1) This can be modeled with just 1 state variable if we are not interested in which repair person is being utilized at any given moment.
We also assume that repairs go to whichever person is free at the moment.
If there are 3 working machines, 2 are used as a spare and only one is in active use.

State variable

w – number of working machines

Events

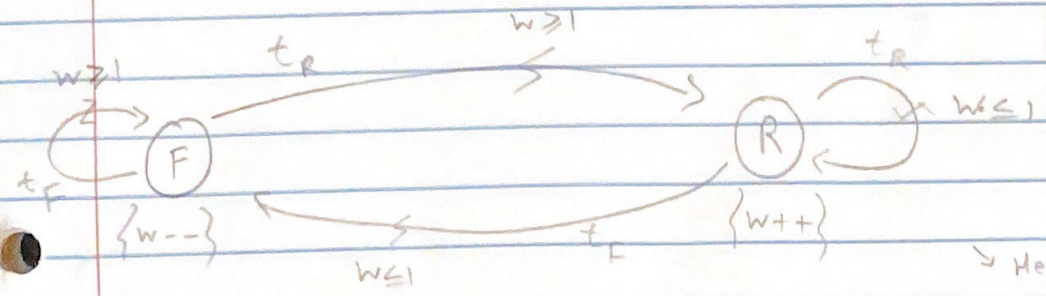
F – Failure of machine

R – Repair of machine

Inputs

t_F – Time to failure (DU(1,6))

t_R – Time to repair (2.5)



Here $w \leq 1$ is same as $w = 1$

2. Generate triangular random variable from $U(0,1)$

For a triangular distribution, the cdf is:

$$F_X(x) = \begin{cases} 0 & \text{if } x \le a \\ \dfrac{(x-a)^2}{(b-a)(c-a)} & \text{if } a < x \le c \\ 1 - \dfrac{(b-x)^2}{(b-a)(b-c)} & \text{if } c < x < b \\ 1 & \text{if } b \le x \end{cases}$$

where the parameters of the triangular distribution are a, b, c with $a \le c \le b$

Let $U \sim U(0,1)$

We need to find

$$X = F_X^{-1}(U)$$

<u>Case 1:</u> $a \le x \le c$

$$U = F_X(x)$$

$$= \frac{(x-a)^2}{(b-a)(c-a)}$$

$$\Rightarrow \quad (x-a)^2 = U(b-a)(c-a)$$

$$\Rightarrow \quad x = a + \sqrt{U(b-a)(c-a)}$$

The limits would be:

when $x = a$

$$U = \frac{(x-a)^2}{(b-a)(c-a)}$$

$$= \underline{\underline{0}}$$

when $x = c$

$$U = \frac{(x-a)^2}{(b-a)(c-a)}$$

$$= \frac{(c-a)^2}{(b-a)(c-a)}$$

$$= \frac{c-a}{b-a}$$

<u>Case 2:</u> $c < x \leq b$

$$U = F_X(x)$$

$$U = 1 - \frac{(b-x)^2}{(b-a)\cancel{(c-a)}(b-c)}$$

$$\Rightarrow (b-x)^2 = (1-U)(b-a)(b-c)$$

$$\Rightarrow x = b - \sqrt{(1-U)(b-a)(b-c)}$$

The limits would be:

$$U = 1 - \frac{(b-x)^2}{(b-a)(b-c)}$$

When $x = c$

$$U = 1 - \frac{(b-c)^2}{(b-a)(b-c)}$$

$$= \frac{c-a}{b-a}$$

When $x = b$

$$U = 1 - \frac{(b-b)^2}{(b-a)(b-c)}$$

$$= \underline{\underline{1}}$$

Thus if $U \sim U(0,1)$, in order to generate the random variable $X \sim \text{Triangular}(a,b,c)$

$$X = \begin{cases} a + \sqrt{U(b-a)(c-a)} & \text{if } 0 < U < \frac{c-a}{b-a} \\ b - \sqrt{(1-U)(b-a)(b-c)} & \text{if } \frac{c-a}{b-a} \leq U < 1 \end{cases}$$

3)

