Mechanical Oscillation

Restoring force

The force with the help of which the body recovers its original configuration (shape and size) after application of deforming force on it, is called restoring force. Due to this force body tends to move to and fro motion and constitute a vibration. It is proportional to displacement of particle from mean position.

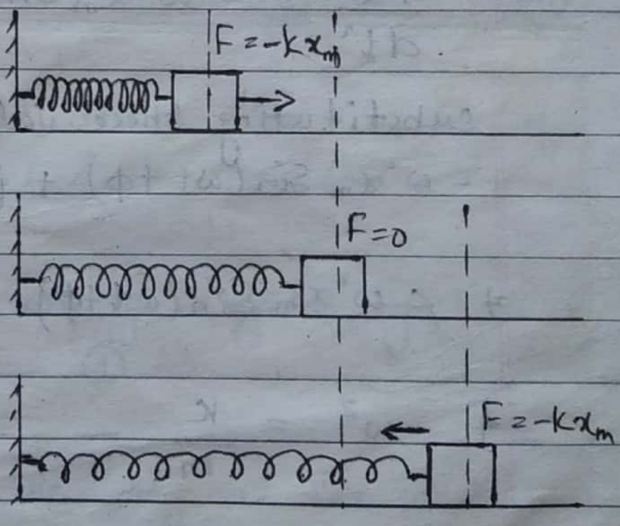
i.e, $F \propto -x$

$\Rightarrow F = -kx$

where, k is a constant of proportionality called force constant

Differential equation of linear simple harmonic motion

Consider a body of mass 'm' attached to a spring of force constant 'k' stretched to a distance 'x' from its mean position then the restoring force acting on it is given by



$F \propto -x$

$\Rightarrow F = -kx$

$\Rightarrow ma = -kx$

$\Rightarrow m\frac{d^2x}{dt^2} = -kx$

$\Rightarrow \frac{d^2x}{dt^2} = -\frac{k}{m}x$

$\Rightarrow \frac{d^2x}{dt^2} + \left(\frac{k}{m}\right)x = 0$ ——— (1)

Equation (1) is differential equation of linear SHM.
Let us consider a function

$x = x_m \sin(\omega t + \phi)$ ——— (2)

be the solution of eq^n (1). For this function to be the solution it must satisfy the equation.
Differentiating eq^n (2) w.r.t. 't', we get,

$\frac{dx}{dt} = \omega x_m \cos(\omega t + \phi)$

and

$\frac{d^2x}{dt^2} = -\omega^2 x_m \sin(\omega t + \phi)$

substituting these values in eq^n (1), we get,

$-\omega^2 x_m \sin(\omega t + \phi) + \left(\frac{k}{m}\right)x_m \sin(\omega t + \phi) = 0$

$\Rightarrow \cancel{-}\omega^2 \cancel{x_m} \cancel{\sin(\omega t + \phi)} = \cancel{+}\left(\frac{k}{m}\right)\cancel{x_m} \cancel{\sin(\omega t + \phi)}$

$\Rightarrow \omega^2 = \frac{k}{m}$ ——— (3)

$\therefore$ from eq^n (1) and (3), we get,

$\boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0}$ ——— (4)

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Equation (4) is the general equation of linear SHM.
If 'T' is the time period of the linear SHM then

$$T = \frac{2\pi}{\omega} \text{ ---------- } \left(\omega = 2\pi f \Rightarrow \omega = \frac{2\pi}{T}\right)$$

$$\Rightarrow T = \frac{2\pi}{\sqrt{\left(\frac{k}{m}\right)}} \text{ ---------- (from eq. (3))}$$

$$\Rightarrow T = 2\pi\sqrt{\left(\frac{m}{k}\right)} \text{ ---------- } (5)$$

if 'f' is the frequency of linear SHM, then

$$f = \frac{1}{T}$$

$$\Rightarrow f = \frac{1}{2\pi}\sqrt{\left(\frac{k}{m}\right)} \text{ ---------- } (6)$$

Differential equation of linear SHM (energy consideration)

Consider a body of mass 'm' attached to a spring of spring constant 'k' stretched to a distance 'x' from its mean position.
At this position, potential energy is given by

$$P.E. = \frac{1}{2}kx^2 \text{ ---------- } (1)$$

if 'v' is the velocity of the body at the same position then its kinetic energy is given by

$$K.E. = \frac{1}{2}mv^2 \text{ ---------- } (2)$$

The total energy is the sum of kinetic energy and potential energy.

$\therefore$ E [total energy] = K.E. + P.E.

$\Rightarrow E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$

During SHM total energy of the system remains constant.

