INTRODUCTION

1.1 Definition and Scope of Applied Mechanics

Definition:

Mechanics can be defined as that science which describes and predicts the conditions of rest or motion of bodies under the action of forces. It is divided into three parts: *mechanics of rigid bodies, mechanics of deformable bodies,* and *mechanics of fluids.* The mechanics of rigid bodies is subdivided into *statics* and *dynamics,* the former dealing with bodies at rest, the latter with bodies in motion.

Scope:

Applied mechanics is the foundation of most engineering sciences and is prerequisite to their study. It can be used to explain and predict natural and physical phenomena and thus forms the foundation for engineering application. Knowledge of applied mechanics is must for the design and analysis of many types of structural members, mechanical components, and electrical devices encountered in engineering.

1.2 Concept of Rigid Body and Deformable Body

A *rigid body* is such type of body which does not deform even if very large force is applied on it. Most of the bodies considered in elementary mechanics are assumed to be rigid. Actual structures and machines, however, are never absolutely rigid and deform under the loads to which they are subjected. But these deformations are usually small and do not appreciably affect the conditions of equilibrium or motion of the structure under consideration.

For the study of statics, it is necessary to assume a body as "perfectly rigid" because if the body is not considered perfect rigid, then there will be some deformation which will lead to unstatic conditions.

Deformable body is such a body, which when on applying a force on it, there is appreciable change in the shape and size of it.

The differences between rigid body and deformable body are tabulated below:

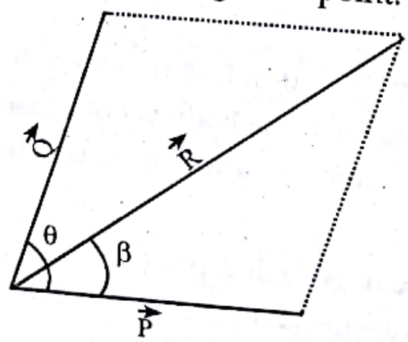
Rigid body	Deformable body
i. In rigid body, particles are so interconnected that they don't change their position however large the forces applied may be.	i. In deformable body, under application of forces, particles change their position.
ii. A perfectly rigid body does not change its shape and size under the application of forces but only changes its position.	ii. Deformable body changes its shape and size.
iii. Since deformation of body is of negligible amount, condition of equilibrium is used to analyze rigid body.	iii. Condition of equilibrium can't be used as deformation is large even under small forces.
E.g., stone, RCC block	E.g., chalk, tennis ball

1.3 Fundamental Concepts and Principles of Mechanics: Newtonian Mechanics

The study of applied mechanics is based on the following six fundamental principles:

i. **The Parallelogram Law of Forces**

It states that if two forces, acting at a point be represented in magnitude and direction by the two adjacent sides of a parallelogram, then their resultant is represented in magnitude and direction by the diagonal of the parallelogram passing though that point.



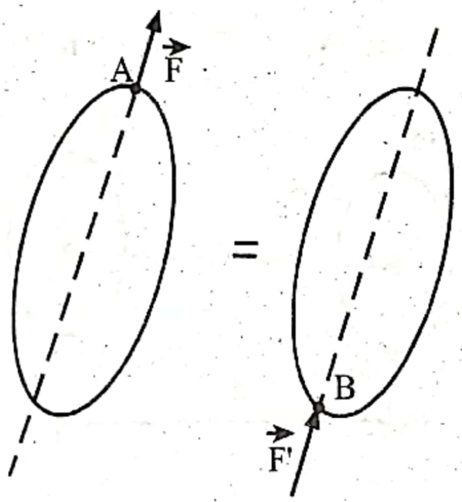
Mathematically, $\vec{R} = \vec{P} + \vec{Q}$

$$|\vec{R}| = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$$

$$\beta = \tan^{-1}\left(\frac{Q\sin\theta}{P + Q\cos\theta}\right)$$

ii. **Principle of Transmissibility of Force**

It states that the condition of equilibrium or of motion of a rigid body will remain unchanged if a force $\vec{F}$ acting at a given point of the rigid body is replaced by a force $\vec{F'}$, of the same magnitude and direction but acting at different point, provided that the two forces have the same line of action.



Here, force $\vec{F}$ acting at point A is replaced by a force $\vec{F'}$ such that $|\vec{F}| = |\vec{F'}|$ at B and having same line of action.

Clearly, $\vec{F'} = \vec{F}$

iii. **Newton's First Law**

It states that every body continues to be in the state of rest or motion unless an external force acts upon it.

iv. **Newton's Second Law**

It states that the rate of change of linear momentum is directly proportional to the applied force and it occurs in the direction of applied force.

Mathematically,

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{dm\vec{v}}{dt} = m\frac{d\vec{v}}{dt} = m\vec{a}$$

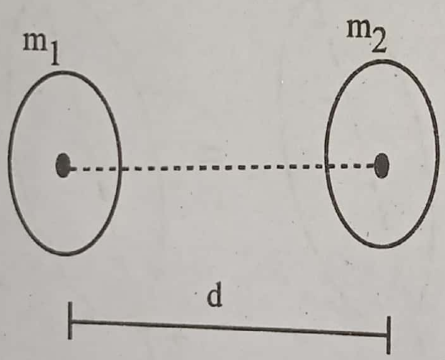
v. **Newton's Third Law**

It states that to every action, there is an equal and opposite reaction.

vi. **Newton's Law of Gravitation**

It states that every body in the universe attracts other body with a force proportional to the product of their masses and inversely proportional to the square of the distance separating them.

$F = \frac{G m_1 m_2}{d^2}$, where $G = 6.67 \times 10^{-12}\ Nm^2/kg^2$



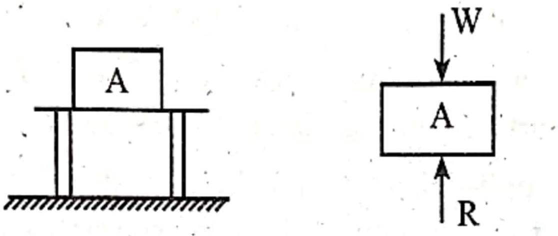
CHAPTER - 2

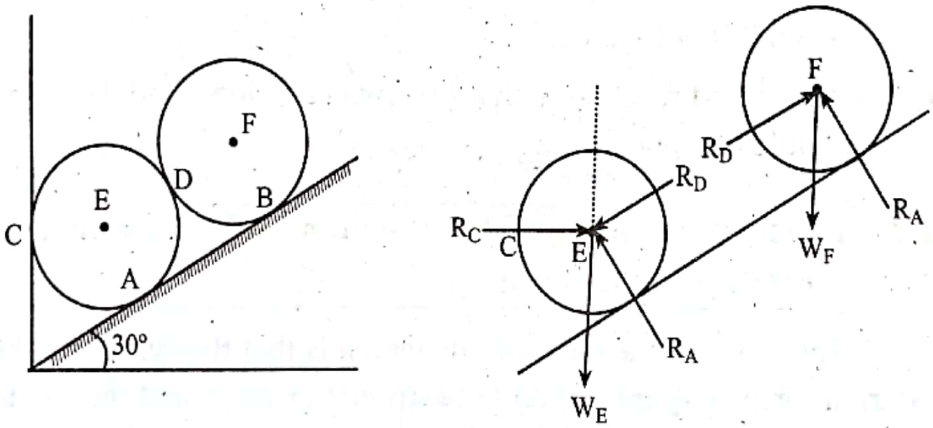
BASIC CONCEPT IN STATICS AND STATIC EQUILIBRIUM

2.1 Concept of Particle and Free Body Diagram

Particle means a very small amount of matter which may be assumed to occupy a single point in space. A *rigid body* is a combination of a large number of particles occupying fixed positions with respect to each other.

Free body diagram is a sketch of the body (space diagram) drawn in such a way that it shows all the reaction forces, applied forces, and moment on the body.





Importance of free body diagram:

i. We can analyze the complex body by isolating it from the system of bodies.

ii. We can easily apply equilibrium equation on free body diagram.

iii. Free body diagram is sketch of the isolated body showing all the forces acting on it by vector.

iv. We can adopt any co-ordinate system whose axes are not only in horizontal and vertical direction.

v. Free body diagram indicates each applied load including the weight of isolated body.

vi. Free body diagram indicates all the dimensions including slope.

Guidelines for drawing good free body diagram:

i. The body to be freed for consideration may be the entire system or any portion of the system. So, it is important to make a clear decision as to which portion of the system is to be freed.

ii. The free body diagram should have no supports or connections.

iii. Any adopted coordinate system whose axes are not in the horizontal or vertical directions should be shown.

iv. Appropriate dimensions are needed for defining appropriate configuration of force system.

v. Each applied load should be indicated with an arrow and labeled either with its known magnitude or with a letter when it is not known.

vi. Action and reaction with an arrow head along with labelling should be done.

2.2 Physical Meaning of Equilibrium and its Essence in Structural Application

The physical meaning of *equilibrium* is that the system of the external forces will impart no translational or rotational motion to the body considered.

Principle of Equilibrium

Principle of equilibrium states that a stationary body which is subjected to coplanar forces (concurrent or parallel) will be in equilibrium if the algebraic sum of all the external forces is zero and also algebraic sum of moments of all the external forces about any point in their plane is zero.

Mathematically,

$\Sigma F = 0, \Sigma M = 0$

Importance of equilibrium in structure analysis:

Condition of equilibrium is the central theme for structural analysis. Civil engineers analyze and design the structures (bridges, buildings, transmission tower) standing on the concept of equilibrium.

2.3 Equation of Equilibrium in Two and Three Dimension

Two-dimensional (2-D) analysis for equilibrium condition:

i. **For a particle**

A particle will be in equilibrium condition if algebraic sum of all the coplanar forces acting on it is zero.

$\Sigma \vec{F}_p = 0$; where p denotes particle

Generally, the forces are resolved into horizontal and vertical components.

$\therefore \Sigma(F_x)_p = 0$ and $\Sigma(F_y)_p = 0$

ii. **For a rigid body**

A rigid body will be in equilibrium condition if the algebraic sum of all coplanar external forces is zero and the algebraic sum of moments of all the forces about any point in their plane is zero.

Mathematically,

$\Sigma\vec{F} = 0, \Sigma\vec{M} = 0$

which can be resolved as:

$\Sigma F_x = 0, \Sigma F_y = 0, \Sigma M = 0$

Three-dimensional (3-D) analysis for equilibrium condition:

i. **For a particle**

A particle will be in equilibrium condition if algebraic sum of all external forces acting on a point is zero. The external forces considered now may not be necessarily coplanar as 3-D analysis is being performed.

Thus, we can write

$$\Sigma\overrightarrow{F}_p = 0$$

On resolving into three reference directions, we have

$$\Sigma(F_x)_p = 0, \Sigma(F_y)_p = 0, \Sigma(F_z)_p = 0$$

ii. **For a rigid body**

For a rigid body to be in equilibrium condition under the action of external space forces, the algebraic sum of all these forces should be equal to zero.

Thus, the equilibrium equations for 3-D are

$$\Sigma\overrightarrow{F} = 0, \Sigma\overrightarrow{M} = 0$$

which can be further resolved as:

$$\Sigma F_x = 0, \Sigma F_y = 0, \Sigma F_z = 0$$

$$\Sigma M_x = 0, \Sigma M_y = 0, \Sigma M_z = 0$$

where ΣM_x = algebraic sum of moment about yz plane.

