Complex Cellular decomposition

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[1]

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Contents

Abstract

The Cellular parametrization theorem developed by Binyamini and Novikov has numerous applications as shown in [1]. We provide a modification to the aforementioned result in the real cell case so that the obtained real cellular cover is compatible with the given functions over the cell and disjoint over the positive real part of the cell. This result in particular helps us obtain a set of real cellular maps that form a partition of semialgebraic sets whose size is polynomial in terms of its complexity. We further show that the improved result implies the preparation theorem by Lion and Rolin in its full strength for bounded semialgebraic sets, with precise bounds on the number of cells and a strong control on the units .

1 Introduction

In their work in [1], Gal Binyamini and Dimitry Novikov introduced the notion of a *complex cell* and developed a whole theory around it. One of the important results they obtained was the Cellular Parametrization Theorem (CPT). As a consequence of the algebraic version of this CPT, they show that a semialgebraic set $S \subset (0,1)^l$ of complexity β can be covered by the image of real part of $\mathrm{poly}_l(\beta, \rho, 1/\sigma)$ many real cellular maps $f_j : \mathcal{C}_j^{\{\sigma\}} \to \mathcal{P}_l^{\{\rho\}}$, each of complexity $\mathrm{poly}_l(\beta)$, where $\mathcal{P}_l$ is the Cartesian product of l unit discs.

Notation: $\alpha = \mathrm{poly}_X(\beta)$ is a shorthand for $\alpha \leq P_X$, where $X \mapsto P_X$ is a universally fixed mapping whose values are univariate polynomials with positive coefficients.

This result in particular raises the following question:

Question. *Can we improve the above result to say that given S, ρ, σ as above, we have $poly_l(\beta, \rho, 1/\sigma)$ many real cellular maps $f_j : \mathcal{C}_j^{\{\sigma\}} \to \mathcal{P}_l^{\{\rho\}}$ each of complexity $poly_l(\beta)$ such that $f_j(\mathbb{R}_+\mathcal{C}_j) \subset S$* ***are pairwise disjoint*** *and $\cup_j (f_j(\mathbb{R}_+\mathcal{C}_j)) = S$.*

We precisely formulate the above informal question and prove it in this Master's thesis.

The implication of our result would be in the *Preparation theorems* Lion and Rolin developed in their work in [2]. In [1], it was shown that the CPT is related to the preparation theorem in [2]. However, [1] only provides a cover, whereas in [2], they obtain a partition. This poses significant issues, for instance, in the integration of functions over semialgebraic sets. While the result from [1] does provide a bound on the number of sets that form a cover, it is not strong enough in this regard as the cover is not disjoint. The result of the thesis solves this issue, allowing us to recover the full generality of the preparation theorem with disjoint covers [2].

We state the preparation theorem (for the special case of subanalytic sets, as the thesis deals with semialgebraic sets) as stated by Van den Dries and Speissegger in [3]:

Theorem 1.1 ([3], Theorem 2.1)**.** *Let $f_1, \ldots, f_k : \mathbb{R}^{l+1} \to \mathbb{R}$ be globally subanalytic functions. Then there is a finite covering $\mathcal{C}$ of $\mathbb{R}^{l+1}$ by globally subanalytic sets and for each set $S \in \mathcal{C}$ there are $\lambda_1, \ldots, \lambda_k \in \mathbb{Q}$ and functions $\theta, a_1, \ldots, a_k : \mathbb{R}^l \to \mathbb{R}$ and $u_1, \ldots, u_k : \mathbb{R}^{l+1} \to \mathbb{R}$, all globally subanalytic, such that graph θ is disjoint from S and for $i = 1, \ldots, k$ and all $(x, y) = (x_1, \ldots, x_n, y) \in S$, we have*

$$f_i(x, y) = |y - \theta(x)|^{\lambda_i} a_i(x) u_i(x, y), |u_i(x) - 1| < \frac{1}{2}$$

We will show section 3, a modified version of the CPT, using which we show that the real cellular maps into a semialgebraic set, can indeed be assumed to be real disjoint, as shown in 3.1.1.

It was already shown in [1, 4.2] that the Real CPT implies the Preparation Theorem1.1. In fact, they obtained a more precise form of the units u_i described above.

Our result strengthens this claim by showing that the cylindrical cells obtained by [1] can be made disjoint, allowing us to obtain a partition as in [2], but with stronger bounds on the units and control on the complexity, therefore strengthening the claims.

2 Complex Cells

In this section, we introduce some notations used throughout the thesis. For a more detailed understanding of complex cells and their properties, the reader is referred to [1].

For $r, r_1, r_2 \in \mathbb{C}$, with $|r| > 0, |r_2| > |r_1| > 0$, we define,

$$D(r) := \{|z| < |r|\} \qquad D_o(r) := \{0 < |z| < |r|\}$$

$$A(r_1, r_2) := \{|r_1| < |z| < |r_2|\} \qquad * := \{0\}$$

For $0 < \delta < 1$ & $0 < \rho < \infty$ define

$$*^\delta := * \qquad D_o^\delta(r) := D_o(\delta^{-1}r)$$

$$A^\delta(r_1, r_2) := A(\delta r_1, \delta^{-1}r_2) \qquad D^\delta(r) := D(\delta^{-1}r)$$

$$\mathcal{F}^{\{\rho\}} := \mathcal{F}^\delta,$$

where,

$$\rho = \frac{2\pi\delta}{1-\delta^2} \quad \text{if } \mathcal{F} = D$$

$$\rho = \frac{\pi^2}{2|log\delta|} \quad \text{if } \mathcal{F} = D_o, A$$

Definition 2.1. *Let $\mathcal{X}, \mathcal{Y}$ be sets and $\mathcal{F} : \mathcal{X} \to 2^{\mathcal{Y}}$ be a map. Define*

$$\mathcal{X} \odot \mathcal{F} := \{(x, y) : x \in \mathcal{X}, y \in \mathcal{Y},\ y \in \mathcal{F}(x)\}$$

We are now in a position to define complex cells

Definition 2.2. *A complex cell $\mathcal{C}$ of length zero is the singleton $\mathbb{C}^0$. The type of $\mathcal{C}$ is the empty word. A complex cell of length l+ 1 has the form $\mathcal{C}_{1\ldots l} \odot \mathcal{F}$ where $\mathcal{C}_{1\ldots l}$ is a complex cell of length l and $\mathcal{F}$ is one of $*, D(r), D_o(r), A(r_1, r_2)$ where $r \in \mathcal{O}_b(\mathcal{C}_{1\ldots l})$ satisfies $|r(z)| > 0$ for $(z_1 \ldots z_l) \in \mathcal{C}_{1\ldots l}$; and $r_1, r_2 \in \mathcal{O}_b(\mathcal{C}_{1\ldots l}$ satisfy $0 < |r_1(z_1 \ldots z_l)| < |r_2(z_1 \ldots z_l)| for (z_1 \ldots z_l) \in \mathcal{C}_{1\ldots l}$ The type $T(\mathcal{C})$ is $T(\mathcal{C}_{1\ldots l})$ followed by the type of the $\mathcal{F}$*

Here, $\mathcal{O}_b(\mathcal{X})$ denotes the space of bounded holomorphic functions on $\mathcal{X}$.

Definition 2.3 (Real complex cells and real part of cells). *The cell of length zero is real and equals its real part. A cell* $\mathcal{C} := \mathcal{C}_{...} \odot \mathcal{F}$ *is real if* $\mathcal{C}_{1...l}$ *is real and the radii involved in* $\mathcal{F}$ *can be chosen to be real holomorphic functions on* $\mathcal{C}_{1...l}$*. The real part* $\mathbb{R}\mathcal{C}$ *(resp. positive real part* $\mathbb{R}_+\mathcal{C}$*) of* $\mathcal{C}$ *is defined to be* $\mathbb{R}\mathcal{C}_{1...l} \odot \mathbb{R}\mathcal{F}$ *(resp.* $\mathbb{R}_+\mathcal{C}_{1...l} \odot \mathbb{R}_+\mathcal{F}$*), where* $\mathbb{R}\mathcal{F} := \mathcal{F} \cap \mathbb{R}$ *(resp.* $\mathbb{R}_+\mathcal{F} := \mathcal{F} \cap \mathbb{R}_+$*), except the case* $\mathcal{F} = *$*, where we set* $\mathbb{R}_+ * = \mathbb{R}* = *$*. A holomorphic function on* $\mathcal{C}$ *is said to be real if it is real on* $\mathbb{R}\mathcal{C}$*.*

Definition 2.4. *For* $\delta \in (0,1)^{l+1}$ *we say that a cell* $\mathcal{C}$ *of length* $l+1$ *admits a* δ*-extension* $\mathcal{C}^\delta := \mathcal{C}_{1...l}^{\delta_{1...l}} \odot \mathcal{F}^{\delta_{l+1}}$ *if* $\mathcal{C}_{1...l}$ *admits a* $\delta_{1...l}$ *extension, and if the function* r *(respectively* r_1, r_2*) involved in* $\mathcal{F}$ *admits holomorphic continuation to* $\mathcal{C}_{1...l}^{\delta_{1...l}}$ *and satisfies* $|r(z_1,\dots,z_l)| > 0$ *(respectively* $0 < |r_1(z_1,\dots,z_l)| < |r_2(z_1,\dots z_l)|$*). The* $\{\rho\}$*-extension* $\mathcal{C}^{\{\rho\}}$ *is defined similarly.*

From now on, whenever we say $\mathcal{C}^\delta$ (or $\mathcal{C}^{\{\rho\}}$) is a complex cell, we mean that $\mathcal{C}$ is a complex cell that admits a δ (or $\{\rho\}$) extension.

Definition 2.5. *A real holomorphic map* $f : \mathcal{C} \to \hat{\mathcal{C}}$ *between two real cells of length l is called a real cellular map if* $f(z_1,\dots z_l) = (\phi_1(z_1), \phi_2(z_1,z_2), \dots \phi_l(z_1,\dots z_l))$*, where each* $\phi_j \in \mathcal{O}_b(\mathcal{C}_{1,\dots j}))$ *is a real holomorphic map and it is a polynomial of positive degree with respect to* z_j*, with leading coefficient* $= \pm 1$*.*

We say f is prepared if $\phi_j(z_1,\dots z_j) = \pm z_j^{d_j} + \hat{\phi}_j(z_1,\dots z_{j-1})$*;* $\hat{\phi}_j \in \mathcal{O}_b(\mathcal{C}_{1,\dots j-1})$*.*

For any $F \in \mathcal{O}_b(\hat{\mathcal{C}})$*, we say* f *is compatible with* F *if* Fof *is either identically zero or nowhere zero on* $\mathcal{C}$

Do note that in [1], the cellular maps were defined with the assumption that ϕ_j were monic in z_j. This would be insufficient for us as indicated in 5.2.6. So we have introduced the signs.

We now state the real cellular parametrization theorem (Real CPT) proved in [1], which is one of the main interests in this thesis.

Theorem 2.6 ([1], Theorem 8). *Let* $\rho, \sigma \in (0,\infty)$*. Let* $\mathcal{C}^{\{\rho\}}$ *be a real cell and* $F_1, \dots F_M \in \mathcal{O}_b(\mathcal{C}^{\{\rho\}})$*, then there exists a set of real cellular maps* $\{f_j : \mathcal{C}_j^{\{\sigma\}} \to \mathcal{C}^{\{\rho\}}\}$ *such that each* f_j *is prepared,* $\mathbb{R}_+\mathcal{C} \subset \cup_j f_j(\mathbb{R}_+\mathcal{C}_j)$ *and* f_j *is compatible with every* F_k*.*

If $\mathcal{C}^{\{\rho\}}, F_1, \dots F_M$ *are algebraic of complexity* β*, then the cover has size* $poly_l(\beta, M, \rho, 1/\sigma)$ *and complexity* $poly_l(M, \beta)$

We also define the notion of a $v-$cover of a cell, which will be used in later sections.

Definition 2.7. *Let* $\mathcal{C}$ *be a cell of length 1. For* $\mathcal{C} = D_o(r)$ *or* $A(r_1, r_2)$*, define the* $v-$*cover* $\mathcal{C}_{\times v}$ *by*

$$D_o(r)_{\times v} := D_o(r^{1/v}) \quad A(r_1,r_2) := A(r_1^{1/v}, r_2^{1/v}).$$

For $\mathcal{C} = D(r)$*, only the* $1-$*cover is defined, and it is the cell itself.*

Define $R_v : \mathcal{C}_{\times v} \to \mathcal{C}$ via $z \mapsto z^v$.

For a cell of length l and for $v = (v_1, \dots v_l) \in \pi_1(\mathcal{C}) \simeq \Pi G_i$ (here G_i is trivial for types $*$ and D, and equals $\mathbb{Z}$ for types D_o and A) such that $v_j | v_k$ whenever $j > k$ and $G_j = G_k = \mathbb{Z}$, we define $\mathcal{C}_{\times v}$ and $R_{\times v}$ by induction. For $\mathcal{C} = \mathcal{C}_{1,\dots,l-1} \odot \mathcal{F}$,

$$\mathcal{C}_{\times v} := (\mathcal{C}_{1,\dots,l-1})_{\times v_{1,\dots,l-1}} \odot R^*_{\times v_{1,\dots,l}} \mathcal{F}_{\times v_l}$$

$$R_{\times v}(z_{1,\dots,l}) := z^v.$$

For more detailed description of the above construction and proof of why this is a well defined cell, the reader is directed to [1, section 2.6].

