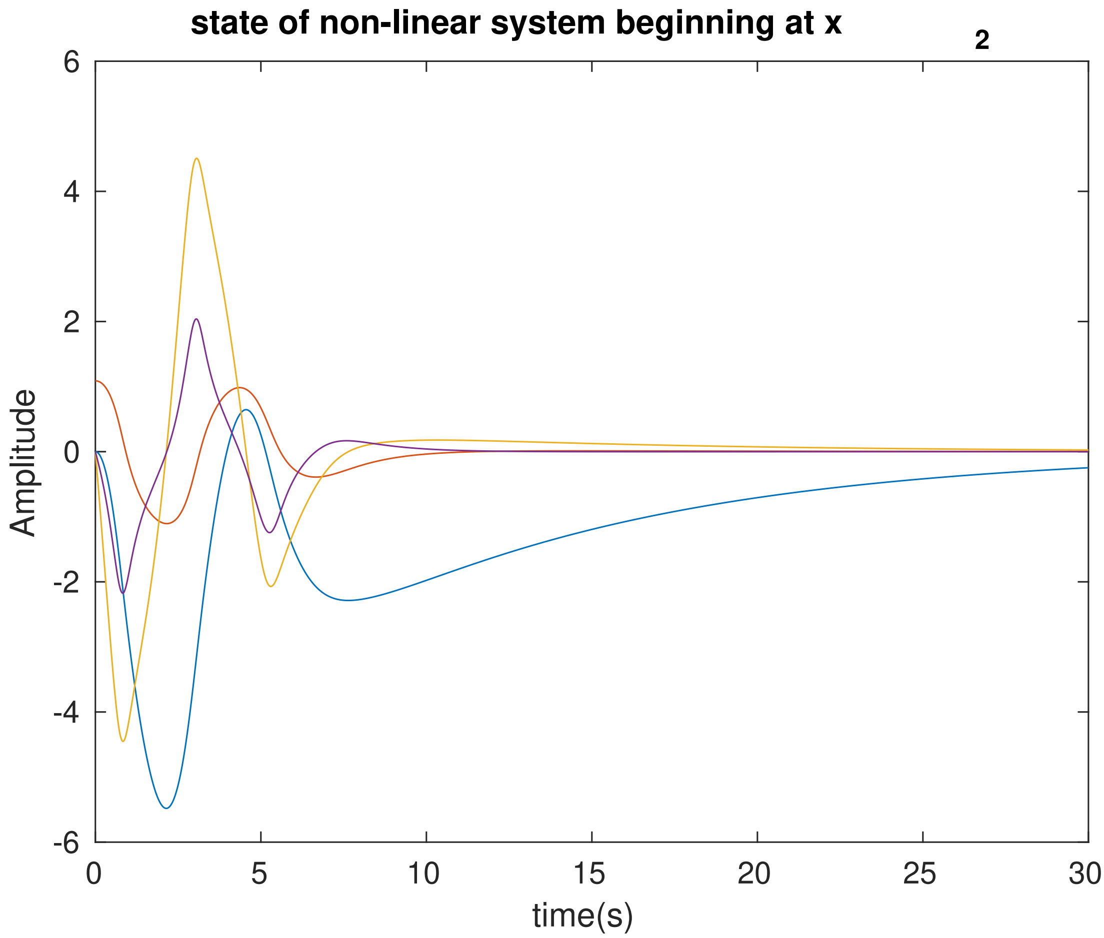
2 e. Repeat 2 d. using non-linearized state equations

The non linear system is computed below:

```
[t,x4] = ode45(@(t,x4)(non_linear_fun(x4, -K*x4)), t, x_0);
plot(t, x4)
title('state of non-linear system beginning at x_0')
ylabel('')
xlabel('time(s)')
```