ΣM_y = algebraic sum of moment about xz plane.

ΣM_z = algebraic sum of moment about xy plane.

CHAPTER - 3

FORCES ACTING ON PARTICLE AND RIGID BODY

3.1 Force

Force is defined as an external agent which changes or tends to change the speed, direction, or shape of system.

The characteristics of force are:

- Force has both magnitude and direction. So, it is a vector quantity.
- Force has point of application.
- Force is a transmissible vector i.e., it can be moved along its line of action.

Forces may be classified as:

1. **Point force**
   - It is a force which is assumed to act through point.
2. **Body force**
   - It is a force which acts on each element of body. Examples: gravitational force, inertia force, electromagnetic force, etc.
3. **Surface force**
   - It is a force which acts on the surface or area elements of the body. When the area considered lies on the actual boundary of the body, the surface force distribution is termed as *surface traction.*

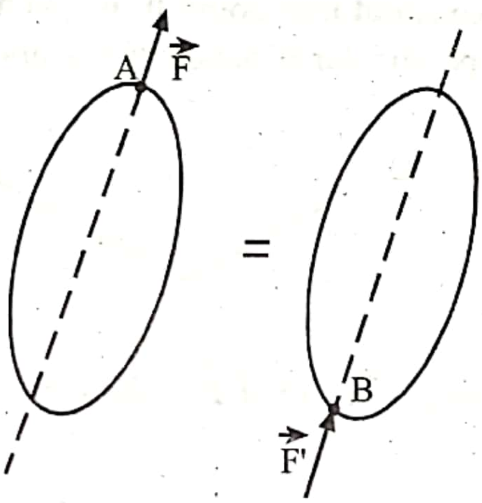
Force may also be classified as:

1. **Translation force**
   - The force which moves or tends to move a body from one point to another point is called *translation force.*
2. **Rotational force**
   - The force which rotates or tends to rotate a body around a central axis is called *rotational force.*

3.2 Principle of Transmissibility and Equivalent Forces

Principle of Transmissibility of Force

It states that the condition of equilibrium or of motion of a rigid body will remain unchanged if a force $\vec{F}$ acting at a given point of the rigid body is replaced by a force $\vec{F'}$, of the same magnitude and direction, but acting at different point, provided that the two forces have the same line of action.



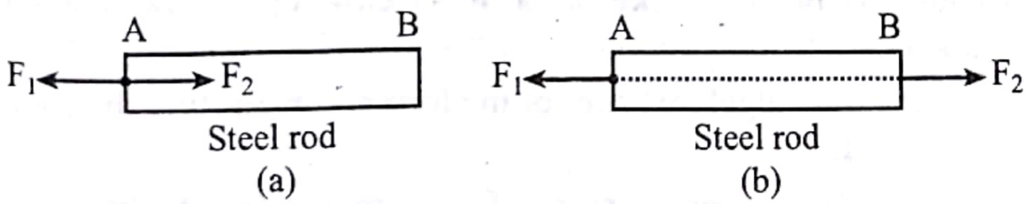
Here, Force $\vec{F}$ acting at point A is replaced by a force $\vec{F'}$ such that $|\vec{F'}| = |\vec{F}|$ at B and having same line of action.

Clearly, $\vec{F'} = \vec{F}$

Limitation:

Consider a steel rod in equilibrium. Apply two forces of same magnitude but in opposite direction at A. The equilibrium condition of steel rod is not disturbed yet.

But as per principle of transmissibility, the force 'F' can be replaced at B without disturbing its line of action as shown in figure (b).

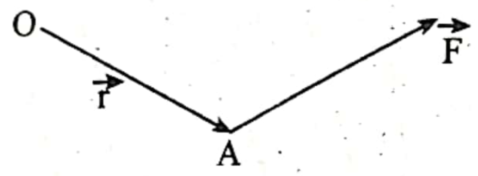


If we observe fig. (b), the rod is in tension and will have some deflections depending upon its area, length, modulus of elasticity magnitude of force. This tensile force will result in the elongation of that steel bar which is contradiction of principle of transmissibility.

Thus, principle of transmissibility is applicable only to the rigid body in which we neglect the internal effects.

3.3 Moments and Couples

Moment of force about a point is defined as the turning tendency of a force about that point. It is given by the product of force and the perpendicular distance of the line of action of force from that point.



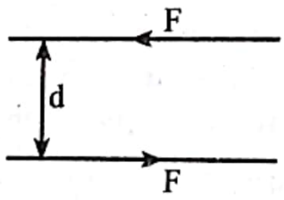
In vector form, moment of $\vec{F}$ about O is expressed as

$$\vec{M_O} = \vec{r} \times \vec{F}$$

where $\vec{r}$ = position vector of A w.r.t. O

A system formed by two parallel forces equal in magnitude but opposite in direction separated by a finite distance is known as *couple.*

Moment of couple = F × d



Characteristics of couple:

A couple (whether clockwise or anticlockwise) has the following characteristics:

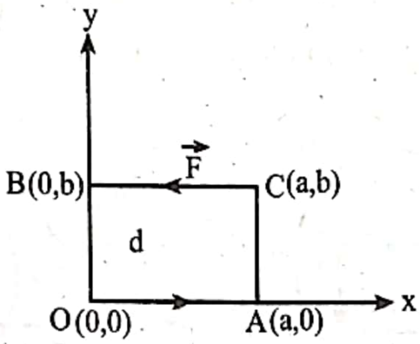
1. The algebraic sum of the forces, constituting the couple is zero.
2. Th algebraic sum of the moments of the forces, constituting the couple, about any point is the same, and equal to the moment of the couple itself.

3. A couple cannot be balanced by a single force, but can be balanced only by a couple, but of opposite sense.
4. Any number of coplanar couples can be reduced to a single couple, whose magnitude will be equal to the algebraic sum of the moments of all the couples.

Couple is a Free Vector

Proof:

To show couple is a free vector, we have to show moment of couple about any point will be same.



Consider a couple as shown in figure acting on a xy-plane. The couple is acting on a rectangular element of length a and breadth b.

Taking moment about O,

$\vec{M}_O = \vec{r}_{OB} \times (-F\hat{i})$ (Note: $\vec{r}_{AB} = \vec{r}_{OB} - \vec{r}_{OA}$)

$= \{(0,b) - (0,0)\} \times (-F\hat{i})$

$= +b\hat{j} \times -F\hat{i} = bF\hat{k}$

Taking moment about A,

$\vec{M}_A = \vec{r}_{AC} \times \vec{F}$

$= \{(a,b) - (a,0)\} \times (-F\hat{i})$

$= (a\hat{i} - b\hat{j}) \times (-F\hat{i}) = bF\hat{k}$

Taking moment about B,

$\vec{M}_B = \vec{r}_{BO} \times F\hat{i}$

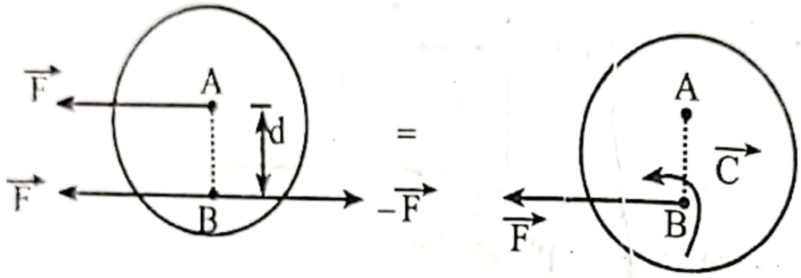
$= \{(0,0) - (0,b)\} \times F\hat{i}$

$= -b\hat{j} \times F\hat{i} = bF\hat{k}$

Since $M_O = M_A = M_B$, it is proved that couple is a free vector.

3.4 Resolution of a Force into Force and a Couple

Any force $\vec{F}$ acting on a rigid body can be moved to an arbitrary point O provided that a couple is added whose moment is equal to the moment of $\vec{F}$ about O. Let a force $\vec{F}$ acts at point A on the body as shown in figure below. If it is desired to transfer it to point B, we replace equal and opposite force $\vec{F}$ at point B. The second equivalent system can be viewed as force $\vec{F}$ acting at B and couple $\vec{C}$ whose magnitude is F×d, where d = distance between A and B.



This couple $\vec{C}$ and $\vec{F}$ are perpendicular to each other

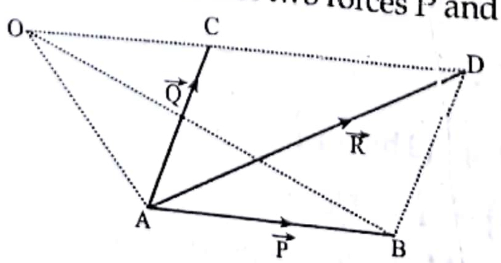
$$|\vec{C}| = F \times d$$

$$\vec{C} = \vec{r}_{BA} \times \vec{F}$$

3.5 Varignon's Theorem

Varignon's theorem states that moment of a resultant of two forces about a point lying in the plane of the forces is equal to the algebraic sum of moments of these forces about the same point.

Consider two concurrent forces P and Q as shown. Let O be the point about which moment is to be calculated. Through O, draw a line parallel to the direction of force P which meets the line of action of force Q at C. Let R be the resultant of two forces P and Q.



Moment of force P about O (M_P) = 2×ΔAOB

Moment of force Q about O (M_Q) = 2 ×ΔAOC

Moment of force P about O (M_R) = 2 ×ΔAOD

From figure,

ΔAOD = ΔAOC + ΔACD

or, ΔAOD = ΔAOC + ΔABD

or, ΔAOD = ΔAOC + ΔAOB

or, 2 × ΔAOD = 2 × ΔAOC + 2 × ΔAOB

$\therefore M_R = M_Q + M_P$

This principle can be extended for number of forces i.e., moment of resultant of a number of forces about a point lying in the plane of forces is equal to the algebraic sum of moments of these forces about the same point.

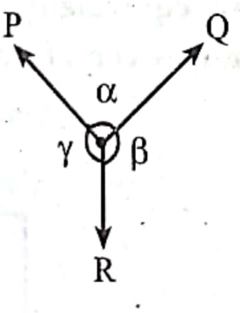
$$\vec{r} \times (\vec{F_1} + \vec{F_2} + \vec{F_3} + \ldots\ldots) = \vec{r} \times \vec{F_1} + \vec{r} \times \vec{F_2} + \vec{r} \times \vec{F_3} + \ldots\ldots$$

$$\therefore \vec{r} \times \vec{F} = \vec{r} \times \vec{F_1} + \vec{r} \times \vec{F_2} + \vec{r} \times \vec{F_3} + \ldots\ldots$$

3.6 Lami's Theorem

Lami's theorem states that if a body is in equilibrium under the action of three forces, then, each force is proportional to the sine of the angle between the other two forces.

