# PHOTONS AND MATTER WAVES



## Quantum Theory of Radiation :

According to quantum theory of radiation the energy from a body is radiated in descrete packets. Each packet of energy is called quantum of radiation. If '$\gamma$' is the frequency of radiated radiation then the energy of a quantum of radiation is given by

$$E = h\gamma \quad \text{——— (1)}$$

where, h is planck's constant.

The quantum of radiation is also called photon. If a incident light consists of 'n' numbers of photons then the total energy is given by

$$E_n = nh\gamma \quad \text{——— (2)}$$

## De-Broglie Hypothesis : Matter Waves

According to de-Broglie a moving particles sometime behaves as a particle and sometime behaves as a wave. The wave associated with moving particle is called particle wave or matter wave or de-Broglie wave. The wavelength of de-Broglie wave is called de-Broglie wavelength.

The wavelength of particle of mass 'm' moving with velocity 'v' is given by

$$\lambda = \frac{h}{mv} = \frac{h}{p} \quad \text{——— (1)}$$

## Proof:

According to quantum theory of radiation, the energy of photon of frequency '$\nu$' is given by

$$E = h\nu \quad \text{——— (1)}$$

where, h is planck's constant.

If a photon is considered as a particle of mass 'm' moving with velocity 'c', then according to Einstein's mass-energy relation

$$E = mc^2 \quad \text{——— (2)}$$

Since energy of photons in two case is same

$$\therefore \; h\nu = mc^2$$

$$\Rightarrow \frac{hc}{\lambda} = mc^2$$

$$\Rightarrow \frac{h}{\lambda} = mc$$

$$\Rightarrow \lambda = \frac{h}{mc}$$

$$\Rightarrow \lambda = \frac{h}{p} \quad \text{——— (3)}$$

As matter also possesses dual nature so the wavelength of the wave associated with a matter wave of mass 'm' and moving with velocity 'v' is

$$\lambda = \frac{h}{p}$$

$$\Rightarrow \lambda = \frac{h}{mv} \quad \text{——— (4)}$$

Scanned with CamScanner

DATE

## Wave Velocity or Phase Velocity of de-Broglie wave

The velocity with which a plane monochromatic wave moves (or speed of wave form) through a medium is known as phase or wave velocity. It is represented by $v_p$. According to de-Broglie hypothesis (1924) every moving particle has a wave associated with it which is known as matter wave. The wavelength of matter wave is given by

$$\lambda = \frac{h}{mv} \Rightarrow \lambda = \frac{h}{p} \quad \text{——— (1)}$$

If $\nu$ is the frequency of matter wave then phase velocity of de-Broglie wave is given by

$$v_p = \lambda \nu$$

Multiplying and dividing R.H.S. by $h$, we get,

$$v_p = \frac{h\nu}{\left(h/\lambda\right)} \Rightarrow v_p = \frac{E}{p} \quad \text{——— (2)}$$

where, $E = h\nu$, is the energy of the particle and $p = \frac{h}{\lambda}$, is the linear momentum associated with moving particle.

For free particle (non-relativistic case)

$$E = \frac{1}{2}mv^2$$

where, $m$ is the mass of the particle and $v$ is the velocity of the particle

and

$$p = mv$$

∴ from eq. (2), we get

$$v_p = \frac{\frac{1}{2}mv^2}{mv} \Rightarrow v_p = \frac{v}{2} \quad \text{——— (3)}$$

i.e., phase velocity is equal to half of particle velocity.

For particle moving with high speed (relativistic case)

$$E = mc^2$$

where, $m = \dfrac{m_0}{\sqrt{\left(1-\frac{v^2}{c^2}\right)}}$ = dynamic mass

Also,

$$p = mv = \frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}}$$

∴ from eq$^n$ ②, we get,

$$v_p = \frac{\dfrac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}}}{\dfrac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}}} \Rightarrow v_p = \frac{c^2}{v} \quad \text{---} \quad (4)$$

From equations ③ and ④, it can be seen that the phase velocity of moving particle may be greater or smaller than particle velocity.