$\therefore \frac{d(E)}{dt} = 0$

$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}mv^2 + \frac{1}{2}kx^2\right) = 0$

$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}mv^2\right) + \frac{d}{dt}\left(\frac{1}{2}kx^2\right) = 0$

$\Rightarrow \frac{1}{2}m\frac{d}{dt}(v^2) + \frac{1}{2}k\frac{d}{dt}(x^2) = 0$

$\Rightarrow \frac{1}{2}m\left(2v\frac{dv}{dt}\right) + \frac{1}{2}k\left(2x\frac{dx}{dt}\right) = 0$

$\Rightarrow mv\frac{dv}{dt} + kx\left(\frac{dx}{dt}\right) = 0$

$\Rightarrow m\left(\frac{dx}{dt}\right)\left(\frac{d^2x}{dt^2}\right) + kx\left(\frac{dx}{dt}\right) = 0$

$\Rightarrow \left(\frac{dx}{dt}\right)\left[m\frac{d^2x}{dt^2} + kx\right] = 0$

$\because \frac{dx}{dt} \neq 0$

$\therefore m\frac{d^2x}{dt^2} + kx = 0$

$$\Rightarrow \frac{d^2x}{dt^2} + \left(\frac{k}{m}\right)x = 0 \quad \text{——— (3)}$$

Equation (3) is differential equation of linear SHM.

Equation of simple harmonic motion (not in differential form)

Consider an oscillating system consisting of a particle. At any instant of time, if the particle is at displacement 'x' away from its mean position then restoring force on the particle is given by

$$F = -kx \quad \text{——— (1)}$$

where, k is a constant called spring constant. If 'm' is the mass of the particle and 'a' is the acceleration at that instant then

$$F = ma \quad \text{——— (2)}$$

from eq? (1) and (2),

$$ma = -kx$$

$$\Rightarrow a = -\left(\frac{k}{m}\right)x$$

$$\Rightarrow \frac{dv}{dt} = -\left(\frac{k}{m}\right)x$$

$$\Rightarrow \frac{dx}{dt} \cdot \frac{dv}{dx} = -\left(\frac{k}{m}\right)x$$

$$\Rightarrow v\frac{dv}{dx} = -\left(\frac{k}{m}\right)x$$

$\Rightarrow v \frac{dv}{dx} = -\omega^2 x$ ------ $\left(\because \omega^2 = \frac{k}{m}\right)$

$\Rightarrow v\, dv = -\omega^2 x\, dx$

Now, integrating both sides, we get,

$$\int_0^v v\, dv = \int_{x_0}^{x} -\omega^2 x\, dx$$

$$\Rightarrow \left[\frac{v^2}{2}\right]_0^v = -\omega^2 \left[\frac{x^2}{2}\right]_{x_0}^{x}$$

$$\Rightarrow \frac{v^2}{2} = -\frac{\omega^2}{2}(x^2 - x_0^2)$$

$$\Rightarrow v^2 = \omega^2 (x_0^2 - x^2)$$

$$\Rightarrow v = \sqrt{\omega^2 (x_0^2 - x^2)}$$

$$\Rightarrow v = \omega \sqrt{(x_0^2 - x^2)}$$

$$\Rightarrow \frac{dx}{dt} = \omega \sqrt{(x_0^2 - x^2)}$$

$$\Rightarrow \frac{dx}{\sqrt{(x_0^2 - x^2)}} = \omega\, dt$$

Again, integrating both sides, we get,

$$\int_{x_0}^{x} \frac{dx}{\sqrt{(x_0^2 - x^2)}} = \int_0^t \omega\, dt$$

$$\Rightarrow \left[\sin^{-1}\left(\frac{x}{x_0}\right)\right]_{x_0}^{x} = \omega t$$

$$\Rightarrow \sin^{-1}\left(\frac{x}{x_0}\right) - \sin^{-1}\left(\frac{x_0}{x_0}\right) = \omega t$$

$$\Rightarrow \sin^{-1}\left(\frac{x}{x_0}\right) - \sin^{-1}(1) = \omega t$$

$$\Rightarrow \sin^{-1}\left(\frac{x}{x_0}\right) - \frac{\pi}{2} = \omega t$$

$$\Rightarrow \sin^{-1}\left(\frac{x}{x_0}\right) = \omega t + \frac{\pi}{2}$$

Here, $\left(\frac{\pi}{2}\right)$ is called initial phase, i.e, phase angle at time $t=0$.

$$\frac{x}{x_0} = \sin\left(\omega t + \frac{\pi}{2}\right)$$

$$\Rightarrow x = x_0 \sin\left(\omega t + \frac{\pi}{2}\right) \quad \text{——— (3)}$$

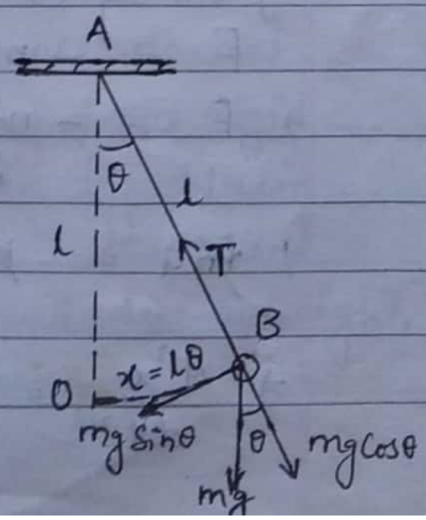
At time $t=0$, if we start from $x = x_0$ then initial phase angle has value of $\left(\frac{\pi}{2}\right)$ but we have option of starting from any position other than x_0 at time $t=0$. In general initial phase angle can be ϕ.