3 The Main Claims

As one may notice in [1, section 4], 1.1 was proven (with stronger units and control on complexity) using the CPT for bounded semialgebraic sets. This motivates us to approach the solution to our question by attempting to modify the Real version of the CPT. Indeed, we claim the following :

Proposition 3.1 (Modified Real CPT)**.** *Let $\rho, \sigma \in (0, \infty)$. Let $\mathcal{C}^{\{\rho\}}$ be a real cell of length l and $F_1, \dots F_M \in \mathcal{O}_b(\mathcal{C}^{\{\rho\}})$ be real holomorphic functions, with $\mathcal{C}^{\{\rho\}}$, $F_1, \dots F_M$ algebraic of complexity β, then there exists a set of real cellular maps $\{f_j : \mathcal{C}_j^{\{\sigma\}} \to \mathcal{C}^{\{\rho\}}\}$ such that each f_j is prepared, $\mathbb{R}_+\mathcal{C} = \bigsqcup_j f_j(\mathbb{R}_+\mathcal{C}_j)$ and each f_j is compatible with every F_k. Moreover this set has size $poly_l(\beta, M, \rho, 1/\sigma)$ and complexity $poly_l(M, \beta)$*

We shall refer to such a set of cellular maps obtained above as a " disjoint real cell cover". This is an extension to the notion of a real cell cover defined in [1].

As a corollary, we obtain the following:

Corollary 3.1.1. *Assume the modified Real CPT holds. Let $\sigma, \rho \in \mathbb{R}_+$. Given a semialgebraic set $S \subset (0,1)^l$ of complexity β, there exist $poly_l(\beta, \rho, 1/\sigma)$ many real cellular maps $f_j : \mathcal{C}_j^{\{\sigma\}} \to \mathcal{P}_l^{\{\rho\}}$, each of complexity $poly_l(\beta)$, such that $f_j(\mathbb{R}_+\mathcal{C}_j^{\{\sigma\}}) \subseteq S$, and $f_j(\mathbb{R}_+\mathcal{C}_j)$ form a disjoint cover of S. Here, $\mathcal{P}_l$ is the Cartesian product of l unit discs.*

Proof. Assuming the modified real CPT described above, we proceed as follows.

Given a semialgebraic set $S \subset (0,1)^l$ with complexity β, we know by definition that

$$S = \{\mathrm{sgn}(P_1) = \sigma_1, \dots \mathrm{sgn}(P_N) = \sigma_N\}; \quad \sigma_1, \dots \sigma_N \in \{-1, 0, +1\}$$

Where $N \leq \beta$ and $P_1, \dots P_N \in \mathbb{R}[x_1, \dots x_l]$ are polynomials of degree at most β.

Thus, applying the modified Real CPT to $\mathcal{P}_l^{\{\rho\}}$ with the polynomials that define S gives us a collection of $\mathrm{poly}_l(\beta, \rho, 1/\sigma)$ real prepared cellular maps

$f_j : \mathcal{C}^{\{\sigma\}} \to \mathcal{P}_l^{\{\rho\}}$ each with complexity $\mathrm{poly}_l(\beta)$, such that each f_j is prepared and each f_j is compatible with every $P_i, i \in \{1, \dots N\}$. Moreover,

$$(0,1)^l = \bigsqcup_j f_j(\mathbb{R}_+\mathcal{C}_j).$$

The compatibility condition above implies that $P_i \circ f_j(\mathbb{R}_+\mathcal{C}_j)$ is of a constant sign, for every i and j. Thus one may choose a subset of the above f_j's, such that

$$f_j(\mathbb{R}_+\mathcal{C}_j) \subset S \text{ and } \bigsqcup_j f_j(\mathbb{R}_+\mathcal{C}_j) = S.$$

This is possible because S was defined in terms of the signs of the P_i.

Thereby, finishing the proof of the above proposition.

□

Thus, we are left with proving this modified CPT. Section 5 is dedicated to the proof of the modified real CPT.

4 Scope of the Result

Here, we indicate how our result in the previous section may be applied to obtain a stronger version of the preparation theorem cited. This section largely retraces the arguments outlined in [1, section 4].

Given a bounded semialgebraic function $F : X \subset (0,1)^l \to (0,1)$. Denote by $G_F \subset (0,1)^{l+1} (\subset \mathcal{P}_l^{1/2} \times \mathcal{P}^{1/2})$, the graph of the function F.

We may now use 3.1.1 and 3.1 in order to obtain real prepared cellular maps $f_j : \mathcal{C}_j^{\{\sigma\}} \to \mathcal{P}_l^{1/2} \times \mathcal{P}^{1/2}$ such that

$$\begin{aligned} \bigsqcup f_j(\mathbb{R}_+\mathcal{C}_j) = G_F \qquad f_j \text{ is compatible with } \tilde{F}, \tilde{G} \\ \tilde{F} : \mathcal{P}_l^{1/2} \times \mathcal{P}^{1/2} \to \mathbb{C}; (x,y) \mapsto y \\ \tilde{G} : \mathcal{P}_l^{1/2} \times \mathcal{P}^{1/2} \to \mathbb{C}; (x,y) \mapsto x_l. \end{aligned} \tag{1}$$

Notice that as G_F is a graph, the type of $\mathcal{C}_j$'s must end with $*$. We now use [1, Monomialization lemma] and the preparedness of the $f_j's$ to conclude that on each $\mathcal{C}_j^{\{\sigma\}}$, either $f_j^*\tilde{F} \equiv 0$ or $f_j^*\tilde{F}$ is non-zero everywhere on $\mathcal{C}_j^{\{\sigma\}}$, and is of the form :

$$\mathbf{z}^{\alpha(j)} U_j(\mathbf{z}). \tag{2}$$

where U_j is a holomorphic map bounded away from zero and infinity on $\mathcal{C}_j$ and $\mathbf{z}$ denotes a point in $\mathcal{C}_j^{\{\sigma\}}$. In fact, we can say even more about $\alpha(j)$ U_j from the monomialization lemma. Indeed,

$|\alpha(j)| = \text{poly}_l(\beta)$

$$\text{diam}(\log U(\mathcal{C}_j;\mathbb{C})) < \text{poly}_l(\beta).\sigma, \text{diam}(\text{Im } \log U(\mathcal{C}_j;\mathbb{R})) < \text{poly}_l(\beta). \quad (3)$$

We rewrite the above functions in terms of the coordinates in $\mathcal{P}_l^{1/2} \times \mathcal{P}^{1/2}$, hereby denoted (x, y).

Since the f_j's are prepared, it follows that:

$$(z_1, \dots z_l) = (x_1^{1/v_1}, \dots, |x_l - \phi_l(z_{1\dots l-1})|^{1/v_l}) \quad (4)$$

Where we restrict $\mathbf{z}_j$ to $\mathbb{R}_+\mathcal{C}_j^{\{\sigma\}}$, and take the positive real branch on the right hand side (The subscript j was dropped above for notational clarity).

It thus follows that in the cylindrical cell $f_j(\mathbb{R}_+\mathcal{C}_j)$, we have

$$F(\mathbf{x}) = \mathbf{z}^\alpha U(\mathbf{z}), \quad (5)$$

where z is of the form described in (4) and α, U are as in (2).

We further notice that since the $f_j's$ are prepared with respect to $\tilde{G}$, we have that whenever ϕ_l is not identically zero, there is a constant $C > 1$, depending on σ, such that

$$\begin{aligned} |x_l - \phi_l(z_{1,\dots l-1})| \leq \phi_l(z_{1,\dots l-1}) \times 1/C, \\ \forall z_{1\dots l-1} \in (\mathcal{C}_j)_{1,\dots l-1}. \end{aligned} \quad (6)$$

This claim is proven in the appendix 7.3.

We therefore see that from (5) we obtain the preparation theorem 1.1 for the bounded semialgebraic case. Moreover, since we obtain a partition instead of just a covering because of 3.1 and (6), we obtain a stronger version of the preparation theorem, analogous to the one stated in [6, Theorem 2.4](also see [2]) for the bounded semialgebraic case.

We do note that (6) implies $x_l \sim_\epsilon \phi_l$ where $\sim_\epsilon$ is as defined in [6]: For any real-valued functions f and g on a set A, we say that f **is equivalent to** g **on** A, written $f \sim_\epsilon g$ on A, if there exists $\epsilon > 1$ such that for all $x \in A$,

$$\begin{cases} \epsilon^{-1} f(x) \leq g(x) \leq \epsilon f(x), & \text{if } f(x) \geq 0, \\ \epsilon f(x) \leq g(x) \leq \epsilon^{-1} f(x), & \text{if } f(x) < 0. \end{cases}$$

One may note that the preparation is done for multiple functions simultaneously in [6] and [2]. Our result can be extended for the case where we have multiple semialgebraic functions $F_i : X \subset (0,1)^n \to (0,1)$ instead of the single function F that we dealt with above. Indeed, the modified Real CPT would give us a cylindrical decomposition of X with the cellular maps satisfying (1) to (6) for all the functions F_i.

Further, the monomialization lemma gives us a more precise nature of the units compared to [2] or [6].

This shows that our result strengthens the results obtained in [1, Section 4].

5 Proof of Modified Real CPT

This section is devoted to proving the Modified Real CPT. As one may notice in the inductive proof of the CPT in [1], the authors use the (real) Refinement Theorem, the (real) Weierestrass Preparation Theorem(WPT) and the (real) Cellular Preparation Theorem (CPrT) in the inductive steps.

Our proof follows a similar construction, and therefore, it is necessary that the maps obtained in the WPT, CPrT are also disjoint. We will provide such modifications in section 6 to these accessory lemmas. For now, we will assume that the WPT, CPrT and the Refinement theorem indeed provide us with real disjoint cellular maps in the algebraic case!

We now prove the modified Real CPT. We follow a similar approach as in [1], with first proving a weak version, where the real cellular maps obtained are not necessarily prepared, but satisfy the other claims. We then use the Modified CPrT to make these cellular maps prepared.

Since we are interested in the algebraic version of the Real CPT, we assume that the cells and the functions F_i defined over them are algebraic of complexity $\mathrm{poly}_l(\beta)$.

5.1 Setup

We assume that the Modified CPT holds for every real cell of length less than l and and every real cell of length l and dimension smaller than l. This is our inductive assumption (the case for l=0,1 can be proved easily, as the reader may notice). We now prove the statement for a real cell of length and dimension l.

We may without loss of generality, assume that our cell is of the form

$$(\mathcal{C} \odot \mathcal{F})^{\{\rho\}},$$

where $\mathcal{F}$ is an annulus (the case for a disc/ punctured disc is similar, while the case $\mathcal{F} = *$ is trivial). We further consider the case where we have a single function $F \in \mathcal{O}_b((\mathcal{C} \odot \mathcal{F})^{\{\rho\}})$ (not identically zero) over the cell. We can do so because, if we prove there is a disjoint real cellular cover compatible with a single function F, we also prove that we can construct a disjoint real cellular cover compatible with M functions. A similar statement was proved in [1, section-8.1.1], albeit for the Real CPT without the disjointness assumption, but we urge the reader to verify that this proof can be modified in a simple way to make it work for the disjoint version as well. We will thus assume that we have a single function F over the Real Cell for which we need a compatible cover.

As we will show in section 6, we have a Modified WPT and Modified Refinement theorem, which are a stronger version of the Real WPT and Refinement theorem in [1], where in our case, the real cellular maps we obtain will be disjoint over the positive real part of the cell. We can therefore assume by the virtue of these results that $\mathcal{C}$ admits $\hat{\rho}$ extension, with $\hat{\rho}^{-1} = \mathrm{poly}_l(\rho, \beta)$, analogous to a similar assumption made in [1]. We will assume below that ρ is already small enough (subject to the above estimate). Similarly, σ may be replaced with $\hat{\sigma} = \text{poly}_l(\beta)$. We therefore assume that σ is large ($= 1/2$ say).

We apply the Modified WPT to cover $\mathcal{C} \odot \mathcal{F}$ by cells of the form $f_j : \mathcal{C}_j^{\{\rho\}} \odot \mathcal{F}_j^{\gamma} \to (\mathcal{C} \odot \mathcal{F})^{\{\rho\}}$ which are Weierstrass for F with gap $\gamma = 1 - 1/\text{poly}_l(\beta)$. It is enough to prove the Modified CPT for each of these cells separately, i.e. we may assume without loss of generality $F \in O_b(\mathcal{C}^{\{\rho\}} \odot \mathcal{F}^{\gamma})$ does not vanish in $\mathcal{C}^{\{\rho\}} \odot (\mathcal{F}^{\gamma} \setminus \mathcal{F})$ - see [1, section 7 and 6.3], also Modified WPT .

It now follows from these assumptions that the zero locus of F is a proper ramified cover of $\mathcal{C}$ under the projection $\pi_{1 \dots l-1} : \mathcal{C}^{\{\rho\}} \times \mathcal{F}^{\gamma} \to \mathcal{C}^{\{\rho\}}$.

We can construct a discriminant polynomial $\mathcal{D}$ over the base cell $\mathcal{C}^{\{\rho\}}$ such that outside $\{\mathcal{D} = 0\}$, the projection $\pi_{1 \dots l-1}$ restricted to the zero locus of F is a proper covering map. For the construction of $\mathcal{D}$ and the proof that it is algebraic of complexity $\text{poly}_l(\beta)$, refer to [1, section-7, Lemma 64]

We may by inductive assumption, obtain a disjoint real cellular cover $\{f_j : \mathcal{C}_j^{\{\rho\}} \to \mathcal{C}^{\{\rho\}}\}$ for $\mathcal{D}$. If $f_j(\mathcal{C}_j^{\{\rho\}}) \subset \{\mathcal{D} = 0\}$, then $\dim(\mathcal{C}_j) \leq \dim(\mathcal{C})$-1, as cellular maps preserve dimensions. In this case, we may construct a disjoint real cellular cover for $(\mathcal{C}_j \odot f_j^* \mathcal{F})^{\{\rho\}}$, compatible with $\hat{f}_j^* F$ (here $\hat{f}_j = (f_j, id)$). This in turn gives us a set of real disjoint cellular cover for $f_j(\mathcal{C}_j) \odot \mathcal{F}$.

We are hence left with covering the real parts of those $f_j(\mathcal{C}_j) \odot \mathcal{F}$, for which $f_j(\mathcal{C}_j^{\{\rho\}}) \cap \{\mathcal{D} = 0\} = \phi$. But now, going to the pullback of $\mathcal{C}$ via f_j, one can see that the projection map onto the first l coordinates is a proper unramified covering map (as the discriminant locus is empty). Hence, we may without loss of generality assume that the natural projection of $\mathcal{C}^{\{\rho\}} \odot \mathcal{F}^{\gamma} \cap \{F = 0\}$ onto $\mathcal{C}^{\{\rho\}}$ is unramified.

