BITS Pilani K K Birla Goa Campus
Department of Computer Science & Information Systems
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Lecture No: 12

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0.1 Introduction

There are many possible definitions of chaos. In fact, there is no general agreement within the scientific community as to what constitutes a chaotic dynamical system. However, this will not deter us from offering one possible definition. This definition has the advantage that it may be readily verified in a number of different and important examples. However, you should be forewarned that there are many other possible ways to capture the essence of chaos. To describe chaos, we need one preliminary notion from topology, that of a dense set and to prove if system is chaotic, we have three satisfiability theorems:

1. Suppose X is a set and Y is a subset of X. We say that Y is dense in X if, for any point x ∈ X, there is a point y in the subset Y arbitrarily close to x.
2. A dynamical system is transitive if for any pair of points x and y and any $\epsilon > 0$ there is a third point z within ϵ of x whose orbit comes within ϵ of y.
3. A dynamical system F *depends sensitively on initial conditions* if there is a $\beta > 0$ such that, for any x and any $\epsilon > 0$, there is a y within ϵ of x and a k such that the distance between $F^k(x)$ and $F^k(y)$ is at least β.

Fixed Point Theorem: *Suppose $F: [a, b] \rightarrow [a, b]$ is continuous. Then there is a fixed point for $Fin[a, b]$*. This theorem asserts the existence of at least one fixed point for $Fin[a, b]$; there may, of course, be more. There are several important hypotheses in this theorem, the first two being continuity and the fact that F takes the interval $[a, b]$ into itself. Violation of either of these may yield a function without fixed points.

Suppose x_0 is a fixed point for F. Then x_0 is an attracting fixed point if $|F'(x_0)| < 1$. The point x_0 is a repelling fixed-point if $|F'(x_0)| > 1$. Finally, if $|F'(x_0)| = 1$, the fixed point is called neutral or indifferent. Attracting Fixed Point Theorem states that Suppose x_0 is an attracting fixed point for F. Then there is an interval I that contains x_0 in its interior and in which the following condition is satisfied: if $x \in I$, then $F^n(x) \in I$ for all n and, moreover, $F^n(x) \rightarrow x_0$ as $n \rightarrow \infty$. Repelling Fixed Point Theorem states that Suppose x_0 is a repelling fixed point for F. Then there is an interval I that contains x_0 in its interior and in which the following condition is satisfied: if $x \in I$ and $x \neq x_0$, then there is an integer $n > 0$ such that $F^n(x) \notin I$. These two theorems combined justify our use of the terminology "attracting" and "repelling" to describe the corresponding fixed points. In particular, they tell us the "local" dynamics near any fixed point x_0 for which $|F'(x_0)| \neq 1$.

These key concepts are basic foundation to understand the topic proposed in this lecture scribe.

0.2 The Logistic Map

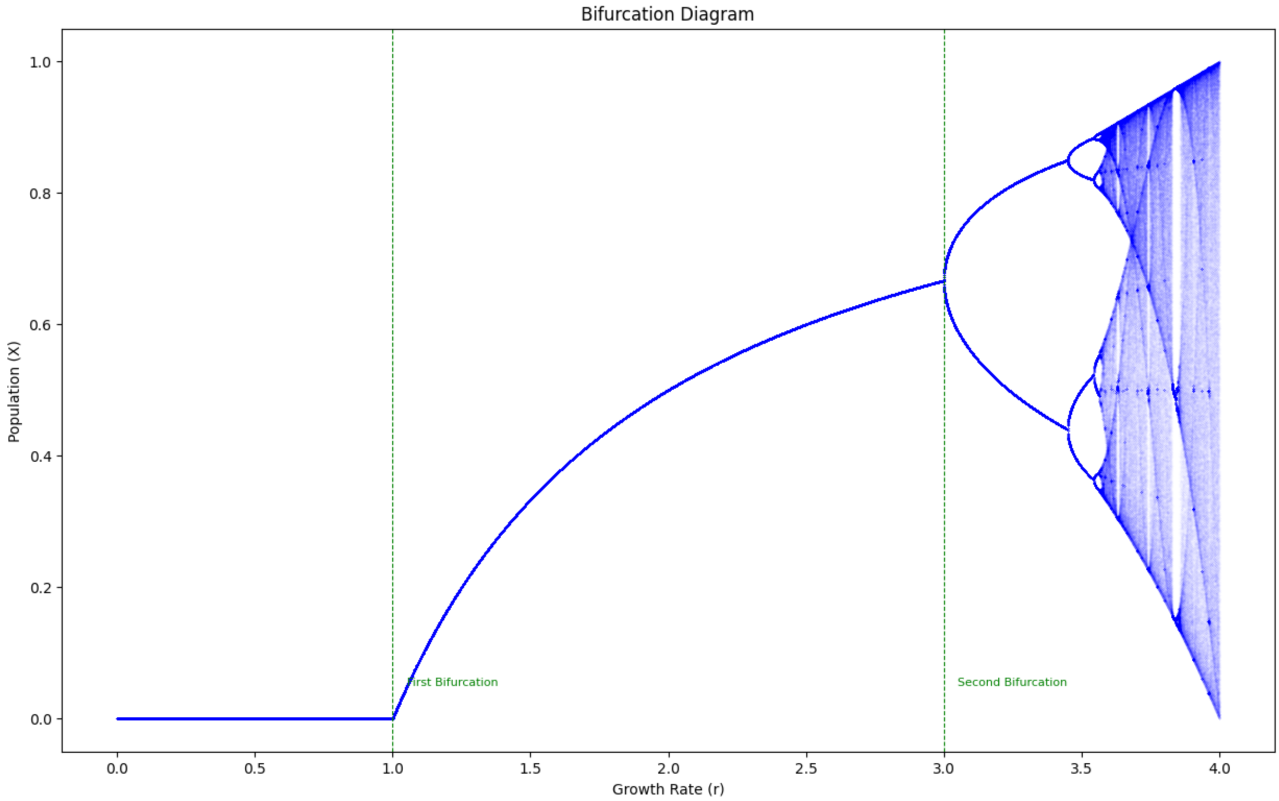
This model is based on the common s-curve logistic function that shows how a population grows slowly, then rapidly, before tapering off as it reaches it's carrying capacity. The logistic function uses a differential equation that treats time as continuous. The logistic map instead uses a nonlinear difference equation to look at discrete time steps. It's called the logistic map because it maps the population value at any time step to its value at the next time step:

$$X_n = r\,X_{n-1}(1 - X_{n-1})$$

This equation defines the rules, or dynamics, of our system: '*x*' represents the population at any given time '*n*', and 'r' represents the growth rate. In other words, the population level at any given time is a function of the growth rate parameter and the previous time step's population level. If the growth rate is set too low, the population will die out and go extinct. Higher growth rates might settle toward a stable value or fluctuate across a series of population booms and busts. As simple as this equation is, it produces chaos at certain growth rate parameters. Below we model this equation using python and matplotlib as follows:

```
def logistic_map(r, x):
    return r * x * (1 - x)

def logistic_bifurcation_diagram(steps_per_parameter, iterations_per_step,
r_steps,transient,r_start,r_end):
    r_values = np.linspace(r_start, r_end, r_steps)
    x = np.random.random(r_steps)

    plt.figure(figsize=(15, 9))

    for _ in range(steps_per_parameter):
        if _ >= transient:
            x = logistic_map(r_values, x)

    for _ in range(iterations_per_step):
        x = logistic_map(r_values, x)
        plt.scatter(r_values, x, s=0.0001, c='b', alpha=0.5)

    plt.xlabel('Growth Rate (r)')
    plt.ylabel('Population (X)')
    plt.title('Bifurcation Diagram')

    # Annotating important features
    # First bifurcation point
    plt.axvline(1, color='green', linestyle='--', linewidth=0.8)
    plt.text(1.05, 0.05, 'First Bifurcation', color='green', fontsize=8)
    # Second bifurcation point
    plt.axvline(3, color='green', linestyle='--', linewidth=0.8)
    plt.text(3.05, 0.05, 'Second Bifurcation', color='green', fontsize=8)
    plt.show()

transient=500 #to remove first 500 transient values
logistic_bifurcation_diagram(10000,500,10000,transient,0,4)
```

Output:



We are executing for 500 generations across 10,000 growth rates points between 0.0 to 4.0. Using something called a bifurcation diagram we will be visualizing this with various steps below:

Here we are removing the first 500 values as transient values, as in many dynamical systems, especially those exhibiting chaotic behaviour, there often exists a period of transient behaviour before the system settles into its long-term behaviour. These initial transient values typically arise due to the system's initial conditions and may not represent the true long-term behaviour of the system. For example, when exploring the logistic map's behaviour across different parameter values (e.g., values of 'r'), the system may initially exhibit transient behaviour before converging to a stable periodic orbit or entering chaotic behaviour.