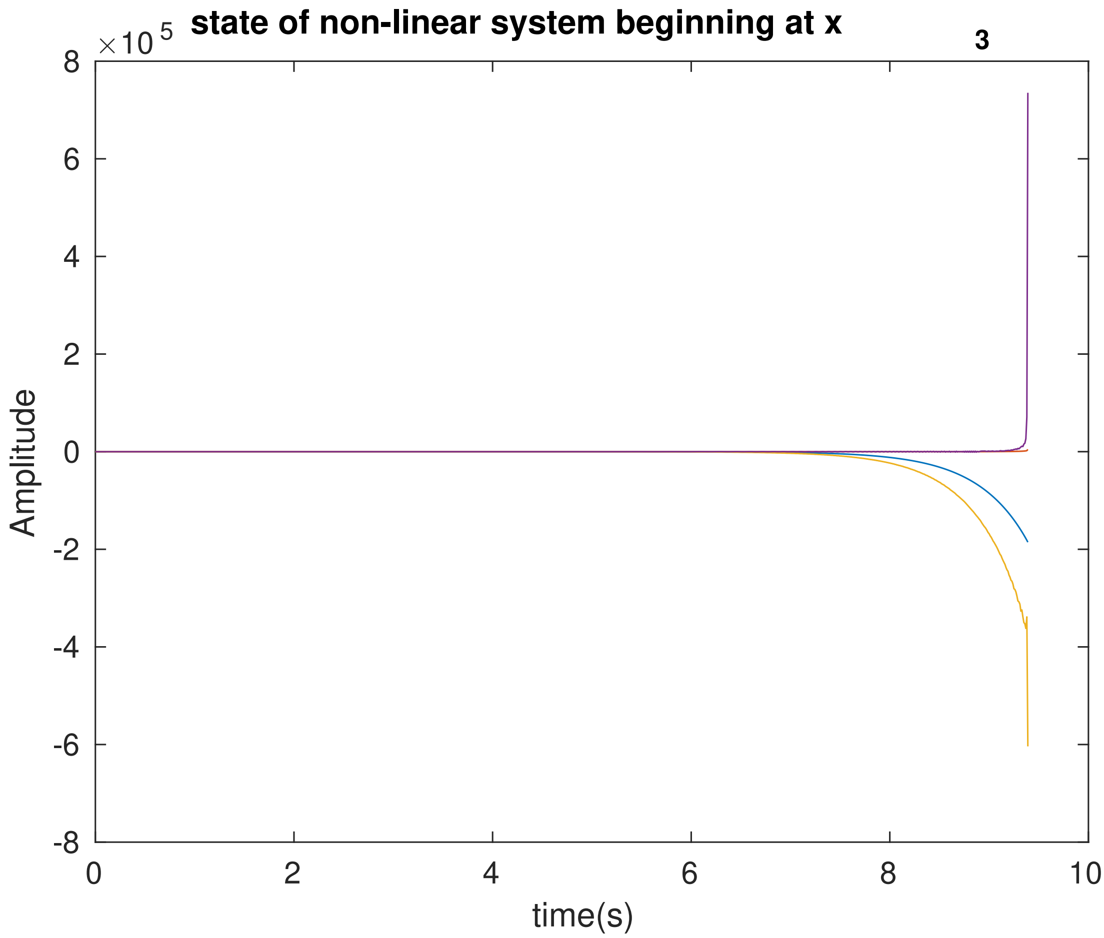

```
[t,x4] = ode45(@(t,x4)(non_linear_fun(x4, -K*x4)), t, x_2);
plot(t, x4)
title('state of non-linear system beginning at x_2')
ylabel('')
xlabel('time(s)')
```




```
[t,x4] = ode45(@(t,x4)(non_linear_fun(x4, -K*x4)), t, x_3);
```

```
Warning: Failure at t=9.394808e+00.  Unable to meet integration tolerances without reducing the step
```

```
plot(t, x4)
title('state of non-linear system beginning at x_3')
ylabel('')
xlabel('time(s)')
```



The non-linear system is not stable when the initial state vector has a high value of x_c as in the case of x_3 = [0; 1.1; 0; 0]

2 f. Find the matrix C to sensor measurements in y are in inches (y = Cx)

y measures only position therefore should have a dimension of (1,) i.e is a scalar

x = [x; ; x'; '] and has shape (4,1)