According to special theory of relativity, the speed of any particle having finite rest mass must be less than the speed of light (c). The information associated with particle (mass, energy, momentum, charge, etc) travels with particle itself. So, phase velocity of a single de-Broglie wave or matter wave is physically meaningless quantity (and not observable).

B Group velocity and particle velocity :

The group velocity is defined as the velocity of center of wave packet or the phase velocity of modulated amplitude.

Mathematically, group velocity is given by

$$v_g = \frac{d\omega}{dk} = \frac{dE}{dp} \quad \text{———} \quad (1)$$

where, $E$ = energy
$p$ = linear momentum
$\omega$ = angular velocity
$k$ = wave number

For free particle :

$E = K.E + 0 \quad \text{- - - -} \quad (\because P.E = 0)$

$\Rightarrow E = \frac{p^2}{2m}$

$\therefore v_g = \frac{dE}{dp} \Rightarrow v_g = \frac{d}{dp}\left(\frac{p^2}{2m}\right)$

$\Rightarrow v_g = \frac{1}{2m} \cdot 2p \Rightarrow v_g = \frac{p}{m}$

$\Rightarrow v_g = \frac{mv}{m} \Rightarrow v_g = v \quad \text{———} \quad (2)$

$\therefore$ For free particle group velocity is equal to the particle velocity.

## Schrodinger's Time Dependent Wave Equation :

The quantity that characterises the de-Broglie wave is called the wave function. It is denoted by $\psi$. It may be a complex function. Let the wave function along $x$-direction is

$$\psi = A e^{-i(\omega t - kx)}$$

$$\Rightarrow \psi = A e^{-i\omega\left(t - \frac{x}{v}\right)}$$

$$\Rightarrow \psi = A e^{-i2\pi\nu\left(t - \frac{x}{v}\right)}$$

$$\Rightarrow \psi = A e^{-i2\pi\left(\nu t - \frac{\nu x}{v}\right)}$$

$$\Rightarrow \psi = A e^{-i2\pi\left(\nu t - \frac{x}{\lambda}\right)} \quad \text{———} \quad (1)$$

Let E be the total energy and p be the momentum of the particle, then

$$E = h\nu$$

$$\Rightarrow \nu = \frac{E}{h} \quad \text{———} \quad (2)$$

and

$$\lambda = \frac{h}{p} \quad \text{———} \quad (3)$$

$\therefore$ from equation (1), (2) and (3), we get,

$$\psi = A e^{-2\pi i\left(\frac{Et}{h} - \frac{px}{h}\right)}$$

$$\Rightarrow \psi = A e^{-\frac{2\pi i}{h}(Et - px)} \quad \text{———} \quad (4)$$

Differentiating eq$^n$ (4) w.r.t. 't', we get,

$$\frac{\partial \psi}{\partial t} = A e^{-\frac{2\pi i}{h}(Et - px)} \cdot \left(-\frac{2\pi i E}{h}\right)$$

$$\Rightarrow \frac{\partial \psi}{\partial t} = -\frac{2\pi i E}{h} A e^{-\frac{i2\pi}{h}(Et-px)}$$

$$\Rightarrow \frac{\partial \psi}{\partial t} = -\frac{2\pi i E}{h} \psi$$

$$\Rightarrow E\psi = -\frac{h}{2\pi i} \frac{\partial \psi}{\partial t}$$

$$\Rightarrow E\psi = \frac{ih}{2\pi} \frac{\partial \psi}{\partial t} \quad \text{——— } (5)$$