$\therefore$ General equation of SHM can be written as

$$x = x_0 \sin(\omega t + \phi) \quad \text{——— (4)}$$

Simple pendulum

A simple pendulum is a heavy point mass suspended by a weightless and inextensible string from a rigid support about which it can oscillate.



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The distance between center of bob to the point where thread is suspended is called effective length of simple pendulum.

Consider a simple pendulum of effective length 'l' and mass of bob 'm'. When bob is at point B two forces act on it

(i) gravitational force acting vertically downward and
(ii) tension T acting along BA.

The gravitational force 'mg' can be resolved into two perpendicular components

(i) mg cosθ directed opposit to BA.
(ii) mg sinθ acting perpendicular to the thread.

Here, the restoring force is

$$F = -mg\sin\theta \quad \text{--- (1)}$$

the negative sign indicates that force (i.e, acceleration) is directed toward mean position.

If angular displacement is small then

$$\sin\theta \approx \theta$$

and $$\theta = \frac{x}{l} \quad \text{--- (2)}$$

from eq. (1) and (2),

$$F = -mg\theta$$

$$\Rightarrow F = -mg\left(\frac{x}{l}\right)$$

$$\Rightarrow \cancel{m}a = -\cancel{m}\left(\frac{g}{l}\right)x \quad \text{--- } (\because F = ma)$$

$$\Rightarrow \frac{d^2x}{dt^2} = -\left(\frac{g}{l}\right)x \quad \text{--- (3)} \quad \left(\because a = \frac{d^2x}{dt^2}\right)$$

$\because \left(\frac{g}{l}\right)$ is always constant for a given pendulum.

$\therefore \frac{d^2x}{dt^2} \propto -x$ or $a \propto -x$

$\therefore$ motion of simple pendulum is simple harmonic motion.

Comparing eq. (8) with general differential equation of SHM, we get,

$$\omega^2 = \left(\frac{g}{l}\right)$$

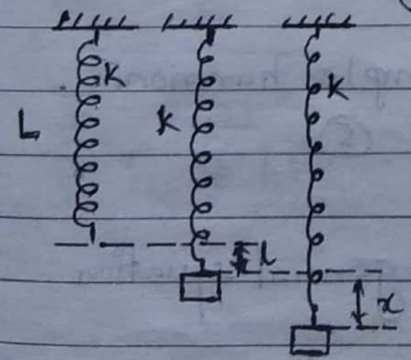
$$\Rightarrow \omega = \sqrt{\left(\frac{g}{l}\right)}$$

$\therefore$ The time period of the pendulum is

$$T = \frac{2\pi}{\omega}$$

$$\Rightarrow \boxed{T = 2\pi\sqrt{\left(\frac{l}{g}\right)}} \quad \text{—— (9)}$$

Suspended Spring-mass system



Consider a spring whose normal length is 'L', hanging from rigid support as shown in fig. When body of mass 'm' is suspended let the spring extends, due to weight, by length 'l'. Therefore, the restoring force is

$$F = -kl \quad \text{--- } (1)$$

where, k is spring constant.

Since at this stage, body is at rest so

$$F = -mg \quad \text{--- } (2)$$

from eq. (1) and (2),

$$-mg = -kl$$

$$\Rightarrow \frac{m}{k} = \frac{l}{g} \quad \text{--- } (3)$$

Now, if we displace a body slightly downward by distance 'x' then the new restoring force develops on the spring given by

$$F' = -k(l+x) \quad \text{--- } (4)$$

$\therefore$ resultant force acting on the body executing up and down motion will be

$$\text{resultant force} = F' - F$$

$$\Rightarrow ma = -k(l+x) - (-kl)$$

$$\Rightarrow ma = -kx$$

$$\Rightarrow a = -\left(\frac{k}{m}\right)x$$

$$\therefore a \propto -x$$

$\therefore$ motion of loaded spring is simple harmonic.

$$\Rightarrow \frac{d^2x}{dt^2} = -\left(\frac{k}{m}\right)x \quad \text{--- } (5)$$