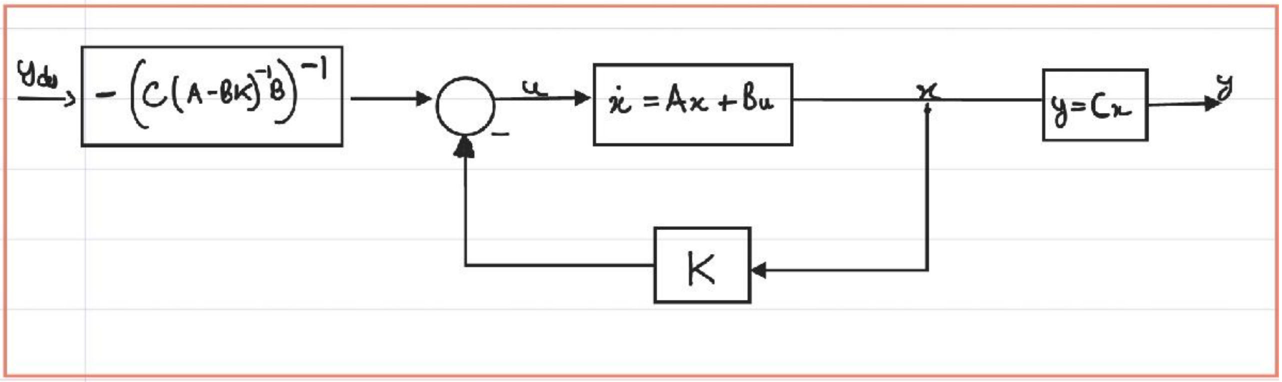
Therefore C must have shape (1,4)

```
% Finding C from above equation

% 1 metre = 39.3701 inches, therefore C must be

C = [39.3701, 0, 0, 0]
```

```
C = 1x4
   39.3701         0         0         0
```

2 g. Create a tracking controller to specify a desired cart position trajectory

A tracking controller is represented as:



At equilibrium: $\dot{x} = 0$

& $x = x_e$

$$\therefore \quad \dot{x} = A x_e + BKx_e + Bv = 0$$

$$\dot{x} = A x_e + B(-Kx_e + v)$$

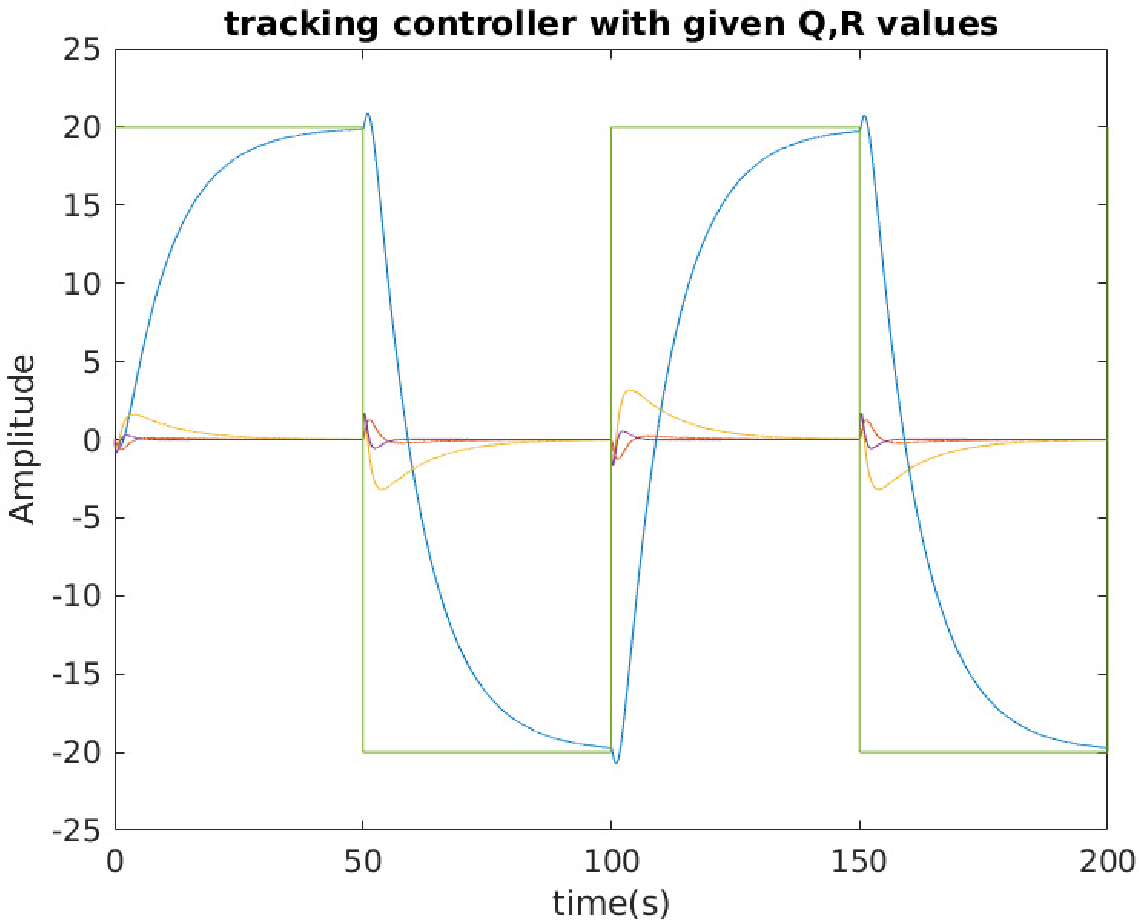
$$\& \quad v = -\left[C(A-BK)^{-1}B\right]^{-1} \cdot y_{des}$$

```
% Defining the specified timesteps and duration
t_5 = 0 : 0.01 : 200;
x_5 = [0; 0; 0; 0];
K % from previous LQR caluculation
```

```
K = 1x4
   -0.3162   10.2723   -6.7857    9.2183
```

```
% define parameters for y_des only for plotting
freq=0.01;
offset=0;
amp=20;
duty=50;

% define y_des as a square wave function
y_des = offset+amp*square(2*pi*freq.*t_5,duty);

v = -1 * inv(C*inv((A-B*K))*B);
```

```
[t,x5] = ode45(@(t,x5)(tracking_controller(x5, -K*x5, v, t)), t_5, x_5);
% since y_des is in inches and x5 is in metres, we multiply the x5 plot
plot(t, x5*39.3701)
hold on
plot(t_5,y_des)
hold off
title('tracking controller with given Q,R values')
ylabel('')
xlabel('time(s)')
```



