Evaluation of residual stresses and strains using the Eigenstrain Reconstruction Method

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ABSTRACT

The study of residual stress has long been an important research field in science and engineering, due to the fact that uncontrolled residual stresses are detrimental to the performance of products. Numerous research contributions have been devoted to the quantification of residual stress states for the purpose of designing engineering components and predicting their lifetime and failure in service. For the purposes of the present study these can be broadly classified into two main approaches, namely, the interpretation of experimental measurements and process modelling. In this paper, a novel approach to residual stress analysis is developed, called here the Eigenstrain Reconstruction Method (ERM). This is a semi-empirical approach that combines experimental characterisation, specifically, residual elastic strain measurement by diffraction, with subsequent analysis and interpretation based on the eigenstrain theory. Three essential components of the ERM, i.e. the residual strain measurement, the solution of the inverse problem of eigenstrain theory, and the Simple Triangle (SIMTRI) method, are described. The ERM allows an approximate reconstruction of the complete residual strain and stress state in the entire engineering component. This is a significant improvement compared to the experimentally obtained limited knowledge of stress components at a selected number of measurement points, or to the simple interpolation between these points.

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1. Introduction

The understanding of the inter-relationship between processing, structure, properties, and performance of a material is a research area of fundamental importance for most structural applications. The study of a material performance in terms of its integrity is important since the prevention of engineering component failure in service is essential for safe execution of our daily lives. Without a study of material processing the accurate prediction of failure is unlikely to be attainable. The fact is that not only do the manufacturing processes (e.g. casting, forming, machining, joining, quenching, peening, etc.) modify the structure and the stress–strain state of a material, but they also invariably give rise to residual stresses within components. Residual stress is one of the mechanisms through which processing significantly affects the quality, durability and mechanical response of engineered components.

The fundamental views taken in the present investigation is that residual stresses in engineering components are caused by incompatible internal permanent strains (i.e. eigenstrain) induced by inhomogeneous inelastic deformation, temperature gradients,

or phase transformations during manufacturing and processing of the components. In service, the associated residual stresses may combine with applied stresses to cause unexpected failure or to shorten the component lifetime.

Uncontrolled residual stresses are detrimental to the reliable prediction of product performance, and it is therefore necessary and important to quantify the residual stresses, for the purposes of improved engineering component design, the prediction of their durability and the avoidance of failure in service. However, in current industrial practice, the presence of the residual stresses is accounted for in the assessment of the engineering component's fitness-for-purpose in one of two very simple ways. Either the residual stresses are ignored entirely or the worst case scenario is assumed of purely tensile, yield level stress. As a result, engineering components are overdesigned for safety reasons, but as a consequence, product performance is reduced. In the context of economical effectiveness, appropriate residual stress assessment can lead to weight savings, cost reductions, and improved integrity.

One of the problems of dealing with residual stresses is that they cannot be measured directly. In fact, one can observe that the very concept of stress is an imaginary tool used for convenience of description. In classical textbooks on continuum mechanics (Timoshenko, 1987; Gere and Timoshenko, 1999), stress is defined as the internal force per unit area, where the force is

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thought to act through an imaginary section. As a consequence of this complexity of the very concept of stress, other quantities related to residual stresses are measured instead; usually some aspect deformation, such as strains or strain increments; or some indirectly related physical quantities, e.g. magnetisation or speed of sound.

In order to quantify the residual stress state in engineering components, two principal approaches have long been used, namely, the interpretation of experimental measurements and deformation process modelling. However, significant limitations still remain in both approaches: reliable process modelling requires extensive and expensive characterisation of material properties and the use of appropriate validation procedures. Moreover, often there is an absence of reliable predictive techniques based on process modelling. On the other hand, measurements always provide a limited level of detail, due to the finite number of discrete data points that restricts the possibility of reconstructing full-field stress distributions. Some measurement techniques, e.g. diffraction, are non-destructive, whilst others, e.g. hole drilling and contour method, are destructive and hence inevitably need to remove material. Although diffraction techniques are truly non-destructive, there is a key limitation which is their sensitivity exclusively to extensional and compressive lattice strains, i.e. strains of elastic or thermal origin, and not to permanent plastic or creep strains. This is due to the fact that plastic deformation occurs either by crystal slip (at small strains), or by large rigid body displacements of material blocks. Both processes occur in such a way that the average lattice spacing in the deformed region remains unchanged (although the width of the statistical distribution of lattice spacings within a considered gauge volume, and with it the diffraction peak broadening, are affected by the plastic deformation).

In order to overcome the above-mentioned limitations, the semi-empirical approach based on the theory of eigenstrain is proposed that combines experimental characterisation in terms of residual elastic strain with subsequent analysis and interpretation. It is useful to clarify at this point that, strictly speaking, the present investigation does not fall into the category of process modelling studies, since no attempt is being made to trace the evolution of material state, e.g. from the two separate pieces and into a welded assembly, through complex thermo-mechanical deformation history. Instead, the approach is aimed at reconstructing as fully as possible the component residual strain/stress state after the processing operation. It could therefore be termed "*post-process analysis*".

The key task of post-process analysis is the reconstruction and decomposition of the complete residual strain state, whereby the two-or three-dimensional spatial distribution of strains is obtained, and the total strain is separated into elastic and inelastic parts. The significance of this approach lies in the fact that, once this reconstruction and decomposition is accomplished, both the deformation (strain) state and the stress state (directly related to the elastic part of total strain) become fully characterised. The only approach that offers a suitable tool for this purpose is the Eigenstrain Reconstruction Method (ERM) that is developed in the present study.