Comparing this equation (5) with general equation of SHM, we get,

$$\omega^2 = \left(\frac{k}{m}\right)$$

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$$\Rightarrow \omega = \sqrt{\left(\frac{k}{m}\right)}$$

$$\Rightarrow 2\pi f = \sqrt{\left(\frac{k}{m}\right)} \quad \text{------} \quad (\because \omega = 2\pi f)$$

$$\Rightarrow \frac{2\pi}{T} = \sqrt{\frac{k}{m}} \quad \text{------} \quad (\because f = \frac{1}{T})$$

$$\Rightarrow \boxed{T = 2\pi\sqrt{\left(\frac{m}{k}\right)}} \quad \text{------} \quad (6)$$

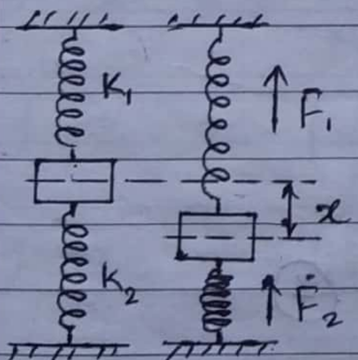
from eqⁿ (1) and (6),

$$\boxed{T = 2\pi\sqrt{\frac{l}{g}}} \quad \text{------} \quad (7)$$

Equations (6) and (7) gives time period of suspended spring-mass system.

Motion of mass suspended by two springs

(i) Mass between springs :



When body is displaced by small displacement 'x' then restoring force on the body due to stretched and compressed spring will be

$$F_1 = -k_1 x \quad \text{------} \quad (1)$$

$$\text{and } F_2 = -k_2 x \quad \text{------} \quad (2)$$

Since the restoring force on the body is due to both springs so total or net force on the body is

$$F = F_1 + F_2$$

$$\Rightarrow F = -k_1 x - k_2 x$$

$$\Rightarrow ma = -(k_1 + k_2)x$$

$$\Rightarrow ma = -kx$$

where, $k = k_1 + k_2$ is called effective spring constant.

$$\therefore a = -\left(\frac{k}{m}\right)x$$

$$\Rightarrow a \propto -x$$

$\therefore$ the motion is simple harmonic.

$$\Rightarrow \frac{d^2x}{dt^2} = -\left(\frac{k}{m}\right)x \quad \text{———} \left(\because a = \frac{d^2x}{dt^2}\right)$$

Comparing this eq^n^ with general equation of SHM, we get,

$$\omega^2 = \left(\frac{k}{m}\right)$$

$$\Rightarrow \omega = \sqrt{\left(\frac{k}{m}\right)}$$

$$\Rightarrow 2\pi f = \sqrt{\left(\frac{k}{m}\right)}$$

$$\Rightarrow \frac{2\pi}{T} = \sqrt{\left(\frac{k}{m}\right)}$$

$$\Rightarrow T = 2\pi\sqrt{\frac{m}{k}}$$

$$\Rightarrow \boxed{T = 2\pi\sqrt{\frac{m}{(k_1+k_2)}}} \quad \text{———} (3)$$

Equation (3) gives the time period of this spring-mass system.

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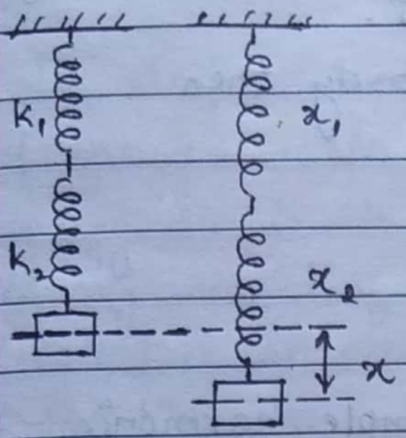
(ii) Mass at one end of strings



Consider a body of mass 'm' attached at end of two springs connected in series having spring constants 'k_1' and 'k_2' resp.. Let 'x' be the displacement of body from its mean position on application of force 'F'. If x_1 and x_2 are elongation of two springs on application of force F then

$$F = -k_1 x_1$$

and $$F = -k_2 x_2$$

$$\Rightarrow \quad x_1 = -\frac{F}{k_1} \quad \text{——— (1)}$$

and $$x_2 = -\frac{F}{k_2} \quad \text{——— (2)}$$

Therefore, total elongation is

$$x = x_1 + x_2$$

$$\Rightarrow x = -\frac{F}{k_1} - \frac{F}{k_2}$$

$$\Rightarrow x = -\left(\frac{1}{k_1} + \frac{1}{k_2}\right) F$$

$$\Rightarrow x = -\left(\frac{k_1 + k_2}{k_1 k_2}\right) F$$

$$\Rightarrow F = -\left(\frac{k_1 k_2}{k_1 + k_2}\right) x$$

$$\Rightarrow F = -kx$$

where, $k = \left(\frac{k_1 k_2}{k_1 + k_2}\right)$ and is called effective spring constant.