We thus have v sections $y_1, \dots y_v$ such that each y_j lifts to a univalued map $y_j : \hat{\mathcal{C}}_j \to \mathbb{C}$, where $\hat{\mathcal{C}}_j := \mathcal{C}_{\times v_j}, 1 \leq v_j \leq v, v_j \in \mathbb{N}$ ([1, Lemma 54]). Further y_j extends holomorphically to $\hat{\mathcal{C}}_j^{\{v_j \cdot \rho\}}$ We also include y_0- the zero map on the base, and the functions r_1, r_2 in the list of sections as in [1]. Note that $v = \text{poly}_l(\beta)$. Notice also that y_j's are algebraic of complexity $\text{poly}_l(\beta)$

We denote by $\hat{\mathcal{C}}$ the cell $\mathcal{C}_{\times v!}$. Notice that all y_i are univalued over $\hat{\mathcal{C}}$. Further, we note that for 4 sections y_a, y_b, y_c, y_d (and similarly for any number of sections), we have a cell $\hat{\mathcal{C}}_{a,b,c,d} := \mathcal{C}_{\times (v_{a,b,c,d})}$ over which these sections are univalued. This cell is of $\text{poly}_l(\beta)$ complexity, and admits a $\{v_{a,b,c,d} \cdot \rho\}$ extension, $v_{a,b,c,d} := \text{lcm}(v_a, v_b, v_c, v_d)$. For more details about these pullbacks over which the y_i are univalued, the reader is referred to [1, 6.3.1].

We note that, in order to make it easy for the reader, we often construct cell covers while considering $\hat{\mathcal{C}}$ (which is a-priori of exponential complexity in β). As we will be using no more than five sections at once in the constructions, It is implicit that these cell maps descend down to a cell map via a cell of the form $\hat{\mathcal{C}}_{a,b,c,d,e}$, which indeed have $\mathrm{poly}_l(\beta)$ complexity.

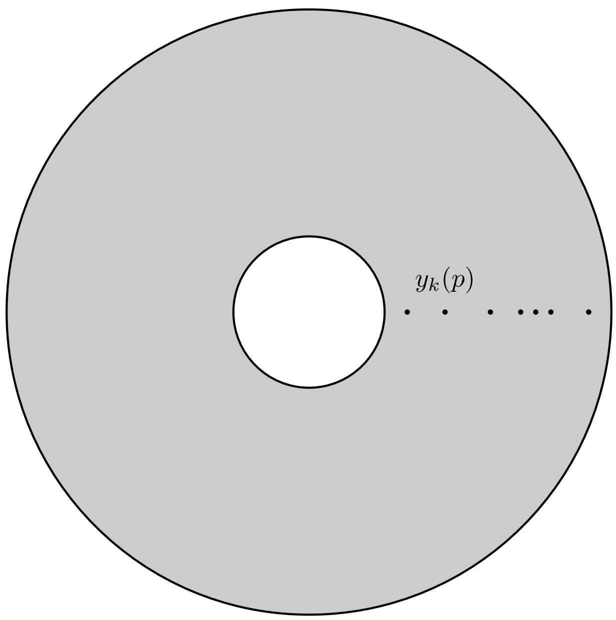
We now have the set-up to prove the claim.

We proceed to prove the Modified Real CPT first in the simple situation where all y_i are real holomorphic functions, and then for the general case, we will show the required modifications for the proof. The arguments in this case is essentially similar to the same steps as in the simplified situation.

5.2 Simple situation: y_i are real

5.2.1 The ordering among the sections

Here, one notices that the y_i are ordered over the positive real part of the base cell $\mathbb{R}_+\hat{\mathcal{C}}$, because we have $y_j \neq y_k$ and that the y_i are real. For the ease of computation, we assume moreover that all these sections are **positive** real over $\mathbb{R}_+\mathcal{C}$. The y_i cover the zeroes of F, and the map $\hat{\mathcal{C}}_i^{\{v_j\rho\}} \odot *$ to the cell $\mathcal{C}^{\{\rho\}} \odot \mathcal{F}^\gamma$ via $(R_{\times v_j}, y_j)$, we have real (disjoint) cellular maps compatible with F. We do not include the sections y_0, r_1, r_2 of course, as they do not cover the zeroes for F in this case.



The y_k are ordered over $p \in \mathbb{R}_+\hat{\mathcal{C}}$, and this order is preserved over $\mathbb{R}_+\hat{\mathcal{C}}$

We now proceed to set $y_0 = 0$ and cluster around it, following similar steps as in [1]. We note that as assumed in [1], we assume [1, Proposition 56] and its analogue, holds over not just the cell, but also $\mathcal{C}^{\{\sigma\}}$ ($\mathcal{C}$ admits such an extension via the inductive hypothesis).

Here, we make a slight distinction in defining the $\mathcal{F}_{0,q}$ and $\mathcal{F}^{+}_{0,q}$. Since we aim to obtain a disjoint cover, we set

$$\mathcal{F}_{0,q} = A(l_{0,q}\hat{y}_{0,q}, r_{0,q}\hat{y}_{0,q}) 1 \leq q \leq m, \quad \mathcal{F}_{0,q^+} = A(r_{0,q}\hat{y}_{0,q}, l_{0,q+1}\hat{y}_{0,q+1}), 1 \leq q \leq m-1$$

and

$$\mathcal{F}_{0,0^+} = D_o(l_{0,1}\hat{y}_{0,1}), \quad \mathcal{F}_{0,m^+} = A(r_{0,m}\hat{y}_{0,m}, \infty)$$

instead of

$$A^\delta(l_{0,q}\hat{y}_{0,q}, r_{0,q}\hat{y}_{0,q}), \quad A^\delta(r_{0,q}\hat{y}_{0,q}, l_{0,q+1}\hat{y}_{0,q+1}),$$

and

$$D_o^\delta(l_{0,1}\hat{y}_{0,1}), \quad A^\delta(r_{0,m}\hat{y}_{0,m}, \infty)$$

described in [1]. Here m denotes the number of clusters around y_0. As shown in 7.1.2, these annuli will be well defined as shown in the Appendix. As in [1], we may take ρ small enough for proposition 7.1.2 to hold over $\mathcal{C}^{\{\rho\}}$.

Also note that, so far, we have only clustered around y_0. We will cluster around the other sections subsequently.

Also, notice that each annulus is described by at most three sections and therefore, they arise as pullbacks over the projection $\hat{\mathcal{C}} \to \hat{\mathcal{C}}_{a,b,c}$ for a suitable choice of a, b, c.

We note from the appendix that the $\mathcal{F}_{0,q}, \mathcal{F}_{0,q^+}(p)$ cover $\mathbb{C} \setminus \{0\}$ except the boundaries $\partial\mathcal{F}_{0,q}$, $\partial\mathcal{F}_{0,q^+}$. However, since we are only concerned with covering the real part of the cell, we introduce m many cells ending with type $*$ to cover the real boundary points.

Say r_1, r_2 are in the q_1, q_2^{th} clusters respectively $(q_1 \leq q_2)$. The $\mathcal{F}^\gamma_{0,q^+}$ do not contain any y_i. Thus they serve as a cell compatible with F, mapping into $\mathcal{C}^{\{\rho\}} \odot \mathcal{F}^\gamma$ -refer the note below.

These are the first type of cells we use to cover the cell. It is easy to see that they do not overlap over $\mathbb{R}_+(\mathcal{C} \odot \mathcal{F})$ -see appendix. We are thus left with covering the real part of the cell so that these covers do not intersect with the already constructed cells over the real part. For this we proceed as follows. As mentioned before, we can and will assume that

$$y_i(x) < y_{i+1}(x), \forall i \in \{0, \ldots, v-1\} \forall x \in \mathbb{R}_+\hat{\mathcal{C}} \tag{7}$$

Of course we ensure that in the next step, we cover only the region between r_1 and r_2 in $\mathcal{F}_{0,q_1}$ and $\mathcal{F}_{0,q_2}$ respectively i.e say $r_1 = \hat{y}_{q_1}, r_2 = \hat{y}_{q_2}$ then we modify $\mathcal{F}_{0,q_1}$ and $\mathcal{F}_{0,q_2}$ to $A(r_1, r_{0,q_1}r_1)$ and $A(l_{0,q_2}r_2, r_2)$ respectively, and ensure that the cells we construct as part of the cover admit a 1/2 extension such that its image lies inside $\mathcal{C}^{\{\rho\}} \odot \mathcal{F}^\gamma$.

Note: When we mention above that the annuli describe a cell, we actually mean that we can construct a (real) cellular map $\hat{\mathcal{C}}_{0,q^+} \odot \mathcal{F}_{0,q^+} \to \mathcal{C} \odot \mathcal{F}$. Here, $\hat{\mathcal{C}}_{0,q^+}$ denotes a pullback of $\mathcal{C}$ such that the sections $y_0, \hat{y}_{0,q}, \hat{y}_{0,q+1}$ are univalued. Notice that the above cells admit a γ extension compatible with F, owing to Proposition 7.1.2.

In the subsections that follow, when we say that a **certain fibre (disc, punctured disc, or an annulus) describes a cell cover**, we mean it in a similar spirit.

5.2.2 Constructing the Simple Groups

We focus our attention to the q^{th} cluster now, $q_1 \leq q \leq q_2$. We may assume without loss of generality that $\rho = O(1/v^{11})$, where v is the number of sections and $\hat{y}_{0,q} = 1$ under a suitable real map (this is a simple scaling map $z \mapsto \frac{z}{\hat{y}_{0,q}}$). We also have $y_0 = 0$ by assumption.

When we say **Normalized coordinates** , we refer to the coordinates under a certain normalization, like in this particular case : $y_0 = 0, \hat{y}_{0,q} = 1$. From this point on until (not including) 5.2.6, we are working with this normalization!

This in particular gives us $\mathcal{F}_{0,q} \subset A(1/2, 2)$ (recall $\gamma = 1 - 1/\text{poly}_l(\beta)$.).

Recall we fixed a point p in $\mathbb{R}_+\hat{\mathcal{C}}$ in the clustering process. We proceed to work over this point again. We saw that if $\varepsilon < \varsigma \in \{1, \dots v\}$ and $y_\varepsilon, y_\varsigma \in \mathcal{F}_{0,q}$, then $y_\varepsilon(p) < y_\varsigma(p)$ from (7).

If y_j is the first section in this cluster, and if y_j, y_{j+1} are more than $1/v^4$ distance apart over p, we consider $\{y_j\}$ as a *simple group*. If not, we proceed to check if y_{j+1}, y_{j+2} are further than $1/v^4$ distance apart over p. We keep doing this until we find $y_{j'}$ in this cluster such that $y_{j'}(p)$ is either the last section in $\mathcal{F}_{0,q}$ or $y_{j'}(p), y_{j'+1}(p)$ are further than $1/v^4$ distance. We thereby obtain a sequence of sections $y_j \dots y_{j'}$, where no two consecutive sections are more than $1/v^4$ distance apart over p. We then proceed to check similarly for the subsequent sections in this cluster.
We refer to each set of these sections obtained this way as a *simple group* from here on. y_j **and** $y_{j'}$ **i.e the first and last sections in this simple group are hereby referred to as the edge sections of this simple group.**

Now, we look at the point $\frac{y_j(p)+y_{j'}(p)}{2}$. There is some y_l in this simple group such that $y_l(p)$ is closest to this point. We fix $y_l(p)$ as our **center** of this simple group. Notice how in a simple group, this chosen center is **at most** $\frac{1}{2v^4}$ distance from $\frac{y_j(p)+y_{j'}(p)}{2}$. Centered around this $y_l(p)$, we draw a disc of radius

$$\max\{y_{j'}(p) - y_l(p), y_l(p) - y_j(p)\} + 1/v^4 \ . \tag{8}$$

We then move on to $y_{j'+1}$ and repeat this same process starting with this section. We call these constructed discs, the **discs defining the simple group**. The simple group we refer to will be clear from the context.

5.2.3 Coalescing the simple groups

From the previous step, we obtain a collection of simple groups of sections where two consecutive sections in these groups are less than $1/v^4$ distance apart over p. Now if the discs around two groups are far enough ($> 1/v^4$ distance) over p, we leave the discs as they are. However, if not, we combine the two groups together and make a bigger group consisting of the sections in both the simple groups. We again define the edge sections of this group in an analogous way as in 5.2.2. The center is also chosen in the exact same way, i.e. the nearest section to the midpoint of the edge sections (at p). Note that in this case, since we are combining two simple groups with their discs $\leq 1/v^4$ distance from each other, the center is now at most $\frac{5}{2v^4}$ distance from the midpoint of the edge sections at p.

With the center as chosen above, we draw a disc of radius $1/v^4 + d_{\max}$, where $d_{\max}$ is the larger of distance from the chosen center to each of the edge sections (at p). Note that this choice of radius is in similar spirit as in (8). We refer to these discs as the disc defining the group.

We keep repeating this process till we obtain groups such that the discs defining the groups are $> 1/v^4$ distance apart over p. This coalescing action is a finite process and has to terminate, because we have finitely many sections.

We therefore we get a collection of groups $G_1, \ldots G_M$ inside this cluster with centers $c_1(p), \ldots c_M(p)$, where the c_k denote the section representing the center chosen for the group G_k. Further, the discs $D_{k,p}$ **defining the** G_k are at least $1/v^4$ distance apart from each other. Each $D_{k,p}$ is of the form $D(c_k(p), 1/v^4 + d_{\max,k})$., where $d_{\max,k}$ is as above (the k denotes the k^{th} group).

We also assume here that the groups are ordered, in the sense that for any pair of sections $y_\varepsilon \in G_i$ and $y_\varsigma \in G_{i+1}$, we have $y_A(x) < y_B(x), \forall x \in \mathbb{R}_+\hat{\mathcal{C}}$, $\forall i \in \{1, \ldots M-1\}$.