2. Eigenstrain theory

2.1. Definition of eigenstrain

The term eigenstrain and the notation ε^* were introduced by Mura (1982), who proposed the word based on the German paper "Eigenspannungen und Eigenspannungsquellen" (Reissner, 1931). The eigenstrain indicates any permanent strain arising in the material due to some inelastic process such as plastic deformation, crystallographic transformation, thermal expansion mismatch

between different parts of an assembly, etc. It thus accounts for all permanent strains that arise in the material exhibiting inelastic behaviour and give rise to residual stresses, but is not merely a sum of the various non-linear strains.

In the small strain approximation, the additive decomposition of the total strain can be expressed via the sum of elastic strain and eigenstrain parts

$$\varepsilon_{total} = \varepsilon_e + \varepsilon^* \tag{1}$$

where ε_e is elastic strain and ε^* eigenstrain.

It is noted here that in the literature another term can be found, *inherent strain*, introduced by Ueda (1975). It appears that this term is equivalent in all respects to the term *eigenstrain*. According to Ueda, from the mechanical viewpoint the residual stress induced in an engineering component after manufacturing or processing, e.g. welding, may be regarded as the consequence of incompatible strains consisting of plastic strain, shrinkage strain (in the case of welding), creep strain and strain due to phase transformation, although originally all of these phenomena are caused by the welding heat input. This means that no residual stress arises within the component if there is no incompatible strain, even e.g. under a thermal cycle generating uniform heating (Yuan and Ueda, 1996). Therefore, we believe that the term inherent strain can be used interchangeably with the term eigenstrain.

2.2. Concept of eigenstrain

Before using the term eigenstrain, the concept of eigenstrain serving as the source of residual stress was developed by Mindlin and Cheng (1950) with the terminology of *nuclei of strain*, and by Eshelby (1957) who referred to them as *stress-free transformation strains*. Based on the concept of eigenstrain, Mura (1982) developed a mathematical framework for the determination of residual stress corresponding to a given eigenstrain distribution, for the case of an infinite three-dimensional body. In spite of the complexity of the framework, it can be physically seen and understood using the concept of eigenstrain with the aid of Fig. 1. The process of introducing an eigenstrain distribution into a solid object is illustrated on the left. For example, it is thought that a phase transformation associated with volume change takes place within a domain (inclusion); for simplicity the eigenstrain can be assumed to be uniform over a sphere or ellipsoid. This gives rise to a residual stress state shown on the right, which is uniform within the ellipsoid (the celebrated Eshelby result (Eshelby, 1957)) and decays to zero at large distances outside the inclusion. However, the limitation of this analytical method is that it cannot be applied to the case of most engineering structures without considerable alteration, due to the important influence of finite dimensions and overall shape.

The problems that arise within the eigenstrain theory of residual stress can be conveniently illustrated in reference to Eq. (1). In a typical residual stress analysis situation some elements of this equation are known at some locations, whilst others remain to be determined. For example, in the Eshelby ellipsoid problem the transformation strain (eigenstrain) is known to be constant within the domain of the inclusion, and zero everywhere else. The corresponding elastic strain and the total strain are, on the other hand, unknown, and need to be determined everywhere.

By way of classification, two kinds of problems can be identified. One is called the *direct problem of eigenstrain theory*, according to the terminology in (Korsunsky, 2006): using a known eigenstrain field, deduce the residual elastic strain (and stress) field everywhere. For a known non-uniform eigenstrain distribution, the analytical expression for finding the residual elastic strains (and hence stresses) within an infinitely extended elastic body containing eigenstrains has been presented by Mura (1982). The
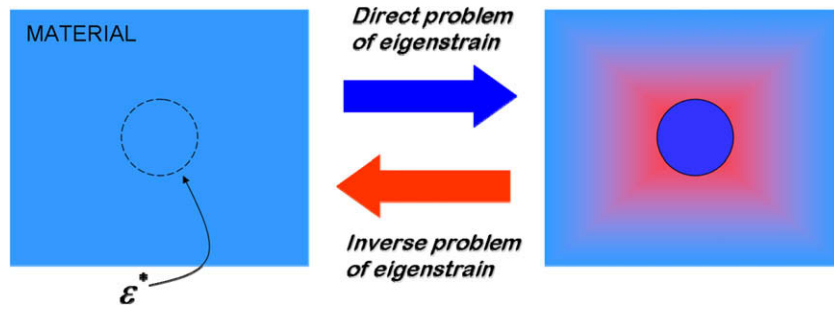
Fig. 1. Illustration of the concept of eigenstrain.

relationship between a given eigenstrain distribution $\varepsilon_{ij}^*(x')$ and the corresponding residual elastic strain $e_{kl}(x)$ is expressed by the following formula:

$$e_{kl}(x) = -\varepsilon_{kl}^*(x) - \int_{-\infty}^{\infty} C_{pqmn}\varepsilon_{mn}^*(x')G_{kp,ql}(x-x')dx' \qquad (2)$$

where C_{ijkl} the elastic stiffness coefficients, and $G_{kp}(x-x')$ denotes the Green's function for the particular geometry, representing the displacement component in the k direction at x when a body force is applied at x' in the p direction in an infinitely extended material. Since the solution for the stress field of a point eigenstrain is singular, it involves the use of finite part integral formulas for strongly singular integrals. The Eshelby solution for residual deformation due to uniform eigenstrain within an ellipsoidal inclusion is an example solution of the *direct problem of eigenstrain theory* (Eshelby, 1957). However, there is a practical limitation in that the Green's function is known explicitly for the special geometries such as infinite space, semi-infinite space bounded by a plane, and the corresponding two-dimensional solution (Korsunsky et al., 2007a). Consequently, it is not practicable to seek analytical expressions for the geometries typical of engineering components of complex shape.

