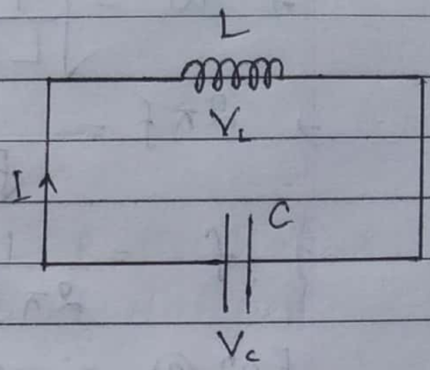
Electromagnetism

LC Oscillations



Let us consider an LC circuit as shown in fig. Let 'L' be the inductance of the inductor and 'C' be the capacitance of the capacitor.
Let, at any instant of time, V_L and V_c be the potential difference across the inductor and capacitor respectively, then

$$V_L + V_c = 0$$

$$\Rightarrow L\frac{dI}{dt} + \frac{Q}{C} = 0$$

$$\Rightarrow L\frac{d}{dt}\left(\frac{dQ}{dt}\right) + \frac{Q}{C} = 0 \quad \ldots\ldots \left(\because I = \frac{dQ}{dt}\right)$$

$$\Rightarrow L\frac{d^2Q}{dt^2} + \frac{Q}{C} = 0$$

$$\Rightarrow \frac{d^2Q}{dt^2} + \left(\frac{1}{LC}\right)Q = 0 \quad \text{——— (1)}$$

Eq. (1) is the differential equation of EM oscillation.
Comparing eq^n (1) with general differential equation of mechanical oscillation given by

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \quad \text{——— (2)}$$

we get,

$$\omega^2 = \frac{1}{LC}$$

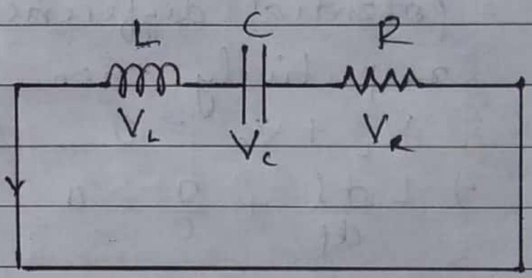
$$\Rightarrow \omega = \sqrt{\frac{1}{LC}}$$

$$\Rightarrow 2\pi f = \frac{1}{\sqrt{LC}}$$

$$\Rightarrow f = \frac{1}{2\pi\sqrt{LC}} \quad \text{——— (3)}$$

Eq^n (3) gives the frequency of electromagnetic oscillation.

Damped LC Oscillations.



Let us consider an LCR circuit as shown in fig. In LCR circuit, the amplitude of oscillation decreases exponentially with time due to presence of ~~resti~~ resistance R which act as resistive factor.

Let, at any instant of time, V_L, V_C and V_R be the potential difference across inductor, capacitor and resistor, respectively, then

$$V_L + V_C + V_R = 0$$

$$\Rightarrow L\frac{dI}{dt} + \frac{Q}{C} + IR = 0$$

$$\Rightarrow L\frac{d}{dt}\left(\frac{dQ}{dt}\right) + R\frac{dQ}{dt} + \frac{Q}{C} = 0$$

$$\Rightarrow L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$$

$$\Rightarrow \frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{Q}{LC} = 0$$

$$\Rightarrow \frac{d^2Q}{dt^2} + 2.\left(\frac{R}{2L}\right)\frac{dQ}{dt} + \left(\frac{1}{LC}\right)Q = 0 \quad \text{———} \quad (1)$$

Eq: (1) is equation of damped LCR oscillation. Comparing with general equation of damped mechanical oscillation, we get,

$$\omega' = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$$

$$\Rightarrow 2\pi f = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$$

$$\Rightarrow f = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2} \quad \text{———} \quad (2)$$

Eq. (2) is required frequency of damped LCR oscillation.

Displacement Current (I_D):

Displacement current, a phenomenon analogous to an ordinary electric current helps to explain magnetic fields that are produced by changing electric fields. Ordinary electric currents, whether steady or varying, produce an accompanying magnetic field in the vicinity of the current. Maxwell predicted that a magnetic field also must be associated with a changing electric field even in the absence of a conduction current.

