Wave Motion

Wave Equation

Mechanical wave is a disturbance produced in a material medium which moves with defined speed without changing its form. For eg, sound wave, water wave, oscillation of stretched string etc.

The wave which advances with time is called travelling wave or progressive wave. The equation of travelling wave is given by,

$$y = A \sin \omega\left(t - \frac{x}{v}\right) \quad \text{———} \quad (1)$$

where,
A is amplitude
ω is angular frequency
and k is angular wave number

$$\Rightarrow y = A \sin\left(\omega t - \frac{\omega}{v} x\right)$$

$$\Rightarrow y = A \sin\left(\omega t - \frac{2\pi f}{v} x\right) \quad \text{- - -} \quad (\because \omega = 2\pi f)$$

$$\Rightarrow y = A \sin\left(\omega t - \frac{2\pi}{\lambda} x\right) \quad \text{- - - -} \quad \left(v = f\lambda \Rightarrow \frac{1}{\lambda} = \frac{f}{v}\right)$$

$$\Rightarrow y = A \sin(\omega t - kx) \quad \text{———} \quad (2)$$

where, $k = \frac{2\pi}{\lambda} = \frac{\omega}{v}$

Equations (1) and (2) are equations of a wave moving from left to right. If a ~~mov~~ wave is moving from right to left, then wave equation is given by

$$y = A \sin \omega\left(t + \frac{x}{v}\right) \quad \text{——— (3)}$$

$$\Rightarrow y = A \sin(\omega t + kx) \quad \text{——— (4)}$$

Wave velocity and particle velocity

Equation of a wave is given by

$$y = A \sin \omega\left(t - \frac{x}{v}\right) \quad \text{——— (1)}$$

where, 'y' is the displacement of particle at a distance 'x' from the origin at any instant 't', 'A' is the amplitude and v is the velocity of the wave.

Differentiating eq. (1) w.r.t. 't', we get,

$$\frac{dy}{dt} = A \omega \cos \omega\left(t - \frac{x}{v}\right) \quad \text{——— (2)}$$

Again, differentiating eq. (1) w.r.t. 'x', we get,

$$\frac{dy}{dx} = -A\left(\frac{\omega}{v}\right) \cos \omega\left(t - \frac{x}{v}\right) \quad \text{——— (3)}$$

Dividing eq. (2) by eq. (3), we get

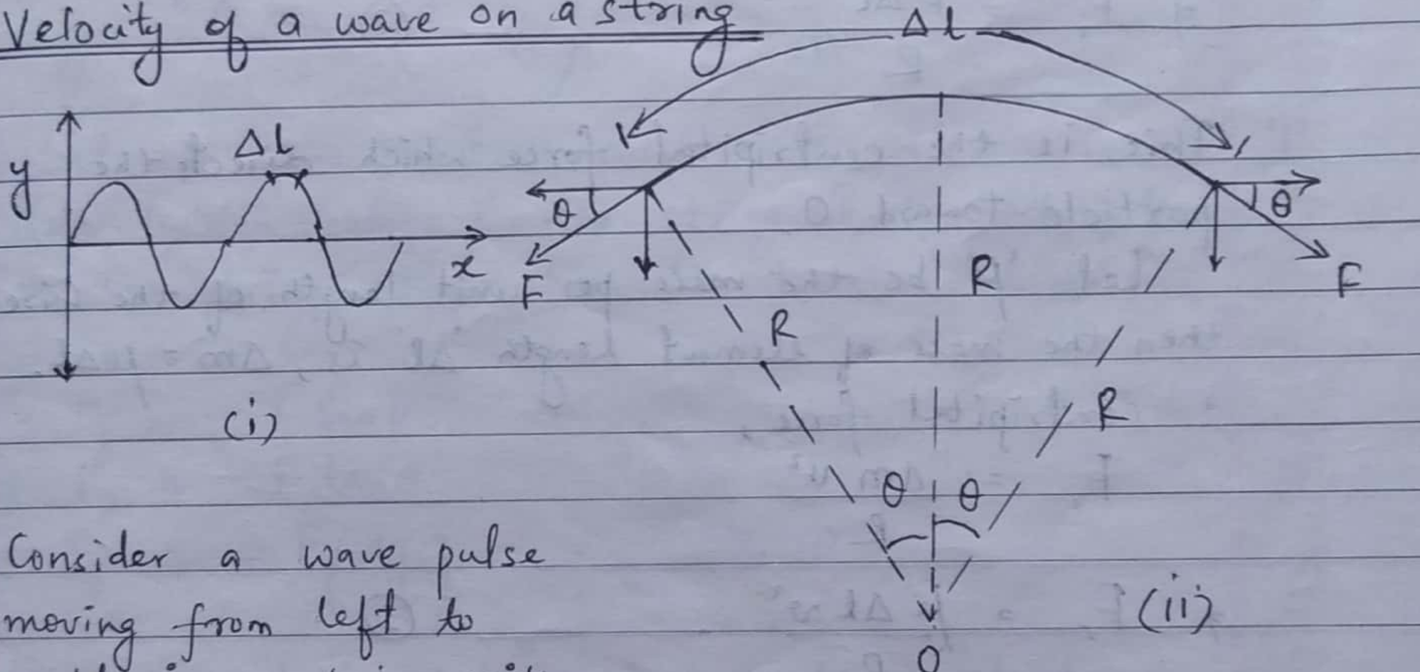
$$\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dy}{dx}\right)} = \frac{A \omega \cos \omega\left(t - \frac{x}{v}\right)}{-A\left(\frac{\omega}{v}\right) \cos \omega\left(t - \frac{x}{v}\right)}$$