The other problem is the *inverse problem of eigenstrain theory*, given some knowledge of the residual deformation state, the underlying eigenstrain distribution needs to be found. This inverse method, when suitably developed, will provide a flexible basis for the analysis of residual stress states in complex shaped engineering components, and even their subsequent evolution. The basis for the analytical approach to the *inverse problem of eigenstrain theory* is the quantitative comparison between point-wise experimental data and the predictions of the reconstructed elastic fields at the same locations. In a series of publications by Korsunsky (2005–2007), various aspects of the inverse problem framework have been developed. The use was made of the robust and efficient least squares approach to the determination of unknown eigenstrain distributions from residual elastic strains measured at a finite number of experimental points.

2.3. Practical use of eigenstrain

The pioneering work using the eigenstrain (or inherent strain) for practical applications was carried out by Fujimoto (1970), who presented a fundamental method of analysing residual stress and deformation based on the inherent strain in two-dimensional welded structures. According to Fujimoto, the inherent strain is generally and commonly induced within the narrow strip-like region that encloses the weld joint. Hence, the analysis of residual stress in welded structures can be simplified using a model with an assumption that the inherent strain is distributed only in that region. However, a way of determining inherent strain itself was not described until Ueda's a series of work was presented with

practical applications to such cases as butt-welded joints (Ueda et al., 1975, 1989a,b, 1993a), long welded joints (Ueda and Fukuda, 1983), axisymmetric shaft (Ueda et al., 1984), and T-and I-joints (Ueda et al., 1993b,c), etc. Ueda introduced a new method of evaluating residual stresses based on the inherent strain with the aid of the finite element method. In the method, the relation of the inherent strain $\{\varepsilon^*\}$ or the source of residual stresses with elastic strains $\{\varepsilon\}$ or stress $\{\sigma\}$ produced by $\{\varepsilon^*\}$ at an arbitrary point of a three-dimensional body can be expressed by the following elastic response equations.

$$\{\varepsilon\} = [H^*]\{\varepsilon^*\} \qquad (3)$$
$$\{\sigma\} = [D]\{\varepsilon\} = [D][H^*]\{\varepsilon^*\} \qquad (4)$$

where $[H^*]$ is the elastic response matrix and $[D]$ the elasticity matrix. Therefore, three-dimensional residual stresses can be accurately measured using the elastic analysis based on the elastic response relation equation, instead of complex calculations involved in the thermal elasto-plastic analysis, provided $\{\varepsilon^*\}$ is accurately estimated. The most remarkable aspect of using the inherent strain theory is that (suitably careful) cutting (or sectioning) of the material produces no changes of inherent strain distributions compared to the original structure, but changes of residual stress distributions. Based on this, certain assumptions were made for this method, such that cutting is accompanied by merely elastic strain changes and produces no further inherent strains, and stresses present in thin slices corresponded the plane stress state. It is worth noting here, in passing, that in principle stress relief induced by cutting can be so significant that reverse plastic flow occurs, introducing eigenstrain modification. However, whether this phenomenon takes place or not can be verified with the help of plastic deformation criteria applied to numerical stress models.

With examples analysed by this method, Ueda showed that the inherent strain can be evaluated from the difference in residual stresses between cut geometries, leading to the determination of residual stress in the original structure prior to cutting. The residual stresses determined by the method were compared with further experimental studies, resulting in a good agreement. Nevertheless, the adoption of the method had been hindered due to its complexity and the need for a large experimental effort.

In order to improve the major drawbacks of Ueda's method, Hill (Hill and Nelson, 1995, 1998; Hill, 1996) developed a localised eigenstrain method for the determination of triaxial residual stress in long welds. The method aims to provide stress results within a specific region of interest whilst only requiring measurements to be made within that region. In this method, a block of material needs to be removed from the sample. Further sectioning process along the longitudinal (perpendicular to the welds) direction is required to make a slice and then a chunk, with the general assumption that for welding such the eigenstrain field is dependent on the transverse (parallel to the welds) and in-plane (through thickness) directions, whilst being independent of the longitudinal direction.

Strain variations were measured: on a slice after cutting from the block; on a chunk cut from the slice; and finally on a dice cut from the chunk. Then, the dice were assumed stress-free, with the assumption that the dice were small enough relative to the spatial gradients of eigenstrain to achieve complete stress relief. Based on the strain relaxation data with elastic stress–strain relations for the case of plane stress, residual stress can be estimated at the free-surface measurement sites on the chunk, slice, and block geometries. The localised eigenstrain method further allows computing the residual stress at any point (even remote from the free-surface) in the region of interest, using estimated eigenstrain distributions obtained from the reduced stress data. Therefore, it is apparent that the method retains the benefits of the original eigenstrain method, with a significant reduction of the experimental effort required to produce estimates of residual stress near the weld bead. Nevertheless, it is thought that the localised eigenstrain method is suitable only for the welded structures with simple geometry of solid bodies, unless the direct problem is employed for general complex finite geometries.

Korsunsky proposed a general method that could be applied to a variety of situations, e.g. shot peening (Korsunsky, 2005; Korsunsky et al., 2006), laser shock peening (Korsunsky, 2006), autofret-taged tubes (Korsunsky, 2007), laser forming (Korsunsky et al., 2008), welding (Korsunsky et al., 2007b, 2009), etc., with eigen-strain field used as the primary unknown of the *inverse problem of eigenstrain theory*. The method uses a robust and efficient least squares approach to the determination of unknown eigenstrain distributions from residual elastic strains measured at a finite number of experimental points. For the strain measurement, diffraction techniques were used due to the fact that they allow the evaluation of a variety of structural and deformational parameters inside real components without the need to remove any material. It turned out that combination of *inverse problem of eigenstrain theory* and diffraction techniques becomes a very powerful method to analyse residual strain/stress state in an engineering component. In order to develop this approach further, the Eigenstrain Reconstruction Method is proposed in this paper and implemented for both simple and complex geometries relevant to engineering components.