To understand the behaviour of logistic map with respect to increasing growth rate we will be dividing it into the subsection of growth rate(r) as $0 \le r < 1$, $1 \le r < 3$, $3 \le r \le 3.4$, $3.44949 < r < 3.54409$, $r > 3.54409$. First, we need to plot trajectories and understand the periodic behaviour. The python code to get and plot the trajectory for any r value and starting from any x value, for below example I took x_0 as the 0.1 and iterating for insight able values.

Python functional code to plot trajectories:

```
def logistic_map_for_trajectory(r, x, generations):
    trajectory = [x]
    for _ in range(generations):
        x = r * x * (1 - x)
        trajectory.append(x)
    plt.figure(figsize=(15, 9))
    plt.plot(range(generations + 1), trajectory, marker='o', linestyle='-')
    plt.xlabel('Generation')
    plt.ylabel('Root/Population (X)')
    plt.title('Logistic Map Trajectory (r='+str(r)+',x0='+str(x0)+')')
    plt.grid(True)
    plt.show()
trajectory = logistic_map_for_trajectory(2.1, 0.1, 40)
```

using this function we are going to plot the trajectories for various growth rates.

0.3 Concept of period doubling

In dynamical systems, periodic orbits represent states where the system's behaviour repeats itself after a certain number of iterations. Depending on the parameters of the system, these periodic orbits can be stable or unstable. Stable orbits attract nearby trajectories, while unstable orbits repel nearby trajectories.

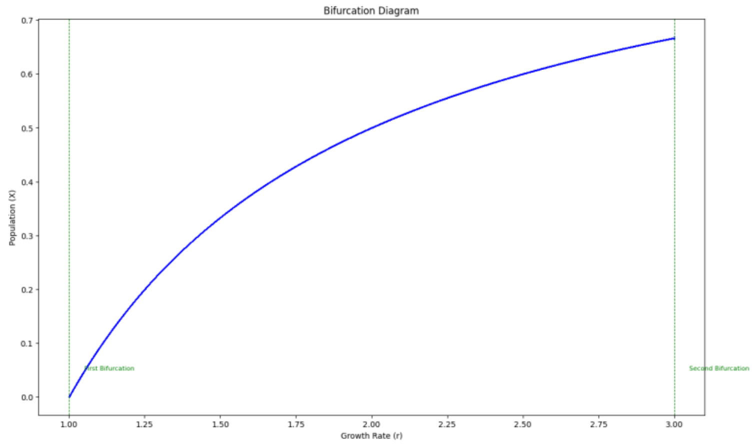
Period-doubling bifurcation occurs when the stability of a periodic orbit changes as a control parameter is varied, leading to the doubling of the period of the orbit. Initially, the system may exhibit a stable periodic orbit with a certain period (e.g., period-1 orbit). As the parameter is varied, the stability of this orbit may change, leading to the creation of a new stable periodic orbit with twice the period (e.g., period-2 orbit)

**Note:* *we scaled [0,4] to [0,10000] so, accordingly represented in the dataset. Column names are those growth rate scaled values and rows are generations.* [*For dataset and codes click here*]().

Period 1:

For the given equation at $r < 3$ value, can find points where $x_{n+1} = x_n$ (meaning points of period one):

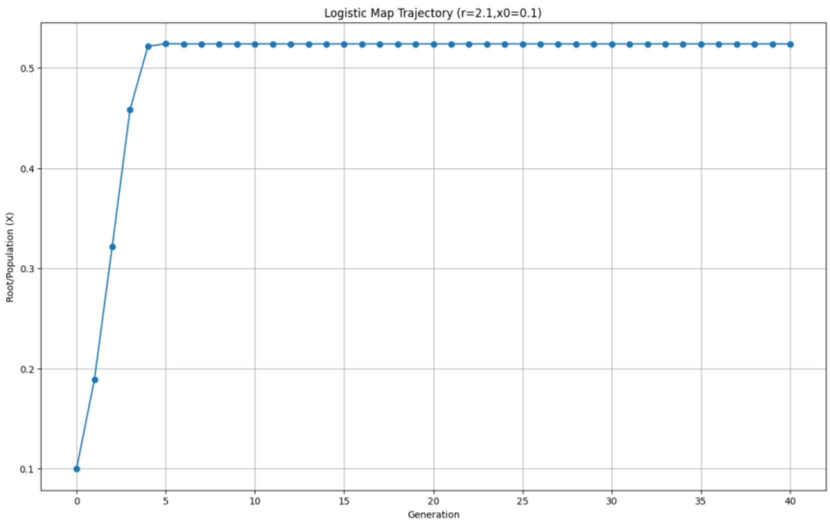
$$X_n = r x_n (1 - \mathrm{x_n})$$
$$(1 - x_n)0 = x_n\,(2.1 - 3.1 x_n)$$
$$X_n = 0, x_n \approx 0.677$$



	KBH	KBI	KBJ	KBK	KBL
1	7495	7496	7497	7498	7499
2	0.666478	0.666522	0.666568	0.666574	0.665341
3	0.666478	0.666522	0.666565	0.666648	0.667965
4	0.666478	0.666522	0.666568	0.666574	0.665341
5	0.666478	0.666522	0.666565	0.666648	0.667965
6	0.666478	0.666522	0.666568	0.666574	0.665341
7	0.666478	0.666522	0.666565	0.666648	0.667964
8	0.666478	0.666522	0.666568	0.666574	0.665342
9	0.666478	0.666522	0.666565	0.666648	0.667964
10	0.666478	0.666522	0.666568	0.666574	0.665342
11	0.666478	0.666522	0.666565	0.666648	0.667964
12	0.666478	0.666522	0.666568	0.666574	0.665342
13	0.666478	0.666522	0.666565	0.666648	0.667963
14	0.666478	0.666522	0.666568	0.666574	0.665343
15	0.666478	0.666522	0.666565	0.666648	0.667963

Ending of Period 1 and Starting of Period 2

Here only one root except $x = 0$ in period one is present, so root is repeated for entire r value generations which result in stable single point (see trajectory diagram below to analyse variations in generations) and at the end of period one we can we can see after column 7497 ($r = 2.9999 \approx 3$) represents two values are repeating, its ending of period 1 and starting of period 2.



	GSX	GSY
1	5249	5250
2	0.523766	0.523857
3	0.523766	0.523857
4	0.523766	0.523857
5	0.523766	0.523857
6	0.523766	0.523857
7	0.523766	0.523857
8	0.523766	0.523857
9	0.523766	0.523857
10	0.523766	0.523857
11	0.523766	0.523857
12	0.523766	0.523857
13	0.523766	0.523857
14	0.523766	0.523857
15	0.523766	0.523857

Trajectory values for r= 2.1

Period 2:

Points of period 2 to be found as follows:

$$f(f(x)) = f^2(x) = x$$

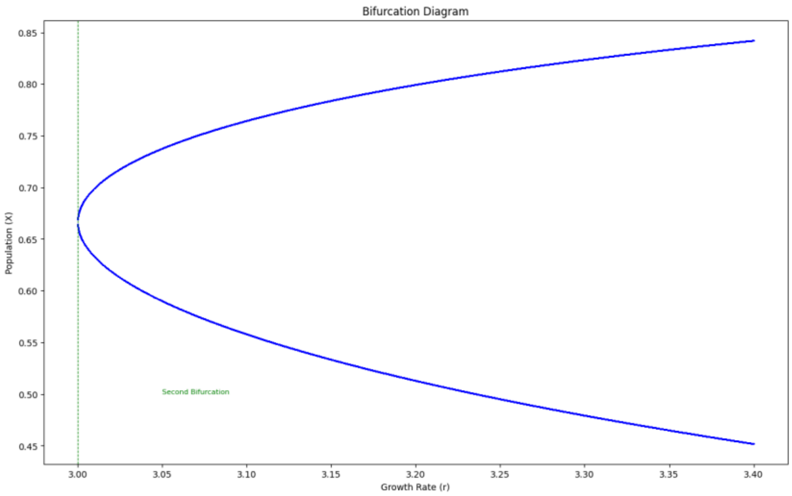
$$x = r^2 x(1-x)\big(1 - rx(1-x)\big)$$

$$0 = x(-r^3x^3 + 2r^3x^2 - (r^3 + r^2)x + r^2 - 1)$$

When $r = 3.1$ has a root at $x = 0$. The other roots may be found using the rather complicated cubic equation, and are $x \approx 0.7645, x \approx 0.5580, x \approx 0.6774$. Note that the two unstable period-1 points are included (see below), but that there are two new points. To see if they are stable, we can check if $|(f^2)'| < 1$, and indeed as