Mathematically,

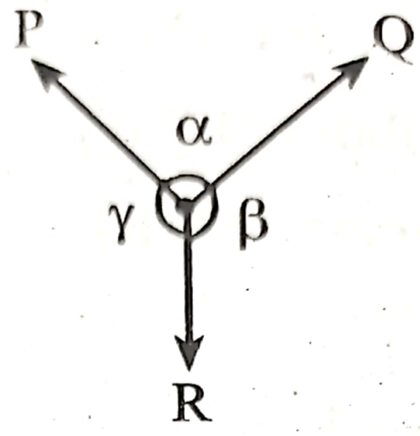


$$\frac{P}{\sin\beta} = \frac{Q}{\sin\gamma} = \frac{R}{\sin\alpha}$$

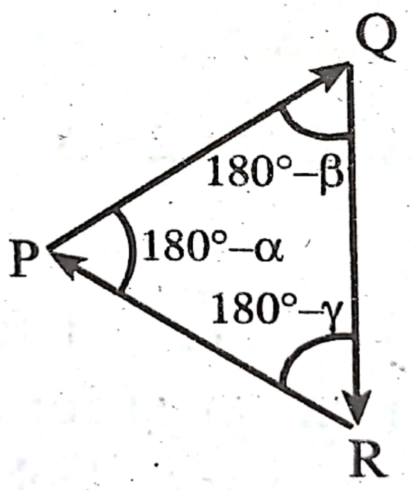
All the forces must be outgoing from a common point.

Proof:

Let P, Q, R be the three concurrent forces in equilibrium as shown in figure.



Forming a triangle with these forces,



Using sine law,

$$\frac{P}{\sin(180°-\beta)} = \frac{Q}{\sin(180°-\gamma)} = \frac{R}{\sin(180°-\alpha)}$$

$$\therefore \frac{P}{\sin\beta} = \frac{Q}{\sin\gamma} = \frac{R}{\sin\alpha} \text{ proved.}$$

CENTRE OF GRAVITY, CENTROID AND MOMENT OF INERTIA

4.1 Centre of Gravity and Centroid

Centre of gravity is defined as the point through which the whole weight of the body is assumed to act. It is denoted by c.g. or G.

Center of mass is the point where the whole mass of body is assumed to act. It differs from the centre of gravity only when the gravitational field is not uniform and non-parallel.

The *centroid* or *centre of area* is defined as the point where the whole area of the figure is assumed to be concentrated. It is analogous to centre of gravity when a body has area but not weight. Centroid may also be defined as a point in a plane area such that the moment of area about any axis through that point is zero.

Centroid of line $(\bar{x}, \bar{y})$ can be calculated as:

$$\bar{x} = \frac{\int x\, dl}{\int dl}, \quad \bar{y} = \frac{\int y\, dl}{\int dl}$$

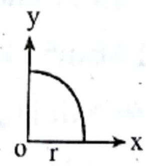
Centroid of area $(\bar{x}, \bar{y})$ can be calculated as:

$$\bar{x} = \frac{\int x\, dA}{\int dA}, \quad \bar{y} = \frac{\int y\, dA}{\int dA}$$

Centroid of common shapes of lines:

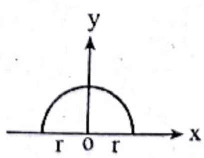
a. Quarter-circular arc

$\bar{x} = \frac{2r}{\pi}$ $\quad \bar{y} = \frac{2r}{\pi}$ $\quad$ Length $= \frac{\pi r}{2}$



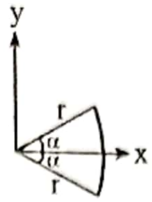
b. Semi-circular arc

$\bar{x} = 0$ $\quad \bar{y} = \frac{2r}{\pi}$ $\quad$ Length $= \pi r$



c. Arc of circle

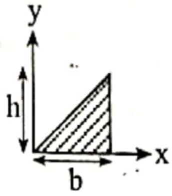
$$\bar{x} = \frac{r \sin \alpha}{\alpha} \qquad \bar{y} = 0 \quad \text{Length} = 2\alpha r$$



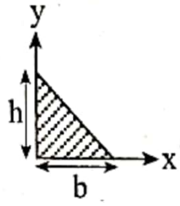
Centroid of common shapes of areas:

a. Triangular area

$$\bar{x} = \frac{2b}{3} \qquad \bar{y} = \frac{h}{3} \qquad \text{Area} = \frac{bh}{2}$$

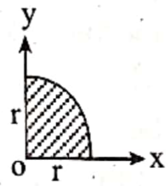


$$\bar{x} = \frac{b}{3} \qquad \bar{y} = \frac{h}{3} \qquad \text{Area} = \frac{bh}{2}$$



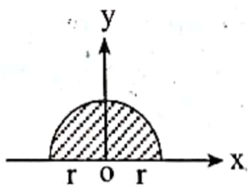
b. Quarter-circular area

$$\bar{x} = \frac{4r}{3\pi} \qquad \bar{y} = \frac{4r}{3\pi} \qquad \text{Area} = \frac{\pi r^2}{4}$$



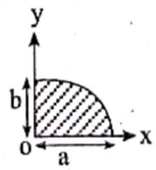
c. Semicircular area

$$\bar{x} = 0 \qquad \bar{y} = \frac{4r}{3\pi} \qquad \text{Area} = \frac{\pi r^2}{2}$$



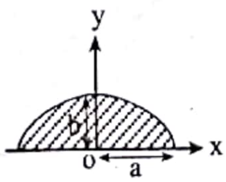
d. Quarter-elliptical area

$$\bar{x} = \frac{4a}{3\pi} \qquad \bar{y} = \frac{4b}{3\pi} \qquad \text{Area} = \frac{\pi ab}{4}$$



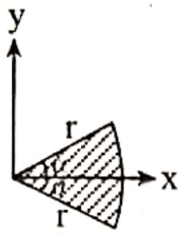
e. Semi-elliptical area

$$\bar{x} = 0 \qquad \bar{y} = \frac{4b}{3\pi} \qquad \text{Area} = \frac{\pi ab}{2}$$



f. Circular sector

$\bar{x} = \dfrac{2r \sin\alpha}{3\alpha}$ $\bar{y} = 0$ Area $= \alpha r^2$



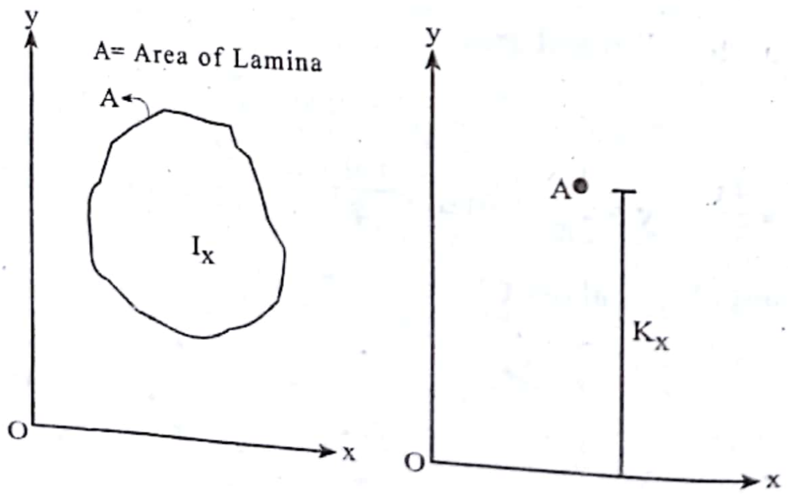
4.2 First Moment of Area, Second Moment of Area/Moment of Inertia, and Radius of Gyration

The integral $\int x \, dA$ is known as the first moment of area A with respect to the y-axis. Similarly, the integral $\int y \, dA$ is known as the first moment of area A with respect to the x-axis.

If the first moment of inertia is again multiplied by the perpendicular distance between axes and centroid, the product so obtained is called *second moment of area* or *moment of inertia.*

Inertia refers to the property of a body by virtue of which body resists any change in its state of rest or of uniform motion. The moment of inertia of a rigid body is a quantity that determines the torque needed for a desired angular acceleration about a rotational axis. So, moment of inertia is essentially a measure of resistance to bending and is applied while dealing with the deflection or deformation of member due to bending.

If the entire area of a lamina is concentrated at a point such that there is no change in moment of inertia about a given axis, then distance of that point from the given axis is called *radius of gyration.*



Suppose MOI of an irregular lamina about x-axis is I_x. If at a distance of k_x from x-axis, its whole area is distributed then, moment of inertia about x-axis can be written as

$$I_x = k_x^2 A$$

$$\text{or, } k_x = \sqrt{\frac{I_x}{A}}$$

This distance k_x is called *radius of gyration w.r.t. x-axis.*

$$\text{Similarly, } k_y = \sqrt{\frac{I_y}{A}}$$

where the distance k_y is called *radius of gyration w.r.t. y-axis.*

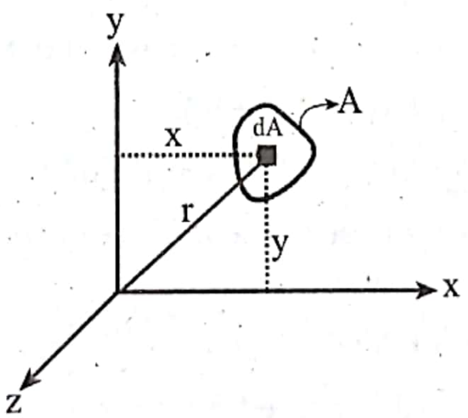
4.3 Perpendicular Axis Theorem

Statement:

If I_x and I_y be the moment of inertia of a plane section about two perpendicular axes, x-axis and y-axis meeting at O respectively, the moment of inertia about the axis z, perpendicular to both x-axis and y-axis is given by

$$I_z = I_x + I_y$$

Proof:



Consider a small lamina of area dA at x and y distance from y-axis and x-axis respectively.

From figure, $r^2 = x^2 + y^2$

From definition, $I_z = \int r^2\, dA$

$$= \int (x^2+y^2)\, dA$$

$$= \int x^2 dA + \int y^2\, dA$$

$$= I_x + I_y \qquad \text{proved.}$$

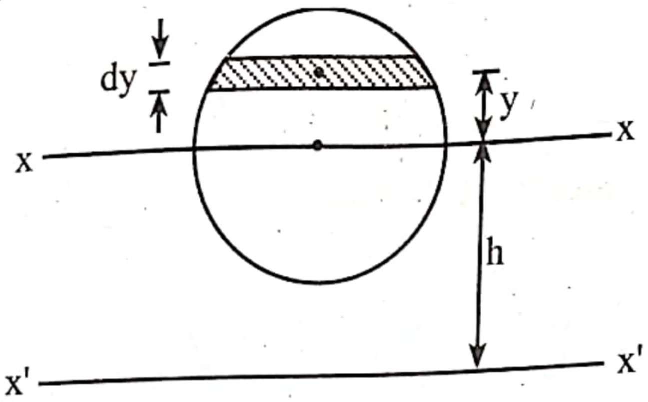
4.4 Parallel Axis Theorem

Statement:

If the moment of inertia of a plane area, A, about an axis through its C.G. is $(I_G)_x$, then moment of inertia of the area about any other axis x'x', parallel to xx and at a distance h from C.G. is given by

$$I_{x'} = (I_G)_x + Ah^2$$

Proof:



Let us consider a strip with elemental area dA at a distance y from centroidal axis xx, then

$$(I_G)_x = \sum dAy^2$$

$$I_{x'} = \sum dA\,(y+h)^2$$

where h = distance between xx-axis and x'x'-axis.

$$= \sum dA\,(y^2 + 2yh + h^2)$$

$$= \sum y^2dA + 2h\sum dAy + h^2 \sum dA$$

Since $\sum dAy$ = First moment of area about centroidal axis = 0, we get

$$I_{x'} = (I_G)_x + 0 + Ah^2$$

$\therefore\ I_{x'} = (I_G)_x + Ah^2$ proved.