3. Description of the Eigenstrain Reconstruction Method

The Eigenstrain Reconstruction Method (ERM) is a methodology for the reconstruction of full-field residual strain and stress distributions within engineering components by matching experimental measurements cast as the *inverse problem of eigenstrain theory*. The ERM possesses great versatility and offers several advantages for the evaluation of residual stresses. It can be applied for the interpretation of experimental results for residual elastic lattice strains measured by diffraction techniques, as well as to the data for strain change or displacement during material removal (slitting (Song et al., 2008), hole drilling (Prime and Hill, 2006), sectioning (Hill and Nelson, 1998), contour method (Kartal et al., 2008), etc.). Once the source of eigenstrain distribution is determined in some form, the residual stresses can be reconstructed by imposing the eigen-strain distribution in a linear elastic finite element model for the component geometry. Note that in principle the task of ERM is very challenging, since detailed spatial variation of each component of the eigenstrain tensor is required. However, note also that in this respect it is similar in its complexity to the task of detailed residual stress analysis (although in principle this problem is often simplified by considering only selected components of interest). Further-more, it is generally true that eigenstrain distributions are more localised than residual stress states: compare e.g. the extent of shot peening and welding eigenstrains (within regions of permanent deformation) and the extent of the corresponding residual stresses.

Once the eigenstrain distribution has been determined in some way, the entire full-field bi-or tri-axial residual stress state can be found at every point within the structure. In order to implement the ERM method, three main bases are essential:

3.1. Residual strain measurement

The purpose of the ERM is to reconstruct residual strain and stress distributions everywhere in a variety of engineering components from limited experimental data. This is possible due to the use of eigenstrain as the source of residual stress. For residual strain measurement, the crucial issue that emerges is not the type of measurements being used, but the size of region where the measurements are carried out. Luckhoo et al. (2009) addressed the effect of limiting the size of region where eigenstrains are installed, demonstrating that the size of the region chosen is extremely influential on the quality of eigenstrain-based reconstruction of residual stresses. It is found empirically that it may be necessary to introduce eigenstrains over a region that is somewhat wider than the region defined by the minima/maxima of measured residual elastic strains found from the experiment. For the use of additional information to identify the size of this eigenstrain domain, it may be useful to utilise additional insight provided by diffraction techniques. Diffraction analysis allows the determination of peak widths (e.g. full width at half maximum, FWHM). Peak broadening is caused by the inhomogeneity of deformation between scattering grains and within each grain as a consequence of dislocation density evolution, which in turn is related to rms elastic strain and ultimately plastic strain. Therefore, the variation of FWHM within the sample can be the basis for the choice of the size of region exposed to eigenstrains.

3.2. The inverse problem of eigenstrain theory

The *inverse problem of eigenstrain theory* makes use of the knowledge of residual elastic strains (or stresses) at a number of measurement points to retrieve the underlying source of eigen-strain field. Once residual strain (or stress) distributions are observed, eigenstrain distributions can then be deduced based on the mathematical framework of the *inverse problem of eigenstrain theory*.

General framework for the *inverse problem of eigenstrain theory* was introduced by Cao et al. (2002), Qian et al. (2004, 2005) and Korsunsky et al. (2004, 2007b). In the ERM, the framework by Korsunsky *et al* is adopted as described below.

First of all, let us consider the formulae for the *direct problem of eigenstrain analysis* by Mura (1982). As shown in Eq. (2), a given eigenstrain distribution $\varepsilon_{ij}^*(x')$ can be used to find the elastic strains and residual stresses arising in an infinitely extended elastic body.

Although the Green's function is often not known for the problems containing complex finite geometries, the calculation of the elastic strains and residual stresses using a given eigenstrain distribution is fairly straightforward, in that the form of Eq. (2) is retained.

Now, let us think of the problem that usually arises in residual strain measurement and interpretation. From the measurement with certain accuracy, residual strains and stresses can be collected at a finite number of points within a bounded sample. This can be thought of the inverse problem in which an unknown eigenstrain distribution $\varepsilon_{ij}^*(x')$ must be determined from the incomplete knowledge of elastic strains and residual stresses.

At this stage it is important to discuss the nature of the inverse problem considered here, and the extent to which the unknown eigenstrain distribution can be determined. Distributions of inelastic strain (eigenstrain) may be compatible or incompatible. Compatible eigenstrain distributions give rise to deformation, but

they do not require elastic strains to accommodate the permanent strain field. Therefore, neither residual stresses, nor residual elastic strain arise. Such compatible eigenstrain distributions are sometimes referred to as "impotent". Since the inverse problem formulation introduced above is based on matching the residual elastic strains, the compatible part of eigenstrain is not reconstructed. However, for the purposes of residual stress analysis this does not represent a restriction.

In order to solve this problem, the quadratic functional of strain mismatch is defined as follows:

$$J = \sum_{i=1,\dots N} w_i \left[e_{kl}(x_i) + \varepsilon_{kl}^*(x_i) + \int_{-\infty}^{\infty} C_{pqmn}\varepsilon_{mn}^*(x')G_{kp,ql}(x_i - x')dx' \right]^2 \tag{5}$$

where w_i denotes the weight factor assigned to each of the N collection points x_i considered. In the brackets, $e_{kl}(x_i)$ means the experimental data consisting of the values of residual elastic strain collected at positions x_i. The remaining two terms rely on the choice of eigenstrain distribution $\varepsilon_{ij}^*(x')$, and can be thought of the predicted elastic strain E_{kl} at a collocation point x_i. Now, Eq. (5) can be written in the alternative form as follows:

$$J = \sum_{i=1,\dots N} w_i[e_{kl}(x_i) - E_{kl}(x_i)]^2 \tag{6}$$

Due to the choice of squared difference to represent the mismatch between measured and predicted elastic strains, the expression on the right of Eq. (6) is non-negative, and is equal to zero only if agreement is perfect. In order to find the minimum value of the functional J, let us assume that the unknown eigenstrain distribution is given by truncated series of basis distribution (functions):

$$\varepsilon^*(x) = \sum_{i=1}^{N} c_i \xi_i(x) \tag{7}$$

where N is the total number of basis distributions used in the prediction, $\xi_i(x)$ corresponds to the basis function, and c_i denotes the unknown coefficients. The result of the direct problem, giving rise to the determination of the elastic strain distribution using an arbitrary eigenstrain distribution $\varepsilon_{ij}^*(x)$, can be applied to each of N basis distributions $\xi_i(x)$ in turn. Note here that the choice of the family of basis functions $\xi_i(x)$ remains the prerogative of the researcher. For example, Korsunsky et al. (2007b, 2008) used Chebyshev polynomials some modulation of the basis functions, so as to reflect the nature of the eigenstrain distribution. Kartal et al. (2008) used Legendre polynomials, and Cao et al. (2002) and Qian et al. (2004, 2005) used a series of smooth basis functions. In practice, however, these approaches are not sufficiently flexible and robust. This is partly due to the fact that the choice may have to be amended depending on the types of samples considered in order to find the best choice of the basis function. Furthermore, often the choice of polynomial basis is restrictive: it would not achieve good fit for samples containing complex or abrupt strain variations, even if the highest order of polynomials were chosen to seek good approximation to the experimental data. In order to overcome these limitations, the SIMTRI method was developed in this study and is explained in the next section.

In any case, $E_{yy,i}(x_j)$, the residual elastic strains at the same locations as in the experiment, are computed from the model containing a particular eigenstrain distribution ("basis function") labelled i.

Due to the linearity of the elastic problem, the superposition principle can be used to represent the residual elastic strains arising from a linear combination of eigenstrain basis functions $i = 1 \dots N$. The overall reconstructed value E_{yy} everywhere can be represented by the sum with coefficients c_i of the individual terms $E_{yy,i}(x)$

$$E_{yy}(x) = \sum_{i=1}^{N} c_i E_{yy,i}(x) \tag{8}$$

Note also that w_i is assumed to be unity for the rest of the development of the mathematical framework.

The goodness of the prediction can be measured by means of the functional J indicating the sum of squares of differences between actual measurements and the predicted values,

$$J = \sum_{j=1}^{M} \left(e_{yy}(x_j) - E_{yy}(x_j) \right)^2 = \sum_{j=1}^{M} \left(e_{yy}(x_j) - \sum_{i=1}^{N} c_i E_{yy,i}(x_j) \right)^2 \tag{9}$$

where M is the number of experimental points.

The search for the best choice of model can be accomplished by minimising J with respect to the unknown coefficients, c_i, by solving

$$\nabla_c J = 0 \text{ or } \frac{\partial J}{\partial c_i} = 0, \quad i = 1, \dots, N \tag{10}$$

Due to the quadratic nature of the functional in (8), the system of linear equations (10) always have a unique solution that corresponds to a minimum of J (for a particular choice of the basis function set). Therefore, for the regularised version of the inverse problem both the existence and uniqueness of the solution are assured. The important issue of stability of the solution is discussed further in the text.

Now, noting that the unknown parameters are $c_1, \dots c_N$, we consider the procedure for finding a unique minimum with respect to the unknown coefficients, c_i. The partial derivatives of J with respect to the coefficient c_i can be written explicitly as

$$\frac{\partial J}{\partial c_i} = 2 \sum_{j=1}^{M} \left[e_{yy}(x_j) - \sum_{k=1}^{N} c_k E_{yy,k}(x_j) \right] \left[-E_{yy,i}(x_j) \right] = 0 \tag{11}$$

Eq. (11) can be rewritten as

$$\sum_{k=1}^{N} c_k \sum_{j=1}^{M} E_{yy,k}(x_j) E_{yy,i}(x_j) = \sum_{j=1}^{M} e_{yy}(x_j) E_{yy,i}(x_j) \tag{12}$$

The above equations can be represented in the matrix form as

$$\begin{vmatrix} E_{yy,1}(x_1)E_{yy,1}(x_1) & \cdots & E_{yy,k}(x_M)E_{yy,1}(x_M) \\ \vdots & \ddots & \vdots \\ E_{yy,1}(x_M)E_{yy,1}(x_M) & \cdots & E_{yy,k}(x_M)E_{yy,i}(x_M) \end{vmatrix} \begin{vmatrix} c_1 \\ \vdots \\ c_i \end{vmatrix} = \begin{vmatrix} e_{yy}(x_1)E_{yy,1}(x_1) \\ \vdots \\ e_{yy}(x_M)E_{yy,i}(x_M) \end{vmatrix} \tag{13}$$

From the above equation, the unknown parameters, $c_1, \dots c_N$, can be obtained. Below we discuss in more detail how this procedure is implemented with the use of the SIMTRI method.

3.3. Simple triangle (SIMTRI) method

SIMTRI method is developed here for the purpose of implementing the Eigenstrain Reconstruction Method within the FE modelling framework. In practice, SIMTRI method is used to generate a formulation of the *inverse problem of eigenstrain theory*. The method consists of FE modelling (normally performed e.g. by the commercial ABAQUS finite element package) and some post-processing analysis (normally performed by MATLAB). The framework of the SIMTRI method classified into four steps and is as follows:

Step 1 – Setup of FE model In the first step of the SIMTRI method, two-or three-dimensional FE model is set up reflecting the sample geometry. If the sample has a complex geometry, CMM (Coordinate Measuring Machine) can be used to obtain

the proper 3D sample shape. Unlike process modelling, only the elastic material properties such as Young's modulus and Poisson's ratio are needed. Due to the characteristic of eigenstrain that exists within a localised region of the sample, fine mesh is recommended to cover the region and coarse mesh to the rest of the region. Note here that level of mesh may affect to the model result and the computation time, so careful judgement is required. Once the model is set up, some dataset would be obtained after the first running.