Now, differentiating eqn. (4) twice w.r.t. $x$, we get,

$$\frac{\partial \psi}{\partial x} = \left(-\frac{2\pi i}{h}\right)(-p) \cdot A e^{-\frac{2\pi i}{h}(Et-px)}$$

$$\Rightarrow \frac{\partial \psi}{\partial x} = \left(\frac{2\pi i p}{h}\right) A e^{-\frac{2\pi i}{h}(Et-px)}$$

and

$$\frac{\partial^2 \psi}{\partial x^2} = \left(-\frac{2\pi i}{h}\right)(-p)\left(\frac{2\pi i p}{h}\right) A e^{-\frac{2\pi i}{h}(Et-px)}$$

$$\Rightarrow \frac{\partial^2 \psi}{\partial x^2} = \left(\frac{2\pi i p}{h}\right)^2 A e^{-\frac{2\pi i}{h}(Et-px)}$$

$$\Rightarrow \frac{\partial^2 \psi}{\partial x^2} = \frac{4\pi^2 (i)^2 p^2}{h^2} A e^{-\frac{2\pi i}{h}(Et-px)}$$

$$\Rightarrow \frac{\partial^2 \psi}{\partial x^2} = -\frac{4\pi^2 p^2}{h^2} A e^{-\frac{2\pi i}{h}(Et-px)}$$

$$\Rightarrow \frac{\partial^2 \psi}{\partial x^2} = -\frac{4\pi^2 p^2}{h^2} \psi$$

$$\Rightarrow p^2 \psi = -\frac{h^2}{4\pi^2} \frac{\partial^2 \psi}{\partial x^2} \quad \text{——— } (6)$$

The total energy of particle is sum of kinetic energy and potential energy.

$$\therefore \quad E = \frac{p^2}{2m} + V$$

Multiplying both sides by $\psi$, we get,

$$E\psi = \frac{p^2\psi}{2m} + V\psi.$$

$$\Rightarrow \frac{ih}{2\pi}\frac{\partial\psi}{\partial t} = \frac{1}{2m}\left(-\frac{h^2}{4\pi^2}\frac{\partial^2\psi}{\partial x^2}\right) + V\psi$$

$$\Rightarrow \frac{ih}{2\pi}\frac{\partial\psi}{\partial t} = -\frac{h^2}{8\pi^2 m}\frac{\partial^2\psi}{\partial x^2} + V\psi \quad \text{———} \quad (7)$$

Equation (7) is called Schrodinger's time dependent wave equation.

## Schrodinger's Time Independent Wave Equation: (Steady State Form)

Let the one dimensional wave function of a particle moving along x-direction is given by

$$\psi = A e^{-\frac{2\pi i}{h}(Et - px)}$$

$$\Rightarrow \psi = A e^{-\frac{2\pi i E t}{h}} \cdot e^{+\frac{2\pi i p x}{h}}$$

$$\Rightarrow \psi = \psi_0 e^{-\frac{2\pi i E}{h}t} \quad \text{———} \quad (1)$$

where,

$$\psi_0 = A e^{\frac{2\pi i p x}{h}} \quad \text{———} \quad (2)$$

Differentiating eq^n (1) w.r.t. 't', we get,

$$\frac{\partial \psi}{\partial t} = -\frac{2\pi i E}{h} \psi_0 e^{-\frac{2\pi i E}{h} t} \quad \text{———} \quad (3)$$

Differentiating eq^n (1) twice w.r.t. x, we get,

$$\frac{\partial^2 \psi}{\partial t^2} = e^{-\frac{2\pi i E}{h} t} \frac{\partial^2 \psi_0}{\partial x^2} \quad \text{———} \quad (4)$$