CHAPTER • 5

FRICTION

5.1 Friction

When a body slides over another body, a force is exerted at a surface of contact by the stationary body on a moving body. This resisting force is called the *force of friction* and acts in the direction opposite to the direction of motion.

Characteristic of friction:

i. It always acts in a direction opposite to that in which motion is intended.

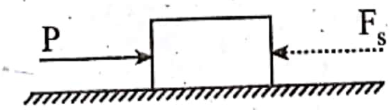
ii. It exists as along as the tractive force acts. So, it is passive force.

iii. It is self-adjusting force i.e., only that much comes to play as is just sufficient to prevent motion.

5.1.1 Static and Dynamic Friction

Static Friction

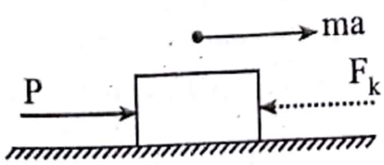
The frictional force that develops between mating surfaces when subjected to external force but there is no relative motion between them is called *static friction*.



If force P acts towards right, then F_s equal to P acts in opposite direction to prevent motion. This force F_s is static friction.

Dynamic Friction

The frictional force that develops between mating surfaces when subjected to external forces and there is relative motion between them is called *dynamic friction* (also known as *kinetic friction*).



If force P acts towards right, then F_k less than P acts in opposite direction and the body moves with acceleration a.

5.1.2 Laws of Solid Friction (Static or Dynamic)

Based on the experiment, following laws of friction have been stated:

i. Friction acts tangential to the surface in contact and is in a direction opposite to that in which motion is to take place.

ii. Frictional force is maximum at the instant of impending motion.

iii. The magnitude of limiting friction bears a constant ratio to the normal reaction between the mating surfaces. This ratio drops to a slightly lower value when motion starts.

iv. Limiting friction is independent of the area and shape of contact surface.

v. Limiting friction depends upon the nature of the surface in contact.

vi. At low velocity, friction is independent of the velocity. But at higher speed, there will be slight reduction in friction.

5.1.3 Advantages and Disadvantages of Friction

Advantages:

i. We can easily walk on road due to good friction which is too difficult to do on ice due to poor friction.

ii. Friction helps to transmit power from the motors and engine to other machines by making use of belts and clutches.

iii. Brakes make use of friction to stop vehicles.

iv. A nail can be easily fixed on the wall due to friction.

Disadvantages:

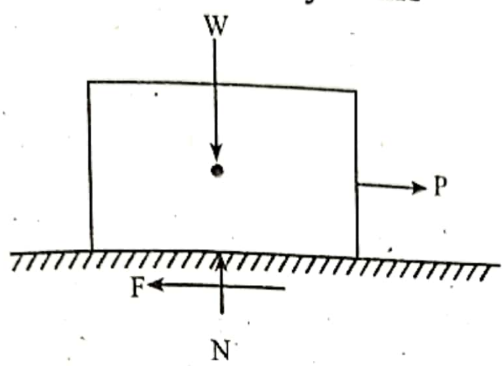
i. Friction causes unnecessary wear and tear of the machinery parts.

ii. Friction causes heat to develop within the machinery which in fact reduces the life of machine.

iii. Efficiency of a machine is decreased due to friction. A part of useful energy is dissipated in overcoming the friction.

5.2 Some Terminologies

Coefficient of Friction: Static and Dynamic



The frictional force is proportional to normal reaction i.e., $F \propto N$. The ratio $\frac{F}{N}$ i.e., the ratio of frictional force to normal reaction is called *coefficient of friction.*

When the system is in state of impending motion (i.e., when the body just comes to motion), the frictional force has maximum value F_s which is known as *static friction*. The ratio $\frac{F_s}{N}$ is called *coefficient of static friction* and is denoted by μ_s.

$$\mu_s = \frac{F_s}{N}$$

When motion starts, the maximum frictional force falls to some lower value F_k which is known as *kinetic friction*. The ratio $\frac{F_k}{N}$ is called *coefficient of kinetic friction* and is denoted by μ_k.

$$\mu_k = \frac{F_k}{N}$$

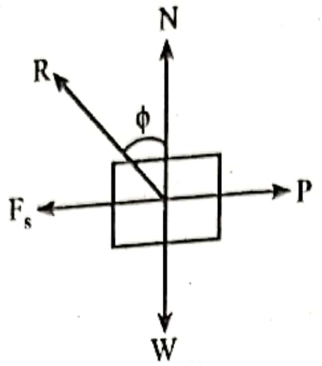
Limiting Friction

The maximum value of frictional force which acts on the body when it just starts to slide over another body is called *limiting friction*.

Mathematically, $F_{max} = F_{lim} = \mu_s N$

Angle of Friction

Angle of friction (ϕ) is defined as the angle which the resultant of normal reaction and limiting frictional force makes with the normal reaction.



where

N = Normal reaction

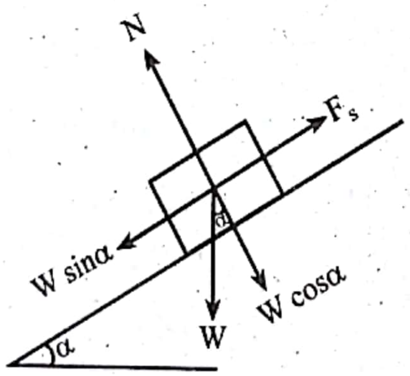
F_s = Limiting friction

R = Resultant of N and $F_s = \sqrt{N^2 + F_s^2}$

$$\tan\phi = \frac{F_s}{N} = \frac{\mu_s N}{N} = \mu_s$$

Angle of Repose

Angle of repose is defined as the angle α of the inclined plane at which the block resting on it is about to slide down the plane.



At equilibrium,

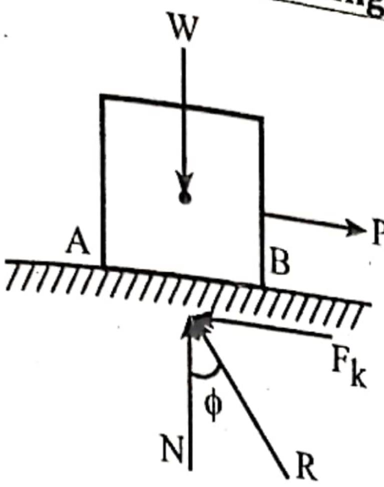
$F_s = W \sin\alpha$

or, $\mu_s N = W \sin\alpha$

or, $\mu_s W \cos\alpha = W \sin\alpha$

or, $\mu_s = \dfrac{W \sin\alpha}{W \cos\alpha} = \tan\alpha$

5.3 Condition of Tipping (Overturning) and Sliding

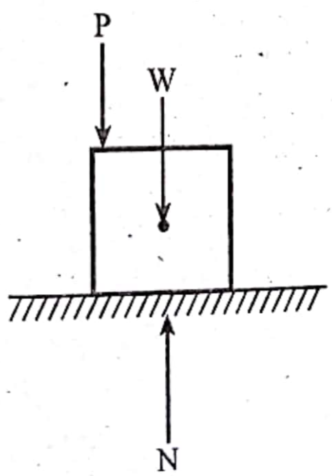


When the applied horizontal force P Increases, frictional force also increases slowly form 0 to F_s (maximum frictional force). Since the applied force is directed towards right end of the block so that couples formed by P and F as well as W and N remain balanced. If R reaches end B of the block before F reaches its maximum value F_s, then the couples will be imbalanced causing overturning or tipping about B. If F reaches to its maximum frictional force F_s earlier than the R reaches end B of the block, the block will slide as P and F will cause more clockwise moment than anticlockwise moment due to N and W.

5.4 No Friction, No Motion, Impending Motion, and Motion

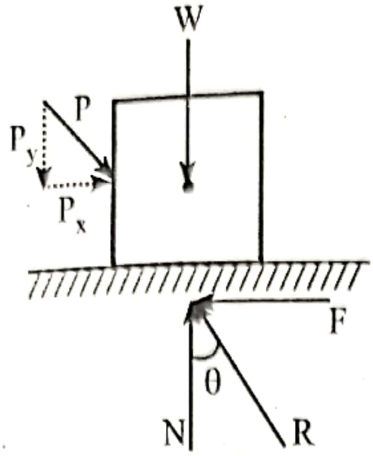
No Friction

When the applied force P is perpendicular to the surface of contact, no friction exists as only vertical forces are present and no horizontal component of force exist.



No Motion

In this case P is in inclined position.

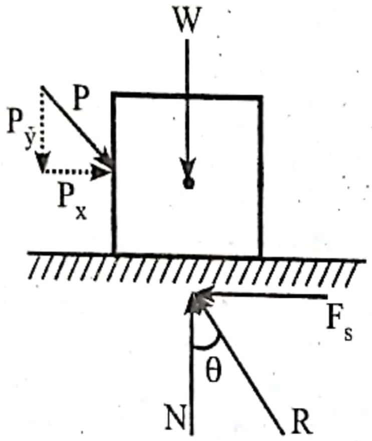


$F = P_x$

$\theta < \phi_s$

This force trends to move the body along the surface but are not large enough to set in motion since angle between N (normal reaction) and resultant (R) i.e., Q is less than angle of static friction (ϕ_s).

Impending Motion



$F_s = P_x$

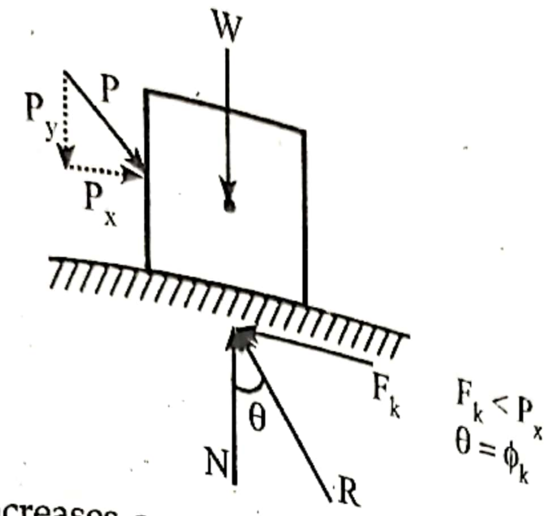
$\theta = \phi_s$

As P_x increases gradually, the resultant R also increases and the angle between N and R becomes ϕ_s. The body is now in the verge of motion.

Here, frictional force = Limiting frictional force

= Maximum frictional force

$(F_s)_{max} = \mu_s N$

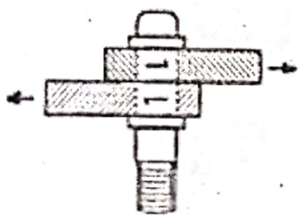


As P_x increases a condition occurs in which the block is in motion. The corresponding frictional force is known as kinetic frictional force F_k and angle is ϕ_k.