Step 2 – Iteration of FE model In this step, the combination of simple triangular pulse functions (see Fig. 2) is employed in the FE model setup. The width and shape of each triangle can be adjusted to control the quality of the reconstructed eigenstrain distribution. The coarseness of the distribution is determined by the triangle base width. For good results this parameter should be made as small as possible, with the consideration that refinement increases the computation time. Regarding the domain over which the triangles are introduced, as a rule-of-thumb one might assume that the eigenstrain domain should be slightly wider than the region between the minima/maxima of residual elastic strains that can be found from the experiment. Now, based on the dataset obtained from the previous step, arbitrary eigenstrains are imposed using the thermo-elastic modelling capacity of ABAQUS, specifically the user-defined sub-routine UEXPAN which is intended for modelling thermal strains due to temperature changes. This allows modelling inelastic strains and their dependence on other field variables. Eigenstrain can be simply considered as the net result of different mechanical and thermal phenomena. Note, however, that modelling it does not require explicit knowledge about its origin. Therefore, a convenient way to introduce eigenstrain is by prescribing an arbitrary coefficient of thermal expansion (CTE) distribution across the model corresponding to the desired eigenstrain and then introducing a uniform unit temperature rise. The result is the imposition of the desired eigenstrain across the sample. Non-uniform CTE distributions are prescribed using ABAQUS UEXPAN subroutine, which assigns a value of CTE in x, y and z directions to every node in the mesh. Within the UEXPAN file the list of coefficients c_k can also be stored for the triangular basis functions, with the corresponding list of x coordinates. The UEXPAN code checks every node, calculates the coefficients for the neighbouring nodes, and interpolates linearly the CTE component in the relevant direction (CTE value in the directions can be set to zero).
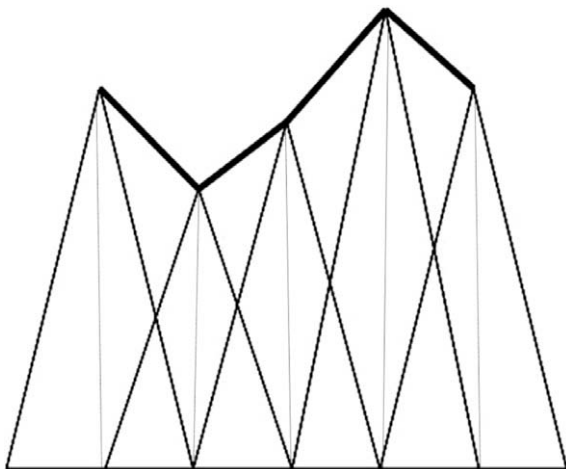

Fig. 2. Illustration of the SIMTRI method: the representation of unknown eigenstrain distribution (bold lines) by the superposition of triangular basis function (thin solid lines) at equally spaced positions spanning the weld cross-section.

UEXPAN files can be generated in this way for each basis function, and the simulation performed repeatedly. Fig. 3 represents an example of a welded structure after the application of a particular UEXPAN file to the 2D FE model. The input of simple localised eigenstrain profile leads to more complex and non-uniform residual strain distributions within the model.

Step 3 – Calculation of deformation parameters for the set of eigenstrain basis functions From the iterative execution of the model, the relevant field data (e.g. residual elastic strains) are extracted, so that the matrix on the left hand side of Eq. (13) can be filled. The equation is then solved (e.g. using MATLAB) and coefficients c_i are obtained.

Step 4 – Final execution of FE model The coefficients c_i are now introduced into the UEXPAN subroutine and the simulation performed once more. The resulting FE reconstruction contains extensive information such as the spatial distribution of residual displacement, strains and stresses, von Mises stress values, etc.

4. Dimensionality of the Eigenstrain Reconstruction Method analysis

One of the most important aspects of the ERM methodology is the *dimensionality* of the different stages of its implementation. Most engineering components and structures must be considered as three-dimensional physical objects. However, very thin plates are often treated as 2D for the purpose of mechanics analysis.

Three types of dimensional measures can be introduced for the ERM. First dimensional measure depends on the nature of the FE model used. The second dimensional consideration concerns the number of eigenstrain components (ε_{ij}^*) introduced into the model. Final, within the ERM method the variation of the chosen eigenstrain components may be associated with a certain number of coordinates or dimensions.

Dimensionality of the entire analysis can therefore be prescribed in terms of three measures defined as follows:

(i) Dimensionality of the FE model (2D or 3D)
(ii) Dimensionality of non-zero eigenstrain components (1D, 2D or 3D)
(iii) Dimensionality of eigenstrain variation with coordinates (1D, 2D or 3D)

In principle, the choice of 3D for each of the above aspects would give the most reliable results. However, fully three-dimensional descriptions of each of the above aspects are difficult to set up, expensive to run, impossible to obtain experimental data for (at least today), and not yet available within the ERM formulation. Since the SIMTRI method is developed for the 1D ERM, the dimensionality of ERM in the present study is fixed to be 1D. Nevertheless, the remains are the choice of dimensionality of the FE model and the eigenstrain components (ε_{ij}^*). Even this reduced choice of analysis dimensionality yields useful insights, as will be demonstrated.