As electric charges do not flow through the insulation from one plate of a capacitor to the other, there is no conduction current but (instead) a displacement current is said to be present to account for the continuity of magnetic effect. The size of the displacement current between the plates of the capacitor being charged and discharged in an alternating current circuit is equal to the size of the conduction current in the wires leading to and from the capacitor.

$\therefore\ I_D = I$

$$\Rightarrow I_D = \frac{dQ}{dt} \quad \text{——— } (1)$$

For capacitor,

$$Q = CV \quad \text{——— } (2)$$

$\therefore$ from eq. ① and ②, we get.

$$I_D = \frac{d(CV)}{dt}$$

$$\Rightarrow I_D = C\frac{dV}{dt} \quad \text{——— ③}$$

Equation ③ gives displacement current in terms of capacitance of capacitor.

Now,

$$I_D = \frac{d}{dt}\left(\frac{\epsilon_0 A}{d} \cdot V\right) \quad \text{---- (for parallel plate capacitor } C = \frac{\epsilon_0 A}{d}\text{)}$$

$$\Rightarrow I_D = \frac{d}{dt}\left(\epsilon_0 A\left(\frac{V}{d}\right)\right)$$

$$\Rightarrow I_D = \epsilon_0 \frac{d}{dt}(EA) \quad \text{---- } \left(\because E = \frac{V}{d}\right)$$

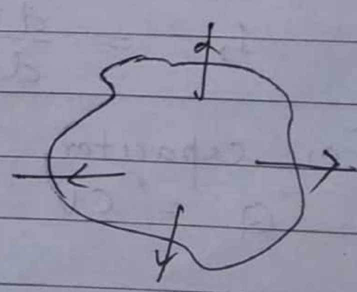
$$\Rightarrow \boxed{I_D = \epsilon_0 \frac{d\phi}{dt}} \quad \text{——— ④ ... } (\because \phi = EA)$$

Hence, displacement current arises due to the change in electric field (i.e. electric flux) in the circuit.

Equation of Continuity:

For a closed surface,

$$I = -\frac{dQ}{dt} \quad \text{——— ①}$$



where, negative sign

indicates current loss.

We have,

$$I = \oint \vec{J} \cdot \vec{ds} \quad \text{———} \quad (2)$$

and $$Q = \oint \rho_v \, dV \quad \text{———} \quad (3)$$

$\therefore$ from eq. (1), (2) and (3), we get,

$$\oint \vec{J} \cdot \vec{ds} = -\frac{\partial}{\partial t} \oint \rho_v \, dV$$

$$\Rightarrow \oint \vec{J} \cdot \vec{ds} = -\oint \left(\frac{\partial \rho_v}{\partial t}\right) dV$$

$$\Rightarrow \oint (\vec{\nabla} \cdot \vec{J}) \, dV = \oint \left(-\frac{\partial \rho_v}{\partial t}\right) dV$$ ——— (by Gauss divergence theorem)

By comparing, we get,

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho_v}{\partial t}$$

$$\Rightarrow \boxed{\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho_v}{\partial t} = 0} \quad \text{———} \quad (4)$$

Equation (4) is called equation of continuity and is another form of law of conservation of electric charge.

For steady state,

$$\frac{\partial \rho_v}{\partial t} = 0$$

$\therefore$ from eq. (4),

$$\boxed{\vec{\nabla} \cdot \vec{J} = 0} \quad \text{—— (5)}$$

Maxwell's Equations in Integral Form :

Following are Maxwell's equations in integral form :

(i) Gauss's law in electrostatics :

$$\oint \vec{E} \cdot \vec{ds} = \frac{q}{\epsilon_0}$$

(ii) Gauss's law in magnetostatics :

$$\oint \vec{B} \cdot \vec{ds} = 0$$

(iii) Faraday's law of electromagnetic induction :

$$\oint \vec{E} \cdot \vec{dl} = -\frac{\partial \phi_B}{\partial t}$$

(iv) Ampere - Maxwell's circuital law :