5.2.4 Transition from the diagram over p to a more general diagram

This whole construction so far has been over the point p. We use the construction in 5.2.3 as a **reference for the actual construction of compatible cells** that will be given now:
In the construction so far (still in the normalized coordinates of 5.2.2), since the centers of $D_{k,p}$ and the radii are fixed, the boundary points of $D_{k,p}$: $c_k(p) \pm (1/v^4 + d_{\max,k})$ are also constants. However, we now make a slight change to this, as we are more interested in the points $c_k \pm (1/v^4 + d_{\max,k})$ i.e. the boundaries of the $D_{k,p}$ need to be allowed to vary with p.

In order to achieve this, we define the D_k as

$$D_k = D(c_k, 1/v^4 + d_{\max,k}). \tag{9}$$

We will show in the appendix that,if ρ is small enough, all sections in G_k lie inside the newly defined D_k (over the entire base cell), and $c_k(x) + (1/v^4 + d_{\max,k}) < c_{k+1}(x) - (1/v^4 + d_{\max,k+1}) \forall x \in \mathbb{R}_+\hat{\mathcal{C}}$ i.e ρ is too small for the D_k, D_{k+1} to ever intersect over the base cell. This will be obtained as a consequence of [1, Fundamental Lemma].

To explain how we use the fundamental lemma briefly, we know for any section y_j inside a cluster, in the normalized coordinates

$$\operatorname{diam}(y_j(\hat{\mathcal{C}}, \mathbb{C} - \{0,1\})) = O_l(v^3\rho). \tag{10}$$

Since we have by [5] that the hyperbolic metric has a lower bound in the region Re(z)< 1/2 (in fact in both regions $\Omega_1 := \{z : |z| \leq 1, |z| \leq |z-1|\}$ and $\Omega_3 := \{z : 1/z \in \Omega_1\}$) given by

$$\frac{1}{2|z|\sqrt{2}[4 + \log(3 + 2\sqrt{2} + |\log|z||)]} \quad ;$$

and using the facts that $z \mapsto 1 - z$ is an isometry in $\mathbb{C} - \{0,1\}$, along with the fact that our cluster is contained in $A(1/2, 2)$ in the normalized coordinates of 5.2.2, one can easily conclude that the hyperbolic diameter is bounded by a constant times the euclidean diameter. It now follows any section is either always contained in D_k or never in it and that the $D_k's$ never intersect over the base cell for small enough ρ.

Further, it is obvious to conclude that the distance between any D_k, D_{k+1} (now with the centres allowed to move as described above) does not vary much over the base cell. We will formulate these statements in the Appendix precisely.

From here on, unless explicitly stated, when we say "group discs" , we mean the discs D_k as in (9) (the group we are referring to will be clear by context) .

5.2.5 Covering the complement of the D_k

Leaving out these discs D_k described in 5.2.4, we cover the rest of the positive real parts of the cluster over the positive real part of the base cell, with polynomially many discs- but with taking additional care that these discs are real disjoint (and also the usual condition that their 1/2 extensions do not meet any sections. A disc of radius $O(1/v^5)$ would satisfy these conditions). These form our second set of cells. To not confuse these discs with the D_k, we shall refer to these discs as the **Complement covering discs** (C.C discs for short).

We see that over $\mathbb{R}_+\hat{\mathcal{C}}$ we are essentially trying to cover line segments uniformly (as we are looking for a real cover), and the end points of these line segments vary as the base point $p \in \mathbb{R}_+\hat{\mathcal{C}}$ changes. We therefore need to define compatible real cellular maps for $\hat{\mathcal{C}} \odot \mathcal{F}^{\gamma}_{0,q}$, that cover these line segment disjointly over $p \in \mathbb{R}_+\hat{\mathcal{C}}$.

Let us initially restrict to $\mathbb{R}_+\hat{\mathcal{C}}$. As seen above the job here is to essentially cover line segments. The end points of each of these line segments are determined using four sections, as the reader may see- two of the sections to determine the normalized coordinates and the two centers c_k, c_{k+1}. In fact, in the normalized coordinates, the line segment is given by

$$(c_k + d_{\max,k} + 1/v^4, c_{k+1} - 1/v^4 - d_{\max,k+1}). \tag{11}$$

The line segment to the left of D_1 (The first group disc in the clsuter) and the right of D_M (the last group disc in that cluster) can be dealt with in an analogous way.

Let us focus on a single line segment as given in (11). In order for the real parts of the C.C discs to not intersect over any real base point and cover a given line segment

1. The centers of the C.C discs would vary with the change of the base point.
2. As the radii and the centers of the C.C discs are determined by four sections, $y_0, \hat{y}_{0,q}, c_k, c_{k+1}$, the cellular map we therefore construct corresponding to this disc will have a base cell of the form $\hat{\mathcal{C}}_{a,b,c,d}$ (the notation is as in 5.1). However, since this cell has complexity $poly_l(\beta)$, these cells can be used as a part of the real disjoint cell cover.

   (A similar constraint holds for when we deal with the line segments to the left of D_1 and to the right of D_M)

It is a simple exercise, for the length of the line segment we are trying to cover does not change a lot by the virtue of ρ being small, and therefore, we may use a constant number of discs to cover the segment uniformly.

We illustrate an example as follows: Take the line segment as in (11). We compute the length of this segment over $p \in \mathbb{R}_+\hat{\mathcal{C}}$. call the length λ_p. Let N:= $\lfloor \lambda_p \times v^5 \rfloor - 1$, where $\lfloor x \rfloor$ denotes the greatest integer $\leq x$. It is clear that $N = \text{poly}_l(\beta)$ as $\mathcal{F}_{0,q} \subseteq A(1/2, 2)$

Set $L = c_k + d_{\max,k} + 1/v^4, \qquad R = c_{k+1} - d_{\max,k+1} - 1/v^4$

These are the following C.C discs:

$$D(L, 1/v^5), D(L + 1/v^5, 1/v^5), \ldots, D(L + \frac{N-1}{v^5}, 1/v^5) D(L + \frac{N}{v^5}, R - L - \frac{N}{v^5}) \tag{12}$$

Notice how the last C.C disc above has radius $\leq 2/v^5$. We notice that the above discs are well defined over the entire base cell, although we restricted ourselves to $\mathbb{R}_+\hat{\mathcal{C}}$, while constructing them.

Moreover, these C.C discs are disjoint over $\mathbb{R}_+\hat{\mathcal{C}}$.

Of course, these C.C discs so far do not yet fully form a real disjoint cover for the line segment, as we are leaving out the boundary points of these C.C discs. But these may be covered with cells ending with type $*$.

Moreover notice that these C.C discs and their $1/2$ extensions do not meet any of the sections y_j as L, R are sufficiently far away from y_j (at least $1/v^4$ at p and since y_j, c_k, c_{k+1} do not move much in this normalization, the claim follows).

We also note that in the clusters containing the boundary sections, we only use those C.C. discs, whose positive real parts lie within the boundary, over the positive real part of the base cell. This is to ensure that the resulting cell cover is real disjoint.

We note from the arguments above that if we use the C.C Discs to form a part of the real cell cover we are interested in, the cells used for describing the C.C discs are of size $\mathrm{poly}_l(\beta, \rho, 1/\sigma)$, as required (the $\rho, 1/\sigma$ factors come from the Refinement Theorem.).

5.2.6 The Recursion

So far, we have covered the complement of the D_k inside the cluster. We now move onto covering these D_k.

Here, we repeat the same steps from 5.2.1 to 5.2.5, by setting c_k to be zero. One subtle point to observe is that we need to cover the negative real part of the fiber when setting $c_k = 0$. It is for this reason we made the modifications in the definition of a real cellular map in 2.5. This essentially helps us cover the points $z_l < c_k$ in the fiber with real cellular maps.

D_k in the new coordinates

Until 5.2.5, there was a normalization $y_0 = 0$, $\hat{y}_{0,q} = 1$. The D_k we defined were in these coordinates. Specifically we had $D_k = D(c_k, d_{\max,k} + 1/v^4)$ where c_k was a section. So replacing the normalization in 5.2.2 with the new coordinate system $c_k = 0$ (i.e a translation), we would have

$$D_k = D(0, (d_{\max,k} + 1/v^4)(\hat{y}_{0,q} - y_0));$$

Let us set

$$b_k := (d_{\max,k} + 1/v^4)(\hat{y}_{0,q} - y_0),$$

this is the boundary of D_k in this coordinate system,as one can see.

In this recursive step, we only need to cover up to the boundary of D_k, for we intend to obtain a disjoint real cover. Note that this boundary does not equal any other section uniformly over the base, by construction. so we may include b_k when clustering around c_k. We also note that we fix the same p as in the previous step when clustering. Setting the same p is only a matter of bookkeeping.

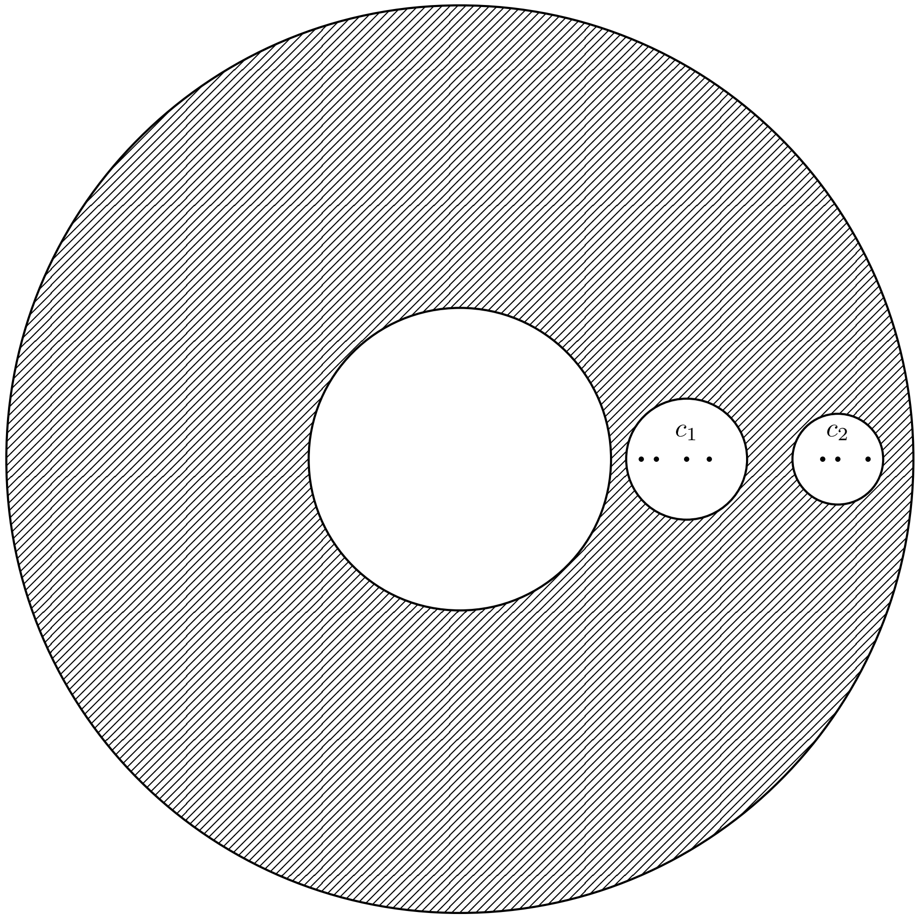
Say b_k is in the α^{th} cluster, then we modify the annulus $\mathcal{F}_{k,\alpha}$ to be the annulus $A(l_{k,\alpha}\hat{y}_{k,\alpha}, b_k)$.

Note: If the above $\mathcal{F}_{k,\alpha}$ has two or more sections, we always ensure that we *do not* set $\hat{y}_{k,\alpha} = b_k$. Moreover, if there is a section that is less than b_k at p in this cluster, then we set this section to be $\hat{y}_{k,\alpha}$ The reason we are careful about this will be clear in the subsequent sections and the appendix.

We illustrate a subtle point here. In [1, Proposition 56], ρ was chosen to be small enough so that all the annuli $\mathcal{F}_{i,q}, \mathcal{F}_{i,q^+}$ were well defined and had desirable properties. However, as one may see, we introduce a new section at every step and then cluster around the respective center. So this process is slightly different, and, apriori the annuli obtained in the recursive steps might not be well defined. We will show in the appendix that if $1/\rho > \mathrm{poly}_l(v, |log\gamma|)$, we do get that the annuli obtained through this recursive clustering are well defined and have the same desirable properties.

From here on, we exactly follow the steps outlined from 5.2.1 to 5.2.5, creating simple groups and coalescing them, and later covering their complements. We keep repeating this recursive process till we cover every such disc obtained in each level of recursion. This process will terminate because at each level of recursion, we lose at least one section (the center), and are left with lesser number of sections to deal with.

This essentially covers all of $\mathbb{R}_+(\mathcal{C} \odot \mathcal{F})$ except for the boundary points of the disc and annuli that we have constructed in the clustering process of 5.2.1 (we alluded to this point there as well). However, since these boundary points are going to be as many as the number of clusters we have constructed , this is $\mathrm{poly}_l(\beta)$(as shown in [1]). We may cover these boundary points with $\mathrm{poly}_l(\beta)$ many cells of type $\mathcal{C}_j \odot *$ (where $\mathcal{C}_j$ is of length $l-1$)



An illustration of the construction in the q^{th} cluster described for the simple case where we have two groups after coalescing. We note that the discs constructed for this move as c_1 and c_2 move but their radius is always a constant. moreover, since ρ is very small, These discs will never intersect, allowing us to individually address each disc at a time

5.3 Complexity of the Real disjoint Cell Cover

We can notice that the above outlined construction gives us a real cellular cover. We further need to show the following:

1. That this is indeed a real disjoint cell cover.
2. That this cover is of size $\text{poly}_l(\rho, 1/\sigma, \beta)$
3. The complexity of the cellular maps forming the cover is $\text{poly}_l(\beta)$.