$$(f^2)'(x) = -4r^3x^3 + 6r^3x^2 - 2(r^3 + r^2)x + r^2$$

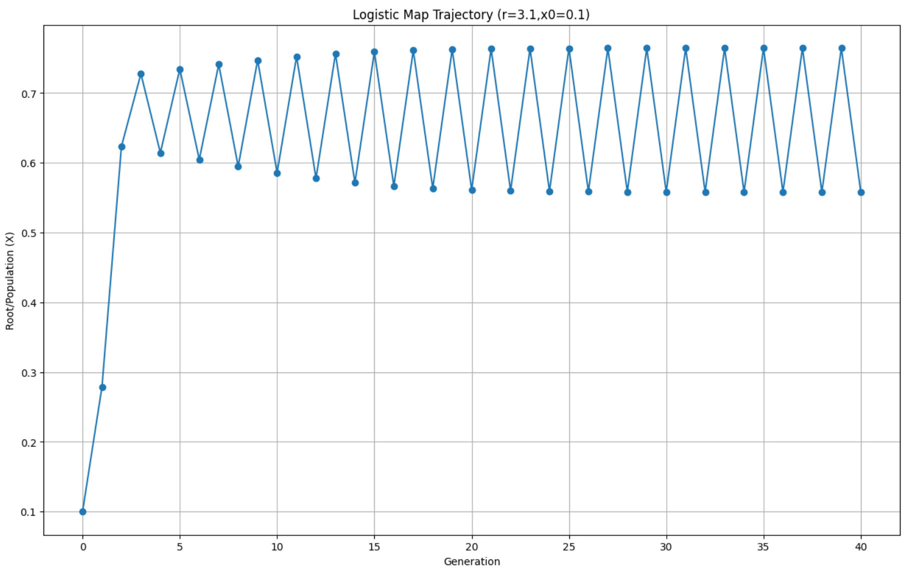
Substituting both $x = 0.5580$ and $x = 0.7645$ yields a number around $|0.589 \ldots| < 1$, meaning that both points are stable. Therefore, both are attractors, as seen in the previous numerical map. On the other hand, $(f^2)'(0.6774) = 1.21$ and $(f^2)'(0) = 9.61$, neither of which are between positive and negative one and thus both points are unstable. The point $x \approx 0.6774$ would be stable if the period had not increased: this is a period-doubling bifurcation. We can verify periodic reptations in dataset.



	KLB	KLC	KLD	KLE	KLF	KLG
1	7749	7750	7751	7752	7753	7754
2	0.558063	0.764704	0.557626	0.765057	0.765233	0.556973
3	0.764527	0.557844	0.764881	0.557408	0.55719	0.765409
4	0.558063	0.764704	0.557626	0.765057	0.765233	0.556973
5	0.764527	0.557844	0.764881	0.557408	0.55719	0.765409
6	0.558063	0.764704	0.557626	0.765057	0.765233	0.556973
7	0.764527	0.557844	0.764881	0.557408	0.55719	0.765409
8	0.558063	0.764704	0.557626	0.765057	0.765233	0.556973
9	0.764527	0.557844	0.764881	0.557408	0.55719	0.765409
10	0.558063	0.764704	0.557626	0.765057	0.765233	0.556973
11	0.764527	0.557844	0.764881	0.557408	0.55719	0.765409
12	0.558063	0.764704	0.557626	0.765057	0.765233	0.556973
13	0.764527	0.557844	0.764881	0.557408	0.55719	0.765409
14	0.558063	0.764704	0.557626	0.765057	0.765233	0.556973
15	0.764527	0.557844	0.764881	0.557408	0.55719	0.765409

Period 2 data at 3.1

We can verify the given period doubling with trajectory of particular r value and can visualize the repeating pattern as below:



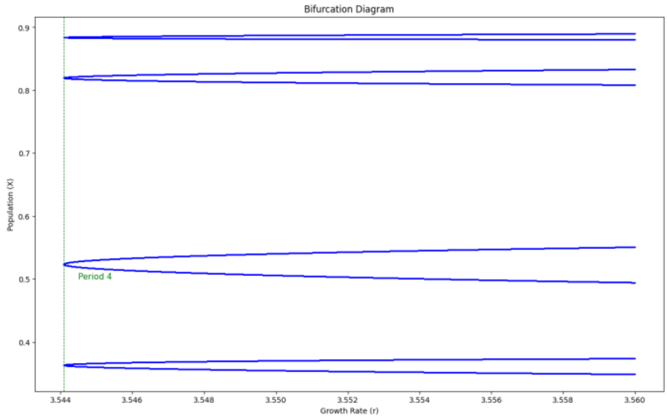
	KLB	KLC
1	7749	7750
2	0.558063	0.764704
3	0.764527	0.557844
4	0.558063	0.764704
5	0.764527	0.557844
6	0.558063	0.764704
7	0.764527	0.557844
8	0.558063	0.764704
9	0.764527	0.557844
10	0.558063	0.764704
11	0.764527	0.557844
12	0.558063	0.764704
13	0.764527	0.557844
14	0.558063	0.764704
15	0.764527	0.557844

Trajectory values for r = 3.1

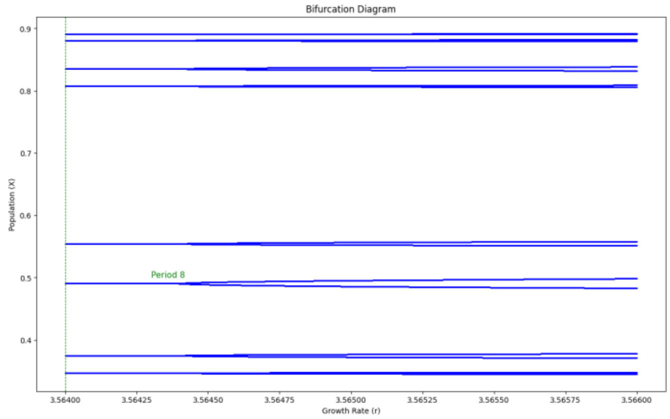
Here we can see trajectory of the value after second bifurcation point have two repeating values verifies the period two.

Period $4, 8, 16 \ldots$:

As r increases, succeeding periods 4,8,16,32,64 ... exist:



Period 4

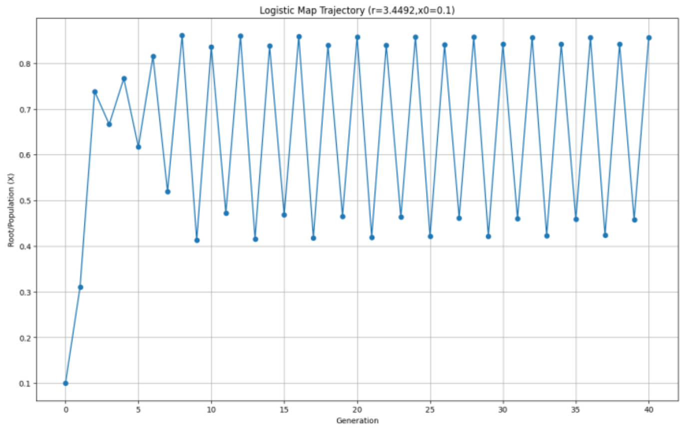


Period 8

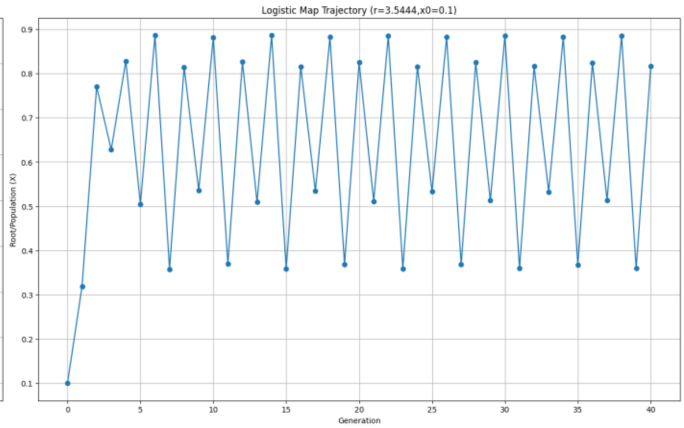
Given any natural number that is the power of two, there is some r value range for which the logistic map trajectory has that number as a periodic cycle. For period four values our equation to be differentiate till fourth order as follows:

$$F\left(F\left(f\left(f(x)\right)\right)\right) = f^4(x) = x$$

and for period eight we need to differentiate the equation till eighth order and similarly for succeeding periods. Trajectories for fourth and eighth period are visualised below:



Period 4



Period 8

	LSQ	LSR
1	8622	8623
2	0.849886	0.442002
3	0.440042	0.850783
4	0.849887	0.437925
5	0.44004	0.849094
6	0.849886	0.442002
7	0.440042	0.850783
8	0.849887	0.437925
9	0.44004	0.849094
10	0.849886	0.442002
11	0.440042	0.850783
12	0.849887	0.437925
13	0.44004	0.849094
14	0.849886	0.442002
15	0.440042	0.850783

Period 4

	MBU	MBV
1	8860	8861
2	0.883462	0.885069
3	0.364917	0.36058
4	0.821413	0.817286
5	0.519934	0.529336
6	0.88468	0.883138
7	0.361599	0.365838
8	0.818197	0.822385
9	0.527225	0.517776
10	0.883462	0.885069
11	0.364917	0.36058
12	0.821413	0.817286
13	0.519934	0.529336
14	0.88468	0.883138
15	0.361599	0.365838

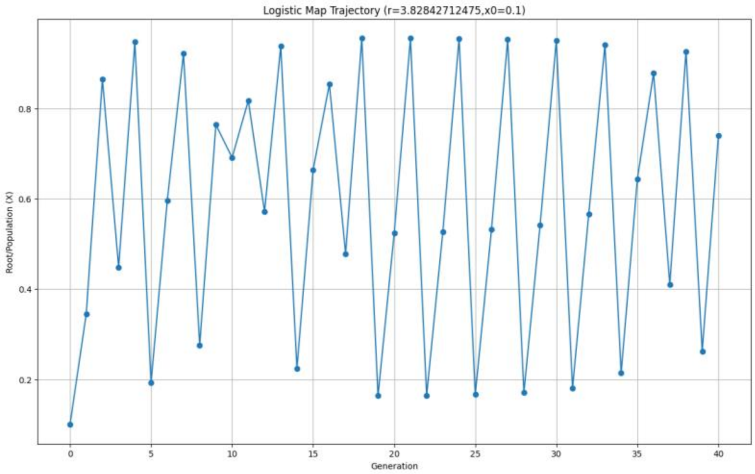
Period 8

Period 4 starts around the $r = 3.449 *$ and period 8 starts around $r = 3.544 *$, also we can observe the period doubling distance between two r values is reducing by $\mathbf{log}(\boldsymbol{n})$.

What about period 3:

However, between successive period-doubling bifurcations, there often exist regions where the system exhibits periodic behaviour with intermediate periods. For example, there may be a region where the system settles into a stable periodic orbit with a period of 3 (period-3 behaviour) before undergoing another period-doubling bifurcation.

When the $r = 3.82842712475$ we notice the instance of three repeating values as shown below trajectory and in dataset.

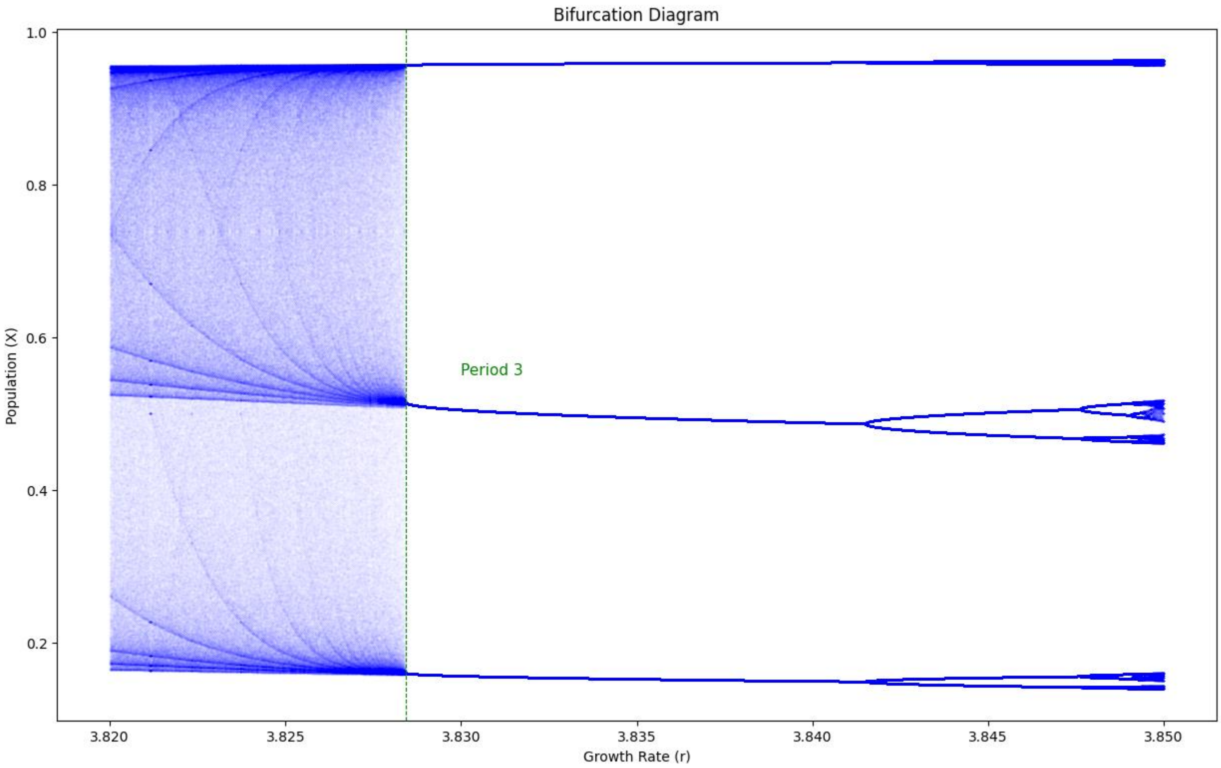


Period 3

	NDD	NDE	NDF	NDG	NDH	NDI
1	9571	9572	9573	9574	9575	9576
2	0.158148	0.507643	0.506052	0.95741	0.957548	0.502493
3	0.509752	0.957072	0.957255	0.15617	0.155706	0.957672
4	0.956832	0.157323	0.156697	0.504719	0.503549	0.155286
5	0.158148	0.507643	0.506052	0.95741	0.957548	0.502493
6	0.509752	0.957072	0.957255	0.15617	0.155706	0.957672
7	0.956832	0.157323	0.156697	0.504719	0.503549	0.155286
8	0.158148	0.507643	0.506052	0.95741	0.957548	0.502493
9	0.509752	0.957072	0.957255	0.15617	0.155706	0.957672
10	0.956832	0.157323	0.156697	0.504719	0.503549	0.155286
11	0.158148	0.507643	0.506052	0.95741	0.957548	0.502493
12	0.509752	0.957072	0.957255	0.15617	0.155706	0.957672
13	0.956832	0.157323	0.156697	0.504719	0.503549	0.155286
14	0.158148	0.507643	0.506052	0.95741	0.957548	0.502493
15	0.509752	0.957072	0.957255	0.15617	0.155706	0.957672

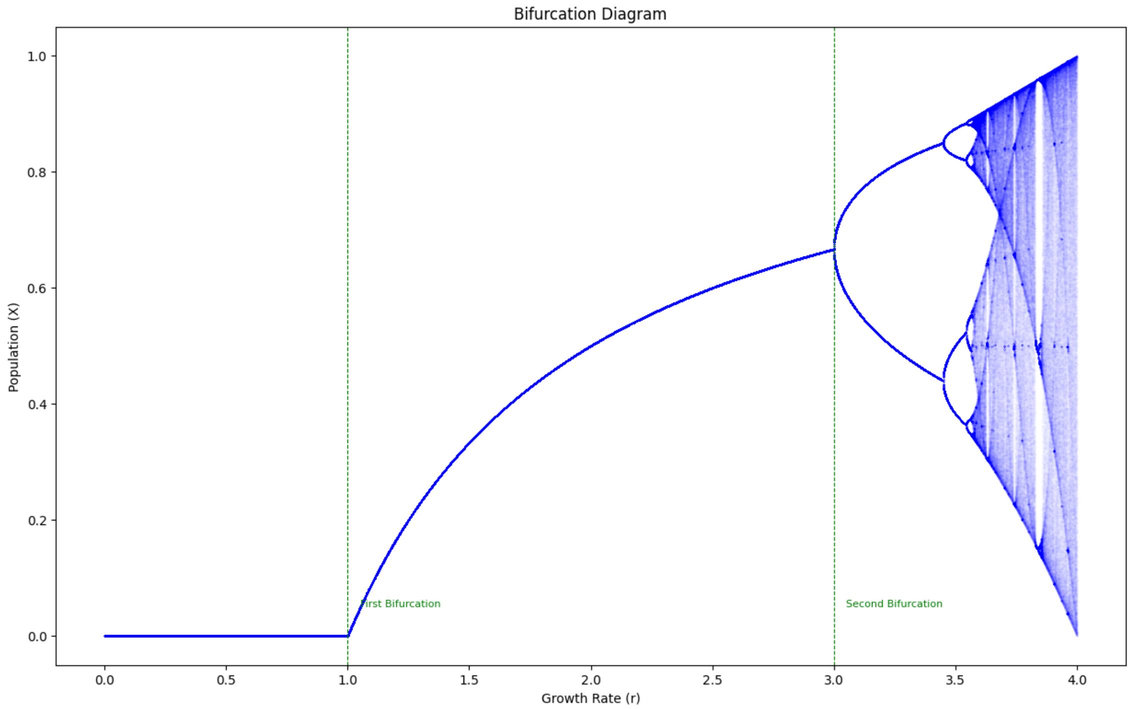
Period 3 data at r=3.82842712475

Period three starts around the $r = 3.828*$ and we can see in column 9572 and bifurcation diagram can be visualised as below:



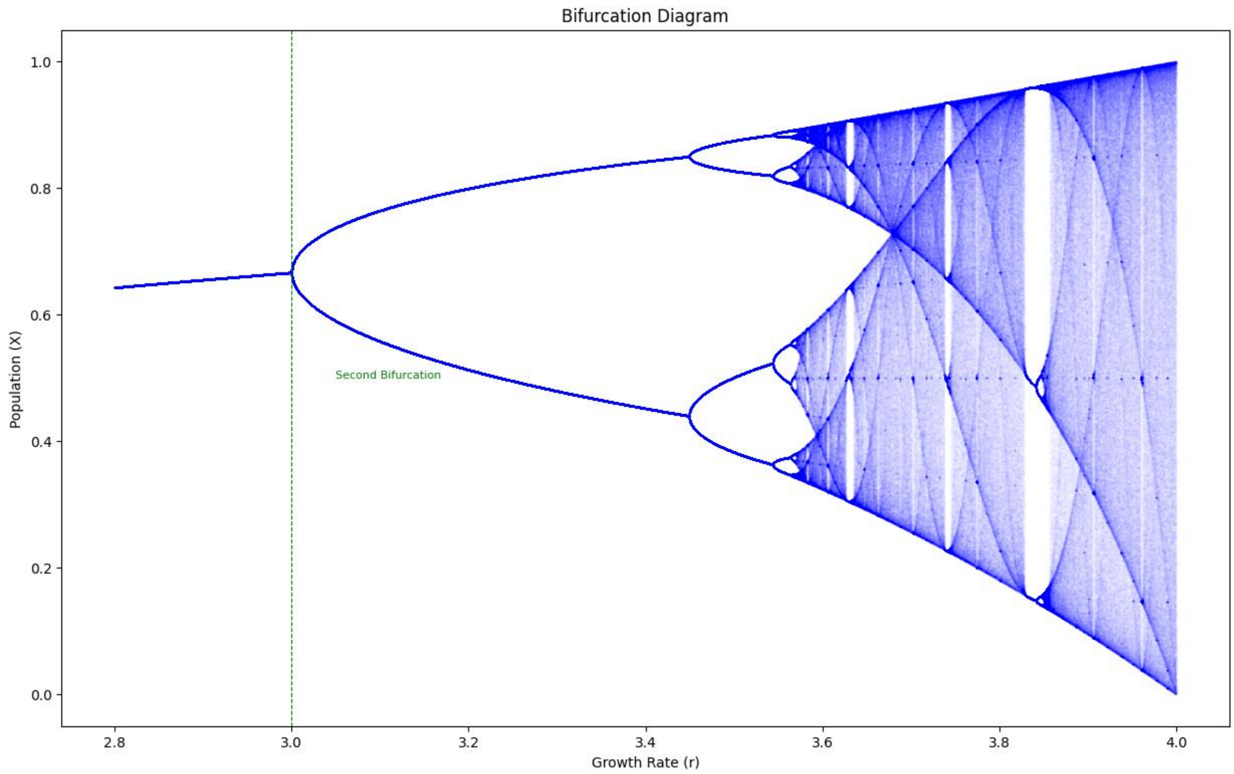
0.4 Understanding Bifurcation diagram

When r is small (less than one, to be specific), the population heads towards 0.



For growth rates less than 1.0, the system always collapses to zero (extinction). For growth rates between 1.0 and 3.0, the system always settles into an exact, stable population level. Look at the vertical slice above growth rate 2.5. There's only one population value represented (0.6) and it corresponds to where the magenta line settles in the line chart shown earlier.

So, why is this called a bifurcation diagram? Let's zoom into the growth rates between 2.8 and 4.0 to see what's happening:



At the vertical slice above growth rate 3.0, the possible population values fork into two discrete paths. At growth rate 3.2, the system essentially oscillates exclusively between two

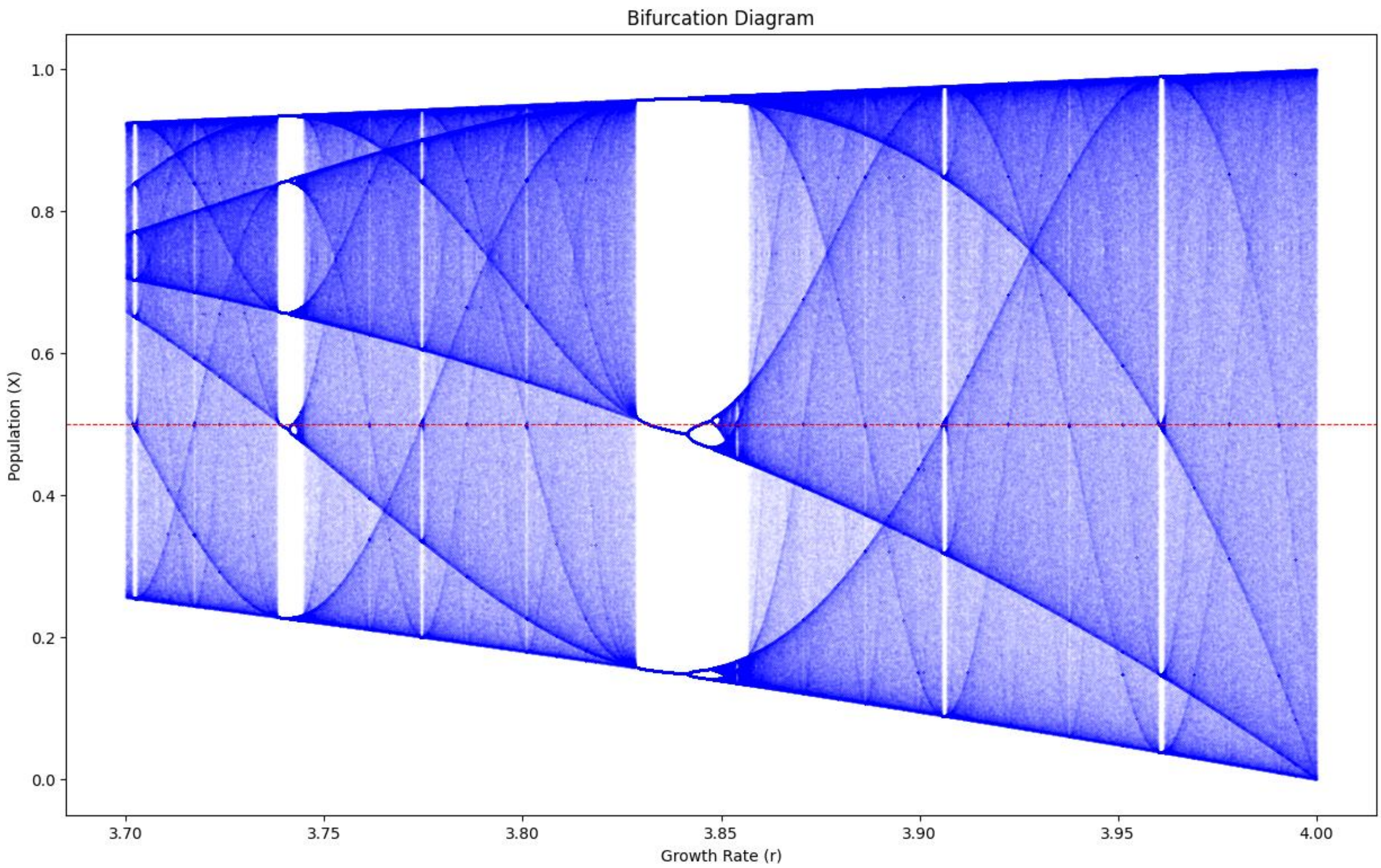
population values: one around 0.5 and the other around 0.8. In other words, at that growth rate, applying the logistic equation to one of these values yields the other.
Just after growth rate 3.4, the diagram bifurcates again into four paths. when the growth rate parameter is set to 3.5, the system oscillates over four population values. Just after growth rate 3.5, it bifurcates again into eight paths. Here, the system oscillates over eight population values. Demonstrates the phenomenon of period doubling.