$$\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dy}{dx}\right)} = -v$$

$$\Rightarrow \left(\frac{dy}{dt}\right) = -v\left(\frac{dy}{dx}\right) \quad \text{——— (4)}$$

$\Rightarrow$ (particle velocity at a point) = $-$ (wave velocity) $\times$ (slope of displacement curve at that point.)

Velocity of a wave on a string



Consider a wave pulse moving from left to right in a string with velocity 'v'. Let 'Δl' be the length of small section or element of the pulse. It may be approximated by a circular arc. Let 'R' be the radius of the small arc element 'Δl'.

Let 'F' be the tension on the string. The tension 'F' acts tangentially to the string. The horizontal component of 'F' at each end of the string cancles each other but vertical components adds up giving net downward force given by

$$F_r = F\sin\theta + F\sin\theta$$

$$\Rightarrow F_r = 2F\sin\theta \quad \text{——— (1)}$$

$\because$ 'Δl' is small so θ is also small.

$$\therefore F_r \approx 2F\theta$$

$$\Rightarrow F_r = 2F\left(\frac{\Delta l/2}{R}\right) \quad \text{———} \quad \left(\because \theta = \frac{\text{arc length}}{\text{radius}}\right)$$

$$\Rightarrow F_r = \frac{F\Delta l}{R} \quad \text{——— (2)}$$

This is the centripital force which directs the particle toward 'O'.

Let 'μ' be the mass per unit length of the wire then the mass of element length 'Δl' is, $\Delta m = \mu\Delta l$.

$\therefore$ centripital force,

$$F_r = \frac{\Delta m\, v^2}{R}$$

$$\Rightarrow F_r = \frac{\mu\Delta l\, v^2}{R} \quad \text{——— (3)}$$

$\therefore$ from eq. (2) and eq. (3), we get,

$$\frac{F\Delta l}{R} = \frac{\mu\Delta l\, v^2}{R}$$

$$\Rightarrow v^2 = \frac{F}{\mu}$$

$$\Rightarrow \boxed{v = \sqrt{\frac{F}{\mu}}} \quad \text{——— (4)}$$

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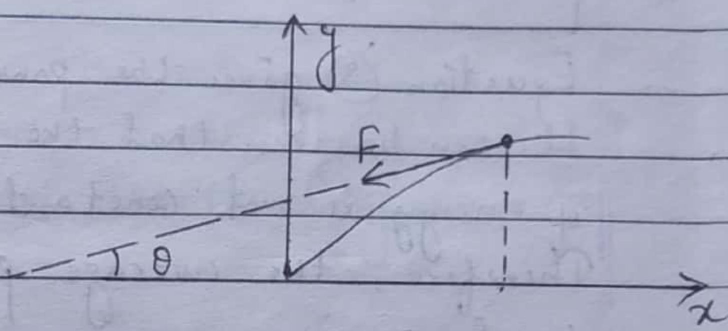
Therefore, the velocity of a wave on the string depends only on the tension (F) and linear mass density (μ).