The rest of this subsection is devoted to proving these three:

5.3.1 We obtain a real disjoint cover

Notice that in the **construction of simple groups** 5.2.2 under the normalized coordinates $y_0 = 0, \hat{y}_{0,q} = 1$, the maximal spacing between any two sections in a simple group is at most $1/v^4$, and we noticed that the center of this group will be in the $1/2v^4$ real neighbourhood of mid point of the edge sections at p. And thus, over p, the boundary of the disc will be exactly $1/v^4$ distance from one of the edge sections and will be at most $2/v^4$ distance from the other edge section.

One can easily notice that the pattern after coalescing is as follows (refer 5.2.3 about the coalescing process): at the point p, the distance between the edge section and the boundary of the disc will be at most $1/v^4+$ the maximum distance between two consecutive sections in that group .
Since we coalesce groups only when their corresponding discs are $< 1/v^4$ distance apart at p, This gives us bound on the max distance between any two consecutive sections in the coalesced group :

If we combine two simple groups with each containing at least two sections, then it is easy to see that the max distance between two consecutive sections has to be less than $2.(2/v^4) + 1/v^4$ (over p). As one can see, when n simple groups with each of size at least 2 are coalesced one after the other in a sequence, the max distance between consecutive section in the resulting group has to be $< \frac{1+4n}{v^4}$.

In fact, it is easy to see that if after a few coalescions, the group has N sections, then the maximal distance between two consecutive sections in this group is bounded by $(3/v^4) \times N$, irrespective of how the coalescion takes place. The simple fact observed here is that if two groups are coalesced, the maximum distance between two consecutive sections in this new group is $d_1 + d_2 + 3/v^4$, where d_1, d_2 represents the maximum distance between two consecutive sections in each of the smaller groups. The conclusion is now clear.

Since there can be at most v sections in a cluster, and since the max distance between consecutive sections is bounded additively with each coalescing step, it follows that under the normalization $y_0 = 0, \hat{y}_{0,q} = 1$ (as in 5.2.2), the boundary of discs $D_{k,p}$ (as defined in 5.2.3) after all the coalescing process is still of $O(1/v^3)$ distance from the edge sections, thereby the radii of these discs being $O(1/v^2)$. These estimates also hold for the group discs D_k .This is because the center c_k and the edge sections do not move much in these normalized coordinates.

We claim that the D_k lie inside the annulus determined by the clusters we started with in 5.2.1.

But for this claim to be true however, we assume here that $1/2(1-\gamma^2) > 1/v$. Without this assumption we may not have that D_k lies inside the cluster. The reader may refer to [1, section6-Proposition 56] for a greater understanding of the gap γ.

This is a valid assumption because $\gamma = 1 - 1/\text{poly}_l(\beta)$ and $v = \text{poly}_l(\beta)$.

Therefore, there must be a $n \in \mathbb{N}$ such that $1/2 - 1/v^{n+1} > \gamma^2/2$. So we may instead have worked with distances of $1/v^{n+4}$ instead of $1/v^4$ in 5.2.2. For now, let's assume $1/2 - 1/v > \gamma^2/2$.

Why do we need this assumption ?

We know that there is an ordering among the sections as mentioned in 5.2.1. Assume that we are currently dealing with the q^{th} cluster around y_0. Without loss of generality assume $\hat{y}_{0,q} = 1$ is the last section in $\mathcal{F}_{0,q}$ with respect to the order mentioned. It is clear that $\hat{y}_{0,q}$ must be an edge section. We know that $\mathcal{F}_{0,q}$ has an outer radius γ^{-2} (refer [1, 6.3.1] to see why). We just saw that the boundary of the D_k is of $O(1/v^3)$ distance from $\hat{y}_{0,q}$. So over any point $x \in \mathbb{R}_+\hat{\mathcal{C}}$, we have that $\hat{y}_{0,q} + 1/v^2 = 1 + 1/v^2 < \gamma^{-2}$, due to the above assumption. A similar argument at the inner boundary also holds, and it is for this reason we we assume $c(1-\gamma^2) > 1/v$. The extra factor of $c = 1/2$ is to account for the fact that the last section in q^{th} cluster is not necessarily $\hat{y}_{0,q}$, but it lies in the annulus $A(1/2, 2)$ after the normalization.- refer [1, Section 7, Section 6.3.1] for why $\mathcal{F}_{0,q} \subset A(1/2, 2)$ in the nromalized coordinates. This confirms that our D_k indeed lie inside the Annuli $\mathcal{F}_{0,q}$.

Notice that inside D_k when we carry out the recursive step 5.2.6, a similar argument as above holds, except possibly at the last fibre inside D_k because we modified the boundary to include only upto the point b_k (see 5.2.6). Say b_k is in the α^{th} cluster around c_k. We only have a problem in the case we have another section less than b_k in this cluster. **Without loss of generality**, let $\hat{y}_{k,\alpha}$ be the last section in the cluster, w.r.t the order. Recall that we are currently working in a setting with $c_k = 0$.

We know in this recursive step as well, the radii of the group discs formed inside D_k (hereby referred to as D_s^2, with 2 indicating the second level of recursion.) are still $O(1/v^2)$ with respect to the normalization $c_k = 0, \hat{y}_{k,\alpha} = 1$, and the distance between the edge section and the boundary of D_s^2 is also of order $O(1/v^3)$ in this normalization.

As $\hat{y}_{k,\alpha}$ is the last section in the α^{th} cluster, it has to be an edge section for some disc D_s^2 Therefore, it would be enough to show that $\hat{y}_{k,\alpha}(x) + 1/v^3 < b_k(x), \forall x \in \mathbb{R}_+\hat{\mathcal{C}}$, when viewed in the above normalized coordinates.

This is easy to see: **In the normalization** $y_0 = 0, \hat{y}_{0,q} = 1$ **(as in 5.2.1), we know from the estimates in 5.3.1 that**

$$(\hat{y}_{k,\alpha} - c_k)(p) = O(1/v^2),$$

and

$$(b_k - \hat{y}_{k,\alpha})(p) = \Omega(1/v^4),$$

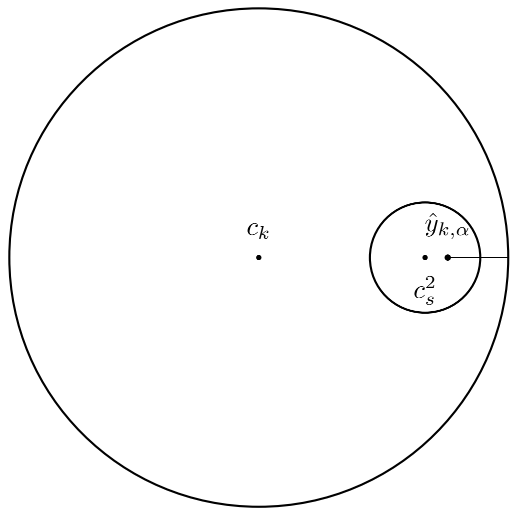
We therefore have that in the new normalization $c_k = 0, \hat{y}_{k,\alpha} = 1$**, that**

$$(b_k - \hat{y}_{k,\alpha})(p) = \Omega(1/v^2).$$

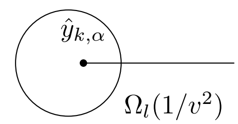
Thus, our claim holds, as b_k does not move much in these coordinates. The picture below is for an illustration.

Note that if $y_{k,\alpha}$ was not the last section, the D_s^2 still lie inside D_k, for distance between the last section and b_k is even bigger in this normalization.

For more about how we came up with groupings of size $1/v^4$, we refer the reader to the appendix.



$\hat{y}_{k,\alpha}$ is far from the boundary b_k of the disc D_k in the new normalized coordinates. So the disc D_s^2 containing $\hat{y}_{\alpha,k}$ will stay inside D_k.



distance between $\hat{y}_{k,\alpha}$ and D_k when($c_k = 0, \hat{y}_{k,\alpha} = 1$). As the boundary of D_s^2 is at most $1/v^3$ from $\hat{y}_{k,\alpha}$, it follows that D_s^2 lies inside D_k.

Thus, it is clear that even in the recursive steps, the D_s^2 lie inside the D_k we start with, thereby ensuring that the construction is indeed disjoint.

5.3.2 We need poly$_l(\beta, \rho, 1/\sigma)$ cells to get the real disjoint cell cover

After construction of the group discs D_k for a specific cluster,we fill the rest of the real part of the annulus with poly$_l(\beta)$ many C.C discs as shown in 5.2.5 (in fact we also showed through the example computation in 5.2.5 that the number of C.C discs required for each cluster, for each recursion, is bounded by a fixed polynomial, say v^8 as a conservative bound).

Say there are $M^{[1]}$ groups (over all the clusters combined. superscript 1 denotes the first level of recursion) in the first level of the recursion with $n_1, \ldots n_{M^{[1]}}$ sections in each group after the coalescing process is complete. In the first step we use $\leq v^8$ many C.C discs to form our cellular maps.

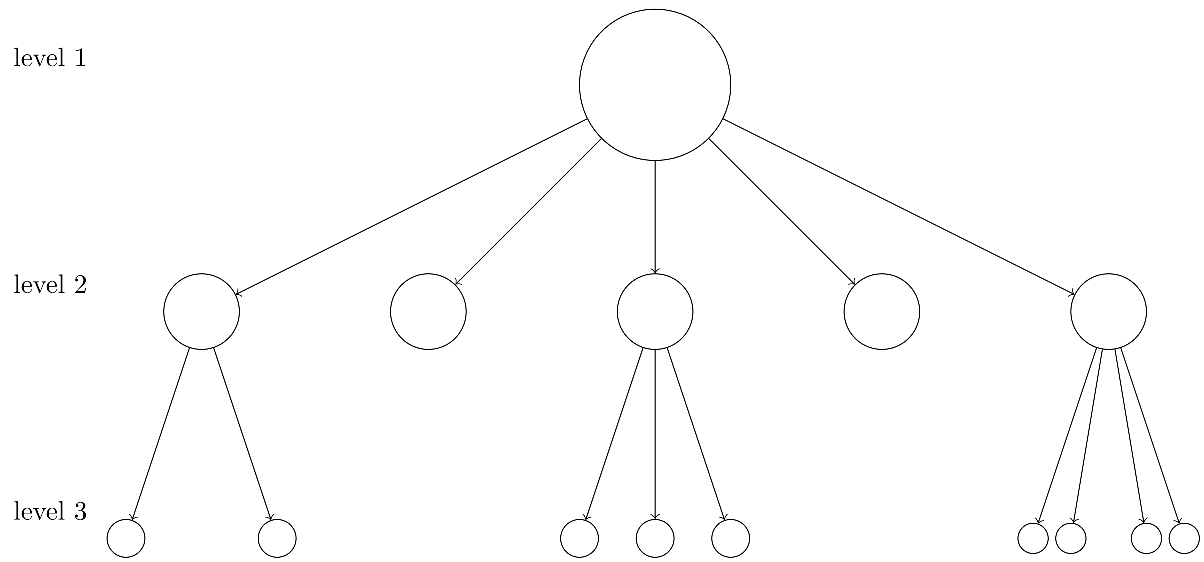
Let's now focus on one of the D_k. Inside this disc, say we form $M_k^{[2]}$ groups. We cover the complement of these groups with $\leq v^8$ many C.C discs. And in this second level of recursion, we have to repeat this for every such D_k. Thus, we would need $\leq M^{[1]} \times v^8$ many C.C discs in a second level of recursion. By $M^{[2]}$, denote the sum $\sum_{\#=1}^{M^{[1]}} M_{\#}^{[2]}$ i.e the total number of groups in the second level of recursion.

A similar pattern is followed in the next level of recursion. Say at each level, there are $M^{[i]}$ many groups. Clearly $M^{[i]} \leq v$ always. Further, there are $h \leq v$ levels of recursion, as we are guaranteed to lose one section in every recursive step.

So overall, we need $\leq v^8 \times (1 + M^{[1]} + ...M^{[h]}) \leq v^8 \times v^2$ many C.C discs in total to obtain a real disjoint cell cover (recall that $v =$ poly$_l(\beta)$).

This essentially finishes the proof of the claim and we indeed get a poly$_l(\beta, \rho, 1/\sigma)$ size cell cover (the factor of σ, ρ appears because of the subsequent refinement performed to obtain the cell covers with appropriate extensions).

(Fig. below) The recursion in display, with $M^{[1]} = 1$. The first recursive step breaks the group into $M^{[2]}$ smaller groups in level 2 and using $\leq v^8$ C.C discs, we cover the complement of these smaller groups. Repeating this process again for each of the smaller discs, we obtain



even smaller groups with smaller discs We need $\leq v^8 \times M^{[2]}$ C.C discs in level 3 (here $M^{[2]} = 5$).

5.3.3 The complexity of the cell covers is poly$_l(\beta)$

Notice that in the construction of the C.C discs 5.2.5, we noted that the discs are determined by four sections, and therefore, the corresponding cells and cellular maps have poly$_l(\beta)$ complexity.

The cellular maps for the C.C discs in the subsequent recursive steps also have a poly$_l(\beta)$ complexity. To show this, we only need to show:

Theorem 5.3.3.1. *Let D_k, $\hat{y}_{k,\alpha}$, b_k ($\hat{y}_{k,\alpha} \neq b_k$), c_k, $d_{max,k}$ be as in 5.2.6. Say D_s^2 is the group disc inside the α^{th} cluster inside D_k obtained via the said recursive step in 5.2.6, then the corresponding boundary of D_s^2 in the given cell, with no normalizations is of the form $c_s^2 \pm (1/v^4 + d_{max,s}^2)(y_{k,\alpha} - c_k)$. Where c_s^2 is the center of D_s^2 and $d_{max,s}^2$ is a constant having an analogous meaning to $d_{max,k}$ (the superscript 2 here indicates the second level of recursion).*

Proof. It is immediate to see that this holds, by tracing the steps in 5.2.3 and 5.2.4.