0.5 The Onset of Chaos

Chaos starts at $r = 3.55995$ value called **Onset of Chaos**. Beyond a growth rate of 3.6, however, the bifurcations ramp up until the system is capable of eventually landing on any population value. This is known as the period-doubling path to chaos. As you adjust the growth rate parameter upwards, the logistic map will oscillate between two then four then eight then 16 then 32 (and on and on) population values. These are periods, just like the period of a pendulum.

As we approach a growth rate of 3.9, the system undergoes numerous bifurcations, resulting in erratic jumps across all possible population values. Although these jumps may appear random, they are not truly random; instead, they adhere to straightforward deterministic rules. Despite its deterministic nature, the model exhibits apparent randomness, demonstrating characteristics of chaos: deterministic yet lacking a predictable pattern.

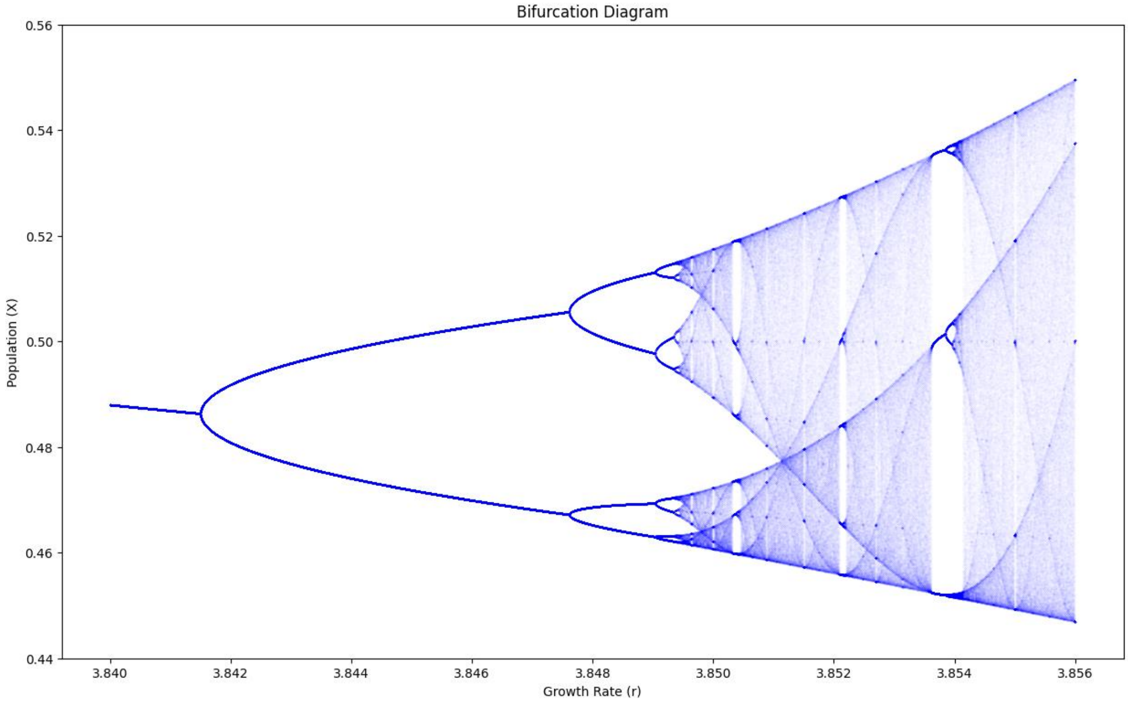
let's focus on the narrow range of growth rates spanning from 3.7 to 3.9:



As we zoom in, we begin to see the beauty of chaos. Out of the noise emerge strange swirling patterns and thresholds on either side of which the system behaves very differently. Between the growth rate parameters of 3.82 and 3.84, the system moves from chaos back into order, oscillating between just three population values (approximately 0.15, 0.55, and 0.95). But then it bifurcates again and returns to chaos at growth rates beyond 3.86.

0.6 Fractals and Strange Attractors

In the plot above, the bifurcations around growth rate 3.85 look a bit familiar. Let's zoom in to the center one:



We see the exact same structure that we saw earlier at the macro-level. In fact, if we keep zooming infinitely in to this plot, we'll keep seeing the same structure and patterns at finer and finer scales, forever. chaotic systems have strange attractors and that their structure can be characterized as fractal. Fractals are self-similar, meaning that they have the same structure at every scale. As you zoom in on them, you find smaller copies of the larger macro-structure. Here, at this fine scale, you can see a tiny reiteration of the same bifurcations, chaos, and limit cycles we saw in the first bifurcation diagram of the full range of growth rates.

0.7 Stability Analysis of Fixed Points

1. For $0 \leq a < 1$

Consider, $a = 0.99 \quad | \quad x_n \to 0$

and $a = 0.35 \quad | \quad x_n \to 0$ *as* $n \to \infty$

Fixed point for $x_n = ax_{n-1}(1 - x_{n-1})$

Are $x = 0$ and $x = \frac{a-1}{a}$

$$x_n = ax_{n-1}(1 - x_{n-1})$$

$$f(x) = ax(1 - x) \text{ where } a = 0.99$$

for $x = 0$ its $f(x) = 0$ and,

for $x = \frac{a-1}{a}$

$$| f'(x) | = | a - 2ax |$$

Put $x = 0$ then,

$$| f'(x) = | a |$$

And we have range where, $a < 1$

So, for $x = p$

$$|F'(p)| < 1$$

It is Attracting Fixed Point

2. For $1 \le a < 3$

$$x_n = a x_{n-1}(1 - x_{n-1})$$

For a > 1

consider, $\quad a = 1 + \epsilon \ \mid \ where \ \epsilon > 0$

$$\Rightarrow x = \frac{a-1}{a}$$

$$\Rightarrow x = \frac{\epsilon}{\epsilon + 1}$$

Now,

$$|f'(x)| = |a - 2ax|$$

$$putting\ x = \frac{a-1}{a} and\ a = 1 + \epsilon$$

$$|f'(x)| = |\ 1 + \epsilon - 2(1 + \epsilon)\frac{\epsilon}{(1+\epsilon)}\ |$$

$$|f'(x) = |1 - \epsilon| < 1$$

Hence, it is Attracting Fixed Point

For $a < 3$

$$a = 3 - \epsilon \quad \mid \quad where\ \epsilon > 0$$

$$x = \frac{a-1}{a} \Rightarrow x = \frac{2-\epsilon}{3-\epsilon}$$

Now,

$$|f'(x)| = |a - 2ax|$$

$$put\quad x = \frac{2-\epsilon}{3-\epsilon}\quad and\ a = 3 - \epsilon$$

$$|f'(x)| = \left| 3 - \epsilon - 2(3 - \epsilon)\frac{(2 - \epsilon)}{(3 - \epsilon)} \right|$$

$$|f'(x) = |\epsilon - 1| < 1$$

Hence, it is Attracting Fixed Point

3. For $3 \le a \le 4$

$Fixed\ point\ for\ x_n = a x_{n-1}(1 - x_{n-1})\ for\ a > 3$

$$\Rightarrow a = 3 + \epsilon \quad \mid \quad where\ \epsilon > 0$$

$$\Rightarrow x = \frac{a-1}{a}$$

$$\Rightarrow x = \frac{2+\epsilon}{3+\epsilon}$$

Now,

$$|f'(x)| = |a - 2ax|$$

$$putting\ x = \frac{2+\epsilon}{3+\epsilon}\ and\ a = 3+\epsilon$$

$$| f'(x) = | 3+\epsilon - 2(3+\epsilon)\ \frac{2+\epsilon}{3+\epsilon}\ |$$

$$| \boldsymbol{f'(x)} | = | -(\mathbf{1}+\boldsymbol{\epsilon}) |$$

$$| \boldsymbol{f'(x)} | = | \mathbf{1}+\boldsymbol{\epsilon}\ | > \mathbf{1}$$

Hence, it is NOT Attracting Fixed Point

$for\ a\ <\ 4$

$$\Rightarrow a = 4-\epsilon \quad | \quad where\ \epsilon > 0$$

$$\Rightarrow x = \frac{a-1}{a}$$

$$\Rightarrow x = \frac{3-\epsilon}{4-\epsilon}$$

Now

$$| f'(x) | = | a - 2ax |$$

$$put\ x = \frac{3-\epsilon}{4-\epsilon} and\ a = 4-\epsilon$$

$$| f'(x) | = | 4\ -\ \epsilon\ -\ 2(4\ -\epsilon)\frac{(3\ -\epsilon)}{(4\ -\ \epsilon)}\ |$$