3.2

In an M/M/1 queue (with $c=1$), the standard error is given by:

$$se = \sqrt{V(\bar{Y}(m))}$$

$$se = \frac{2\sqrt{\rho}}{(1-\rho)^2\sqrt{m}}$$

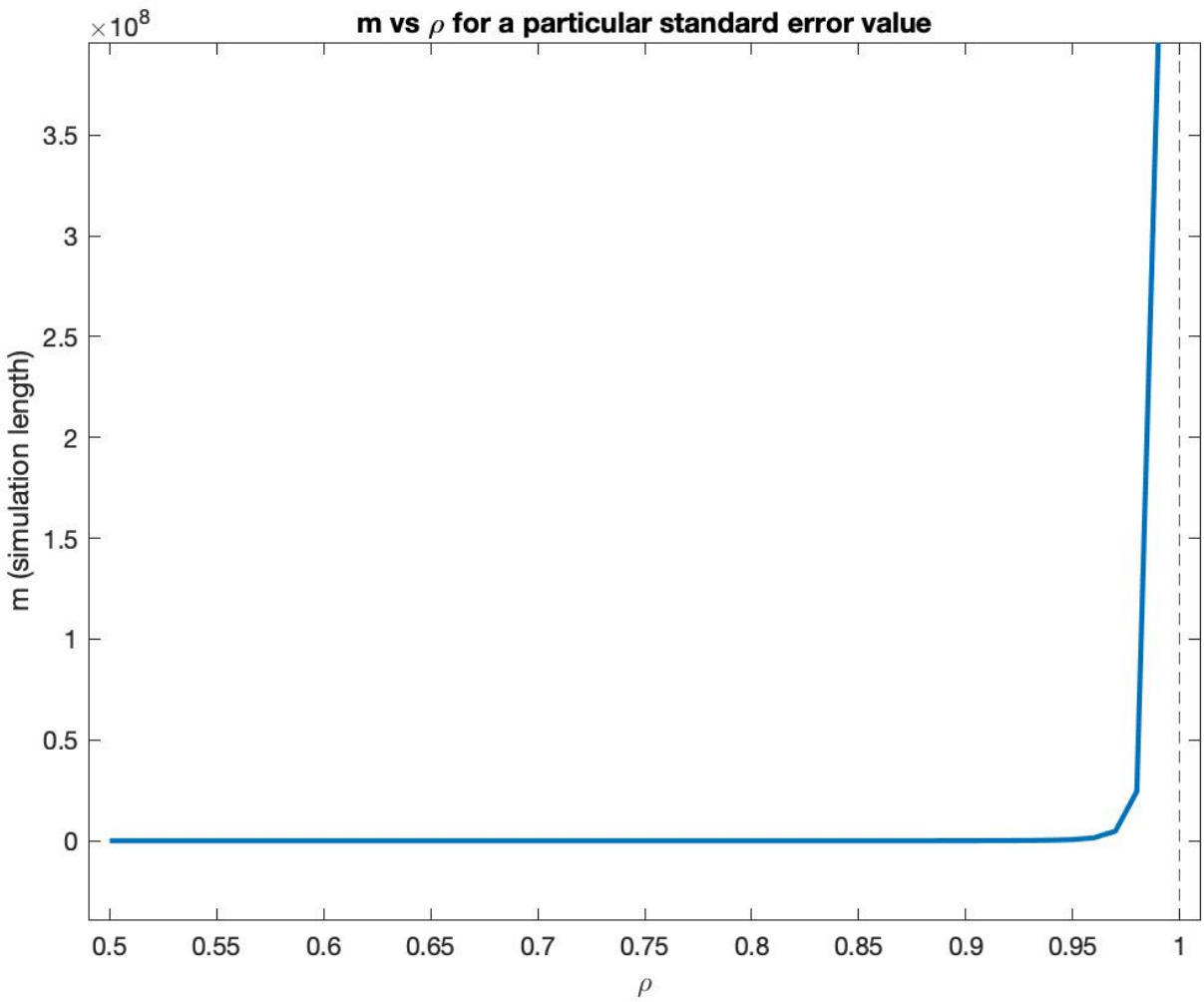
$$\Rightarrow \quad m = \frac{4\rho}{(1-\rho)^4 (se)^2}$$

Now we plot m vs ρ for a particular value of standard error.

