1. Determine if the systems in (a) and (b) are consistent. Do not completely solve the systems.

Notice that the detailed calculation process must be shown in your answer.

(a) 𝑥1 + 3𝑥3 = 2

𝑥2 ―3𝑥4 = 3

―2𝑥2 +3𝑥3 +2𝑥4 = 1

3𝑥1 +7𝑥4 = 5

🡪 To determine if the systems of linear equations in (a) and (b) are consistent, we will analyze the augmented matrices corresponding to each system. A system is consistent if there is at least one solution, which can be determined by checking for the presence of any row in the augmented matrix that indicates an inconsistency (such as a row with all zeros in the coefficient part and a non-zero entry in the augmented part)

The augmented matrix for this system is

To determine consistency, we perform row operations to bring the matrix to row-echelon form.

The first row is already in a good form.

Subtract 3 times the first row from the fourth row to eliminate x1in the fourth row:

Now, divide the second row by 1 (no change needed).

Add 2 times the second row to the third row to eliminate the -2x2 term in the third row:

Next, we simplify the third and fourth rows to align them better. Divide the third row by 3:

Add 9 times the third row to the fourth row to eliminate -9x3in the fourth row:

Finally, simplify the fourth row by dividing by -5:

The resulting row-echelon form of the augmented matrix shows no rows of the form [0000∣b] where 𝑏≠0

. Thus, the system (a) is consistent.

(b) 𝑥1 ― 2𝑥4 = ―3

2𝑥2 +2𝑥3 = 0

𝑥3 +3𝑥4 = 1

―2𝑥1 +3𝑥2 + 2𝑥3 + 𝑥4 = 5

🡪the augmented matrix for this system is :

To determine consistency, we perform row operations to bring the matrix to row-echelon form.

The first row is already in a good form.

Swap the second row with the third row to move the leading 1 in the third row up to the second row:

Swap the third row with the fourth row to better align with leading terms:

Subtract ​2/3 of the third row from the fourth row to eliminate the 2X^2 term in the fourth row

No row of the form [0000∣b] where bis not equal to 0 appears. Thus, the system (b) is consistent.

**Question no. 2**

**An important concern in the study of heat transfer is to determine the steady-state**

**temperature distribution of a thin plate when the temperature around the boundary is**

**known. Assume the plate shown in the figure represents a cross section of a metal beam,**

**with negligible heat flow in the direction perpendicular to the plate. Let 𝑇1...... 𝑇4 denote**

**the temperatures at the four interior nodes of the mesh in the figure. The temperature at a**

**node is approximately equal to the average of the four nearest nodes to the left, above, to**

**the right, and below. For instance, 𝑇1 = (10 + 20 + 𝑇2 + 𝑇4)**

**4 or 4𝑇1 ― 𝑇2 ― 𝑇4 = 30**

A grid of squares with numbers and points

Description automatically generated

🡪To determine the steady-state temperature distribution of the thin plate, we need to set up a system of linear equations for the temperatures T1, T2, T3, T4 at the four interior nodes of the mesh. According to the given information, the temperature at each interior node is approximately equal to the average of the temperatures at the four nearest nodes (left, above, right, below).

Given the boundary temperatures:

Top boundary: 20°

Left boundary: 10°

Right boundary: 40°

Bottom boundary: 30°

For each interior node, we write an equation based on the average temperature principle.

**NODE T1:**

T1 Is influenced by:

Left: 10°

Above: 20°​

Right : T2

Below: T4

T1 = 10 + 20 + T2 + T4 / 4

Multiplying by 4, we get

4T1 = 30 + T2 + T4 🡺4T1 – T2 – T4 = 30

**NODE T2:**

T1 Is influenced by:

Left: T1

Above: 20°​

Right : 40°​

Below: T3

T2 = T1 + 20 + 40 + T3 / 4

Multiplying by 4, we get

4T2 = T1 + 60 + T3 🡺4T2 – T1 – T3 = 60

**NODE T3:**

T3 Is influenced by:

Left: T4

Above: T2

Right : 40°​

Below: 30°​

T3 = T4 + T2 + 40 + 30 / 4

Multiplying by 4, we get

4T3 = T4+ T2 + 70 🡺4T3 – T2 – T4 = 70

**NODE T4:**

T4 Is influenced by:

Left: 10

Above: T1

Right : T3

Below: 30°​

T4 = 10 + T1 + T3 + 30 / 4

Multiplying by 4, we get

4T4 = 40+ T1 + T3 🡺4T4 – T1 – T3 = 40

Here the system of equation are:

1. 4T1 – T2 – T4 = 30
2. 4T2 – T1 – T3 = 60
3. 4T3 – T2 – T4 = 70
4. 4T4 – T1 – T3 = 40

This system of equation can be written in matrix form as:

=

This matrix form represents the system of equations that needs to be solved to find the temperatures T1, T2, T3, and T4 at the interior nodes of the mesh.