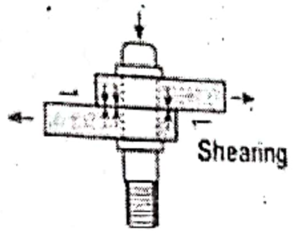
$$F_k = \mu_k N$$

5.5 High Tension Friction Grip Bolts

High tension friction grip bolts are high strength structural bolts which have been tightened such as to induce tension in the bolt shank. Due to the tension in the bolt, the interface between the piles (steel member in joint) cannot move relative to each other because of frictional resistance. The bolt act differently than normal bolts or rivets. Friction along interface takes load in case of high tension friction grip bolt subject to shear.



(a) Ordinary bolt



(b) High tension friction grip bolt

ANALYSIS OF BEAMS AND FRAMES

6.1 Beams and Frames

A structural member designed to support loads applied at various points along the member is known as a *beam*. In most cases, the loads are perpendicular to the axis of the beam and will cause only shear and bending in the beam. When the loads are not at a right angle to the beam, they will also produce axial forces in the beam.

Classification of beam:

i. Cantilever beam
ii. Simply supported beam
iii. Overhanging beam
iv. Fixed beam
v. Continuous beam

A *frame* is a network of beams and columns joined together to carry loads and transfer it to the support.

Classification of frame:

i. Perfect frame
ii. Imperfect frame
  - Deficient
  - Redundant

6.2 Different Types of Load and Support

6.2.1 Types of Load

Loads are classified as:

i. Point load /concentrated load

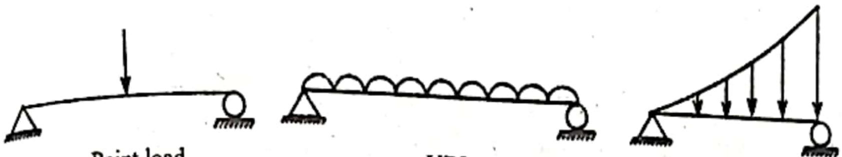
Loads which act on smaller areas are called *concentrated* or *point loads*.

ii. Uniformly distributed load (UDL)

Load which is distributed uniformly along the length of member is called *uniformly distributed load*.

iii. **Uniformly varying load (UVL)**

Load that varies along the length of the structure is called *uniformly varying load*. Load variation may be in linear, parabolic, cubic fashion, etc.



iv. **Couple**

It is the combination of two equal and opposite forces separated by a certain distance.

v. **Moment**

It is the product of force and distance.

vi. **Static and dynamic load**

The loads which do not vary with time are called *static loads* whereas the loads which change with time are *dynamic loads*.

vii. **Imposed load**

Imposed load may be defined as the load that is applied to the structure which is not permanent. Examples: Snow load, wind load, earthquake load.

viii. **Dead and live load**

Dead loads are the self weight of the structures i.e., self weight of slab, beam, column, finishing loads. Such loads do not change their position.

Live load is the load imposed on the structure when it fulfills its design purpose. The position and magnitude of live load may change.

6.2.2 Types of Support

Supports are classified as:

i. **Roller support**

It gives rise to one force reaction which is perpendicular to the plane on which it rests.

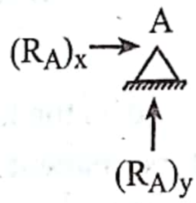
Roller support

- Free to move in x-direction
- Free to rotate about z-direction (axis)
- Degree of freedom = 2

Since it is restrained in y-direction, reaction is developed in this direction only.

ii. Hinge support

It gives rise to one force reaction whose direction is unknown and can be resolved into two forces along x and y axes.

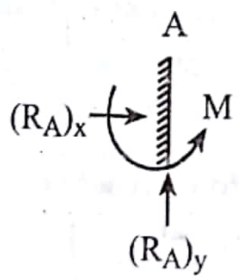


Hinge support

- Free to rotate about z-direction.
  So, its degree of freedom = 1
- Restrained in x and y-direction.
  So, no. of reactions developed = 2

iii. Fixed support

It gives rise to one force reaction having two components and one moment reaction.



Fixed support

- No translation in x and y-direction
- No rotation about z-direction
- Restrained in all three possible directions.

So, degree of freedom = 0

No. of reactions = 3

6.3 Axial Force, Shear Force, and Bending Moment

On application of external loads on a structure, following forces are developed:

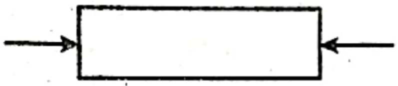
1. **Axial force**

   A force lying along the longitudinal axis of the member which produces either compression or tension is the *axial force.*

   Tensile force is positive axial force and compression is negative axial force.

+ve

Tensile axial force

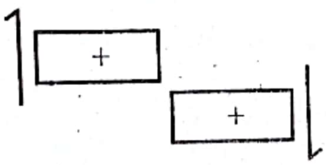


–ve

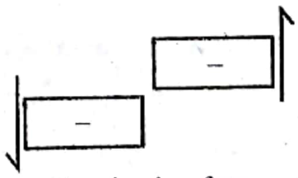
Compressive axial force

2. **Shear force**

   Unbalanced force along the transverse direction is known as *shear force*. Shearing force having an upward direction to the left side of the section is positive or downward force to the right side of the section is positive. The reverse is negative.



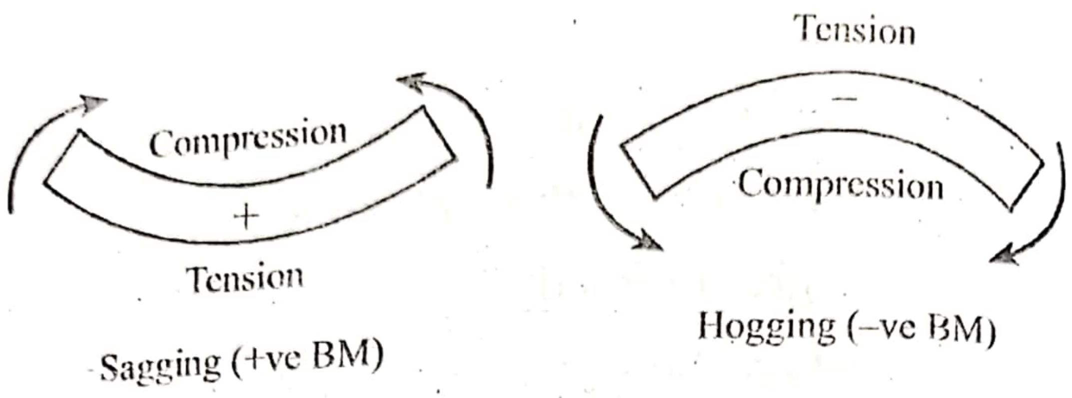
Positive shear force



Negative shear force

3. **Bending moment**

   Unbalanced moment of any section is known as *bending moment* for a particular section. Sagging bending moment is taken as +ve while hogging bending moment is taken as –ve.



6.4 Static Determinacy and Indeterminacy

A structure is said to be *statically determinate* if all the internal member forces and reactions can be determined using the equation of static equilibrium.

For beam,

External indeterminacy = r − (3 + c)

Internal indeterminacy = 0

Total degree of static indeterminacy = r − (3+c) + 0 = r − (3+c)

For frame,

External indeterminacy = r − (3 + c)

Internal indeterminacy = 3 × total no. of cuts required to have open configuration

Total degree of static indeterminacy = (3m + r) − (3j + c)

where r = no. of support reactions, c = no. of equations due to special condition (internal hinge), j = no. of joints.

Note: Internal indeterminacy = Total indeterminacy − External indeterminacy

KINEMATICS OF PARTICLE AND RIGID BODY

Dynamics is the part of mechanics that deals with the analysis of bodies in motion. Dynamics is further classified into *kinematics* and *kinetics*.

Kinematics is the study of the geometry of motion. It is used to relate displacement, velocity, acceleration, and time, without reference to the cause of the motion.

8.1 Rectilinear Kinematics

Rectilinear Motion of Particles

A particle moving along a straight line is said to be in *rectilinear motion*.

The *average velocity* of the particle over the time interval Δt is defined as the quotient of the displacement Δx and the time interval Δt.

$$\text{Average velocity} = \frac{\Delta x}{\Delta t}$$

The *instantaneous velocity* v of the particle at the instant t is obtained from the average velocity by choosing shorter and shorter time intervals Δt and displacements Δx.

$$\text{Instantaneous velocity } (v) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

The *average acceleration* of the particle over the time interval Δt is defined as the quotient of Δv and Δt.

$$\text{Average acceleration} = \frac{\Delta v}{\Delta t}$$

The *instantaneous acceleration* a of the particle at the instant t is obtained from the average acceleration by choosing smaller and smaller values for Δt and Δv.

$$\text{Instantaneous acceleration } (a) = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$$

The acceleration a can also be expressed as:

$$a = \frac{d^2x}{dt^2}$$

$$a = v\frac{dv}{dx}$$

Uniform Rectilinear Motion

In *uniform rectilinear motion*, the acceleration of the particle is zero for every value of t. The velocity v is therefore constant.

$$\frac{dx}{dt} = v = \text{constant}$$

Uniformly Accelerated Rectilinear Motion

In this motion, the acceleration a of the particle is constant.

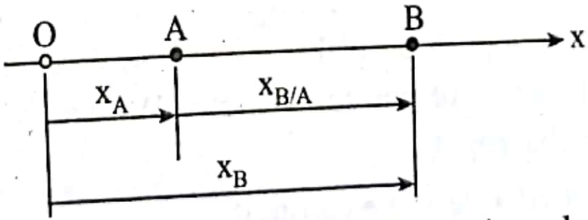
$$\frac{dv}{dt} = a = \text{constant}$$

Also, we can write

$$v\frac{dv}{dx} = a = \text{constant}$$

8.2 Motion of Several Particles

8.2.1 Relative Motion of Two Particles



Consider two particles A and B moving along the same straight line. Let position coordinate of A = x_A, position coordinate of B = x_B.

The *relative position of B w.r.t. A* is $x_{B/A}$. From figure, we can write

$$x_{B/A} = x_B - x_A$$

or, $x_B = x_A + x_{B/A}$ ....(i)

Differentiating equation (i) w.r.t. t, we get

$v_B = v_A + v_{B/A}$ ....(ii) where $v_{B/A}$ is known as relative velocity of B w.r.t. A.

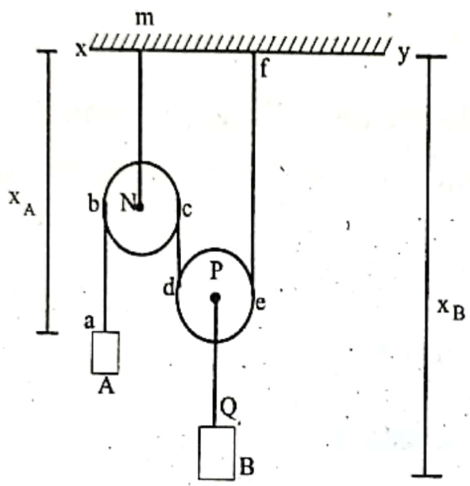
Again, differentiating equation (ii) w.r.t. t, we get

$a_B = a_A + a_{B/A}$ ....(ii)

where $a_{B/A}$ is known as relative acceleration of B w.r.t. A.

8.2.2 Dependent Motion

When the position of a particle depends upon the position of another or several other particles, the motion is known as *dependent motion*. Example: Pulley system



Explanation:

Let the position of A and B be x_A and x_B with reference to xy-line.