2 f. Improving the Tracking Controller

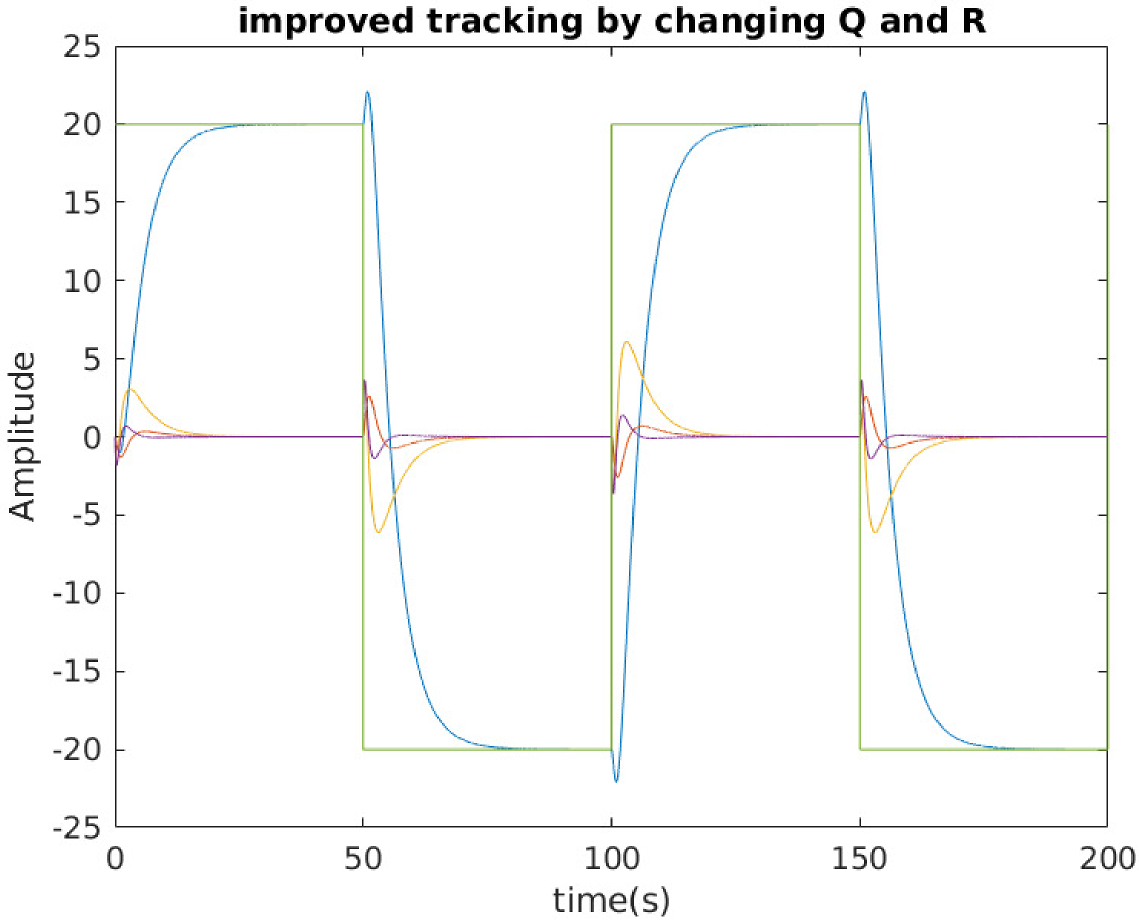
The Q and R values can be tuned to track the square wave desired output. It can be improved in the following ways