If 'a' is the acceleration of body then

$$ma = -kx$$

$$\Rightarrow a = -\left(\frac{k}{m}\right)x$$

$$\Rightarrow a \propto -x$$

$\therefore$ the motion of the body is simple harmonic.

$$\Rightarrow \frac{d^2x}{dt^2} = -\left(\frac{k}{m}\right)x$$

Comparing this equation with general differential equation of SHM, we get,

$$\omega^2 = \left(\frac{k}{m}\right)$$

$$\Rightarrow 2\pi f = \sqrt{\frac{k}{m}}$$

$$\Rightarrow \frac{2\pi}{T} = \sqrt{\frac{k}{m}}$$

$$\Rightarrow T = 2\pi\sqrt{\frac{m}{k}}$$

$$\Rightarrow \boxed{T = 2\pi\sqrt{m\left(\frac{k_1+k_2}{k_1 k_2}\right)}} \text{ ——— (3)}$$

Equation (3) gives time-period of this spring-mass system.

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Energy conservation in simple harmonic motion

In simple harmonic motion a body possesses two types of energy, kinetic energy due to velocity and potential energy due to its displacement from its mean position. At any instant of time the total energy of the oscillator (body) is the sum of these two energies.

$\therefore$ total energy = kinetic energy + potential energy

$\Rightarrow$ $E = K.E. + P.E.$ ——— (1)

Equation of body in SHM is given by

$x = x_m \sin(\omega t + \phi)$ ——— (2)

Now,

$$K.E. = \frac{1}{2} m v^2$$

$$\Rightarrow K.E. = \frac{1}{2} m \left(\frac{dx}{dt}\right)^2$$

$$\Rightarrow K.E. = \frac{1}{2} m \left[\frac{d}{dt}\left(x_m \sin(\omega t + \phi)\right)\right]^2$$

$$\Rightarrow K.E. = \frac{1}{2} m \left(x_m \omega \cos(\omega t + \phi)\right)^2$$

$$\Rightarrow K.E. = \frac{1}{2} m \omega^2 x_m^2 \cos^2(\omega t + \phi) \quad \text{——— (3)}$$

And,

$$P.E. = \frac{1}{2} k x^2$$

$$\Rightarrow P.E. = \frac{1}{2} k x_m^2 \sin^2(\omega t + \phi)$$

$$\Rightarrow P.E. = \frac{1}{2} m\omega^2 x_m^2 \sin^2(\omega t + \phi) \quad \text{--- (4)} \quad \left(\because \omega^2 = \frac{k}{m}\right)$$

from eq. (1), (3) and (4), we get,

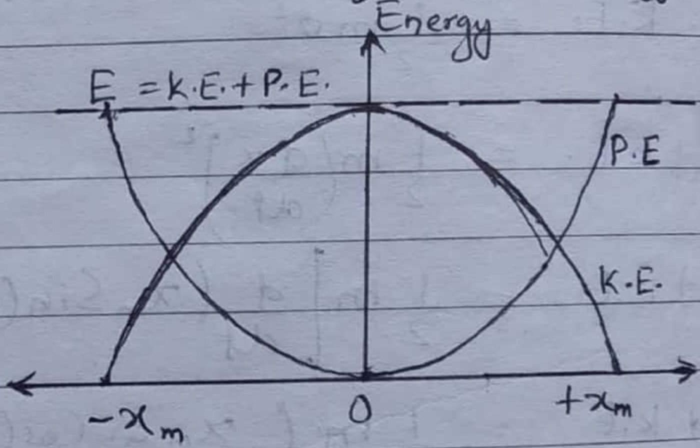
$$E = \frac{1}{2} m\omega^2 x_m^2 \cos^2(\omega t + \phi) + \frac{1}{2} m\omega^2 x_m^2 \sin^2(\omega t + \phi)$$

$$\Rightarrow E = \frac{1}{2} m\omega^2 x_m^2 \left[\cos^2(\omega t + \phi) + \sin^2(\omega t + \phi)\right]$$

$$\Rightarrow E = \frac{1}{2} m\omega^2 x_m^2$$

$$\Rightarrow E = \frac{1}{2} k x_m^2 \quad \text{--- (5)}$$

$\because$ k and x_m are constants so the total energy is also constant. The maximum value of total energy is $\frac{1}{2} k x_m^2$.



At the maximum displacement the kinetic energy is zero, but the potential energy has the value of $\frac{1}{2} k x_m^2$. At equilibrium position the potential energy is zero but the kinetic energy has the value of $\frac{1}{2} k x_m^2$. At other positions, the kinetic energy and potential energy each contributes such that the total energy is always equal to $\frac{1}{2} k x_m^2$ as shown in fig.

Angular Simple harmonic motion

A body is free to rotate about a given axis can make angular oscillation. The angular oscillation is said to be angular SHM if the torque on the body is
(i) directly proportional to the angular displacement of the body from its mean position, and
(ii) always directed toward mean position.