For the clarification of ERM implementations, the model code [FEM-ε_{ij}^*-ERM] will be used to specify the *dimensionality*. For example, if a 3D FE model is used, one component of eigenstrains is introduced, and allowed to vary with one coordinate, then the model code is [3-1-1]. If a 2D FE model were used with two components of eigenstrains varying in one direction, the model code is [2-2-1].

5. Implementations of the Eigenstrain Reconstruction Method

The Eigenstrain Reconstruction Method was applied to particular cases of interest: shot peening and friction stir welding. Shot peening and welding are chosen as the engineering processes of interest because they are representative of deformation processing
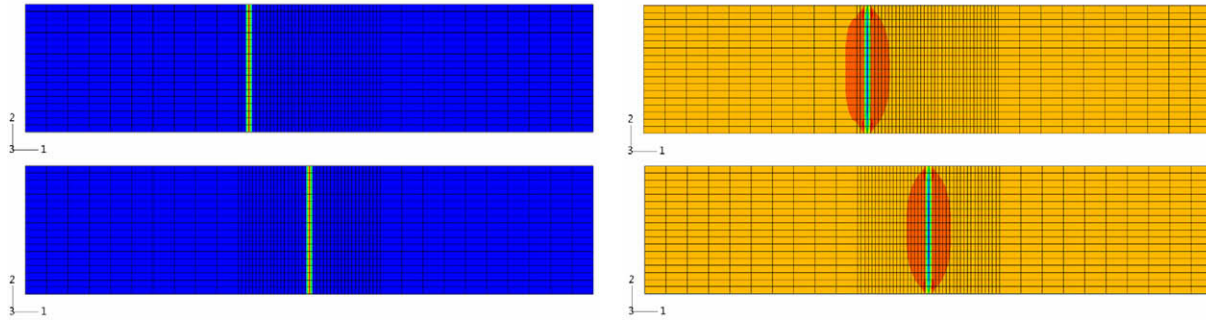
Fig. 3. Contour plots of (left) localised eigenstrain input at different positions and (right) the corresponding residual strain distributions within welded structure.

and thermo-mechanical processing respectively, and are very widely used in industry. Details of these studies can be found in literatures by Jun et al. (in press-b) for the peening and Jun et al. (in press-a) for the welding. Herein, summarisations are given in the context of the ERM.

5.1. Cone-shaped shot-peened sample

For the eigenstrain analysis in the shot-peened slice, the 3D FE model (model code [3-1-1]) was set up that captured the sample geometry. Fig. 4 shows the elastic strain distributions within the 3D FE model. Based on the SIMTRI method, eigenstrains were applied in vicinity of the peened surface in the model. It is seen from the measured and reconstructed elastic strains that the distributions in the middle and bottom regions in the sample show good agreement, whilst slight mismatch could be found in the top region where high strains become more concentrated. Overall, however, it can be concluded that excellent quality match was achieved

between the measured residual elastic strain profiles at three distinct locations within the slice and the variations predicted by introducing a *single* eigenstrain depth profile everywhere within the sample. Remaining mismatch may be explained by local material property variations and some pre-existing residual strains present in the sample prior to shot peening treatment. There is also the possibility that peening small sections of material near the tip causes additional local plasticity that modifies the eigenstrain characterised, to good approximation, by a particular distribution of eigenstrain introduced during treatment.

5.2. Friction stir welding of AA5083/AA6082

For the eigenstrain analysis in the friction stir welds, the 2D plane stress FE model (model code [2-1-1]) was set up reflecting the geometry of the experiment. Fig. 5 shows the reconstructed full-field 2D map of residual elastic strains in the welds. For the validation of the ERM, the middle region of the weld, as indicated
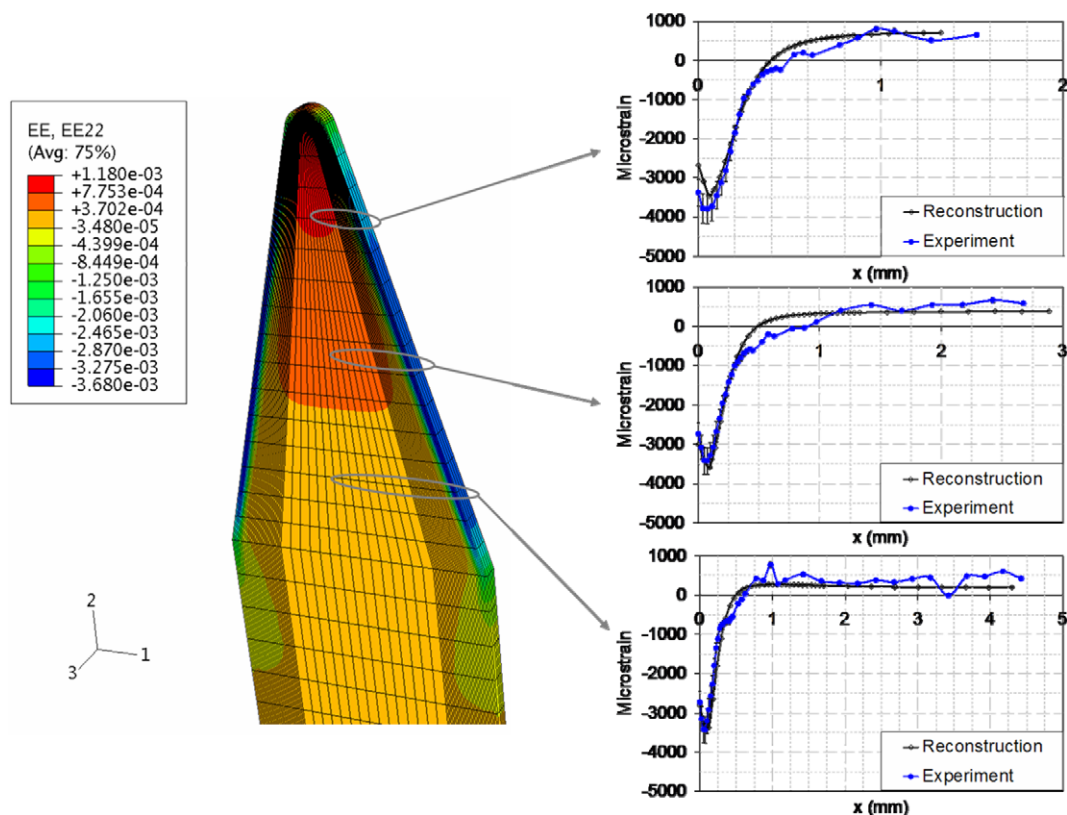