What it does show, is that the $b_s^2 := c_s^2 + (1/v^4 + d_{\max,s}^2)(y_{k,\alpha} - c_k)$ does not depend on $y_0, \hat{y}_{0,q}$ that define b_k. This is important because if b_s^2 depends on b_k instead of $\hat{y}_{k,\alpha}$, then one could obtain a cascading increase in complexity for the C.C discs at each step of the recursion, and therefore end up with an exponential complexity in some of the cells, which is not desirable.

By normalizing $\hat{y}_{k,\alpha} \neq b_k$ to 1 in our recursive step, we avoid this problem and we notice that the C.C discs depend on at most 5 sections to cover the region between b_s^2 and b_k, and that the subsequent recursions also require C.C discs depending on at most 5 sections (which drops to 4 when we are not dealing with the region such as in between b_s^2 and b_k).

□

5.4 Extension to the general case: y_i are not real

In this case, we have a set $\Sigma_{\mathbb{R}}$ of real sections and a set $\Sigma_{\mathbb{C}}$ of sections that are not real over $\mathbb{R}_+\hat{\mathcal{C}}$.

Here, We effectively follow the steps in 5.2, but the process is slightly more complicated because we do not have an ordering of the y_i as in the previous case. However, since the non-real sections occur in conjugate pairs $(y_i, y_{\tilde{i}})$- refer [1, 6.3.2], we may take their centers $\mu_i := \frac{y_i + y_{\tilde{i}}}{2}$, which are real, and study them in place of the non-real sections as follows:

We may without loss of generality, assume that μ_i never equal each other or any of the real sections $y_j \in \Sigma_{\mathbb{R}}$. In what follows, we will show why one can make this assumption:

Consider the function

$$\mathcal{G} \in \mathcal{O}_b(\mathcal{C}^{\{\rho\}} \odot \mathcal{F}^{\gamma}) \quad \mathcal{G}(x_{1\ldots l-1}, y) := \prod_{i,j} (y - (y_i + y_j)). \tag{13}$$

Notice that $\mathcal{G}$ is well defined over the cell due to its symmetric nature.

We claim the following:

Lemma 5.4.1. *The zero locus of the function $\mathcal{G}$ is algebraic and has complexity $\mathrm{poly}_l(\beta)$. There is a polynomial $Q \in \mathbb{C}[x_{1,\ldots l}]$ of degree $\mathrm{poly}_l(\beta)$ such that $\{\mathcal{G} = 0\} \subset \{Q = 0\}$. Moreover, there exists a polynomial $D_Q \in \mathbb{C}[x_{1,\ldots l-1}]$ of degree $\mathrm{poly}_l(\beta)$, such that*

$$\pi : \mathcal{C}^{\{\rho\}} \odot \mathcal{F}^{\gamma} \cap \{Q = 0\} \to \mathcal{C}^{\{\rho\}} \textit{ is a proper covering map outside } \{D_Q = 0\}$$

Proof. We know that the function F (as defined in 5) is algebraic and we assume that F does not vanish identically over the cell $\mathcal{C}^{\{\rho\}} \odot \mathcal{F}^{\gamma}$. We note that the set $\{F = 0\}$ inside the cell is contained in an algebraic hypersurface of degree $\mathrm{poly}_l(\beta)$ (see [1, Lemma 64]), say $\{P = 0\}, P \in \mathbb{C}[x_{1\ldots l}]$, not identically vanishing on $\mathcal{C}^{\{\rho\}} \odot \mathcal{F}^{\gamma}$.

We define the polynomials $P_1, P_2 \in \mathbb{C}[x_{1\dots l-1}, x_l, z]$ given by

$$P_1(x_{1\dots l-1}, x_l, z) := P(x_{1\dots l-1}, x_l); \quad P_2(x_{1\dots l-1}, x_l, z) := P(x_{1\dots l-1}, z - x_l) \tag{14}$$

Taking the resultant $\mathrm{Res}_{x_l}(P_1, P_2)$ gives us a polynomial $Q \in \mathbb{C}[x_{1\dots l-1}, z]$ of degree $\mathrm{poly}_l(\beta)$.

We know that $Q(x_{1\dots l-1}, z) = 0$ if and only if $P_1(x_{1\dots l-1}, x_l, z) = P_2(x_{1\dots l-1}, x_l, z)$ for some $x_l \in \mathbb{C}$. Further note that $(x_{1\dots l-1}, y_i(x_{1\dots l-1}) + y_j(x_{1\dots l-1})), \forall x_{1\dots l-1} \in \mathcal{C}^{\{\rho\}}$ lies inside $\{Q = 0\}$ which is a hypersurface of degree $\mathrm{poly}_l(\beta)$. Thus $\{\mathcal{G} = 0\} \subset \{Q = 0\}$.

We may, without loss of generality, assume that Q is square-free when considered as a polynomial in the last coordinate, for if not, we may always replace Q by one. Define D_Q to be the product of the classical discriminant of Q (w.r.t z_l) and the leading coefficient of Q with respect to the last coordinate. The claim now follows.

□

Due to the above lemma, we may assume that the μ_i never meet any of the real sections or each other. The set $\{D_Q = 0\}$ may be dealt with in a similar way as in the setup 5.1 using the inductive hypothesis.

We note that we only prove that the zero locus of $\mathcal{G}$ is algebraic. But this argument may be easily extended to show that $\mathcal{G}$ is algebraic too. For our purposes however, $\{\mathcal{G} = 0\}$ being algebraic would suffice.

After this step, we can again use the ordering amongst the real sections and the centers μ_i to essentially repeat the steps in 5.2, with two small changes:

- As expected, we consider the sections in $\Sigma_{\mathbb{C}}$ as well as the centers μ_i for the clustering process (similar to how it is done in [1, 6.3.2]).
- We proceed with the grouping steps 5.2.2 to 5.2.4 only considering the sections in $\Sigma_{\mathbb{R}}$ and the centers μ_i, meaning we do not consider the sections in $\Sigma_{\mathbb{C}}$.

  The intuitive reason for doing this is because by keeping track of the centers, we are roughly keeping track of the non real sections as well, so we might as well proceed without considering $\Sigma_{\mathbb{C}}$. The construction of the C.C discs can be done in exactly the same way as 5.2.5, including in the recursion steps 5.2.6.

We highlight the major ideas, using which the claims above follow. The details are left for the reader to verify.

Consider the normalization in the first step, i.e in 5.2.1 ($y_0 = 0, \hat{y}_{0,q} = 1$). Observe that over the real point $p \in \mathbb{R}_+\hat{\mathcal{C}}$ (the same p we used to construct the groups 5.2.2), the real part of a section $y_i \in \Sigma_{\mathbb{C}}$ equals the value at its corresponding center, i.e. $\mathrm{Re}(y_i(p)) = \mu_i(p)$.

Let the group disc D_k contain μ_i, and let $y_i \in \mathcal{F}_{0,q}$ (for if y_i is not in the cluster, we might proceed as usual). Clearly, over p, we have that in the above normalized coordinates, the boundary of D_k is at least $1/v^4$ distance from $\mu_i(p)$. Thus, using the fact that y_i does not change much over the base cell, we conclude that the real part of y_i, $\mathrm{Re}(y_i)$ lies inside the disc D_k and is $\Omega(1/v^4)$ distance from the boundary. Thus, it follows that the C.C discs, in the first level of recursion, do not intersect the $y_i \in \Sigma_{\mathbb{C}}$.

We are left to show a similar result for the next levels of recursion. This is clear for when $c_k = y_j$ for some $y_j \in \Sigma_{\mathbb{R}}$, as the same arguments in the previous paragraph work.

So assume $c_k = \mu_i$. Set $\mu_i = 0$. The fibre $\mathcal{F}_{i,1}$ around μ_i we get from the clustering process in 5.2.1 is of the form $A(l_{i,1}\hat{y}_{i,1}, r_{i,1}\hat{y}_{i,1})$. We further saw that once we normalize to $\hat{y}_{i,1} = 1$, we have $\mathcal{F}_{i,1} \subset A(1/2, 2)$ (refer [1, Section 8]).

Without loss of generality assume that $y_i, y_{\tilde{i}}$ lie in $\mathcal{F}_{i,1}$. Now, under the normalized coordinates $\mu_i = 0, \hat{y}_{i,1} = 1$, at the point $p \in \mathbb{R}_+\hat{\mathcal{C}}$, both $y_i(p), y_{\tilde{i}}(p)$ are purely imaginary. , since $l_{i,1} \geq 1/2$ from above, we have that over the entire base cell $y_i, y_{\tilde{i}}$ do not move much to be at a $1/v^4$ distance from the real line in these normalized coordinates, and therefore, the C.C discs we construct (which have radius $O(1/v^5)$) do not meet $y_i, y_{\tilde{i}}$. These C.C discs do not meet any other sections as well from the previous arguments.

We leave it for the reader to verify the simple fact that indeed this means that the C.C discs we construct will form valid real disjoint cellular cover, with the same complexity estimates as in 5.3

6 CPrT, WPT and the Refinement theorem

As mentioned earlier, we now illustrate the modifications that are needed to obtain a disjoint real version of CPrT, WPT and the refinement theorem. We shall refer to these as the Modified CPrT, WPT and refinement theorem respectively.

We note that all the proofs are based on induction. In particular, as one notes in [1], The proof for the weak version of the CPT (where the maps are not necessarily prepared) for a cell of length= $l+1$ depends on the CPT and WPT for the cells of length$\leq l+1$ and dimension $\leq l$. And similarly for the CPrT.
We therefore note that our inductive proofs for the modified CPT, WPT and CPrT have a similar dependence.

We highlight the exact sequence of induction:

for $l = 0, 1$, modified WPT(MWPT), modified CPT(MCPT), modified CPrT(MCPrT) are true and can be verified easily.

$$\begin{aligned} \ldots \Rightarrow \text{Weak } MCPT(&\text{length} \leq l, \text{or length} = l+1, \text{dimension} \leq l) \\ \Rightarrow MCPrT(&\text{length} \leq l, \text{or length} = l+1, \text{dimension} \leq l) \\ \Rightarrow MCPT(&\text{length} \leq l, \text{or length} = l+1, \text{dimension} \leq l) \\ \Rightarrow MWPT(\text{length} \leq l+1, &\text{or length} = l+2, \text{dimension} \leq l+1) \Rightarrow \ldots \end{aligned} \tag{15}$$

Owing to the above sequence, when proving the MWPT and MCPrT for a cell, we will assume that the weak modified CPT holds for cells of smaller length or same length and smaller dimension.

6.1 WPT

Claim 6.1.1 (Modified WPT)**.** *Let $\rho, \sigma > 0$. Let $\mathcal{C}^{\{\rho\}}$ be a real cell and $F \in \mathcal{O}_b(\mathcal{C}^{\{\rho\}})$ a real function such that $\mathcal{C}^{\{\rho\}}, F$ are algebraic of complexity β. Then there exist $N = poly_l(\beta, \rho, 1/\sigma)$ real Weierstrass maps $f_j : \mathcal{C}_j^{\{\sigma\}} \odot \mathcal{F}_j^{\gamma} \to \mathcal{C}^{\{\rho\}}$ for F with gap $\gamma = 1 - 1/poly_l(\beta)$ such that $\mathbb{R}_+\mathcal{C} = \bigsqcup_j f_j(\mathbb{R}_+(\mathcal{C}_j \odot \mathcal{F}_j))$.*

Proof. We follow in the footsteps of the proof of the Algebraic version of WPT described in [1, Section 7], highlighting the changes required so as to obtain a real disjoint cellular cover. For a detailed content on WPT and its proof, the reader is encouraged to refer [1, section 7].

We proceed by induction on the length of the cell. Modified WPT for $l = 0$ is trivial. Assume the Modified WPT is true for all cells of length$\leq l+1$ and dimension $\leq l$. Now assume the cell is $(\mathcal{C} \odot \mathcal{F})^{\{\rho\}}$ of length $l+1$, and assume without loss of generality $\mathcal{F} = A(r_1, r_2)$.

We are given a function $F \in \mathcal{O}_b((\mathcal{C} \odot \mathcal{F})^{\{\rho\}})$, for which we need to show the modified WPT holds. Assuming the modified refinement theorem, we may assume ρ is small enough, as long as $\rho = \text{poly}(1/\sigma, \beta)$.

We may without loss of generality, assume F is a polynomial P which is square-free in $\mathbb{C}(z_{1..l})[z_{l+1}]$, and in particular has no multiple roots for a generic value of $z_{1..l}$. We may also assume P vanishes when $z_{l+1} = 0,\ r_1(z_{1,\dots l}),$ or $r_2(z_{1,\dots,l})$. Corresponding to this polynomial there is a discriminant function $\mathcal{D}$. We apply our modified real CPT to $\mathcal{C}$ with respect to $\mathcal{D}$- as described in [1, Lemma 64] (as in [1], we assume that $\mathcal{D}$ is multiplied with the leading coefficient of P w.r.t z_{l+1} in order to avoid the zeroes of P from escaping to infinity). Let $f_j : \mathcal{C}_j^{\{\rho\}} \to \mathcal{C}^{\{\rho\}}$ be the corresponding cell cover. The condition where $f_j(\mathcal{C}_j) \subset \{\mathcal{D} = 0\}$ can be handled using inductive assumption, in exactly the same way as in [1], except now using the modified CPT in the inductive step. We therefore obtain real disjoint cellular maps in this step for the Discriminant set.

We are thus effectively reduced to the case , where we can without loss of generality, assume that the projection $\pi : \mathcal{C}^{\{\rho\}} \times \mathbb{C} \cap \{F = 0\} \to \mathcal{C}^{\{\rho\}}$, is a proper covering map.

Here again, following the proof of the algebraic WPT in [1]. As usual, we assume that the constructions in [1, proposition 56] and Proposition 7.1.2 hold over $\mathcal{C}^{\{\sigma\}}$, by setting $\{\rho\} = \{\sigma\}\{\hat{\rho}\}$ using the refinement theorem. We form clusters around $y_0 = 0$, say r_1, r_2 are in the q_1, q_2^{th} clusters respectively $q_1 \leq q_2$.