The time dependent wave equation is

$$\frac{ih}{2\pi} \frac{\partial \psi}{\partial t} = -\frac{h^2}{8\pi^2 m} \frac{\partial^2 \psi}{\partial x^2} + V \cdot \psi$$

$$\Rightarrow \frac{ih}{2\pi} \left( -\frac{2\pi i E}{h} \psi_0 e^{-\frac{2\pi i E}{h} t} \right) = -\frac{h^2}{8\pi^2 m} e^{-\frac{2\pi i E}{h} t} \frac{\partial^2 \psi_0}{\partial x^2} + V \psi_0 e^{-\frac{2\pi i E}{h} t}$$

$$\Rightarrow -i^2 E \psi_0 = -\frac{h^2}{8\pi^2 m} \frac{\partial^2 \psi_0}{\partial x^2} + V \psi_0$$

$$\Rightarrow E \psi_0 = -\frac{h^2}{8\pi^2 m} \frac{\partial^2 \psi_0}{\partial x^2} + V \psi_0$$

$$\Rightarrow \frac{h^2}{8\pi^2 m} \frac{\partial^2 \psi_0}{\partial x^2} + E \psi_0 - V \psi_0 = 0$$

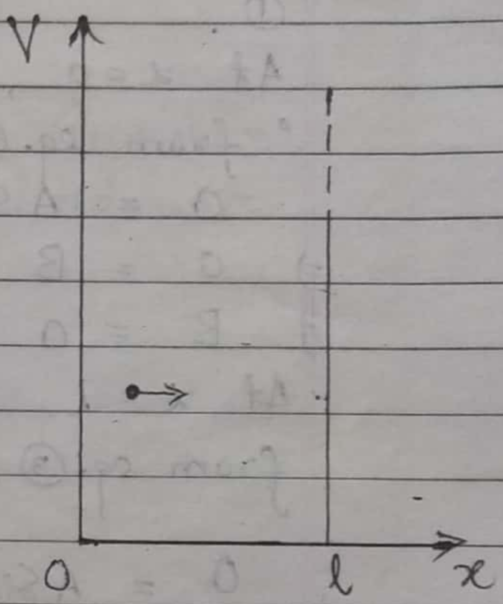
$$\Rightarrow \frac{h^2}{8\pi^2 m} \frac{\partial^2 \psi_0}{\partial x^2} + (E - V) \psi_0 = 0$$

$$\Rightarrow \frac{\partial^2 \psi_0}{\partial x^2} + \frac{8\pi^2 m}{h^2} (E - V) \psi_0 = 0 \quad \text{———} \quad (5)$$

Eq. (5) is Schrodinger's time independent wave equation.

## One - Dimensional Potential Well

Consider a particle moving in a box along the $x$-direction. A particle is bounced back and forth between the walls of the box. The box has insurmountable potential barrier at $x = 0$ and $x = l$. Let the particle has mass 'm' and position '$x$' at any instant of time, where $x$ lies between $0 < x < l$.




The boundary conditions imposed in this problem is

$$V = 0 \quad ; \quad 0 < x < l$$
$$V = \infty \quad ; \quad 0 \geq x \geq l$$

Within box, Schrodinger's wave equation is

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{8\pi^2 m}{h^2} E \psi = 0$$

$$\frac{\partial^2 \psi}{\partial x^2} + k^2 \psi = 0 \quad \text{—— (1)}$$

where, $$k^2 = \frac{8\pi^2 m E}{h^2} \quad \text{—— (2)}$$

The general solution of this equation is

$$\psi = A \sin(kx) + B \cos(kx) \quad \text{--- (3)}$$

The boundary conditions can be used to evaluate the constant A and B in equation (1).