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 (I + I_D)$$

$$\Rightarrow \oint \vec{B} \cdot \vec{dl} = \mu_0 \left(I + \epsilon_0 \frac{\partial \phi_E}{\partial t} \right)$$

Conversion of Maxwell's Equations from Integral to Differential Form

(i) Gauss's law in electrostatics :

We have,

$$\oint \vec{E} \cdot \vec{ds} = \frac{q}{\epsilon_0} \text{ ——— } (1)$$

Here,

$$q = \oint \rho_v \cdot$$

$$\therefore \oint \vec{E} \cdot \vec{ds} = \frac{1}{\epsilon_0} \oint \rho_v \, dV$$

$$\Rightarrow \oint \vec{E} \cdot \vec{ds} = \oint \left(\frac{\rho_v}{\epsilon_0}\right) dV$$

$$\Rightarrow \oint (\vec{\nabla} \cdot \vec{E}) \, dV = \oint \left(\frac{\rho_v}{\epsilon_0}\right) dV$$ ——— (by Gauss divergence theorem)

by comparing, we get,

$$\boxed{\vec{\nabla} \cdot \vec{E} = \frac{\rho_v}{\epsilon_0}} \text{ ——— } (2)$$

(ii) Gauss's law in magnetostatics :

We have,

$$\oint \vec{B} \cdot \vec{ds} = 0$$

$$\Rightarrow \oint (\vec{\nabla} \cdot \vec{B})\, dV = 0 \quad \text{--- (by Gauss divergence theorem)}$$

By comparing, we get,

$$\boxed{\vec{\nabla} \cdot \vec{B} = 0} \quad \text{———— (1)}$$

(iii) Faradays law of electromagnetic induction:

We have,

$$\oint \vec{E} \cdot \vec{dl} = -\frac{\partial \phi_B}{\partial t} \quad \text{———— (1)}$$

Here,

$$\phi_B = \oint \vec{B} \cdot \vec{ds}$$

∴ from eq. (1),

$$\oint \vec{E} \cdot \vec{dl} = -\frac{\partial}{\partial t} \oint \vec{B} \cdot \vec{ds}$$

$$\Rightarrow \oint \vec{E} \cdot \vec{dl} = -\oint \frac{\partial \vec{B}}{\partial t} \cdot \vec{ds}$$

$$\Rightarrow \oint (\vec{\nabla} \times \vec{E}) \cdot \vec{ds} = \oint \left(-\frac{\partial \vec{B}}{\partial t}\right) \cdot \vec{ds} \quad \text{--- (by Stoke's law)}$$

∴ By comparing, we get,

$$\boxed{\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}} \quad \text{———— (2)}$$

(iv) Ampere-Maxwell's Circuital law:

We have,

$$\oint \vec{B}\cdot\vec{dl} = \mu_0\left(I + \epsilon_0 \frac{\partial \phi_E}{\partial t}\right) \text{ —— } ①$$

Here,

$$I = \oint \vec{J}\cdot\vec{ds} \text{ —— } ②$$

and

$$\phi_E = \oint \vec{E}\cdot\vec{ds} \text{ —— } ③$$

∴ from eq. ①, ② & ③,

$$\oint \vec{B}\cdot\vec{dl} = \mu_0\left[\oint \vec{J}\cdot\vec{ds} + \epsilon_0 \frac{\partial}{\partial t}\oint \vec{E}\cdot\vec{ds}\right]$$

$$\Rightarrow \oint \vec{B}\cdot\vec{dl} = \mu_0\left[\oint \vec{J}\cdot\vec{ds} + \epsilon_0 \oint \frac{\partial \vec{E}}{\partial t}\cdot\vec{ds}\right]$$

$$\Rightarrow \oint \vec{B}\cdot\vec{dl} = \mu_0\left[\oint \vec{J}\cdot\vec{ds} + \oint\left(\epsilon_0 \frac{\partial \vec{E}}{\partial t}\right)\cdot\vec{ds}\right]$$

$$\Rightarrow \oint \vec{B}\cdot\vec{dl} = \mu_0 \oint\left(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right)\cdot\vec{ds}$$

$$\Rightarrow \oint (\vec{\nabla}\times\vec{B})\cdot\vec{ds} = \mu_0 \oint\left(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right)\cdot\vec{ds} \text{ --- (by stoke's law)}$$

$$\Rightarrow \oint (\vec{\nabla}\times\vec{B})\cdot\vec{ds} = \oint \mu_0\left(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right)\cdot\vec{ds}$$