I use $se=1$, then

$m=32$ when $\rho = 0.5$

$m = 3.96 \times 10^8$ when $\rho = 0.99$


m vs ρ for a particular standard error value
$\times 10^8$
3.5
3
2.5
2
1.5
1
0.5
0
m (simulation length)
0.5
0.55
0.6
0.65
0.7
0.75
0.8
0.85
0.9
0.95
1
ρ

3.3

$$E(Y) = \frac{\lambda(\sigma^2 + \tau^2)}{2(1-\lambda\tau)}$$

$$\boxed{\sigma = \tau}$$

$$= \frac{2\lambda\tau^2}{2(1-\lambda\tau)} = \frac{\lambda\tau^2}{(1-\lambda\tau)}$$

$$\boxed{\rho = \lambda\tau}$$

$$\boxed{\tau = 1}$$

$$= \frac{\rho}{1-\rho}$$

Relative error, $re = \dfrac{\sqrt{V(\bar{Y}(m))}}{E(Y)}$

$$re = \frac{2\sqrt{\rho}}{(1-\rho)^2\sqrt{m}} \times \frac{1-\rho}{\rho}$$

$$\Rightarrow \quad m = \frac{4}{p(1-p)^2 \, \underline{\underline{(re)}}^2}$$

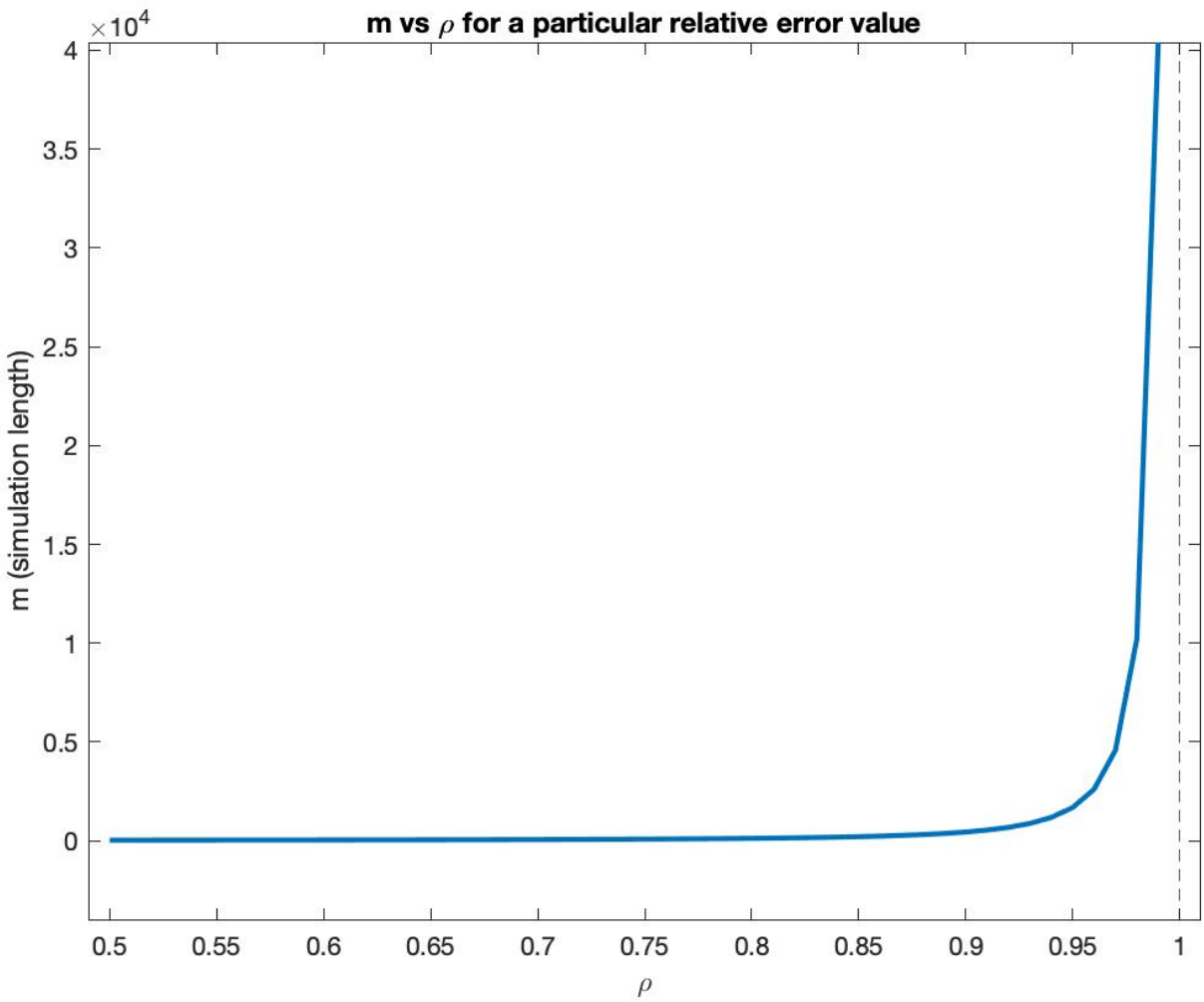
Again we plot m vs p for a particular value of relative error.