At $x = 0$, $\psi = 0$

$\therefore$ from eq. (3), we get,

$$0 = A \sin 0 + B \cos 0$$

$$\Rightarrow 0 = B$$

$$\Rightarrow B = 0 \quad \text{--- (4)}$$

At $x = l$, $\psi = 0$,

from eq. (3), we get,

$$0 = A \sin(kl) + B \cos(kl)$$

$$\Rightarrow 0 = A \sin(kl) + 0 \quad \text{--- (from eq. (4))}$$

$$\Rightarrow A \sin(kl) = 0$$

$$\because A \neq 0$$

$$\therefore \sin(kl) = 0$$

$$\Rightarrow kl = n\pi$$

$$\Rightarrow k = \frac{n\pi}{l} \quad \text{--- (5)}$$

$\therefore$ eq. (3) becomes

$$\psi = A \sin\left(\frac{n\pi x}{l}\right) \quad \text{--- (6)}$$

The energy of particle is given by

$$k^2 = \frac{8\pi^2 m}{h^2} E \quad \text{--- (from eq. (2))}$$

$$\Rightarrow E = \frac{k^2 h^2}{8\pi^2 m}$$

$$\Rightarrow E = \left(\frac{n\pi}{l}\right)^2 \frac{h^2}{8\pi^2 m}$$

$$\Rightarrow E = \frac{n^2 h^2}{8ml^2} \quad \text{——— (7)}$$

It is certain that the particle is somewhere inside the box. Hence for a normalized wave function.

$$\int_0^l |\psi|^2 dx = 1$$

$$\Rightarrow \int_0^l A^2 \sin^2\left(\frac{n\pi x}{l}\right) dx = 1$$

$$\Rightarrow A^2 \int_0^l \left[\frac{1 - \cos\left(\frac{2n\pi x}{l}\right)}{2}\right] dx = 1$$

$$\Rightarrow \frac{A^2}{2} \int_0^l \left[1 - \cos\left(\frac{2n\pi x}{t}\right)\right] dx = 1$$

$$\Rightarrow \frac{A^2}{2} \left[\int_0^l dx - \int_0^l \cos\left(\frac{2n\pi x}{l}\right) dx\right] = 1$$

$$\Rightarrow \frac{A^2}{2} \left[l - 0\right] = 1$$

$$\Rightarrow \frac{A^2}{2} l = 1$$

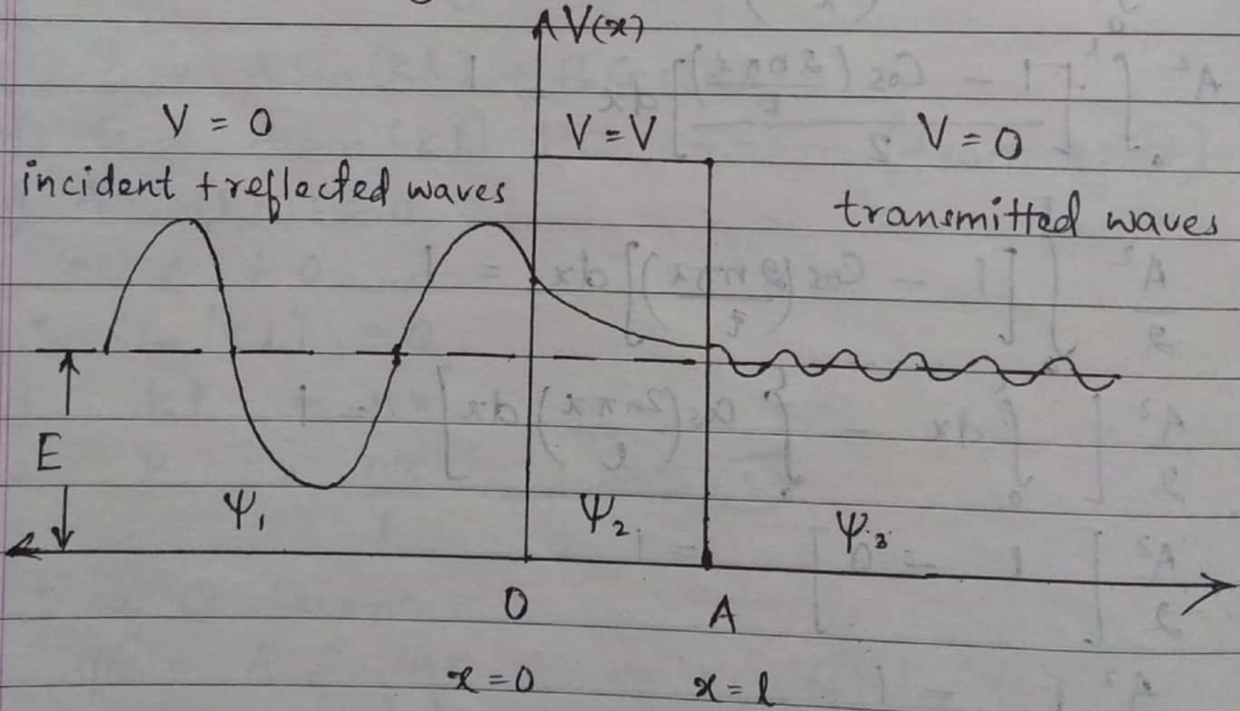
$$\Rightarrow A = \sqrt{\frac{2}{l}} \quad \text{——— (8)}$$