Clearly, The position of B depends on A. From figure, the rope abcdef is of constant length (We are considering an inextensible rope)

ab + bc + cd + de + ef = constant

The length of portion of rope bc and de wrapped around the pulley remains always same. So,

ab + cd + ef = constant.

$\Rightarrow (x_A + MN) + (x_B - MN - PQ) + x_B - PQ = \text{constant}$

We note that the length of MN, PQ remain constant

$\therefore x_A + 2x_B = \text{constant}$ ....(i)

On differentiating two times successively,

$v_A + 2v_B = 0$ ....(ii)

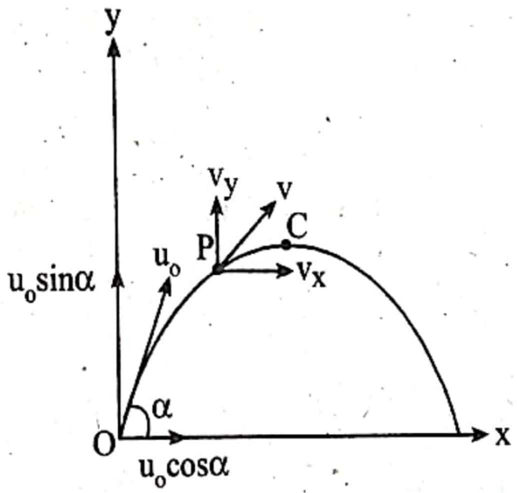
$a_A + 2a_B = 0$ ....(iii)

These equations suggest that motion of A is dependent on motion of B.

8.3 Curvilinear Motion

The motion of a particle along a curved path, other than a straight line is known as *curvilinear motion*. Examples: projectile motion, satellite motion.

8.3.1 Equations for Projectile Motion



Consider a particle projected with velocity u at an angle of α with the horizontal as shown in figure.

Suppose the particle reaches at point P after time t. Let the velocity of the particle at point P be v.

For motion in x-direction,

$a_x = 0,\ u_x = u\cos\alpha$

$x = u_x t = (u\cos\alpha)\,t$ ....(i)

For motion in y-direction,

$a_y = -g,\ v_y = u_y + a_y t = u\sin\alpha - gt$

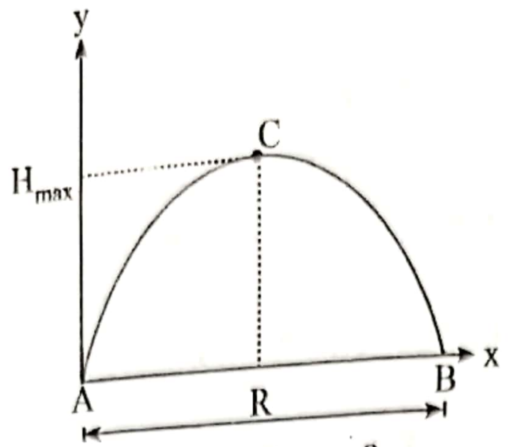
$y = (u_y)t + \frac{1}{2}a_y t^2 = (u\sin\alpha)t - \frac{1}{2}gt^2$

From equation (i),

$x = (u\cos\alpha)t \Rightarrow t = \frac{x}{u\cos\alpha}$

$y = (u\sin\alpha)\times\frac{x}{u\cos\alpha} - \frac{1}{2}\times g\times\left(\frac{x}{u\cos\alpha}\right)^2$

$$= x\tan\alpha - \frac{1}{2}\frac{gx^2}{u^2\cos^2\alpha}$$



At maximum height i.e., at C, $v_y = 0$

or, $u\sin\alpha - gt = 0$

If T is the time of flight, then it takes $\frac{T}{2}$ to reach C.

or, $u\sin\alpha - g\left(\frac{T}{2}\right) = 0$

or, $T = \frac{2u\sin\alpha}{g}$

When $t = T$, $x = R$

or, $(u\cos\alpha)T = R$

or, $R = u\cos\alpha \times \frac{2u\sin\alpha}{g} = \frac{u^2\sin 2\alpha}{g}$

When $t = \frac{T}{2}$, $y = H_{max}$

or, $(u\sin\alpha)\frac{T}{2} - \frac{1}{2}\times g \times \left(\frac{T}{2}\right)^2 = H_{max}$

or, $H_{max} = (u\sin\alpha)\frac{(2u\sin\alpha)}{2g} - \frac{1}{2}\times g \times \left(\frac{2u\sin\alpha}{2g}\right)^2$

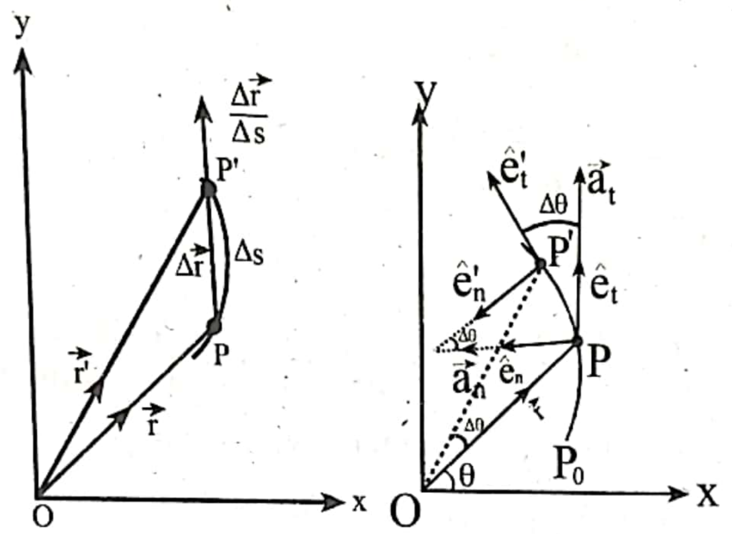
$$= \frac{u^2\sin^2\alpha}{g} - \frac{1}{2}\frac{u^2\sin^2\alpha}{g}$$

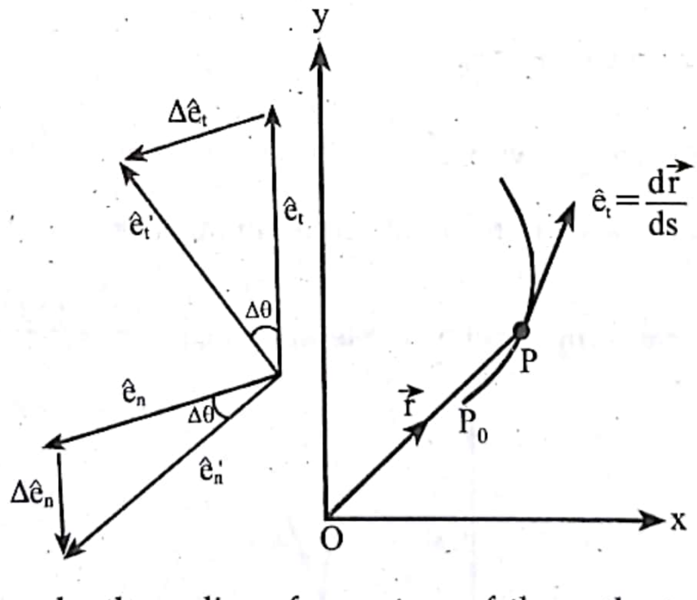
$$= \frac{u^2\sin^2\alpha}{g} - \frac{u^2\sin^2\alpha}{2g}$$

$\therefore H_{max} = \frac{u^2\sin^2\alpha}{2g}$

8.3.2 Tangential and Normal Components of Acceleration

Consider a particle moving in xy-plane as shown in figure. Let the particle be at P and P' at time t and t+Δt respectively. Let $\hat{e}_t$ and $\hat{e}_n$ be the unit vectors, directed along the tangent and along the normal or towards the center of curvature of the path respectively. Unit vector $\hat{e}_t$ and $\hat{e}_n$ will generally change in their directions as the particle moves from one point to another along the curved path.





Let ρ be the radius of curvature of the path at point P, $\hat{e}_t$ and $\hat{e}_t'$ be the tangent unit vectors at P and P'.

$$\Delta s = PP' = \rho\Delta\theta$$

$$\Delta\hat{e}_t = \hat{e}_t' - \hat{e}_t \cong \Delta\theta\hat{e}_n$$

$$\frac{d\theta}{ds} = \lim_{\Delta s \to 0} \frac{\Delta\theta}{\Delta s} = \lim_{\Delta s \to 0} \frac{\Delta\theta}{\rho\Delta\theta} = \frac{1}{\rho} \text{ ....(i)}$$

$$\frac{d\hat{e}_t}{d\theta} = \lim_{\Delta\theta \to 0} \frac{\Delta\hat{e}_t}{\Delta\theta} = \lim_{\Delta\theta \to 0} \frac{\Delta\theta\hat{e}_n}{\Delta\theta} = \hat{e}_n \text{ ....(ii)}$$

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \frac{\rho\Delta\theta}{\Delta t} = \frac{\rho d\theta}{dt}$$

or, $v = \rho\dot{\theta}$ ....(iii)

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt}$$

where, $\frac{d\vec{r}}{ds} = \lim_{\Delta s \to 0} \frac{\Delta\vec{r}}{\Delta s} = \hat{e}_t$, $\frac{ds}{dt} = v$

So, $\vec{v} = \hat{e}_t v = v\hat{e}_t$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d(v\hat{e}_t)}{dt} = \frac{dv}{dt}\hat{e}_t + \frac{v d\hat{e}_t}{dt} = \frac{dv}{dt}\hat{e}_t + \frac{v d\hat{e}_t}{d\theta}\frac{d\theta}{ds}\frac{ds}{dt} \text{ ....(iv)}$$

Using equations (i), (ii), (iii) in (iv), we have

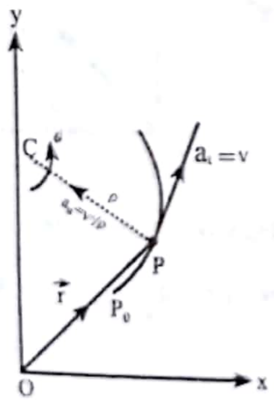
$$\vec{a} = \dot{v}\,\hat{e}_t + v(\hat{e}_n)\left(\frac{1}{\rho}\right)(v)$$

or, $\vec{a} = \dot{v}\,\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n$

Comparing it with $\vec{a} = a_t\hat{e}_t + a_n\hat{e}_n$, we get

tangential component of acceleration $(a_t) = \dot{v}$

normal component of acceleration $(a_n) = \frac{v^2}{\rho} = \frac{(\rho\dot{\theta})^2}{\rho} = \rho\,\dot{\theta}^2$



If y = f(x) is the equation of the path, then the radius of curvature of the path can be calculated as

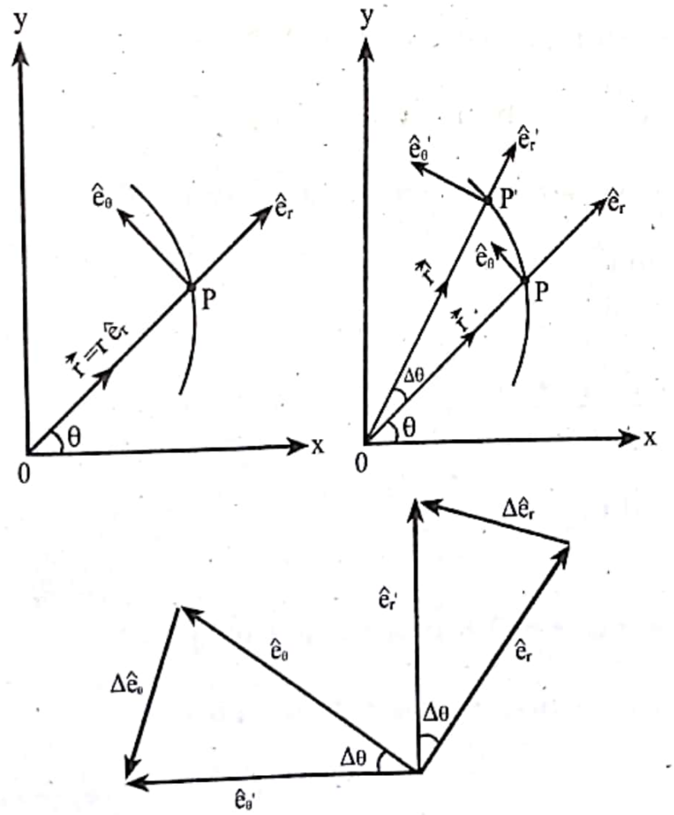
$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

The tangential component of the acceleration reflects a change in speed of the particle, while its normal component reflects a change in the direction of motion of the particle.