I use relative error = 1, then

$m = 32$ when $p = 0.5$

$m = 4.04 \times 10^4$ when $p = 0.99$

The figures may look similar, but note the difference in scale.


m vs ρ for a particular relative error value
$\times 10^4$
4
3.5
3
2.5
2
1.5
1
0.5
0
m (simulation length)
0.5
0.55
0.6
0.65
0.7
0.75
0.8
0.85
0.9
0.95
1
ρ

4) Prove

$$Y_i = \mu + \varphi^i (Y_0 - \mu) + \sum_{j=0}^{i-1} \varphi^j X_{i-j}$$

(for the AR(1) model)

Using induction.

We know that:

$$Y_i = \mu + \varphi (Y_{i-1} - \mu) + X_i \qquad ; i = 1, 2 \ldots$$

<u>Basis step</u>

Show that it holds for $i = 1$

$$Y_1 = \mu + \varphi^1 (Y_0 - \mu) + \sum_{j=0}^{0} \varphi^j X_{1-j}$$

$$= \mu + \varphi (Y_0 - \mu) + X_0$$

This is true since

$$Y_i = \mu + \varphi (Y_{i-1} - \mu) + X_{i-1}$$

$$\Rightarrow Y_1 = \mu + \varphi (Y_0 - \mu) + X_0$$

This holds for $i = 1$

Induction step:

Assume it holds for $i = k$

Need to prove it holds for $i = k+1$

$$y_k = \mu + \varphi^k (y_0 - \mu) + \sum_{j=0}^{k-1} \varphi^j x_{k-j}$$

$$y_{k+1} = \mu + \varphi (y_k - \mu) + x_{k+1}$$

$$= \mu + \varphi \left(\mu + \varphi^k (y_0 - \mu) + \sum_{j=0}^{k-1} \varphi^j x_{k-j} - \mu \right) + x_{k+1}$$

$$= \mu + \varphi^{k+1} (y_0 - \mu) + \varphi \sum_{j=0}^{k-1} \varphi^j x_{k-j} + x_{k+1}$$

$$= \mu + \varphi^{k+1} (y_0 - \mu) + \varphi \left[\varphi^0 x_k + \varphi^1 x_{k-1} + \dots + \varphi^{k-1} x_1 \right] + x_{k+1}$$

$$= \mu + \varphi^{k+1} (y_0 - \mu) + \left[\varphi^0 x_{k+1} + \varphi^1 x_k + \varphi^2 x_{k-1} + \dots + \varphi^k x_1 \right]$$

$$= \mu + \varphi^{k+1} (y_0 - \mu) + \sum_{j=0}^{k} \varphi^j x_{k+1-j}$$

Thus it holds for $k+1$.

Since it holds in base case and inductive step, by mathematical induction, the equation is true.