$$| f'(x) | = | \epsilon - 2 | <\ 0$$

Hence, it is NOT Attracting Fixed Point

4. For $3\ \leq\ \boldsymbol{a}\ \leq\ 3.4$

$$\text{Fixed point for } \mathrm{x_n} = \mathrm{ax_{n-1}}(1-\mathrm{x_{n-1}})$$

$$\mathrm{x_n} = \mathrm{ax_{n-1}}(1-\mathrm{xn}-1)$$

for a ≥ 3

$$\Rightarrow a = 3+\epsilon \quad | \quad where\ \epsilon > 0$$

$$\Rightarrow x = \frac{a-1}{a}$$

$$\Rightarrow x = \frac{2+\epsilon}{3+\epsilon}$$

Now,

$$| f'(x) | = | a - 2ax |$$

$$\text{putting } \mathrm{x} = \frac{2+\epsilon}{3+\epsilon} \text{and } \mathrm{a} = 3+\epsilon$$

$$| f'(x) = | 3+\epsilon - 2(3+\epsilon)\frac{(2+\epsilon)}{(3+\epsilon)}$$

$$| f'(x) = | 1\ +\ \epsilon | <\ 0$$

Now,

$$f(x_n) = ax_n(1-x_n)$$

$$f(f(x_n)) = af(x_n)(1 - f(x_n))$$

Substituting $f(x_n)$ in the equation above, we get-

$$f(f(x_n)) = a(ax_n(1 - x_n))(1 - ax_n(1 - x_n))$$

$$f(f(x_n)) = x_n[a^2(ax_n(1 - x_n))(1 - ax_n(1 - x_n)) - 1] = 0$$

Therefore,

$$x_n = 0$$

Or,

$$a^2(ax_n(1 - x_n))(1 - ax_n(1 - x_n)) - 1 = 0$$

But,

$$\Rightarrow x = \frac{a - 1}{a}$$

So,

$$a^2x^2 - a^2x - ax + a + 1 = 0$$

Therefore, the roots of the above equation are-

$$x_n = \frac{1}{2a}\left(a + 1 \pm \sqrt{(a + 1)^2 - 4(a + 1)}\right)$$

$$x_n = \frac{1}{2a}\left(a + 1 \pm \sqrt{(a - 3)(a + 1)}\right)$$

Now, Differentiating value of $f(f(x))$ with respect to dx-

$$\left|\frac{d\left(f(f(x))\right)}{dx}\right| = \frac{d}{dx}a^2x(1 - x)(1 - ax(1 - x))$$

$$\left|\frac{d\left(f(f(x))\right)}{dx}\right| = \frac{d}{dx}a^2x(1 - x)(1 - ax(1 - x)) - a^2x(1 - ax(1 - x)) + a^2x(1 - x)(-a(1 - x) + ax)$$

$$\left|\frac{d\left(f(f(x))\right)}{dx}\right| = -\frac{d}{dx}a^2(-1 + 2a)(2ax^2 - 2ax + 1)$$

For $x = \frac{1}{2a}\left(a + 1 \pm \sqrt{(a - 3)(a + 1)}\right)$, putting it in the above equation

$$\left|\frac{d(f^2(x))}{dx}\right| = -\frac{d}{dx}\ a^2\left(-1 + 2\frac{\left(a + 1 \pm \sqrt{(a - 3)(a + 1)}\right)}{2a}\right)\left(\frac{\left(a + 1 \pm \sqrt{(a - 3)(a + 1)}\right)^2}{2a} - a - \sqrt{(a - 3)(a + 1)}\right)$$

$$\left|\frac{d(f^2(x))}{dx}\right| = 4 - a^2 + 2a$$

For Checking the Stability, $a > 3$

$$\Rightarrow 4 - a^2 + 2a < 1$$
$$\Rightarrow a^2 - 2a - 3 > 0$$
$$\Rightarrow (a - 3)(a + 1) > 0$$
$$\text{as, } a > 3$$

For , $a < 3.4$

$$\Rightarrow 4 - a^2 + 2a < 1$$
$$\Rightarrow a^2 - 2a + 5 >$$
$$\Rightarrow a < 1 + \sqrt{6}$$
$$\Rightarrow a < 3.4$$

Hence, Period 2 is stable for $3 < a < 3.4$

0.8 Modelling Examples

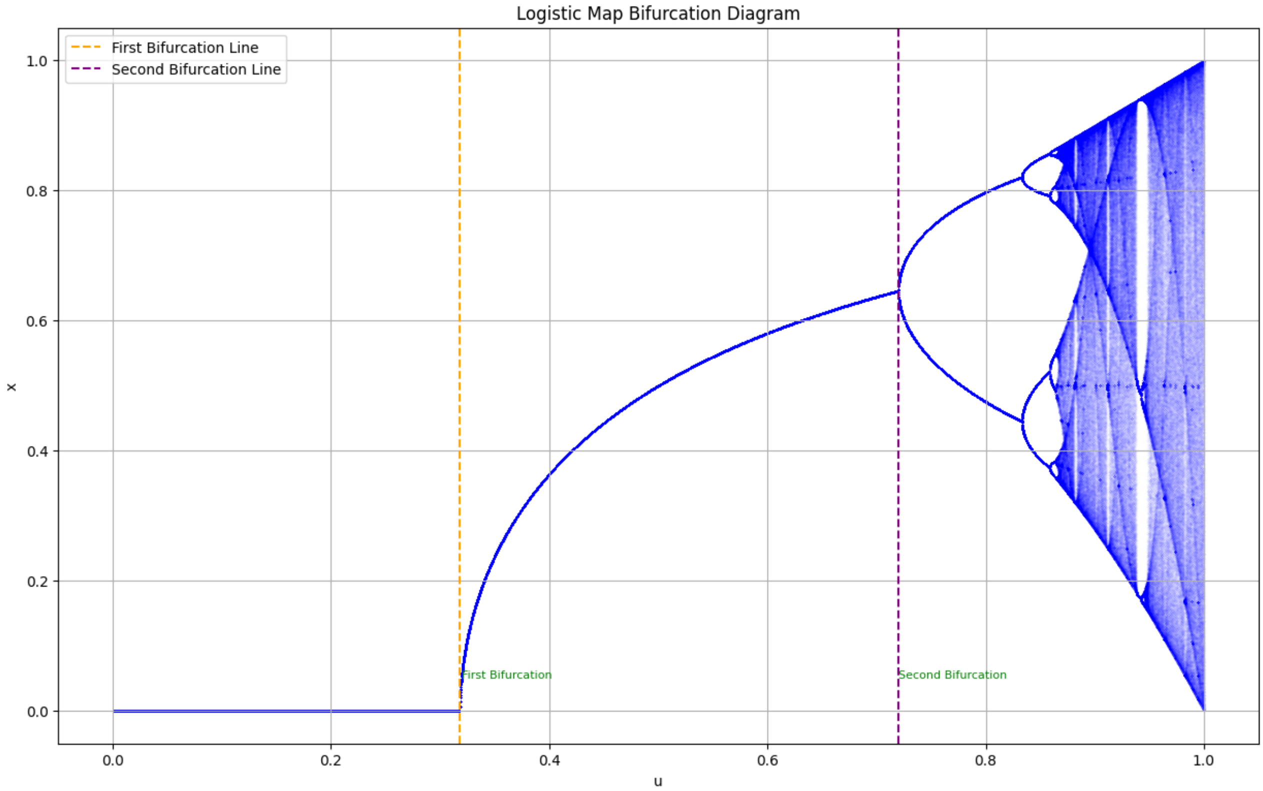
1. $F_\mu(x) = \mu sin(\Pi x)$

When you multiply μ by $sin(\Pi x)$, you're essentially modulating the amplitude of the sine wave by the value of u. A bifurcation graph of this function would typically plot the behaviour of the system (i.e., the values of the function) as the parameter μ is varied. The *x-axis* would represent the values of μ, and the *y-axis* would represent the values of the function $\mu sin(\Pi x)$.

To create a bifurcation diagram, we typically iterate the system for various values of μ, recording the behaviour of the system (e.g., its steady-state values, periodic orbits, chaotic behaviour) at each value of μ, and then plot these behaviours on the bifurcation diagram. This can help reveal the overall structure of the system's dynamics and identify regions of stability and instability as the parameter μ is varied.