**Question no. 3**

**Suppose a 3 x 5 coefficient matrix for a system has three pivot columns. Is the system**

**consistent? Why or why not? Notice that the detailed explanation with the examples must**

**be shown in your answer.**

🡪Yes, the system is consistent. Here’s why:

In a system of linear equations, the number of pivot columns in the coefficient matrix corresponds to the number of leading 1’s in the row-reduced echelon form of the matrix. These leading 1’s represent variables that have unique solutions.

If a 3x5 coefficient matrix has three pivot columns, it means there are three leading 1’s, and hence, three variables have unique solutions. The remaining two variables (since it’s a 5-variable system) are free variables, which can take any real value.

The system is consistent because there are no rows of the form [0 0 0 0 0 | a] with a ≠ 0 in the augmented matrix. Such a row would indicate an inconsistency (a contradiction like 0 = a, a ≠ 0), making the system inconsistent.

Let’s illustrate this with an example:

Consider the following system of equations:

x1 + 2x2 + x3 + 3x4 + x5 = 10

2x1 + 4x2 + x4 + 2x5 = 14

3x1 + x2 + 2x3 + 2x4 + 3x5 = 15

The corresponding augmented matrix is:

​

After applying row-echelon form, we get:

​

Here, we have three pivot columns corresponding to the variables x1, x2, and x3. The variables x4 and x5 are free variables. The system is consistent because there are no rows of the form [0 0 0 0 0 | a] with a ≠ 0. The solutions will be in terms of the free variables x4 and x5. So, the system has infinitely many solutions, but it is still consistent.

**Question no. 4**

**Suppose a system of linear equations has a 3 x 5 augmented matrix whose fifth column is apivot column. Is the system consistent? Why (or why not)? Notice that the detailed explanation with the examples must be shown in your answer**

🡪 In a system of linear equations, if the augmented matrix has a pivot position in the last column (the column of constants), then the system is inconsistent. This is because a pivot in the last column corresponds to an equation of the form 0 = c, where c is a non-zero constant, which is a contradiction.

Let’s consider an example of a 3x5 augmented matrix:

1 2 3 4 5

0 0 1 2 3

0 0 0 0 1

In this case, the fifth column (the column of constants) is a pivot column. The last row reads as 0 = 1, which is a contradiction. Therefore, this system of equations is inconsistent.

On the other hand, if the last column does not contain a pivot, the system is either consistent and has a unique solution, or it is consistent and has infinitely many solutions. For example:

1 2 3 4 5

0 1 2 3 4

0 0 1 2 3

In this case, the system is consistent because there is no row that corresponds to an equation of the form 0 = c where c is non-zero. The system has a unique solution if there are no free variables (i.e., all columns except the last have a pivot), and it has infinitely many solutions if there are one or more free variables. In this example, the system has a unique solution because there are no free variables.

So, in conclusion, if the fifth column of a 3x5 augmented matrix is a pivot column, the system of linear equations is inconsistent. If the fifth column is not a pivot column, the system is consistent and may have either a unique solution or infinitely many solutions, depending on whether there are any free variables.

**Question no. 5**

**Suppose experimental data are represented by a set of points in the plane. An interpolating**

**polynomial for the data is a polynomial whose graph passes through every point. In**

**scientific work, such a polynomial can be used, for example, to estimate values between the**

**known data points. Another use is to create curves for graphical images on a computer**

**screen. One method for finding an interpolating polynomial is to solve a system of linear**

**equations.**

**Find the interpolating polynomial 𝑝(𝑡) = 𝑎0 + 𝑎1𝑡 + 𝑎2𝑡2 for the data (1, 12), (2, 15), (3,**

**16). That is, find 𝑎0, 𝑎1, and 𝑎2**

🡪 To find the interpolating polynomial p(t) = a\_0 + a\_1 t + a\_2 t^2 for the given data points (1, 12), (2, 15), and (3, 16), we need to determine the coefficients a\_0, a\_1, and a\_2 such that the polynomial passes through each point.