Power and Intensity of a wave.

Consider a wave travelling along positive x-direction given by



$$y = A \sin \omega \left(t - \frac{x}{v}\right) \quad \text{——— (1)}$$

Let at some point or position 'x' at any instant 't' the tension acting on the string be 'F', along tangent to the string at point x. The tension along the y-axis is

$$F_y = - F \sin \theta$$

$$\Rightarrow F_y \approx - F \tan \theta$$

$$\Rightarrow F_y = - F \left(\frac{dy}{dx}\right) \quad \text{——— (2)}$$

The power delivered is defined as the product of tension or force and the velocity due to this tension or force. Therefore, the power delivered by the tension F is

$$P = \left(- F \frac{dy}{dx}\right) \cdot \left(\frac{dy}{dt}\right)$$

putting value of y from eq. (1), we get,

$$P = -F \frac{d}{dx}\left\{A \sin \omega\left(t-\frac{x}{v}\right)\right\} \frac{d}{dt}\left\{A \sin \omega\left(t-\frac{x}{v}\right)\right\}$$

$$\Rightarrow P = -F\left\{\left(-\frac{\omega}{v}\right) A \cos \omega\left(t-\frac{x}{v}\right)\right\}\left\{\omega A \cos \omega\left(t-\frac{x}{v}\right)\right\}$$

$$\Rightarrow \boxed{P = \frac{F \omega^2 A^2}{v} \cos^2 \omega\left(t-\frac{x}{v}\right)} \quad (3)$$

Equation (3) gives the power transmitted through x. It can be seen that the power or rate of flow of energy is not constant as input power oscillates. Therefore, the average power transmitted across point x is

$$P_{avg} = \frac{1}{2} \frac{F \omega^2 A^2}{v}$$

$$\because v = \sqrt{\frac{F}{\mu}}$$

$$\Rightarrow F = \mu v^2$$

$$\therefore P_{avg} = \frac{1}{2} \frac{\mu v^2 \omega^2 A^2}{v}$$

$$\Rightarrow \boxed{P_{avg} = \frac{1}{2} \mu v \omega^2 A^2} \quad (4)$$

The intensity of the travelling wave is defined as the average power passing per unit area held perpendicular to the direction in which wave is travelling.

$$\therefore I = \frac{P_{avg}}{\text{area}} = \frac{P_{avg}}{a}$$

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$$\Rightarrow I = \frac{1}{2}\mu v \omega^2 A^2 / a$$

$$\Rightarrow I = \frac{1}{2}\left(\frac{\mu}{a}\right) v \omega^2 A^2$$

$$\Rightarrow \boxed{I = \frac{1}{2}\rho v \omega^2 A^2} \text{ ——— } (5) \quad \left(\because \rho = \frac{\mu}{a}\right)$$

i.e., $I \propto A^2$

$\therefore$ intensity of travelling wave is directly proportional to square of amplitude of the travelling wave.

Standing Wave

When two waves of same amplitude and frequency travelling in the opposit direction superimpose the standing wave is formed.

Consider two waves travelling in opposit directions given by

$$y_1 = A \sin(\omega t - kx) \text{ ——— } (1)$$

and $y_2 = A \sin(\omega t + kx)$ ——— (2)

If 'y' is the resultant of the two waves then

$$y = y_1 + y_2$$

$$\Rightarrow y = A \sin(\omega t - kx) + A \sin(\omega t + kx)$$

$$\Rightarrow y = 2A \sin\left(\frac{(\omega t - kx) + (\omega t + kx)}{2}\right) \cdot \cos\left(\frac{(\omega t - kx) - (\omega t + kx)}{2}\right)$$

$$\Rightarrow y = 2A \sin(\omega t) \cos(kx)$$

$\Rightarrow y = 2A\cos(kx)\sin(\omega t)$ ——— (3)

$\therefore$ each particle of the string vibrates in SHM with an amplitude $2A\cos(kx)$. The amplitude for each particle is different.