Let's plot the bifurcation diagram for the $F_\mu(x) = \mu sin(\Pi x)$ with python and matplotlib as follows:

```
def logistic_map(u, x):
    return u * np.sin(np.pi * x)

def logistic_bifurcation_diagram(steps_per_parameter=10000,
iterations_per_step=500, u_steps=10000):
    us = np.linspace(0, 1, u_steps)
    x = np.random.random(u_steps)

    for _ in range(steps_per_parameter):
        x = logistic_map(us, x)

    plt.figure(figsize=(15, 9))

    for _ in range(iterations_per_step):
        x = logistic_map(us, x)
        plt.scatter(us, x, s=0.00034, c='b', alpha=0.5)

    plt.xlabel('u')
    plt.ylabel('x')
    plt.title('Logistic Map Bifurcation Diagram')
    # First bifurcation line
    plt.axvline(x=0.318, color='orange', linestyle='--', label='First
Bifurcation Line')
    plt.text(0.32, 0.05, 'First Bifurcation', color='green', fontsize=8)
    # Second bifurcation line
    plt.axvline(x=0.72, color='purple', linestyle='--', label='Second
Bifurcation Line')
    plt.text(0.72, 0.05, 'Second Bifurcation', color='green', fontsize=8)
    plt.legend()
    plt.grid(True)
    plt.show()

logistic_bifurcation_diagram()
```



2. $F_b(x) = \begin{cases} \dfrac{x}{b}, & 0 \leq x < b \\ \dfrac{1-x}{1-b}, & b \leq x < 1 \end{cases}$

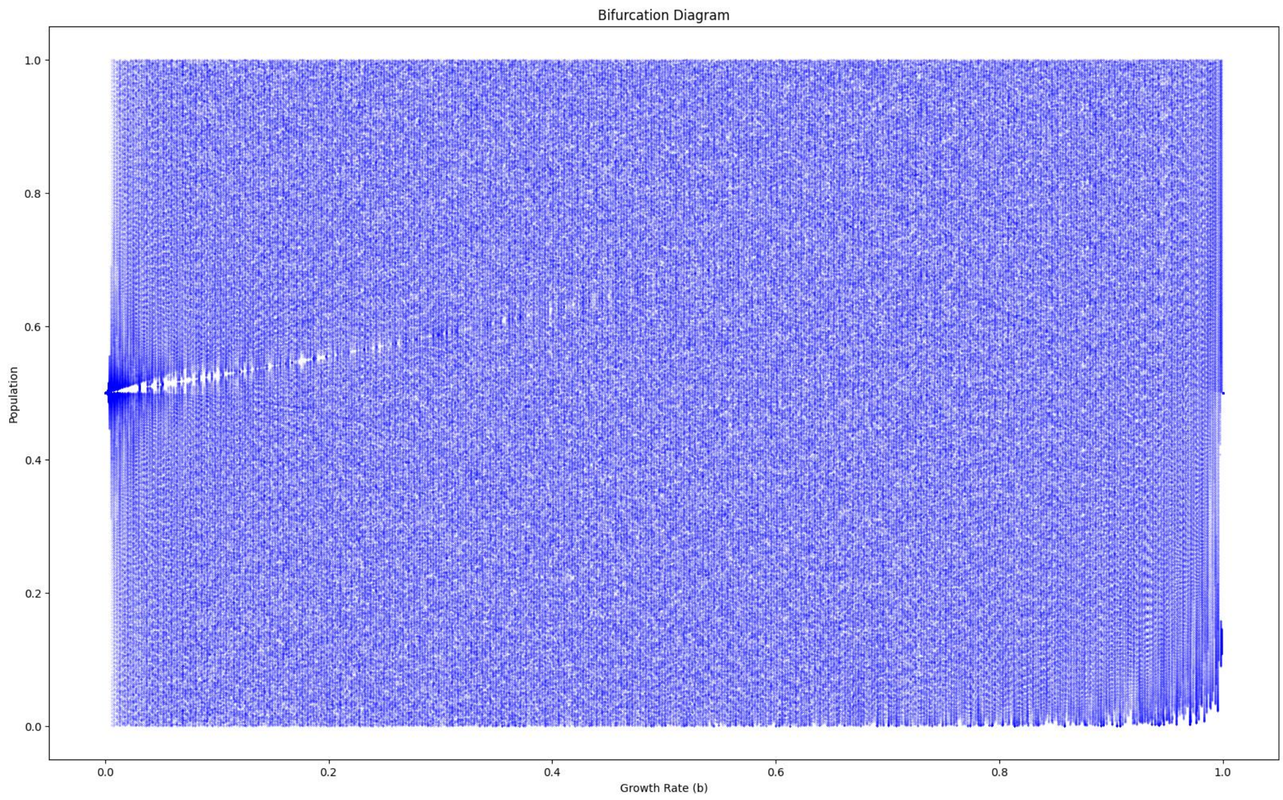
For this equation the parameter b essentially determines the transition point between these two linear regimes. It controls the point at which the function $T(x)$ switches from one linear relationship to the other. As b varies between 0 and 1, it affects the overall behaviour of the system. Below we plot the bifurcation diagram for the above system, it shows the behaviour of the system as the parameter b is varied and how the population X behaves as b changes.

Python and matplotlib code to plot bifurcation diagram for above system:

```
def logistic_map(b, x):
    if 0 <= x < b:
        return x / b
    elif b <= x < 1:
        return (1 - x) / (1 - b)

def logistic_map_bifurcation(b_values, x0, num_iterations):
    bifurcation_diagram = []
    for b in b_values:
        x = x0
        for _ in range(num_iterations):
            x = logistic_map(b, x)
            bifurcation_diagram.append((b, x))
    return np.array(bifurcation_diagram)


# Define parameters
b_values = np.linspace(0, 1, 1000);x0 = 0.5
num_iterations = 1000
```

```
bifurcation_diagram = logistic_map_bifurcation(b_values, x0, num_iterations)
# Plot
plt.figure(figsize=(20, 12))
plt.scatter(bifurcation_diagram[:, 0], bifurcation_diagram[:, 1], s=0.05, c='b',
marker='.')
plt.xlabel('Growth Rate (b)')
plt.ylabel('Population')
plt.title('Bifurcation Diagram')
plt.show()
```

Output:



0.9 Feigenbaum's Constant

Goal: Using the orbit diagram, we have seen in the previous exercises and experiments that the quadratic function $Q_c(x) = x^2 + c$ the logistic function $F_c(x) = cx(1 - x)$, and the sine function c * sin(x) all undergo a sequence of period-doubling bifurcations as the parameter tends to the chaotic regime.

We have also seen that magnifications of the orbit diagram tend to look "the same." In this experiment, we will see that there really is some truth to this: we will see that these period doubling bifurcations always occur at the same rate.

Procedure: In this experiment you will work with either the quadratic or the logistic family. We first need a definition:

Definition: ***Suppose x_0 is a critical point for F, that is, $F'(x_0) = 0$. If x_0 is also a periodic point of F with period n, then the orbit of x_0 is called superstable. The reason for this terminology is that $(F^n)'(x_0) = 0$.***

An approach to compute the first 2^n points on the orbit of the critical point, and then seeing how close you come to this point. Then modify the parameter repeatedly to try to come closer to the value for which the critical point is periodic with the right period. After finding the seven c-values for your function, record these numbers in tabular form:

1. c_0 = c-value for period 2^0
2. c_1 = c-value for period 2^1
3. c_2 = c-value for period 2^2

4. c_3 = c-value for period 2^3
5. c_4 = c-value for period 2^4
6. c_5 = c-value for period 2^5
7. c_6 = c-value for period 2^6

Now use a calculator or computer to compute the following ratios:

$$f_0 = \frac{c_0 - c_1}{c_1 - c_2}, f_1 = \frac{c_1 - c_2}{c_2 - c_3}, \dots, f_4 = \frac{c_4 - c_5}{c_5 - c_6}$$

List these numbers in tabular form, too. Do you notice any convergence? You should, at least if you have carried out the above search to enough decimal places.
The number this ratio converges to is called Feigenbaum's constant. It turns out that this number is "universal"-it appears whenever a typical family undergoes the period-doubling route to chaos.

Feigenbaum's constant = **δ = 4.6692016...**

0.10 References

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3. Boeing, G. Visual Analysis of Nonlinear Dynamical Systems: Chaos, Fractals, Self-Similarity and the Limits of Prediction. Systems 2016, 4, 37. https://doi.org/10.3390/systems4040037
4. Blbadger, Periodic trajectories in the logistic map https://blbadger.github.io/logistic-map#period-three