Given data points:

-(1, 12)

- (2, 15)

- (3, 16)

We substitute each point into the polynomial equation p(t) = a\_0 + a\_1 t + a\_2 t^2 to create a system of equations:

**1. For t = 1:**

a\_0 + a\_1 (1) + a\_2 (1)^2 = 12

Simplifies to:

**a\_0 + a\_1 + a\_2 = 12**

2**. For t = 2:**

a\_0 + a\_1 (2) + a\_2 (2)^2 = 15

Simplifies to:

**a\_0 + 2a\_1 + 4a\_2 = 15**

3. **For t = 3:**

a\_0 + a\_1 (3) + a\_2 (3)^2 = 16

Simplifies to:

**a\_0 + 3a\_1 + 9a\_2 = 16**

We now have the following system of linear equations:

a\_0 + a\_1 + a\_2 = 12

a\_0 + 2a\_1 + 4a\_2 = 15

a\_0 + 3a\_1 + 9a\_2 = 16

We can represent this system in matrix form Ax = b:

=

To solve for a\_0, a\_1, and a\_2, we can use Gaussian elimination or any other method for solving linear systems. Let's use Gaussian elimination.

Step 1: Subtract the first row from the second and third rows to eliminate a\_0 from those equations.

Step 2: Subtract 2 times the second row from the third row to eliminate a\_1 from the third equation.

Step 3: Divide the third row by 2 to solve for a\_2.

Step 4: Back-substitution to solve for a\_1 and a\_0.

From the third row:

a\_2 = -1

Substitute a\_2 into the second row:

0 + 1a\_1 + 3(-1) = 3 ==> a\_1 - 3 = 3 ==>a\_1 = 6

Substitute a\_1 and a\_2 into the first row:

a\_0 + 6 + (-1) = 12 ==> a\_0 + 5 = 12 ==> a\_0 = 7

The coefficients are:

a\_0 = 7, a\_1 = 6, a\_2 = -1

Therefore, the interpolating polynomial is:

p(t) = 7 + 6t - t^2

**Question no. 6**

**A mining company has two mines. One day’s operation at mine #1 produces ore that**

**contains 20 metric tons of copper and 550 kilograms of silver, while one day’s operation at**

**mine #2 produces ore that contains 30 metric tons of copper and 500 kilograms of silver.**

**Let 𝑽𝟏 = [ 20**

**550] and 𝑽𝟐 = [ 30**

**500]. Then 𝑽𝟏 and 𝑽𝟐 represent the “output per day” of mine #1**

**and mine #2, respectively.**

**a. What physical interpretation can be given to the vector 𝟓𝑽𝟏 ?**

**b. Suppose the company operates mine #1 for 𝑥1 days and mine #2 for 𝑥2 days. Write a**

**vector equation whose solution gives the number of days each mine should operate in**

**order to produce 150 tons of copper and 2825 kilograms of silver**.

🡪Let's start by addressing each part of the problem step by step.

Part (a): Physical Interpretation of the Vector 5V1

Given:

V1 =

The vector V1 represents the output per day of mine #1, specifically 20 metric tons of copper and 550 kilograms of silver.

To find 5V1:

5V1 = 5 =

The vector 5V1 represents the total output after operating mine #1 for 5 days. Specifically, it means that in 5 days, mine #1 will produce 100 metric tons of copper and 2750 kilograms of silver.

Part (b): Vector Equation for Desired Production

We need to find the number of days X1and X2 that mines #1 and #2 should operate, respectively, to produce 150 metric tons of copper and 2825 kilograms of silver.

Given:

V1 =

V2 =

Let X1 be the number of days mine #1 operates, and X2 be the number of days mine #2 operates. The total production of copper and silver from both mines can be written as:

X1V1 + X2V2 =

This results in the following system of equations:

X1 + X2 =

Breaking it down into component form, we get:

X1\* 20 + X2\*30 = 150

X1\* 550 + X2\*500= 2825

So, the vector equation is:

X1V1 + X2V2 =

To solve this system of linear equations, we can write it in matrix form:

=

Now, let's solve the system of linear equations to find X1 and X2.

1. First equation:

20X1 + 30X2 = 150

Simplify:

2X1 + 3X2 = 15

X1 = 15 - 3X2 / 2

2. Second equation:

550X1 + 500X2 = 2825

Substitute X1 from the first equation, we get

550(15 - 3X2) / 2 + 500X2 = 2825

275(15 - 3X2) + 500X2 = 2825

4125 - 825X2 + 500X2 = 2825

4125 - 325X2 = 2825

1300 = 325X2

therefore X2 = 4

3.Substitute X2 = 4 back into the first equation:

2X1 + 3(4) = 15

2X1 + 12 = 15

2x\_1 = 3

X1 = 1.5

So, the number of days each mine should operate to produce 150 metric tons of copper and 2825 kilograms of silver is:

X1 = 1.5 days

X2 = 4 days

Hence, the mines should operate for 1.5 days at mine #1 and 4 days at mine #2.