$\therefore$ the normalized wave function of the particle is

$$\psi = \sqrt{\frac{2}{l}} \sin\left(\frac{n\pi x}{l}\right) \quad \text{——— (9)}$$

## Tunneling Effect

The passage of a particle through a potential barrier when the particle's total energy (which remaine unchanged in the tunneling effect) is less than the height of the barrier, is called tunneling effect.




Consider a beam of particle of kinetic energy 'E' incident from left on a potential barrier of height 'V' and width 'l'. The potential is described by

$V = 0 \quad ; \quad x < 0 \quad$ (region I)

$V = V \quad ; \quad 0 < x < l \quad$ (region II)

$V = 0 \quad ; \quad x > l \quad$ (region III)

Let $\psi_1$, $\psi_2$ and $\psi_2$ be the respective wave functions in regions I, II and III as shown in fig. The corresponding wave equations are

$$\frac{\partial^2 \psi_1}{\partial x^2} + \frac{8\pi^2 mE}{h^2}\psi_1 = 0$$

$$\frac{\partial^2 \psi_2}{\partial x^2} + \frac{8\pi^2 m}{h^2}(E-V)\psi_2 = 0$$

and $$\frac{\partial^2 \psi_3}{\partial x^2} + \frac{8\pi^2 mE}{h^2}\psi_3 = 0$$

$$\Rightarrow \frac{\partial^2 \psi_1}{\partial x^2} + \alpha^2\psi_1 = 0$$

$$\frac{\partial^2 \psi_2}{\partial x^2} + \beta^2\psi_2 = 0$$

and $$\frac{\partial^2 \psi_3}{\partial x^2} + \alpha^2\psi_3 = 0$$

where,

$$\alpha^2 = \frac{8\pi^2 mE}{h^2}$$

and $$\beta^2 = \frac{8\pi^2 m(V-E)}{h^2}$$

The solution of these equations are

$$\psi_1 = Ae^{i\alpha x} + Be^{-i\alpha x}$$

Scanned with CamScanner

$$\psi_2 = F e^{i\beta x} + G e^{-i\beta x}$$

and $$\psi_3 = C e^{i\alpha x} + D e^{-i\alpha x}$$

where, A, F and C are amplitudes of incident waves in regions I, II and III resp. and B, G and D are amplitudes of reflected waves in regions I, II and III respectively.

Since the probablity density associated with wave function is proportional to the square of the amplitudes of that function therefore the barrier transmission coefficient is given by

$$T = \frac{|C|^2}{|A|^2}$$

and reflection coefficient for the barrier surface at $x = 0$ given by

$$R = \frac{|B|^2}{|A|^2}$$

If a barrier is high, compared to the total energy of the particle, or thick compared to the wavelength of the wavefunction then the is probablity that the particle incident on the barrier from one side will appear on the other side. Such a probablity is zero classically. But a finite quantity in quantum mechanics. Therefore, if a particle with energy E is incident on thin energy barrier of height greater than E, there is finite probablity of particle penentrating the barrier. This phenomenon is called tunneling effect.