∴ By comparing, we get.

$$\boxed{\vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right)} \text{ —— } ④$$

Wave Equation in free space :

Maxwell's equations are

$$\vec{\nabla}\cdot\vec{E} = 0$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

and

$$\vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J} + \epsilon_0\frac{\partial \vec{E}}{\partial t}\right)$$

In free space, there is no charge.

so

$$\rho_v = 0 \quad \text{and} \quad J = 0$$

∴ Maxwell's equation for free space are

$$\vec{\nabla}\cdot\vec{E} = 0 \quad \text{———} \quad (1)$$

$$\vec{\nabla}\cdot\vec{B} = 0 \quad \text{———} \quad (2)$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{———} \quad (3)$$

and

$$\vec{\nabla}\times\vec{B} = \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t} \quad \text{———} \quad (4)$$

Now, taking curl on both sides of eq. (3),

we get,

$$\vec{\nabla}\times\vec{\nabla}\times\vec{E} = \vec{\nabla}\times\left(-\frac{\partial \vec{B}}{\partial t}\right)$$

$$\Rightarrow \vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - (\vec{\nabla}\cdot\vec{\nabla})\vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B})$$

$$\Rightarrow 0 - \nabla^2\vec{E} = -\frac{\partial}{\partial t}\left(\mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t}\right) \quad \text{———} \quad (\text{from eq.}(4))$$

$$\Rightarrow -\nabla^2\vec{E} = -\mu_0\epsilon_0 \frac{\partial^2\vec{E}}{\partial t^2}$$

$$\Rightarrow \nabla^2\vec{E} = \mu_0\epsilon_0 \frac{\partial^2\vec{E}}{\partial t^2}$$

$$\Rightarrow \boxed{\nabla^2\vec{E} - \mu_0\epsilon_0 \frac{\partial^2\vec{E}}{\partial t^2} = 0} \quad \text{——— (5)}$$

Again, taking curl on both sides of eq. (4), we get,

$$\vec{\nabla}\times\vec{\nabla}\times\vec{B} = \vec{\nabla}\times \mu_0\epsilon_0 \frac{\partial\vec{E}}{\partial t}$$

$$\Rightarrow \vec{\nabla}(\vec{\nabla}\cdot\vec{B}) - (\vec{\nabla}\cdot\vec{\nabla})\vec{B} = \mu_0\epsilon_0\, \vec{\nabla}\times\frac{\partial\vec{E}}{\partial t}$$

$$\Rightarrow 0 - \vec{\nabla}^2\vec{B} = \mu_0\epsilon_0 \frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E})$$

$$\Rightarrow -\vec{\nabla}^2\vec{B} = \mu_0\epsilon_0 \frac{\partial}{\partial t}\left(-\frac{\partial\vec{B}}{\partial t}\right) \quad \text{- - - (from eq. (3))}$$

$$\Rightarrow -\nabla^2\vec{B} = -\mu_0\epsilon_0 \frac{\partial^2\vec{B}}{\partial t^2}$$

$$\Rightarrow \nabla^2\vec{B} = \mu_0\epsilon_0 \frac{\partial^2\vec{B}}{\partial t^2}$$

$$\Rightarrow \boxed{\nabla^2\vec{B} - \mu_0\epsilon_0 \frac{\partial^2\vec{B}}{\partial t^2} = 0} \quad \text{——— (6)}$$