Fig. 4. Contour plot of elastic strain and a comparison with experimental results at 3, 8, and 13 mm from the tip within the shot-peened slice sample.
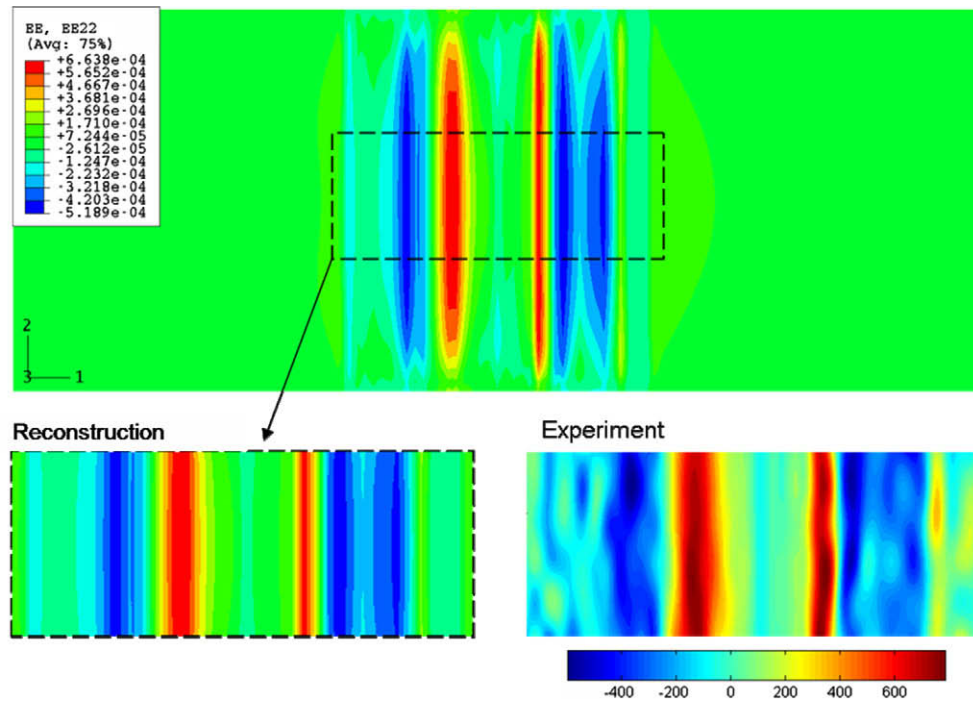
Fig. 5. Contour map of reconstructed residual elastic strains, and comparison with the experimental results.

by dashed rectangle, was compared with the experimental results, which were mapped with six lines on the strain measurement. It is seen that good agreement was found in terms of both strain variation and magnitude.

6. Discussion and conclusions

In this paper, the attention was focused on the methodological developments in the eigenstrain theory. The framework of the Eigenstrain Reconstruction Method was presented in terms of three essential stages: residual strain measurement, formulation of the *inverse problem of eigenstrain theory*, and the SIMTRI method. The latter approach was developed in order to improve the flexibility and the ease of implementation for the determination of the unknown eigenstrain distribution. Unlike the previously reported use of various polynomial functions by other researchers, the use of a series of triangular pulses as the basis functions to represent eigenstrain provides a more stable and generic methodology for matching the residual elastic strain distributions from the experimental data by the use of ERM.

The issue of the solution stability is of considerable importance for inverse analysis, as it is related to the consistency and trustworthiness of the conclusions derived from the analysis. At least two important aspects of solution stability can be identified, namely, the dependence on the order of approximation (the number of eigenstrain basis functions used), and the dependence on the choice of experimental data (e.g. the number of data points used). Both effects have been investigated in our studies. The solution was found to be stable with respect to the number of basis functions. Increasing the "mesh" density (the number of collocation points) lead to an improvement in the level of detail, and saturation in solution quality was achieved when the density of experimental measurements was reached. The choice of experimental data used in the inverse analysis was of particular importance. The solution was found to be stable in that respect, too. However, if the density of measurement points was low, and some key data were removed, the fidelity of the solution was reduced with

respect to the reconstruction based on the complete experimental data set. This behaviour was thought to be satisfactory and as expected.

For the validation of the ERM, the applications of the method were considered to shot peening and friction stir welding. Based on the successful results obtained, the reconstruction of full-field bi-or tri-axial residual strain and stress distributions within the samples now appears possible. By way of independent validation of the method, additional measurements were made and compared with the reconstructed distributions. The very satisfactory agreement obtained showed that the Eigenstrain Reconstruction Method allows good reconstruction of the complete strain/stress state in entire samples and components. The resulting description offers a more convenient description of the residual stress state than the raw input data. It is *continuous* across the body, due to the nature of the finite element modelling approach. The description is also *complete*, in that *full* spatial distributions of *all* components of inelastic and elastic strains (and hence stresses) were reconstructed. Finally, the description is *consistent*, in that all the continuum mechanics requirements of traction-free surface conditions, stress balance and strain compatibility are satisfied.

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