If $q_1 < q_2$ then we define

$$\tilde{\mathcal{F}} := A(r_{0,q_1} r_1, l_{0,q_2} r_2), \tag{16}$$

This acts as a real weierstrass cell map due to [1, Proposition 56]. We are yet to cover the positive real part of the annuli $A(l_{0,q_1} r_1, r_1), A(l_{0,q_2} r_2, r_2)$. But these may be dealt with using the groupings introduced in 5.2.2. We just need to ensure that we only include those C.C discs whose positive real part lie within the bounds described by r_2 and r_1.

If $q_1 = q_2$, then we simply proceed with the groupings. Notice that since by choice $\gamma^{5v} > 1/2$, a C.C disc admitting a 1/2 extension admits a γ extension too. Thus, they serve as compatible Weierstrass cells, that

□

6.2 CPrT

Claim 6.2.1 (Modified CPrT)**.** *If $f : \mathcal{C}^{\{\rho\}} \to \hat{\mathcal{C}}$ be a real cellular map, with $f, \mathcal{C}^{\{\rho\}}, \hat{\mathcal{C}}$ algebraic of complexity β. Then, there exists a disjoint real cellular cover $\{g_j : \mathcal{C}_j^{\{\rho\}} \to \mathcal{C}^{\{\rho\}}\}$ of size $\text{poly}_l(\beta, \rho)$ and complexity $\text{poly}_l(\beta)$, such that each $f o g_j$ is prepared.*

Here,we again follow the same steps outlined in [1, section 8], using the Modified CPT and Modified WPT instead of the usual real CPT and WPT respectively. Note that the authors in [1] use the fact that the cellular maps in the CPT are a translate in the last variable and that the maps obtained through WPT are translates in the last two variables.

In our case, we note that while the maps in the modified CPT are indeed translates in the last variable, the maps in the modified WPT, are translates in the last vraiable and prepared in the second last variable: This is because after the initial CPT and Refinement on the base cell, we obtain the WPT maps via going to a suitable $\times v$ cover of the base cell (which results in a prepared map in the second last variable). Since we then compose this with the CPT and refinement maps for the base cell(which are translates in their last variable) to get the resulting cell map, We are doing the following in the second last coordinate:

$$\begin{aligned} &\text{prepared in second last variable} \\ \to\ &\text{translate in second last variable} \\ \to\ &\text{translate in second last variable} \end{aligned} \tag{17}$$

Clearly, this results in a prepared map in the second last variable.

In the last variable, these maps are just translates, as seen through the proof

We note here that all we need to prove is that the real cell cover obtained in [1, Lemma 76] can be, in fact, made into a disjoint real cell cover. The reader may notice that this claim would imply the modified CPrT. We, therefore, state and prove this lemma instead:

Lemma 6.2.1.1. *If $f : \mathcal{C}^{\{\rho\}} \to \hat{\mathcal{C}}$ be a real cellular map, with $f, \mathcal{C}^{\{\rho\}}, \hat{\mathcal{C}}$ algebraic of complexity β. Then, there exists a real cellular cover $\{g_j : \mathcal{C}_j^{\{\rho\}} \to \mathcal{C}^{\{\rho\}}\}$ of size $poly_l(\beta, \rho)$ and complexity $poly_l(\beta)$, such that each fog_j is prepared in the last variable. Moreover, this cover can be assumed to be disjoint over the real part, i.e. $g_i(\mathbb{R}_+\mathcal{C}_i) \cap g_j(\mathbb{R}_+\mathcal{C}_j) = \phi, \forall i, j$, and $\bigsqcup g_j(\mathbb{R}_+\mathcal{C}_j) = \mathbb{R}_+\mathcal{C}$.*

Proof. We only indicate the changes that are needed to the proof of Lemma 76 in [1] to get a disjoint real cover. For a detailed description of the proof, the reader is directed to [1]. we make sure to follow the same conventions as in [1], in order to make it convenient for the reader to refer their proof and compare the changes we have made to it.

Let $\mathcal{C} = \mathcal{C}_{1,\dots,l} \odot \mathcal{F}$. We know the last coordinate of f is of the form $P(z_{1,\dots,l}; z_{l+1})$, where P is a monic polynomial in z_{l+1} of degree μ with coefficients holomorphic in $\mathcal{C}^{\{\rho\}}_{1,\dots l}$.

As seen in [1], the case where $\mathcal{F} = *$ is trivial, and f is already prepared in the last variable.

So we assume $\mathcal{F} \neq *$.

We denote by $\Sigma \subset \mathcal{C}$ denote the set of critical points of P with respect to z_{l+1}.

$$\Sigma := \{\frac{\partial P}{\partial z_{l+1}} = 0\} \tag{18}$$

It is clear that Σ is a hypersurface with zero-dimensional fibres over $\mathcal{C}_{1,\ldots l}$.

We now apply the modified Real CPT to $\mathcal{C}$ and Σ to obtain a disjoint real cellular cover $\{g_j : \mathcal{C}_j^{\{\rho\}} \to \mathcal{C}^{\{\rho\}}\}$ compatible with Σ . The case where $g_j(\mathcal{C}_j^{\{\rho\}}) \subset \Sigma$ can be dealt with in a similar way as in [1] using inductive assumptions (albeit with the Modified CPT), leaving out the case where $g_j(\mathcal{C}_j^{\{\rho\}})$ is disjoint form Σ.

Notice that g_j are translates in its last variable w_{l+1}, and therefore $\frac{\partial P \circ g_j}{\partial w_{l+1}} = \frac{\partial P}{\partial z_{l+1}}(g_j(w_{1\ldots l+1}))$.

We may therefore assume without loss of generality that $\Sigma = \{\}$ to begin with.

We know that P is bounded in absolute value by some constant N on $\mathcal{C}^{\{\rho\}}$. Set $D := D(N)$. Consider the cell

$$\tilde{\mathcal{C}} := \mathcal{C}_{1\ldots l} \odot D \odot \mathcal{F}$$

with coordinates, $z_{1\ldots l}, w, z_{l+1}$, which also admits a $\{\rho\}$ extension. Let $\Gamma \subset \tilde{\mathcal{C}}^{\{\rho\}}$ be the hypersurface

$$\Gamma := \{w = P(z_{1\ldots l}; z_{l+1})\}.$$

We now apply the Modified WPT to $\tilde{\mathcal{C}}^{\{\rho\}}$ and Γ to obtain a real disjoint cellular cover

$$\{g_\alpha : \mathcal{C}_\alpha^{\{\rho/\mu\}} \odot \mathcal{F}_\alpha^\gamma \to \tilde{\mathcal{C}}^{\{\rho\}}\}$$

of $\tilde{\mathcal{C}}$ by weierstrass cells for Γ (i.e for the function $w - P$).

In the case where $g_\alpha(\mathcal{C}_\alpha \odot \mathcal{F}_\alpha) \subset \Gamma$, since Γ has zero-dimensional fibres over $\mathcal{C}_{1\ldots l} \odot D$, we have $\mathcal{F}_\alpha = *$, and

$$\psi_\alpha(z_{1\ldots l}, w, *) := g_\alpha(z_{1\ldots l}, w, *) = (\ldots, w^n + \phi_\alpha(z_{1\ldots l}, \zeta_\alpha(z_{1\ldots l}, w))),$$

where $\psi_\alpha : \mathcal{C}_\alpha \odot * \to \Gamma$ is a cellular map admitting a $\{\rho/\mu\}-$extension.

In the case where $g_\alpha(\mathcal{C}_\alpha \odot \mathcal{F}_\alpha) \not\subset \Gamma$, as observed in [1], $g_\alpha^*\Gamma$ forms a cover of $\mathcal{C}_\alpha$. We also note that since $\Sigma = \phi$ by assumption, that this is in fact a covering map of degree $\nu_\alpha \leq \mu$. Then [1] claim that $g_\alpha^*\Gamma$ admits $\nu(\alpha)$ multivalued sections $s_{\alpha,j}$. We in-fact claim that these sections, when viewed as univalued functions over a suitable pull back, are indeed disjoint due to the assumption that $\Sigma = \phi$. Each section $s_{\alpha,j} : \mathcal{C}_\alpha \odot * \to g_\alpha^*\Gamma$ takes the form

$$s_{\alpha,j}(z_{1..l}, w, *) = (z_{1..l}, w, \hat{\zeta}_{\alpha,j}(z_{1..l}, w)) \tag{19}$$

$s_{\alpha,j}$ is algebraic of degree $\text{poly}_l(\beta)$ since it is the section of a hypersurface of complexity $\text{poly}_l(\beta)$.

set $\mathcal{C}_{\alpha,j} \odot * := (\mathcal{C}_\alpha \odot *)_{\times v(\alpha,j)}$. Pulling back $s_{\alpha,j}$ by $R_{v(\alpha,j)} : \mathcal{C}_{\alpha,j} \odot * \to \mathcal{C}_\alpha \odot *$ we obtain a univalued cellular map $s_{\alpha,j} \circ R_{v(\alpha,j)} : \mathcal{C}_{\alpha,j} \odot * \to g_\alpha^*\Gamma$ and finally composing with g_α we obtain a cellular map

$$\psi_{\alpha,j} := g_\alpha \circ s_{\alpha,j} \circ R_{v(\alpha,j)} : \mathcal{C}_{\alpha,j} \odot * \to \Gamma \tag{20}$$

Note that crucially since $s_{\alpha,j}$ is identity in the second last variable, and $R_{v(\alpha,j)}$ is a prepared monomial in the second last variable, and since g_α is prepared in the second last variable, we have that the above map is prepared in the second last variable.

From here on, we note that the exact same steps in [1, Lemma 76] (for the real case) can be followed in the construction of the resultant real maps $f_\beta : \mathcal{C}_\beta^{\{\rho\}} \to \mathcal{C}^{\{\rho\}}$.

We note that since the f_β are constructed via these sections $s_{\alpha,j}$, and composed with the maps obtained via the modified WPT and modified CPT, it follows that the images of the real cellular maps f_β are disjoint over $\mathbb{R}_+\mathcal{C}$ and $f \circ f_\beta$ are prepared in the last variable.

$\square$

6.3 The Refinement Theorem

We give a description of the modifications that are needed in the proof of the Refinement Theorem.

Theorem 6.3.1 (Modified Refinement theorem for Real Cells)**.** *Given $\mathcal{C}^{\{\rho\}}$ a real cell with algebraic complexity β, and $0 < \sigma < \rho$, there exists a disjoint real cellular cover $\{f_j : \mathcal{C}_j^{\{\sigma\}} \to \mathcal{C}^{\{\rho\}}\}$ of size $poly_l(\rho, 1/\sigma)$, with each $\mathcal{C}_j$ and f_j algebraic of complexity $poly_l(\beta)$ and further, each f_j is a cellular translate map.*

Proof. We follow the same steps outlined in [1], while highlighting the places where we need to make changes in the proof.

Consider the cell $\mathcal{C} \odot \mathcal{F}$, applying the inductive hypothesis on the base cell $\mathcal{C}$, one may assume that $\mathcal{C}$ admits an E-extension by going to the pullback (E is determined later). Note that this cell cover for the base will be real disjoint.

When we describe the covering of $\mathcal{F}$ we allow ourselves to use discs centered at a point $p \in \mathcal{O}_b(\mathcal{C}^{\{\sigma\}})$ where it is understood that such discs will be centered at the origin in the covering cells $\mathcal{C}_j$ that we construct, and the maps f_j will translate the origin to p.

We consider two different cases as in [1], noting that proving these two cases proves it for all cases by the composition of the constructions outlined in the two cases.

- case 1: $\rho > 1, \sigma = 1$

  When $\mathcal{F}$ is of type D, we are essentially trying to disjointly cover $\mathbb{R}_+ D(1)$ (radius of $\mathcal{F}$ is 1 upto rescaling by a real holomorphic function with comlplexity $\text{poly}_l(\beta)$ on the base cell) with real part of the discs whose $\{1\}$ extensions remain in $D^{\{\rho\}} = D(1 + \Theta(\rho^{-1}))$. However, this easily achieved by using $\text{poly}(\rho)$ discs of radius $O(\rho^{-1})$ with real centers and $\text{poly}(\rho)$ many real fibres of type $*$. A similar argument can be made when $\mathcal{F}$ is $D_o(1)$ (upto rescaling). Here , we first choose a disc D_o such that $D_o^{\{1\}}$ remains in $D_o(1)$. It's easy to see that the inner boundary of the annulus $D_o(1) \backslash D_o$ is $\Omega(1)$. Thus, the annulus can be covered as above for the case of a disc. In both the above cases, we may take $E = \{1\}$

We are left with the case where $\mathcal{F}$ is of type $A(r,1)$ (upto rescaling). We assume that r is positive on $\mathbb{R}_+\mathcal{C}$ in the above constructions. Notice that due to connectedness of $\mathbb{R}_+\mathcal{C}^{\{\sigma'\}}$, either r or $-r$ must be positive on $\mathbb{R}_+\mathcal{C}$ uniformly. Thus, the choice is merely a matter of convention.

Set $E = \{1\}\{\hat{\rho}\}$, with $\{\hat{\rho}\}$ chosen later. We note that as a consequence of the fundamental lemma [1, Lemma 22] for $\mathbb{D} \setminus \{0\}$, applied to the function r over $\mathcal{C}^{\{1\}}$, we have

$$r(\mathcal{C}^{\{1\}}) \subset B(\{0\}, e^{-\Omega_l(1/\hat{\rho})}; \mathbb{C}) \text{ or } \operatorname{diam}(r(\mathcal{C}^{\{1\}}); \mathbb{D} \setminus \{0\}) = O_l(\hat{\rho}). \quad (21)$$

We have the following two cases that we tackle with (21).