Comparing eq. (5) and eq. (6) with general equation of wave,

$$\nabla^2\vec{A} + \frac{1}{v^2}\frac{\partial^2\vec{A}}{\partial t^2} = 0 \quad \text{——— (7)}$$

we get,

$$\frac{1}{v^2} = \mu_0 \epsilon_0$$

$$\Rightarrow v^2 = \frac{1}{\mu_0 \epsilon_0}$$

$$\Rightarrow v = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

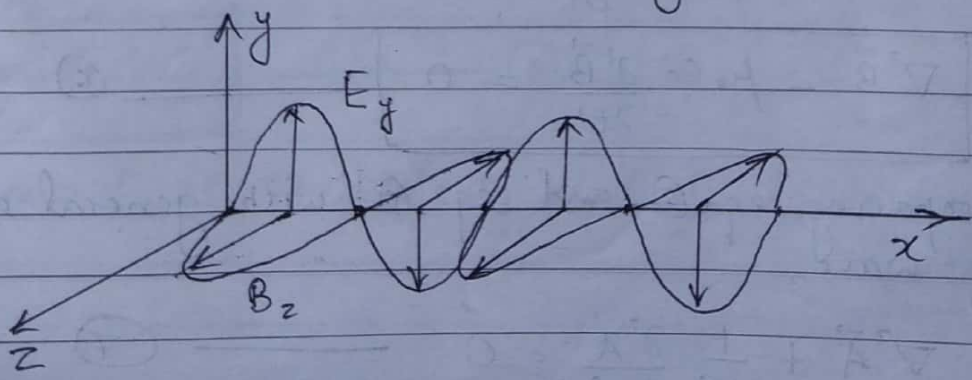
$$\Rightarrow v = \frac{1}{\sqrt{4\pi \times 10^{-7} \times 8.85 \times 10^{-12}}}$$

$$\Rightarrow v = 3 \times 10^{8} \ ms^{-1}$$

$$\Rightarrow v = c$$

$\therefore$ speed of electromagnetic wave is equal to the speed of light in free space or vacuum.

Relation Between Electric Field and Magnetic Field in Electromagnetic Wave:



Consider an electromagnetic wave propogating in the x-direction. Let electric field is along y-axis and magnetic field is changing along z-axis. Therefore, the wave equations are

$$\frac{\partial^2 E_y}{\partial x^2} - \mu_0 \epsilon_0 \frac{\partial^2 E_y}{\partial t^2} = 0$$

and

$$\frac{\partial^2 B_z}{\partial x^2} - \mu_0 \epsilon_0 \frac{\partial^2 B_z}{\partial t^2} = 0$$

Whose solutions are given by

$$E_y = E_m \sin(\omega t - kx) \quad \text{—— (1)}$$

and

$$B_z = B_m \sin(\omega t - kx) \quad \text{—— (2)}$$

where, E_m and B_m are amplitudes of electric and magnetic fields.

Now, from Faradays Law of electromagnetic induction,

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = -\frac{\partial}{\partial t}(B_x \hat{i} + B_y \hat{j} + B_z \hat{k})$$

Here,

$$E_x = E_z = 0$$

and

$$B_x = B_y = 0$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_y & 0 \end{vmatrix} = -\frac{\partial}{\partial t}\left(0.\hat{i} + 0.\hat{j} + B_z\hat{k}\right)$$

$$\Rightarrow \hat{i}\left(0 - \frac{\partial E_y}{\partial z}\right) + \hat{j}\left(0 - 0\right) + \hat{k}\left(\frac{\partial E_y}{\partial x} - 0\right) = -\frac{\partial}{\partial t}\left(0\hat{i} + 0\hat{j} + B_z\hat{k}\right)$$

Comparing the coefficients of $\hat{k}$, we get,

$$\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}$$

$$\Rightarrow \frac{\partial}{\partial x}\left(E_m \sin(\omega t - kx)\right) = -\frac{\partial}{\partial t}\left(B_m \sin(\omega t - kx)\right)$$

$$\Rightarrow -kE_m \cos(\omega t - kx) = -\omega B_m \cos(\omega t - kx)$$

$$\Rightarrow kE_m = \omega B_m$$

$$\Rightarrow \frac{E_m}{B_m} = \frac{\omega}{k}$$

$$\Rightarrow \frac{E_m}{B_m} = \frac{2\pi f}{(2\pi/\lambda)}$$

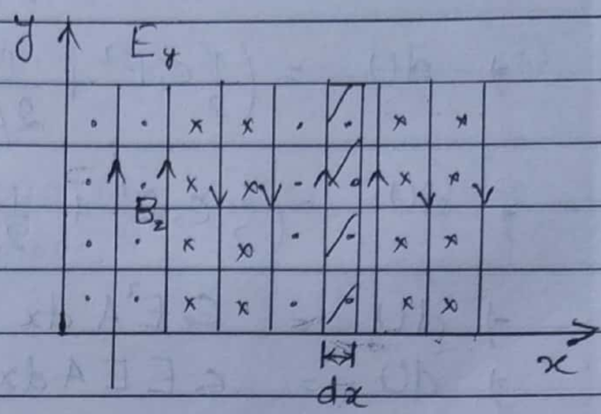
$$\Rightarrow \frac{E_m}{B_m} = f\lambda$$