8.3.3 Radial and Transverse Components of Acceleration

In certain problems of plane motion, the position of the particle is defined by its polar coordinates r and θ. It is then convenient to resolve the velocity and acceleration of particle into radial and transverse components.

Consider that the position of the particle at P is defined by the polar coordinates r and θ as shown in figure. Let $\hat{e}_r$ and $\hat{e}_\theta$ denote the unit vector in radial and transverse direction respectively.



$$\dot{\hat{e}}_r = \frac{d\hat{e}_r}{dt} = \frac{d\hat{e}_r}{d\theta}\frac{d\theta}{dt} = \hat{e}_\theta\,\dot{\theta}$$

$$\dot{\hat{e}}_\theta = \frac{d\hat{e}_\theta}{dt} = \frac{d\hat{e}_\theta}{d\theta}\frac{d\theta}{dt} = -\hat{e}_r\,\dot{\theta}$$

$$\vec{v} = \frac{d\vec{r}}{dt}$$

$$= \frac{d(r\hat{e}_r)}{dt}$$

$$= \frac{r\,d\hat{e}_r}{dt} + \hat{e}_r\frac{dr}{dt}$$

$$= r\dot{\hat{e}}_r + \dot{r}\,\hat{e}_r$$

$$= r\dot{\theta}\,\hat{e}_\theta + \dot{r}\,\hat{e}_r$$

$$= \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta \text{ ....(i)}$$

Comparing equation (i) with $\vec{v} = v_r\hat{e}_r + v_\theta\,\hat{e}_\theta$, we have

Radial component of velocity $(v_r) = \dot{r}$

Transverse component of velocity $(v_\theta) = r\dot{\theta}$

$$\vec{a} = \frac{d\vec{v}}{dt}$$

$$= \frac{d(\dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta)}{dt}$$

$$= \frac{d(\dot{r}\,\hat{e}_r)}{dt} + \frac{d(r\dot{\theta}\,\hat{e}_\theta)}{dt}$$

$$= (\ddot{r}\,\hat{e}_r + \dot{r}\dot{\hat{e}}_r) + (\dot{r}\dot{\theta}\,\hat{e}_\theta + r\ddot{\theta}\,\hat{e}_\theta + r\dot{\theta}\dot{\hat{e}}_\theta)$$

$$= \ddot{r}\,\hat{e}_r + \dot{r}\dot{\theta}\,\hat{e}_\theta + \dot{r}\,\dot{\theta}\,\hat{e}_\theta + r\ddot{\theta}\,\hat{e}_\theta - r\,\dot{\theta}^2\,\hat{e}_r$$

$$[\because \dot{\hat{e}}_r = \hat{e}_\theta\dot{\theta},\ \dot{\hat{e}}_\theta = -\hat{e}_r\dot{\theta}\,]$$

or, $\vec{a} = (\ddot{r} - r\,\dot{\theta}^2)\,\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{e}_\theta$ ....(ii)

Comparing equation (ii) with $\vec{a} = a_r\,\hat{e}_r + a_\theta\,\hat{e}_\theta$, we have

Radial component of acceleration $(a_r) = \ddot{r} - r\,\dot{\theta}^2$

Transverse component of acceleration $(a_\theta) = r\ddot{\theta} + 2\dot{r}\dot{\theta}$

If we take the case of a particle moving along a circular path with its centre at the origin (O), we have

r = constant implies $\dot{r} = 0$ and $\ddot{r} = 0$

KINETICS OF PARTICLE AND RIGID BODY

Kinetics is the study of the relation existing between the forces acting on a body, the mass of the body, and the motion of the body. It is used to predict the motion caused by given forces or to determine the forces required to produce a given motion.

9.1 Newton's Second Law of Motion and Momentum

Newton's second law states that if the resultant force acting on a particle is not zero, the particle will have an acceleration proportional to the magnitude of the resultant and in the direction of the resultant force.

If a is the acceleration of particle when subjected to force F, then

$$F \propto a$$

or, $F = ma$

where m is proportionality constant known as mass of the particle.

In vector form,

$$\vec{F} = m\vec{a}$$

When a particle is subjected simultaneously to several forces, then

$$\Sigma \vec{F} = m\vec{a} \quad \text{............(i)}$$

where $\Sigma \vec{F}$ represents the resultant of all the forces acting on the particle.

Equation (i) can be expressed as

$$\Sigma \vec{F} = m\frac{d\vec{v}}{dt}$$

Since the mass m of the particle is constant, we have

$$\Sigma \vec{F} = \frac{d(m\vec{v})}{dt} \quad \text{..........(ii)}$$

The vector $m\vec{v}$ is called the *linear momentum* or simply the *momentum*, of the particle.

Denoting by $\vec{L}$ the linear momentum of the particle,

$$\vec{L} = m\vec{v}$$

Now, we can express equation (ii) as

$$\Sigma \vec{F} = \dot{\vec{L}}$$

Thus, Newton's second law may also be stated as:

The resultant of the forces acting on the particle is equal to the rate of change of the linear momentum of the particle.

9.2 Equations of Motion and Dynamic Equilibrium

Equations of Motion

From Newton's second law,

$$\Sigma \vec{F} = m\vec{a} \quad \text{............(i)}$$

In order to solve problems involving the motion of a particle, it is more convenient to replace equation (i) by equivalent equations involving scalar quantities.

(i) Rectangular components

$\Sigma F_x = ma_x$, $\Sigma F_y = ma_y$, $\Sigma F_z = ma_z$

(ii) Tangential and normal components

$$\Sigma F_t = ma_t = m\frac{dv}{dt}$$

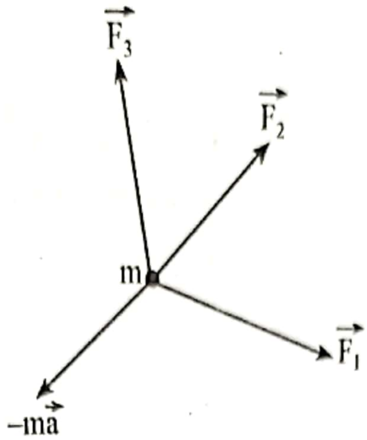
$$\Sigma F_n = ma_n = m\frac{v^2}{\rho}$$

(iii) Radial and transverse components

$$\Sigma F_r = ma_r = m(\ddot{r} - r\,\dot{\theta}^2)$$

$$\Sigma F_\theta = ma_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$$

Dynamic Equilibrium



From Newton's second law,

$$\sum \vec{F} = m\vec{a}$$

$$\text{or, } \sum \vec{F} - m\vec{a} = 0$$

which expresses that if we add the vector $-m\vec{a}$ to the forces acting on the particle, we obtain a system of vectors equivalent to zero. The vector $-m\vec{a}$, of magnitude ma and of direction opposite to that of the acceleration, is called an *inertia vector*. The particle may thus be considered to be in equilibrium under the given forces and the inertia vector. The particle is said to be in *dynamic equilibrium*.

In scalar quantities, we have:

(i) Rectangular components

$$\Sigma F_x - ma_x = 0$$

$$\Sigma F_y - ma_y = 0$$

$$\Sigma F_z - ma_z = 0$$

(ii) Tangential and normal components

$$\Sigma F_t - ma_t = 0$$

$$\Sigma F_n - ma_n = 0$$

(iii) Radial and transverse components

$$\Sigma F_r - ma_r = 0$$

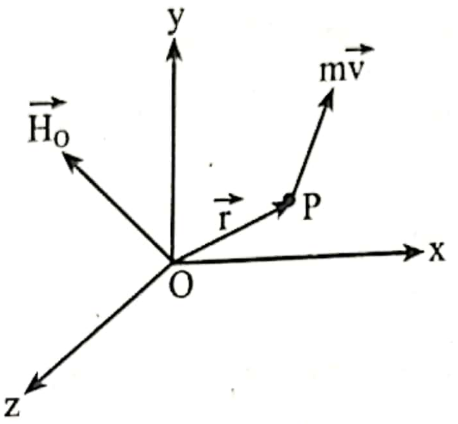
$$\Sigma F_\theta - ma_\theta = 0$$

9.3 Angular Momentum and Rate of Change

Angular Momentum

The moment about O of the vector $m\vec{v}$ is called the *moment of momentum*, or the *angular momentum*, of the particle about O at that instant and is denoted by $\vec{H}_O$.

$$\vec{H}_O = \vec{r} \times m\vec{v}$$



Here,

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$m\vec{v} = mv_x\hat{i} + mv_y\hat{j} + mv_z\hat{k}$$

So,

$$\vec{H}_O = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ mv_x & mv_y & mv_z \end{vmatrix}$$

The components of $\vec{H}_O$ are:

$$H_x = m(yv_z - zv_y)$$

$$H_y = m(zv_x - xv_z)$$

$$H_z = m(xv_y - yv_x)$$

In the case of a particle moving in the xy plane, we have $z = 0$ and hence, $H_x = H_y = 0$. The angular momentum is thus perpendicular to the xy plane and has the magnitude $H_O = H_z = m(xv_y - yv_x)$.

Rate of Change of Angular Momentum

Consider a particle moving in the xy plane i.e., $z = 0$ plane.