5)

1) $Y_i = \mu + \theta X_{i-1} + X_i \quad , i = 1, 2, \dots m$

$X_1 X_2 \dots$ are iid $(0, \sigma^2) \quad , |\theta| < 1$

$i \neq 1$

$$V(Y_i) = V(\mu + \theta X_{i-1} + X_i)$$

$$= V(\mu) + \theta^2 V(X_{i-1}) + V(X_i)$$

$$= 0 + \theta^2 \sigma^2 + \sigma^2$$

$$= \underline{\underline{\sigma^2 (\theta^2 + 1)}}$$

$i = 1$

$$V(Y_1) = V(\mu + \theta \cdot 1 + X_1)$$

$$= V(X_1)$$

$$= \underline{\underline{\sigma^2}}$$

(only if $X_0 = 1$, used in next part)

Equation 1

$$E(Y_i) = E(\mu + \theta X_{i-1} + X_i)$$

$$= E(\mu) + \theta E(X_{i-1}) + E(X_i) \quad = \mu + 0 + 0$$

$$= \underline{\underline{\mu}}$$

$$C(Y_i Y_j) = E(Y_i Y_j) - E(Y_i) \cdot E(Y_j)$$

$$= E(Y_i Y_j) - \mu^2$$

$$= E\left((\mu + \theta X_{i-1} + X_i)(\mu + \theta X_{j-1} + X_j)\right) - \mu^2$$

$$= E(\mu^2) + \mu\theta E(X_{i-1}) + \mu E(X_i)$$
$$+ \mu\theta E(X_{j-1}) + \theta^2 E(X_{i-1} X_{j-1}) + \theta E(X_i X_{j-1})$$
$$+ \mu E(X_j) + \theta E(X_{i-1} X_j) + E(X_i X_j) - \mu^2$$

(since $E(X_i) = 0$ $\forall i = 1, 2 \ldots$

$$= \mu^2 + 0 + 0$$
$$+ 0 + \theta^2 E(X_{i-1} X_{j-1}) + \theta E(X_i X_{j-1})$$
$$+ 0 + \theta E(X_{i-1} X_j) + E(X_i X_j) - \mu^2$$

Since $i \neq j$ and $E(X_k X_\ell) = E(X_k) \cdot E(X_\ell)$ when X_k and X_ℓ are independent

$$= \mu^2 + \theta^2 E(X_{i-1}) E(X_{j-1}) + \theta E(X_i X_{j-1})$$
$$+ \theta E(X_{i-1} X_j) + E(X_i) \cdot E(X_j) - \mu^2$$

$$= \theta E(X_i X_{j-1}) + \theta E(X_{i-1} X_j)$$

We have 3 cases

1) $i = j-1$

Let $i = j-1 = k$

$$E(X_i X_{j-1}) = E(X_k \cdot X_k) = E(X_k)^2 + V(X_k)$$
$$= \sigma^2$$

$E(X_{i-1} X_j) = E(X_{i-1}) \cdot E(X_j) = \underline{\underline{0}}$ (since $i-1 \neq j$)

$\Rightarrow C(Y_i Y_j) = \underline{\underline{\theta \sigma^2}}$

2) $j = i-1$

Similarly

$C(Y_i Y_j) = \theta \sigma^2$

3) Otherwise

$C(Y_i Y_j) = \theta E(X_{i-1} X_j) + \theta E(X_{j-1} X_i)$

$= \theta E(X_{i-1}) \cdot E(X_j) + \theta E(X_{j-1}) \cdot (X_i)$

$= \underline{\underline{0}}$

$$\Rightarrow \quad C(Y_i Y_j) = \begin{cases} \theta \sigma^2 & \text{if } |i-j| = 1 \\ 0 & \text{otherwise} \end{cases}$$

(Equation)

5

2)

Let:

$m \rightarrow$ length of simulation

$n \rightarrow$ no of replications.

$$\bar{\bar{Y}} = \frac{1}{n} \sum_{r=1}^{n} \bar{Y}_r \qquad \bar{Y}_r = \frac{1}{m} \sum_{i=1}^{m} Y_i$$