If 'θ' is the angular displacement of the body from its mean position at any instant of time so as to develop restoring torque 'τ' then

$$\tau \propto -\theta$$

$$\Rightarrow \tau = -k\theta \quad \text{——— (1)}$$

where, k is a constant called torsion constant.

If 'I' is the moment of inertia about point of suspension and 'α' the angular acceleration of body then

$$\tau = I\alpha \quad \text{——— (2)}$$

from eq. (1) and (2),

$$I\alpha = -k\theta$$

$$\Rightarrow I\frac{d^2\theta}{dt^2} = -k\theta$$

$$\Rightarrow \frac{d^2\theta}{dt^2} = -\left(\frac{k}{I}\right)\theta$$

$$\Rightarrow \frac{d^2\theta}{dt^2} + \left(\frac{k}{I}\right)\theta = 0 \quad \text{——— (3)}$$

Equation (3) is the differential equation of angular simple harmonic motion.

Let us consider a function

$$\theta = \theta_m \sin(\omega t + \phi) \quad \text{——— (4)}$$

be the solution of equation (3), For this function to be solution it must satisfy equation (3).

Differentiating eq. (4), we get,

$$\frac{d\theta}{dt} = \omega \theta_m \cos(\omega t + \phi)$$

and

$$\frac{d^2\theta}{dt^2} = -\omega^2 \theta_m \sin(\omega t + \phi) \quad \text{——— (5)}$$

substituting values in eq^n (3), we get,

$$-\omega^2 \theta_m \sin(\omega t + \phi) + \left(\frac{k}{I}\right) \theta_m \sin(\omega t + \phi) = 0$$

$$\Rightarrow \cancel{+}\omega^2 \cancel{\theta_m} \cancel{\sin(\omega t + \phi)} = \cancel{+}\left(\frac{k}{I}\right) \cancel{\theta_m} \cancel{\sin(\omega t + \phi)}$$

$$\Rightarrow \omega^2 = \left(\frac{k}{I}\right) \quad \text{——— (6)}$$

$\therefore$ from eq. (3) and (6),

$$\boxed{\frac{d^2\theta}{dt^2} + \omega^2 \theta = 0} \quad \text{——— (7)}$$

Equation (7) is called general differential equation of angular SHM.

From eq^n (6),

$$\omega = \sqrt{\left(\frac{k}{I}\right)}$$

$$\Rightarrow 2\pi f = \sqrt{\left(\frac{k}{I}\right)}$$

$$\Rightarrow \frac{2\pi}{T} = \sqrt{\left(\frac{K}{I}\right)}$$

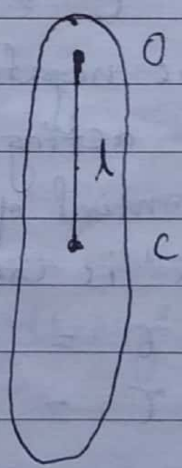
$$\Rightarrow \boxed{T = 2\pi\sqrt{\frac{I}{K}}} \quad \text{———} \quad (8)$$

Equation (8) gives time-period of angular oscillation.

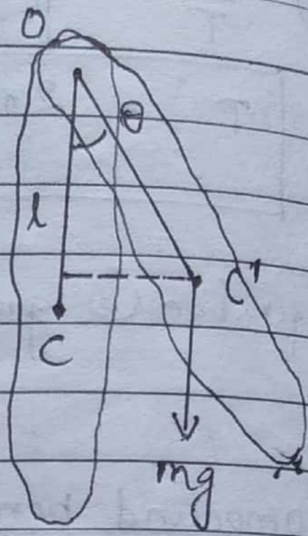
Compound pendulum or Physical pendulum

Any rigid body which can oscillate about a fixed horizontal axis is called compound pendulum. A compound pendulum is also called physical pendulum. A bar pendulum is a type of compound pendulum.

The point 'O' about which the body oscillates is called point of suspension. The distance between the point of suspension and the center of gravity of the pendulum is called the length of compound pendulum. When the center of mass of the body 'C' lies exactly vertically below the point of suspension the body is said to be in its normal position of rest.



Time period of compound pendulum



Let us consider a compound pendulum whose center of suspension is 'O' and center of gravity at rest is 'C'. When the body is displaced by small angle 'θ' let the position of c.g. shifts to 'C''. When the body is at new position the weight 'mg' acts vertically downward at C'. The weight 'mg' at C' and its reaction at the point of suspension constitutes a couple.