Then, we can write

$$H_O = H_z = mv_y x - mv_x y$$

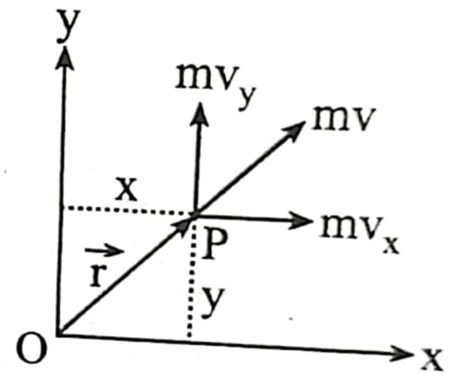
$$\dot{H}_O = \frac{d}{dt}(H_O) = \frac{d}{dt}(mv_y x - mv_x y)$$

$$\text{or, } \dot{H}_O = m(\dot{x}v_y + x\dot{v}_y - \dot{y}v_x - y\dot{v}_x)$$

$$\text{or, } \dot{H}_O = m(v_x v_y + xa_y - v_y v_x - ya_x)$$

$$\text{or, } \dot{H}_O = m(xa_y - ya_x)$$

$$\text{or, } \dot{H}_O = xma_y - yma_x$$

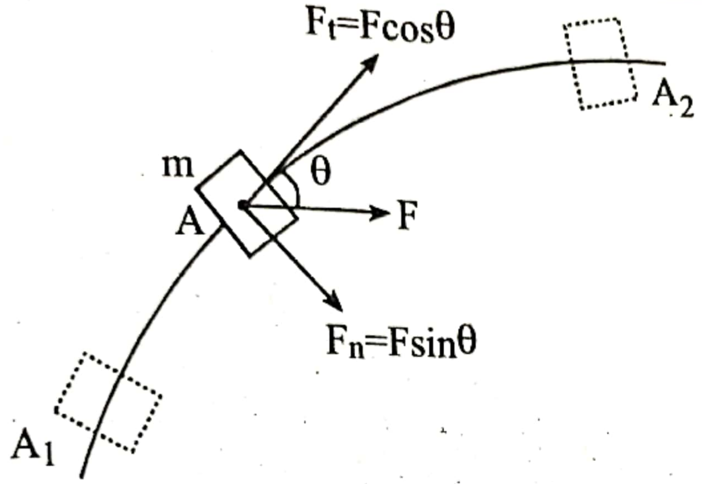


or, $\dot{H}_O = xF_y - yF_x$ i.e., moment of force about O.

So, rate of change of angular momentum of a particle about any point at any instant is equal to the moment of forces about that point.

MOMENT AND ENERGY IN RIGID BODY

10.1 Kinetic Energy of a Particle: Principle of Work and Energy



Consider a particle of mass m acted upon by a force $\vec{F}$ and moving along a path which is either rectilinear or curved.

The tangential component of the force is

$$F_t = ma_t = m\frac{dv}{dt}$$

where v is the speed of the particle.

$$\text{or, } F_t = m\frac{dv}{ds}\frac{ds}{dt} = mv\frac{dv}{ds}$$

$$\text{or, } F_t\, ds = mv\, dv$$

Integrating from A_1 where $s = s_1$ and $v = v_1$ to A_2 where $s = s_2$ and $v = v_2$, we have

$$\int_{s_1}^{s_2} F_t\, ds = m\int_{v_1}^{v_2} v\, dv = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 \quad \text{..........(i)}$$

The expression $\frac{1}{2}mv^2$ is the kinetic energy of the particle and is denoted by T.

From equation (i),

$$U_{1\text{-}2} = T_2 - T_1 \quad \text{.....................(ii)}$$

Above equation expresses that the work of the force $\vec{F}$ is equal to the change in kinetic energy of the particle. This is known as the *principle of work and energy*. Rearranging equation (ii), we write

$$T_1 + U_{1\text{-}2} = T_2$$

Thus, the kinetic energy of the particle at A_2 can be obtained by adding to its kinetic energy at A_1 the work done during the displacement from A_1 to A_2 by the force $\vec{F}$ exerted on the particle.

10.2 Conservation of Linear and Angular Momentum for a System of Particles

Conservation of Linear Momentum

Consider a particle of mass m acted upon by a force $\vec{F}$. From Netwon's 2nd law of motion,

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt}$$

$$\text{or, } \vec{F} = \frac{d(m\vec{v})}{dt} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

where $m\vec{v}$ is called *linear momentum* of the particle.

Equation (i) suggests that the force applied on the particle is equal to the rate of change of linear momentum of the particle.

$$\text{or, } \vec{F}\,dt = d(m\vec{v})$$

Integrating,

$$\int_{t_1}^{t_2} \vec{F}\,dt = \int_{v_1}^{v_2} d(m\vec{v})$$

$$\text{or, } \overrightarrow{Imp}_{1\text{-}2} = m\vec{v}_2 - m\vec{v}_1$$

$$\text{or, } m\vec{v}_1 + \overrightarrow{Imp}_{1\text{-}2} = m\vec{v}_2$$

$$\text{or, } \vec{L}_1 + \overrightarrow{Imp}_{1\text{-}2} = \vec{L}_2 \quad \ldots\ldots\ldots\ldots\text{(ii)}$$

where

$\overrightarrow{Imp}_{1-2}$ = impulse during the interval t_1 to t_2

$\vec{L}_1$ = initial linear momentum = $m\vec{v}_1$

$\vec{L}_2$ = final linear momentum = $m\vec{v}_2$

Equation (ii) represents *impulse-momentum principle.*

If the particle of mass m is acted upon by several forces at the same time, then equation (ii) will be

$$\vec{L}_1 + \Sigma\overrightarrow{Imp}_{1-2} = \vec{L}_2$$

where,

$\Sigma\overrightarrow{Imp}_{1-2}$ = resultant impulse due to number of forces.

If $\Sigma\overrightarrow{Imp}_{1-2} = 0$, then $\vec{L}_1 = \vec{L}_2$, that is, initial linear momentum = final linear momentum. Thus, the total momentum of the particles is conserved.

Conservation of Angular Momentum

From linear impulse momentum principle, we have

$$\Sigma (m\vec{v})_1 + \Sigma (\overrightarrow{Imp}_{1-2})_{ext} = \Sigma(m\vec{v})_2 \qquad \text{..............(i)}$$

Again, from angular Impulse momentum principle,

$$(\vec{H}_o)_1 + \Sigma\int_{t_1}^{t_2} (\vec{M}_o)_{ext}\, dt = (\vec{H}_o)_2 \qquad \text{..............(ii)}$$

From equations (i) and (ii),we conclude that

(system momenta)$_1$+ (system extI_{mp})$_{1-2}$ = (system momenta)$_2$

Thus, the momenta of the particles at time t_1 and the impulses of the external forces from t_1 and t_2 form a system of vector equipollent (not actually equivalent for a system of particles but will be equivalent for rigid body) to the system of the momenta of the particles at time T_2

When no external forces act on the system of particles impulse due to external forces vanish then moment a are conserved.

(system momenta)$_1$ = (system momenta)$_2$

Thus, angular momentum is conserved.

COORDINATE SYSTEM

11.1 Scalar and Vector Quantities

The term "scalar" refers to a quantity whose value may be represented by a single (positive or negative) real number. In other words, the quantities having only magnitude are the scalar quantities. Examples of scalar quantities include mass, density, pressure, volume, temperature, etc. Vector quantities refer to those which have both a magnitude and a direction in space. Examples are force, velocity, acceleration, magnetic flux density, current density, etc.

11.2 Types of Coordinate System

i. Cartesian Coordinate System

- In this system, we set up three coordinate axes mutually at right angles to each other namely x, y, and z axes.
- Any point P is specified as P (x, y, z) and the point is the intersection of three mutually perpendicular planes namely x = constant, y = constant, and z = constant.

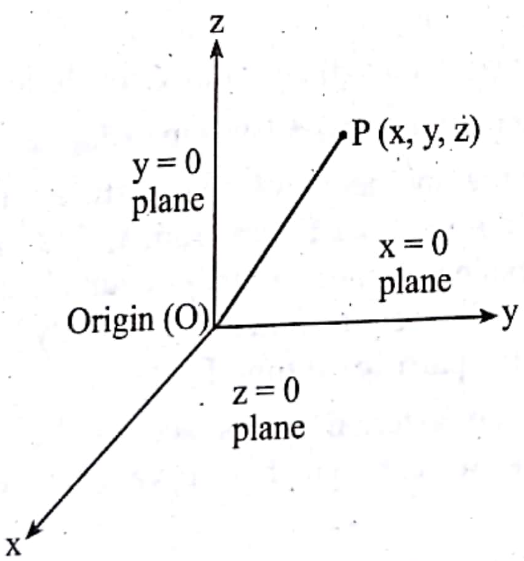


Fig.: Cartesian co-ordinate system

ii. **Cylindrical Coordinate System**

- It is the three dimensional version of the polar coordinates of analytic geometry. The circular cylindrical coordinate system is very convenient whenever we are dealing with problems having cylindrical symmetry.
- Any point P is specified as P (ρ, ϕ, z) and the point is the intersection of three mutually perpendicular surfaces namely a circular cylinder (ρ = constant), a plane (ϕ = constant), and another plane (z = constant).

  where ρ = radius of the circular cylinder

  ϕ = angle between the x-axis and the projection of the line joining origin and point P in the z = 0 plane.

  z = height of the cylinder.

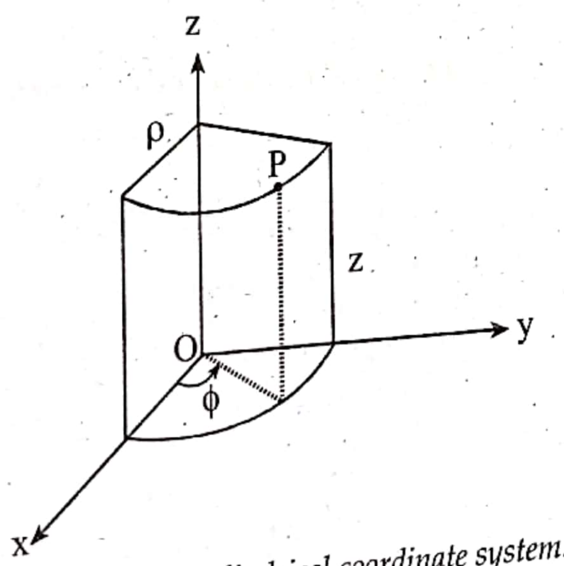


Fig.: The cylindrical coordinate system.

iii. **Spherical Coordinate System**

- The spherical coordinate system is most appropriate when one is dealing with problems having a degree of spherical symmetry.
- Any point P is specified as P(r, θ, ϕ) and the point is the intersection of three mutually perpendicular surfaces namely a sphere, a cone, and a plane.

where r = radius of the sphere

θ = angle between the line joining origin and point P, and the z-axis.

ϕ = angle between the x-axis and the projection of line joining origin and point P in the z = 0 plane.

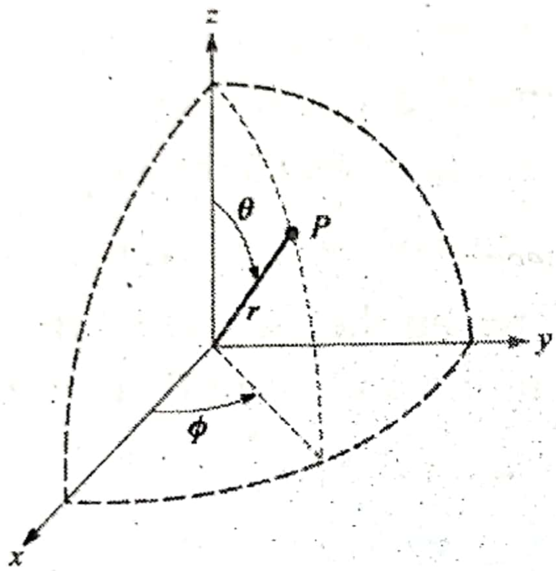


Fig.: The three spherical coordinates.