<u>Bias</u>

$$E(Y_i) = E(\mu + \theta X_{i-1} + X_i)$$

$$= \mu + \theta E(X_{i-1})$$

$$E(Y_i) = \begin{cases} \mu + \theta & \text{if } i = 1 \\ \mu & \text{if } i > 1 \end{cases}$$

$$E(\bar{Y}_r) = \frac{1}{m} \sum E(\bar{Y}_r) = \frac{1}{m}(\mu + \theta + (m-1)\cdot\mu)$$

$$= \mu + \frac{\theta}{m}$$

$$E(\bar{\bar{Y}}) = \frac{1}{n} \sum_{r=1}^{n} E(\bar{Y}_r) = E(\bar{Y}_r) = \mu + \frac{\theta}{m}$$

$$\text{Bias} = E(\bar{\bar{Y}}) - \mu = \frac{\theta}{m}$$

Variance

$$V(\bar{\bar{y}}) = V\left(\frac{1}{n}\sum_{r=1}^{n} \bar{y}_r\right)$$

$$= \frac{1}{n^2} \sum V(\bar{y}_r)$$ since replications are independent

$$= \frac{1}{n^2}\, n\, V(\bar{y}_r)$$

$$= \frac{1}{n} V(\bar{y}_r)$$

$$V(\bar{y}_r) = V\left(\frac{1}{m}\sum_{i=1}^{m} y_i\right)$$

$$= \frac{1}{m^2} V\left(\sum y_i\right) = \frac{1}{m^2}\left(\sum V(y_i) + \sum_{i \neq j} C(y_i\, y_j)\right)$$

where $\sum V(y_i) = \sigma^2 + (m-1)(\theta^2\sigma^2 + \sigma^2)$ (from ~~part 1~~ equation 1)

$$\sum_{i \neq j} C(y_i\, y_j) = 2(m-1) \cdot \theta\sigma^2$$ ~~since~~ (from equation 2)

$$\Rightarrow V(\bar{y}_r) = \frac{1}{m^2}\left(\sigma^2 + (m-1)(\theta^2\sigma^2 + \sigma^2) + 2(m-1)\theta\sigma^2\right)$$

$$V(\bar{Y}) = \frac{1}{n} V(\bar{Y}_r)$$

$$= \frac{1}{nm^2}\left(\sigma^2 + (m-1)(\theta^2\sigma^2 + \sigma^2) + 2(m-1)\theta\sigma^2\right)$$

Case a

$n = 1 \qquad m = 10$

Bias $= \frac{\theta}{m} = \underline{\underline{\frac{\theta}{10}}}$ $\qquad$ Mean $= \mu + \underline{\underline{\frac{\theta}{10}}}$

Variance $= \frac{1}{100}\left(\sigma^2 + 9(\theta^2\sigma^2 + \sigma^2) + 18\theta\sigma^2\right)$

Case b

$n = 2 \qquad m = 5$

Mean $= \mu + \frac{\theta}{m} = \mu + \underline{\underline{\frac{\theta}{5}}}$ $\qquad$ Bias $= \underline{\underline{\frac{\theta}{5}}}$

Variance $= \frac{1}{50}\left(\sigma^2 + 4(\theta^2\sigma^2 + \sigma^2) + 8\theta\sigma^2\right)$

Case a and b

When $\theta > 0$ $\qquad$ When $\theta < 0$

Bias is +ve $\qquad$ Bias is −ve

Variance of case a

$$= \frac{1}{10}\sigma^2 + \frac{9}{100}\theta\sigma^2(2+\theta)$$

Variance of case b

$$= \frac{1}{10}\sigma^2 + \frac{8}{100}\theta\sigma^2(2+\theta)$$

$(2+\theta)$ is always +ve

$\Rightarrow V(a) > V(b)$ when θ is +ve

$V(b) > V(a)$ when θ is -ve

3)

Bias $= \frac{\theta}{m}$ $\therefore$ Magnitude of bias increases as m decreases.