∴ The moment of restoring couple,

$$\tau = -mg.l\sin\theta \quad \text{——— (1)}$$

the negative sign indicates that the restoring couple is acting towards the mean position. If 'I' is the moment of inertia about the point of suspension and 'α' is the angular acceleration of pendulum then

$$\tau = I\alpha$$

$$\tau = I\frac{d^2\theta}{dt^2} \quad \text{——— (2)}$$

from eq: (1) and (2),

$$I\frac{d^2\theta}{dt^2} = -mgl\sin\theta$$

For small angular displacement

$$\sin\theta \approx \theta$$

$\therefore I \frac{d^2\theta}{dt^2} = -mgl\theta$

$\Rightarrow \frac{d^2\theta}{dt^2} = -\left(\frac{mgl}{I}\right)\theta$

$\Rightarrow \frac{d^2\theta}{dt^2} + \left(\frac{mgl}{I}\right)\theta = 0$ ——— (3)

Comparing this equation with general equation of angular SHM, we get,

$\omega^2 = \left(\frac{mgl}{I}\right)$

$\Rightarrow \omega = \sqrt{\frac{mgl}{I}}$

$\Rightarrow 2\pi f = \sqrt{\frac{mgl}{I}}$

$\Rightarrow \frac{2\pi}{T} = \sqrt{\frac{mgl}{I}}$

$\Rightarrow T = 2\pi\sqrt{\frac{I}{mgl}}$ ——— (4)

If 'I_0' is the moment of inertia of pendulum about an axis through its c.g., parallel to the axis through O, then by parallel axis theorem,

$I = I_0 + ml^2$ ——— (5)

If 'k' is the radius of gyration then

$I_0 = mk^2$

$\therefore$ from eq^n (5),

$I = mk^2 + ml^2$

$\Rightarrow I = m(k^2 + l^2)$ ——— (6)

$\therefore$ from eq. (4) and (6),

$$T = 2\pi \sqrt{\frac{m(k^2+l^2)}{mgl}}$$

$$\Rightarrow T = 2\pi \sqrt{\frac{(k^2+l^2)}{gl}}$$

$$\Rightarrow T = 2\pi \sqrt{\frac{\left(\frac{k^2}{l}+l\right)}{g}} \text{ ——— (7)}$$

Equation (7) is the time period of compound pendulum in terms of radius of gyration.

Now, if we substitute $\left(\frac{k^2}{l}+l\right) = L$ in eq. (7), then

$$T = 2\pi \sqrt{\frac{L}{g}} \text{ ——— (8)}$$

which is the relation of time period of simple pendulum having length 'L'. Therefore, 'L' is called length of equivalent simple pendulum or reduced length of simple pendulum.

A point 'P' at a distance $\left(\frac{k^2}{l}+l\right)$ away from center of suspension along the direction of center of gravity is called center of oscillation.

Interchangablity of center of suspension and center of oscillation

We have, time period of compound pendulum as

$$T = 2\pi\sqrt{\frac{\left(\frac{k^2}{l}+l\right)}{g}} \quad \text{——} \; (1)$$

Let $\frac{k^2}{l} = l'$ —— (2)

$\Rightarrow k^2 = ll'$ —— (3)

$\therefore$ from eqn (1) and (2),

$$T = 2\pi\sqrt{\frac{(l'+l)}{g}} \quad \text{——} \; (4)$$

Now, if we oscillate pendulum about the axis through center of oscillation the time period will change. Let T' be the new time period of the pendulum. In this case the length of the pendulum is $\frac{k^2}{l} = l'$. Therefore, the new time period of the pendulum is

$$T' = 2\pi\sqrt{\frac{\left(\frac{k^2}{l'}+l'\right)}{g}}$$

$\because k^2 = ll'$ — — — (from eq. (3))

$$\therefore T' = 2\pi\sqrt{\frac{\left(\frac{ll'}{l'}+l'\right)}{g}}$$

$$\Rightarrow T' = 2\pi\sqrt{\frac{(l+l')}{g}} \quad \text{——} \; (5)$$

from eqn (4) and (5), we get,

$$T = T'$$

i.e., the time period is same. So, the center of suspension and center of oscillation are interchangable.

Maximum and minimum time period of compound pendulum

The time period of compound pendulum is

$$T = 2\pi\sqrt{\frac{\left(\frac{k^2}{l}+l\right)}{g}}$$

Squaring both sides, we get

$$T^2 = \frac{4\pi^2}{g}\left(\frac{k^2}{l}+l\right)$$

Differentiating w.r.t. l, we get,

$$2T\frac{dT}{dl} = \frac{4\pi^2}{g}\left(-\frac{k^2}{l^2}+1\right) \text{ ——— (1)}$$

For maxima or minima,

$$\frac{dT}{dl} = 0$$

$$\Rightarrow \frac{4\pi^2}{g}\left(-\frac{k^2}{l^2}+1\right) = 0$$

$$\because \frac{4\pi^2}{g} \neq 0$$

$$\therefore \left(-\frac{k^2}{l^2}+1\right) = 0$$

$$\Rightarrow \frac{k^2}{l^2} = 1$$

$$\Rightarrow l^2 = k^2$$

$$\Rightarrow l = \pm k \text{ ——— (2)}$$

$\because -k$ is meaningless

$\therefore \; l = k$ ——— (3)

On differentiating eqⁿ (1), we get,

$$\frac{d^2T}{dl^2} = (+)ve$$

Therefore, time period T of a compound pendulum is minimum when the length of the pendulum is equal to radius of gyration and its value is

$$T_{min} = 2\pi\sqrt{\frac{\left(\frac{k^2}{k}+k\right)}{g}}$$

$$\Rightarrow T_{min} = 2\pi\sqrt{\frac{2k}{g}} \quad \text{———} \quad (4)$$

Now, if $l = 0$ or $l = \infty$.

then

$$T = \infty$$

i.e, maximum value of time period.