$$\Rightarrow \boxed{\frac{E_m}{B_m} = c} \quad \text{——— (8)}$$

where, c is the velocity of light in vacuum.

Poynting Vector

The flow of energy per unit time per unit area in a plane electromagnetic wave is called poynting vector. It is denoted by S and



$$S = \frac{1}{A}\frac{dU}{dt} \quad \text{——— ①}$$

where, dU is the energy flowing in time 'dt' from area A held perpendicular to direction of propogation of electromagnetic wave.

Let wave coveres 'dx' distance in time 'dt'.

The energy dU is given by

$$dU = dU_E + dU_B$$

$$\Rightarrow dU = u_E A dx + u_B A dx$$

$$\Rightarrow dU = (u_E + u_B) A dx$$

where,

$u_E = \frac{1}{2}\epsilon_0 E^2$ and $u_B = \frac{B^2}{2\mu_0}$ are energy densities in electric and magnetic fields.

$$\therefore dU = \left(\frac{1}{2}\epsilon_0 E^2 + \frac{B^2}{2\mu_0}\right) A dx$$

$\Rightarrow dU = \left(\frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}\frac{E^2}{c^2}\right)A\,dx$ ---- $\left(\because c = \frac{E}{B} \Rightarrow B = \frac{E}{c}\right)$

$\Rightarrow dU = \left(\frac{1}{2}\epsilon_0 E^2 + \frac{E^2}{2\mu_0\left(\frac{1}{\mu_0\epsilon_0}\right)}\right)A\,dx$ --- $\left(c = \frac{1}{\mu_0\epsilon_0}\right)$

$\Rightarrow dU = \left(\frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\epsilon_0 E^2\right)A\,dx$

$\Rightarrow dU = \epsilon_0 E^2 A\,dx$

$\Rightarrow dU = \epsilon_0 E E A\,dx$

$\Rightarrow dU = \epsilon_0 E c B A\,dx$ --- ② $\left(c = \frac{E}{B} \Rightarrow E = cB\right)$

So from eq. ① and ②,

$$S = \frac{1}{A}\cdot\frac{\epsilon_0 E c B A\,dx}{dt}$$

$$\Rightarrow S = \frac{1}{\cancel{A}}\cdot \epsilon_0 E B \cancel{A} c\frac{dx}{dt}$$

$\Rightarrow S = \epsilon_0 E B c^2$ ---- $\left(\because c = \frac{dx}{dt}\right)$

$$\Rightarrow S = \epsilon_0 E B\cdot\frac{1}{\mu_0\epsilon_0}$$

$\Rightarrow \boxed{S = \frac{1}{\mu_0} E B}$ ——— ③

In vector form,

$\boxed{\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B})}$ ——— ④

Here,

$E = E_m \sin(\omega t - kx)$

and $B = B_m \sin(\omega t - kx)$

$\therefore \quad S = \frac{1}{\mu_0} E_m \sin(\omega t - kx) \, B_m \sin(\omega t - kx)$

$\Rightarrow \quad S = \frac{1}{\mu_0} E_m B_m \sin^2(\omega t - kx)$ ——— (5)

The average value of poynting vector is also called intensity of the wave.

$\therefore$ Intensity is given by

$I = S_{avg}$

$\Rightarrow \quad I = \frac{1}{\mu_0} E_m B_m \left[\frac{1}{T} \int_0^T \sin^2(\omega t - kx) \, dt \right]$

$\Rightarrow \quad \boxed{I = \frac{E_m B_m}{2\mu_0}}$ ——— (6)