For minimum amplitude,

$$A\cos(kx) = 0$$

$$\because A \neq 0$$

$$\therefore \cos(kx) = 0$$

$$\Rightarrow \cos(kx) = \cos\left((2n+1)\frac{\pi}{2}\right)$$

$$\Rightarrow kx = (2n+1)\frac{\pi}{2}$$

$$\Rightarrow \frac{2\pi}{\lambda}x = (2n+1)\frac{\pi}{2}$$

$$\Rightarrow x = (2n+1)\frac{\lambda}{4} \quad \text{——— (4)}$$

i.e., $x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \ldots$

For these particles the displacement is always zero as if they are fixed. These points are called nodes.

For maximum,

$$A\cos(kx) = \text{maximum}$$

for $A\cos(kx)$ to be maximum,

$$\cos(kx) = 1$$

$$\Rightarrow \cos(kx) = \cos(n\pi)$$

$$\Rightarrow \cancel{\cos}\ kx = n\pi$$

$$\Rightarrow \frac{2\pi}{\lambda}x = n\pi$$

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$\Rightarrow x = \frac{n\lambda}{2}$ ——— (5)

i.e., $x = \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}, 2\lambda, \ldots$

In these points the particles vibrate in SHM with amplitude 2A. These points are called anti-nodes.

Standing wave on a string fixed at both ends (resonance)

Let the equation of the wave travelling along positive x direction is

$y_1 = A \sin(\omega t - kx)$ ——— (1)

After reflection at the boundary the reflected wave is given by

$y_2 = -A \sin(\omega t + kx)$ ——— (2)

When these two waves superimpose the resultant displacement is given by

$y = y_1 + y_2$

$\Rightarrow y = A \sin(\omega t - kx) - A \sin(\omega t + kx)$

$\Rightarrow y = A[\sin(\omega t - kx) - \sin(\omega t + kx)]$

$\Rightarrow y = 2A \sin\left(\frac{(\omega t - kx) - (\omega t + kx)}{2}\right) \cdot \cos\left(\frac{(\omega t - kx) + (\omega t + kx)}{2}\right)$

$\Rightarrow y = 2A \sin(-kx) \cos(\omega t)$

$\Rightarrow y = -2A \sin(kx) \cos(\omega t)$ ——— (3)

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Since the string is fixed at both ends so the displacement $y = 0$ at $x = l$.
Applying this boundary condition in eq. ③, we get,

$$0 = -2A \sin(kl) \cos(\omega t)$$

Here,

$$-2A \cos(\omega t) \neq 0$$

$$\therefore \sin(kl) = 0$$

$$\Rightarrow \sin(kl) = \sin(n\pi)$$

$$\therefore kl = n\pi$$

$$\Rightarrow \frac{2\pi}{\lambda} l = n\pi$$

$$\Rightarrow \frac{2l}{\lambda} = n$$

$$\Rightarrow 2l = n\lambda$$

$$\Rightarrow 2l = n\frac{\cancel{f}}{f} \quad \text{----} \left(v = f\lambda \Rightarrow \lambda = \frac{v}{f}\right)$$

$$\Rightarrow 2l = \frac{n\sqrt{(F/\mu)}}{f} \quad \text{---} \left(\because v = \sqrt{\frac{F}{\mu}}\right.$$

$$\Rightarrow \boxed{f = \frac{n}{2l}\sqrt{\left(\frac{F}{\mu}\right)}} \text{———— ④}$$

Equation ④ gives the frequency of standing wave on a stretched string having mass per unit length 'μ' and tension 'F'.

For $n = 1$, the frequency obtained is minimum given by

$$f_1 = \frac{1}{2l}\sqrt{\frac{F}{\mu}}$$

This frequency is also called fundamental frequency.

For $n = 2$,

$$f_2 = \frac{2}{2l}\sqrt{\frac{F}{\mu}} = 2f_1$$, first overtone

For $n = 3$,

$$f_3 = \frac{3}{2l}\sqrt{\frac{F}{\mu}} = 3f_1$$, second overtone

and so on.