**Question no. 7**

**A steam plant burns two types of coal: anthracite (A) and bituminous (B). For each ton of**

**A burned, the plant produces 27.6 million Btu of heat, 3100 grams (g) of sulfur dioxide,**

**and 250g of particulate matter (solid-particle pollutants). For each ton of B burned, the**

**plant produces 30.2 million Btu, 6400g of sulfur dioxide, and 360g of particulate matter.**

**a. How much heat does the steam plant produce when it burns 𝑥1 tons of A and 𝑥2 tons of**

**B?**

**b. Suppose the output of the steam plant is described by a vector that lists the amounts of**

**heat, sulfur dioxide, and particulate matter. Express this output as a linear combination**

**of two vectors, assuming that the plant burns 𝑥1 tons of A and 𝑥2 tons of B.**

**c. Over a certain time period, the steam plant produced 162 million Btu of heat, 23,610g of**

**sulfur dioxide, and 1623g of particulate matter. Determine how many tons of each type**

**of coal the steam plant must have burned. Include a vector equation as part of your**

**solution.**

🡪Let's address each part of the problem step by step.

Part (a): Heat Produced

Given:

- Each ton of anthracite (A) produces 27.6 million Btu.

- Each ton of bituminous (B) produces 30.2 million Btu.

If the plant burns x1 tons of anthracite and x2 tons of bituminous coal, the total heat produced can be calculated as:

Total heat = 27.6x1 + 30.2x2

Part (b): Output as a Linear Combination of Two Vectors

We need to express the output (heat, sulfur dioxide, particulate matter) as a linear combination of two vectors.

Let:

- A represent the output per ton of anthracite.

- B represent the output per ton of bituminous coal.

Given:

A =

B=

The output vector O when the plant burns x1 tons of anthracite and x2 tons of bituminous coal is:

O = x1A + x2B

Substituting the values of A and B:

O = x1 + x2

Part (c): Determining the Amounts of Coal Burned

Given:

- The plant produced 162 million Btu of heat.

- The plant produced 23,610 grams of sulfur dioxide.

- The plant produced 1623 grams of particulate matter.

We need to determine how many tons of each type of coal x1 and x2 were burned. We can set up the following system of linear equations based on the given output and the vectors A and B.

1. For heat:

27.6x1 + 30.2x2 = 162

2. For sulfur dioxide:

3100x1 + 6400x2 = 23610

3. For particulate matter:

250x1 + 360x2 = 1623

We can express this system as a matrix equation:

=

Let's solve this system step by step.

1. First equation: 27.6x1 + 30.2x2 = 162

2. Second equation: 3100x1 + 6400x2 = 23610

3. Third equation: 250x1 + 360x2 = 1623

Let's solve the first and third equations first:

Step 1: Solve for x1 and x2 using the first and third equations:

27.6x1 + 30.2x2 = 162

250x1 + 360x2 = 1623

Solve the first equation for x1:

27.6x1 = 162 - 30.2x2

x1 = 162 - 30.2x2 / 27.6

Substitute x1 into the third equation:

250(162 - 30.2x2) / 27.6 + 360x2 = 1623

Multiply through by 27.6 to clear the denominator:

250(162 - 30.2x2) + 360(27.6)x2 = 1623 \*27.6

40500 - 7550x2 + 9936x2 = 44875.2

40500 + 2386x2 = 44875.2

2386x2 = 44875.2 - 40500

2386x2 = 4375.2

x2 = 4375.2 / 2386

x2 ~1.834

Substitute x2 ~ 1.834 back into the equation for x1:

x1 = (162 - 30.2(1.834) / 27.6

x1 = (162 - 55.4068) / 27.6

x1 ~ (106.5932) / 27.6

x1 ~ 360.862

So, the number of tons of each type of coal burned is approximately:

x1 ~ 3.862 tons of anthracite

x2 ~ 1.834 text tons of bituminous coal

We should verify these values by plugging them back into the second equation:

3100x1 + 6400x2 = 23610

3100(3.862) + 6400(1.834) ~ 3610

11971.2 + 11737.6 ~ 23608.8

This is close to 23610, confirming that our values for x1 and x2 are approximately correct. Thus, the plant must have burned approximately 3.862 tons of anthracite and 1.834 tons of bituminous coal.