Therefore, time period of compound pendulum is maximum when the length of pendulum is zero (i.e, center of suspension passes through center of gravity) or infinity.

Radius of gyration (k)

The time period of compound pendulum is

$$T = 2\pi\sqrt{\frac{\left(\frac{k^2}{l}+l\right)}{g}}$$

squaring both sides, we get,

$$T^2 = \frac{4\pi^2}{g}\left(\frac{k^2}{l}+l\right)$$

$$\Rightarrow gT^2 = \frac{4\pi^2(k^2+l^2)}{l}$$

$$\Rightarrow (gT^2)\,l = (4\pi^2k^2)+(4\pi^2)\,l^2$$

$$\Rightarrow (4\pi^2)\,l^2 - (gT^2)\,l + (4\pi^2k^2) = 0 \quad \text{——— (1)}$$

This is quadric equation in 'l' whose root is of the form α and β and given by

$$\alpha+\beta = -\frac{b}{a}$$

and $$\alpha\cdot\beta = \frac{c}{a}$$

If 'l_1' and 'l_2' are roots of the eq^n (1) then

$$l_1+l_2 = \frac{T^2g}{4\pi^2} \quad \text{——— (2)}$$

and

$$l_1\cdot l_2 = \frac{4\pi^2k^2}{4\pi^2} \quad \text{——— (3)}$$

from eq^n (3),

$$l_1\cdot l_2 = \frac{4\pi^2k^2}{4\pi^2}$$

$$\Rightarrow l_1\cdot l_2 = k^2$$

$$\Rightarrow \boxed{k = \sqrt{l_1\cdot l_2}} \quad \text{——— (4)}$$

Equation (4) gives the radius of gyration of compound pendulum.

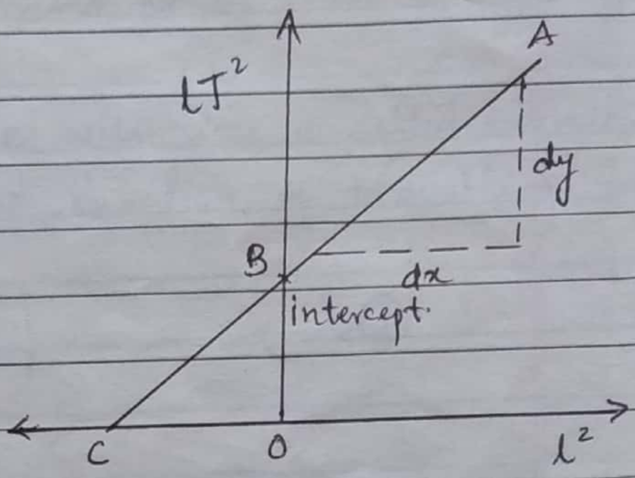
(cont. from "Time period of compound pendulum")

The time period of compound pendulum is

$$T = 2\pi\sqrt{\frac{\left(\frac{k^2}{l}+l\right)}{g}}$$

Squaring both sides, we get,

$$T^2 = \frac{(2\pi)^2}{g}\left(\frac{k^2}{l}+l\right)$$



$$\Rightarrow \cancel{g}T^2 = \frac{4\pi^2}{g}\left(\frac{k^2+l^2}{l}\right)$$

$$\Rightarrow (lT^2) = \frac{4\pi^2k^2}{g} + \frac{4\pi^2l^2}{g}$$

$$\Rightarrow (lT^2) = \left(\frac{4\pi^2}{g}\right)l^2 + \left(\frac{4\pi^2k^2}{g}\right) \quad \text{——— (1)}$$

Equation (1) is of the form

$$y = mx + c$$

Here,

slope of graph $= \dfrac{4\pi^2}{g}$

$$\Rightarrow g = \frac{4\pi^2}{(\text{slope of graph})} \quad \text{——— (2)}$$

and

intercept $= \dfrac{4\pi^2k^2}{g}$

$$\Rightarrow k = \frac{(g \times \text{intercept})^{1/2}}{2\pi} \quad \text{——— (3)}$$

With the help of equations (2) and (3), the values of 'g' and 'k' can be calculated from $lT^2 - l^2$ graph.