1. Currently the controlloer is reaching the required amplitude very late. Therefore Q can be increased to reduce convergence time
2. However, increasing Q caused an overshoot when the square wave changed amplitude from +20 to -20
3. To reduce this overactuation the R value was then increased to increase the cost of actuation

```
% using t_5 timespan and x_5 initial state specified in 2.d

Q = [[10,0,0,0];[0,5,0,0];[0,0,1,0];[0,0,0,5]];
R = 20;
K = lqr(A, B, Q, R)
```

```
K = 1x4
   -0.7071   11.2732   -7.7079   10.2244
```

```
v = -1 * inv(C*inv((A-B*K))*B);

[t,x5] = ode45(@(t,x5)(tracking_controller(x5, -K*x5, v, t)), t_5, x_5);
% since y_des is in inches and x5 is in metres, we multiply the x5 plot
plot(t, x5*39.3701)
hold on
plot(t_5,y_des)
hold off
title('improved tracking by changing Q and R')
ylabel('')
xlabel('time(s)')
```



Helper functions defined for ode45

```
function x_dot = tracking_controller (x, F, v, t)
xc = x(1);
phi = x(2);
xcdot = x(3);
phidot = x(4);

% constants in the system
gamma = 2;
alpha = 1;
beta = 1;
D = 1;
mu = 3;

% define parameters for y_des
```

```
freq=0.01;
offset=0;
amp=20;
duty=50;

% define y_des as a square wave function
y_des = offset+amp*square(2*pi*freq.*t,duty);
u = F + v*y_des;

divisor = ((gamma*alpha) - (beta*beta*cos(phi)*cos(phi)))^(-1);

x_dot = [xcdot;
        phidot;
        divisor*((u*alpha) - (beta*phidot*phidot*sin(phi)*alpha) - (alpha*mu*xcdot) + (beta*D*sin(phi
        divisor*((u*beta*cos(phi)) - (beta*beta*phidot*phidot*sin(phi)*cos(phi)) - (beta*cos(phi)*mu*


% divisor_1 = alpha / ((gamma*alpha) - (beta*beta*cos(phi)*cos(phi)));
% divisor_2 = (beta*cos(phi)) / ((gamma*alpha) - (beta*beta*cos(phi)*cos(phi)));

% x_dot = [xcdot;
%         phidot;
%         divisor_1*(u - (beta*phidot*phidot*sin(phi)) - (mu*xcdot) + ((beta*D*cos(phi)*sin(phi))/alph
%         divisor_2*(u - (beta*phidot*phidot*sin(phi)) - (mu*xcdot) + ((gamma*D*sin(phi)) / (beta*cos(

end
```

```
function x_dot = non_linear_fun (x, u)
xc = x(1);
phi = x(2);
xcdot = x(3);
phidot = x(4);

% constants in the system
gamma = 2;
alpha = 1;
beta = 1;
D = 1;
mu = 3;

divisor = ((gamma*alpha) - (beta*beta*cos(phi)*cos(phi)))^(-1);

x_dot = [xcdot;
        phidot;
        divisor*((u*alpha) - (beta*phidot*phidot*sin(phi)*alpha) - (alpha*mu*xcdot) + (beta*D*sin(phi
        divisor*((u*beta*cos(phi)) - (beta*beta*phidot*phidot*sin(phi)*cos(phi)) - (beta*cos(phi)*mu*


% divisor_1 = alpha / ((gamma*alpha) - (beta*beta*cos(phi)*cos(phi)));
% divisor_2 = (beta*cos(phi)) / ((gamma*alpha) - (beta*beta*cos(phi)*cos(phi)));

% x_dot = [xcdot;
%         phidot;
%         divisor_1*(u - (beta*phidot*phidot*sin(phi)) - (mu*xcdot) + ((beta*D*cos(phi)*sin(phi))/alph
%         divisor_2*(u - (beta*phidot*phidot*sin(phi)) - (mu*xcdot) + ((gamma*D*sin(phi)) / (beta*cos(

end
```