Variance is not as straight forward.
Since the covariance term $= 2(m-1)\theta\sigma^2$

If θ is +ve then variance increases for higher m
if θ is -ve then variance decreases for higher m
(atleast in these 2 example cases)

$|\text{Bias}(a)| < |\text{Bias}(b)|$
and

$V(a) > V(b)$	if $\theta > 0$	$V(a) = V(b)$
$V(a) < V(b)$	if $\theta < 0$	if $\theta = 0$

6) In an $M/M/\infty$ queue, the arrival rate is poisson (λ) as opposed to a $M(t)/M/\infty$ queue which has a Poisson $(\lambda(t))$ arrival rate.

a) The service rate is Expo $\left(\frac{1}{\tau}\right)$.
Thus the departure rate when there are $m(t)$ cars in the system $= \dfrac{m(t)}{\tau}$

The arrival rate $= \lambda$

Then we get:

$$\frac{d}{dt} m(t) = \lambda - \frac{m(t)}{\tau}$$

At steady state, $\dfrac{d}{dt} m(t) = 0$

$$\Rightarrow \quad \lambda = \frac{m(t)}{\tau}$$

$$\Rightarrow \quad m(t) = \underline{\underline{\lambda \cdot \tau}}$$

$$\lim_{t \to \infty} E(N(t)) = \underline{\underline{\lambda \tau}}$$

(At steady state)

b) In a $M/M/\infty$ queue, we know that $m(t)$ follows a Poisson distribution at steady state.

Since $E(N(t)) = \lambda \tau$

$\Rightarrow N(t) \sim \text{Poisson}(\lambda \tau)$

$V(N(t)) = \lambda \tau$

Due to law of large numbers:

$E(\bar{N}(t)) = m(t) = \lambda \tau$

$V(\bar{N}(t)) = \frac{V(N(t))}{N} = \frac{\lambda \tau}{10{,}000}$

$Sd = \sqrt{\frac{\lambda \tau}{10{,}000}}$

~~If we approximate the Poisson distribution as normal~~:

$\cancel{N(t) \sim N(\lambda \tau, \sqrt{\lambda \tau})}$

Then the confidence interval, $CI = \mu \pm Z \sigma / \sqrt{10{,}000}$

$= \mu \pm Z_{0.99} \sigma / \sqrt{10{,}000}$

$= \lambda \tau \pm 2.576 \sqrt{\lambda \tau} / \sqrt{10{,}000}$

(In this problem, $\lambda = 10$, $\tau = 1$)

$= 10 \pm \frac{2.576 \sqrt{10}}{\sqrt{10{,}000}} = (9.918, 10.0814)$ ~~(1.854, 18.146)~~

This CI does not cover the range of values seen in the simulation.

If there were a discrepancy, it may be due to initial

This discrepancy is due to the initial bias. The system starts with zero cars in the system and the simulation only runs for 24 hrs.
The mean we get in the simulation is 9.585 with 10,000 replications.

In part C, we initialize 10 cars and thus by reducing the initial bias, we get the mean as 10.0016 which is within the C I. Thus we see initial bias is the problem.

6

c) The simulation already runs for 24 hours. So no change in that part.

We make sure these 10 cars leave in a statistically correct amount of time by scheduling their departures. If we fail to schedule 10 departure events as well, these 10 cars will remain in the system for an infinite amount of time. The rate of departure is Exponential(1).

I add a section of code right after VBASim is initialized. It loops 10 times and in each loop, 1 car is added to the parking lot and its departure is also scheduled. I do not use the Arrival() function since it will schedule an arrival at a future time; but we want 10 cars at the start itself.

```
MaxCars = 10
for i in range(0,10):
        NewCar = bc.entity(Clock)
        ParkingLot.Add(NewCar, Clock)
        VBASim.Schedule(Calendar, "Departure", RNG.Expon(MeanPT,2)
                                                , Clock)
```

```python
for reps in range(0,10,1):
    Clock = 0.0
    MaxCars = 0


    VBASim.VBASimInit(Calendar,TheQueues,TheCTStats,TheDTStats,TheResources,Clock)

    # This section was added
    # It adds 10 cars in the system and then schedules the departure for those 10 cars as well
    MaxCars = 10
    for i in range(0,10):
        NewCar = bc.Entity(Clock)
        ParkingLot.Add(NewCar,Clock)
        VBASim.Schedule(Calendar,"Departure", RNG.Expon(MeanPT,2), Clock)

        #Print to ensure code is working correctly
        #print(ParkingLot.NumQueue())
        #print(Clock)

    VBASim.Schedule(Calendar,"Arrival",RNG.Expon(MeanTBA,1),Clock)
    VBASim.Schedule(Calendar,"EndSimulation",24,Clock)
```