Say we are in the first case of (21). In this case we choose an annulus A such that $A^{\{1\}} = A(r,1)$ (with choosing $\hat{\rho} = O_l(1)$). We are thus left with covering two annuli components of $A(r,1)\backslash A$. Clearly the logarithmic width of each of the components is $O(1)$

Now we note that for $1 \le \beta < \infty$ and $r\beta \le 1$ uniformly over $\mathcal{C}^{\{1\}}$, a disc of radius $O(r\beta/\rho)$ centered around $r\beta$ admits a $\{1\}$ extension that lies inside $(\mathcal{C} \odot \mathcal{F})^{\{\rho\}}$. A similar conclusion holds for $0 < \alpha \le 1$ with $r < \alpha$ uniformly over the cell $\mathcal{C}^{\{1\}}$, and a disc of radius $O(\alpha/\rho)$ centered at α .

This is a direct conclusion from [1, Lemma 49], stated in a way that is useful to us. We note that the real part of the two annuli with constant logarithmic width can be covered disjointly with discs as described above (the α case for the outer annulus and the $r\beta$ case for the inner annulus) and fibres of type $*$. Since the Logarithmic width is a constant, we immediately see that we need $\operatorname{poly}(\rho)$ many discs.

Say we are in the second case of (21). If we can uniformly over $\mathcal{C}^{\{1\}}$, choose an annulus A such that $A^{\{1\}} = A(r,1)$, we can proceed as before, with $\hat{\rho} = \Omega_l(1)$.

If this is not possible, it implies that $r(x) = r_0, log|r_0| \ge -\pi^2$ for some $x \in \mathcal{C}^{\{1\}}$.

In this case, as noted in [1], one may with $\hat{\rho}$ a suitable constant, assume that.

$$\frac{\log|r(p)|}{\log|r(q)|} \in (9/10, 10/9) \quad \forall p, q \in \mathcal{C}^{\{1\}} \quad (22)$$

We note that for a smaller $\hat{\rho}$, this bound only gets tighter, so we are allowed to make $\hat{\rho}$ a smaller constant if we wish.

Here we diverge from the proof in [1].

We need a stronger estimate on the hyperbolic distance between two points in a punctured disc. In particular, we need a good control on $arg(r)$.

We note that the hyperbolic distance between two points $z_1, z_2, arg(z_1) = 0, arg(z_2) = \theta \in [0, \pi]$, with $\tau_j = \log(1/|z_j|)$. Without loss of generality, say $\tau_1 \geq \tau_2$, then the hyperbolic diatance between the points $h_{\mathbb{D}\setminus\{0\}}(z_1, z_2)$ is given by (refer [7])

$$h_{\mathbb{D}\setminus\{0\}}(z_1, z_2) = \operatorname{arctanh}\sqrt{\frac{\theta^2 + (\tau_1 - \tau_2)^2}{\theta^2 + (\tau_1 + \tau_2)^2}} \tag{23}$$

In particular, one sees that by applying (21)

$$h_{\mathbb{D}\setminus\{0\}}(z_1, z_2) < c\hat{\rho} \implies \sqrt{\frac{\theta^2 + (\tau_1 - \tau_2)^2}{\theta^2 + (\tau_1 + \tau_2)^2}} < \frac{e^{2c\hat{\rho}} - 1}{e^{2c\hat{\rho}} + 1}$$

Setting $H = \frac{e^{2c\hat{\rho}} - 1}{e^{2c\hat{\rho}} + 1}$, we see that

$$\frac{\theta^2 + (\tau_1 - \tau_2)^2}{(\tau_1 + \tau_2)^2 - (\tau_1 - \tau_2)^2} < \frac{H^2}{1 - H^2} \implies$$
$$\theta^2 < ((\tau_1 + \tau_2)^2 - (\tau_1 - \tau_2)^2) \times (\frac{H^2}{1 - H^2}) - (\tau_1 - \tau_2)^2,$$

in particular, we have

- $$\theta^2 < ((\tau_1 + \tau_2)^2) \times (\frac{H^2}{1 - H^2})$$

We now note that from (22), that if z_1, z_2 are in $r(\mathcal{C}^{\{1\}})$, then

$$\begin{aligned} &\theta^2 < (19/9)^2 (\log)^2 (1/|z_2|) \times (\frac{H^2}{1 - H^2}) \implies \\ &\theta^2 < (190/81)^2 (\log)^2 (1/|r_0|) \times (\frac{H^2}{1 - H^2}). \end{aligned} \tag{24}$$

Where we used $|z_2| > |r_0|^{10/9}$ from (22) (a crude bound, but this will suffice). We might as well set $r = z_2$, then θ here denotes the argument of r, which assumes values between $-\pi$ and π (as we assumed in our calculations above that the **relative** angle between two points is in $[0, \pi]$)

Recall that we assumed $r > 0$ on $\mathbb{R}_+\mathcal{C}^{\{1\}}$

We notice that because of 22, and the fact that $|r_0| > e^{-\pi^2}$, we conclude $\frac{|r|}{|1-r|}$ has a lower bound given by B, given in terms of r_0 (in fact $B = (|r_0|^{10/9}/(1 - |r_0|^{9/10}))$ works. Notice that $1/B < e^{10\pi^2/9}$, from the bound on r_0). It therefore follows that a disc of radius $B|1 - r|/k\rho, (k > 1$ fixed constant) centered at the point r, admits a $\{1\}$ extension that lies inside $A^{\{\rho\}}$. We describe this disc as a translation of $D(B(1-r)/k\rho)$ by r. Notice that described this way, the function describing the radius of the disc is holomorphic, as needed.

Moreover, we note that by (24), for $\hat{\rho} = \Theta_l(1)$, one may achieve $cos\theta > |r_0|^{9/5}, \theta = arg(r)$. To prove this, let us investigate the case $arg(r) = \theta \in [0, \pi]$; the case $arg(r) \in [-\pi, 0]$ is the same by symmetry.

In this case $cos\theta > |r_0|^{9/5}$ is same as $\theta < arccos(|r_0|^{9/5})$

Now because $|r_0| > e^{-\pi^2}$, and

$$log^2(1/|r_0|) < arccos^2(|r_0|^{9/5}), \forall a < |r_0| < 1 \tag{25}$$

where $a > 0$ is a constant. By adjusting the value of H (i.e setting $\rho = \Theta_l(1)$) in (24), one may indeed achieve this bound for theta as $|r_0|$ is bounded away from zero ($|r_0| > e^{-\pi^2}$), and $\log(1/x)$ has an upper bound in $a < x < 1$.

This bound on $cos(\theta)$ in particular gives us that the line joining the points 1 and r, lies inside the annulus $A(r, 1)$. The strategy is to place discs of radius $B(1-r)/k\rho$, at the points $r, r+B(1-r)/k\rho, \ldots r+(n-1)B(1-r)/k\rho$ where n is the largest integer less than $k\rho/B$, Following this, we place a disc of radius $(1-r) \times (1 - nB/\rho k)$ centered at $r + nB(1-r)/k\rho$. We further place fibres of type $*$ at the boundary points.

This implies $r + \frac{NB(1-r)}{k\rho} \in A(r,1) \forall N \in \mathbb{N} \leq k\rho/B$. And since $|1-r|B < |r| < 1$, it implies that a disc of radius $B(1-r)/k\rho$ placed anywhere along the line joining r to 1 admits a $\{1\}$ extension inside $A^{\{\rho\}}$ (as in general [1, Lemma 49], states that a disc of radius $\alpha/k\rho, k > 1$ fixed constant centered around α admits a $\{1\}$ extension for $|r| \leq \alpha$ uniformly over the base cell).

Thus we need poly(ρ) many discs and fibres of type $*$ to cover positive real part of the annulus.

It therefore follows that we obtain a disjoint real cover.

This proves the claim for the case $\rho > 1, \sigma = 1$.

- Case 2 : $\rho = 1, \sigma < 1$

  In this case, we note that $\mathcal{F}^{\{\sigma\}} = \mathcal{F}^{\varepsilon}$, where

  $$\varepsilon = \Theta(\sigma), \mathcal{F} = D; \quad \varepsilon = e^{-\Theta(1/\sigma)}, \mathcal{F} = D_0, A$$

  Say $\mathcal{F}$ is type D, upto rescaling $\mathcal{F} = D(1)$. Set $E = \{\sigma\}$. One can notice that discs of radius $O(\sigma)$ centered on a real point in $D(1)$, has a $\{\sigma\}$ extension that lies inside $D(1)^{\{1\}}$. Thus using poly($1/\sigma$) many discs of such radius and fibres of type $*$, one may obatin a disjoint real cover of $D(1)$.

We note the following conclusion that we draw from [1, Lemma 50] for $\mathcal{F} = A(r, 1), D_0(1)$:

For $0 < \alpha \leq 1$ and $r < \alpha$ uniformly over $\mathcal{C}^{\{1\}}$, and a disc of radius $O(\alpha\sigma)$ centered at α admits a $\{\sigma\}$ extension that lies inside $(\mathcal{C} \odot \mathcal{F})^{\{1\}}$. A similar conclusion holds for $1 \leq \beta < \infty$ with $r\beta \leq 1$ uniformly over the cell $\mathcal{C}^{\{1\}}$, a disc of radius $O(r\beta\sigma)$ centered around $r\beta$ when $\mathcal{F}$ is of type $A(r, 1)$.

We proceed to the case $\mathcal{F} = D_0(1)$. We again set $E = \{\sigma\}$. We can embed a disc $D_0(e^{-\Theta(1/\sigma)})$ inside $D_0(1)$ such that its $\{\sigma\}$ extension still remains inside $D_0(1)$. We are thus left with covering the real part of $A(e^{-\Theta(1/\sigma)}, 1)$ and the boundary point of the disc $D_0(e^{-\Theta(1/\sigma)})$.

The real part of this annulus above may be covered disjointly using the above conclusion with discs and fibres of type $*$ (to cover the boundaries). Notice that the logarithmic width of the annulus is $\Theta(1/\sigma)$. We also note that we use the α case from the above conclusion to cover the annulus $A(T, 1)$, with $log|T| = -\Theta(1/\sigma)$. We therefore see that since a point α admits a disc of radius $\sim \alpha\sigma/10$, we need N discs and fibres of type $*$ to cover the real part of the annulus, where N is such that $|T|(1+\sigma/10)^N = 1$. This bound is obtained by stacking a disc of radius $T\sigma/10$ at T, then a disc of radius $T(1 + \sigma/10)\sigma$ at $T(1 + \sigma/10)$, and so on, with possibly an adjustment in the last disc so that it covers exactly upto 1. Clearly N can be made poly $(1/\sigma)$ (We use the fact $0 < \sigma < 1$)

Now $\mathcal{F} = A(r, 1)$. Set $E = \{\sigma\}\{\hat{\rho}\}, \hat{\rho}$ will be chosen later.

Using the fundamental lemma for $\mathbb{D} \setminus \{0\}$, applied to r on the cell $\mathcal{C}^{\{\sigma\}}$, one of the following holds:

$$r(\mathcal{C}^{\{\sigma\}}) \subset B(\{0\}, e^{-\Omega_l(1/\hat{\rho})}; \mathbb{C}) \text{ or } \operatorname{diam}(r(\mathcal{C}^{\{\sigma\}}); \mathbb{D} \setminus \{0\}) = O_l(\hat{\rho}). \quad (26)$$

Suppose we are in the first case of (26), with $r(\mathcal{C}^{\{\sigma\}}) \subset B(0, e^{-\Omega_l(1/\hat{\rho})}; \mathbb{C})$. Then choosing $\hat{\rho} = \Omega_l(\sigma)$, enables us to choose an annulus A uniformly over the base cell such that $A^{\{\sigma\}} = A(r, 1)$. What remains are two subannuli that need to be covered (along with the boundary points). As these annuli are of log width $O(1/\sigma)$, one may use the same argument as before to cover it, using poly $(1/\sigma)$ many discs.

Say now we are in the second case of (26). If we can uniformly over $\mathcal{C}^{\{1\}}$ choose an annulus A such that $A^{\{\sigma\}} = A(r, 1)$, we can proceed as before with $\hat{\rho} = \Omega_l(1)$.

If not, we have that for some base point p in $\mathcal{C}^{\{\sigma\}}$, $r(p) = r_0$, with $\log|r_0| \geq -\pi^2/\sigma)$. We now proceed as in case 1. The equation stated in the form (24)is still valid here.

Again recall we assumed $r(\mathbb{R}_+\mathcal{C}^{\{\sigma\}}) > 0$

Say that $\hat{\rho}$ is small enough that

$$\frac{\log|r(p)|}{\log|r(q)|} \in (9/10, 10/9), \forall p, q \in \mathcal{C}^{\{\sigma\}} \tag{27}$$

We may make this $\hat{\rho}$ smaller if we wish, as in case, but this time, subjected to the estimate $\hat{\rho}^{-1} = \text{poly}(1/\sigma)$.
We note that the line joining r to 1 is in $A(r, 1)$, whenever $cos(\theta) > |r|^2, \theta = arg(r)$. It is thus enough to show as in case 1, under the assumption $arg(r) = \theta \in [0, \pi]$ that $\theta^2 < arccos^2(|r_0|^{9/5})$

We note that from (24), the above requirement would be satisfied if $(190/81)^2(\log)^2(1/|r_0|) \times (\frac{H^2}{1-H^2}) < arccos^2(|r_0|^{9/5})$. We note again

$$\log^2(1/|r_0|) < arccos^2(|r_0|^{9/5}), \forall |r_0| \in (a, 1), a > 0. \tag{28}$$

In this case as $|r_0| \geq e^{-\pi^2/\sigma}$, we have $\log^2(1/|r_0|) \leq \pi^4/\sigma^2$. Thus setting $(190/81)^2 \times (\frac{H^2}{1-H^2}) \leq \sigma^2/\pi^4$ would imply

$$(190/81)^2(\log)^2(1/|r_0|) \times (\frac{H^2}{1 - H^2}) \leq 1, \quad e^{-\pi^2/\sigma} < |r_0| < 1.$$

Since the left hand side above is monotonically decreasing with respect to $|r_0| < 1$, and the function $arccos$ is monotonically decreasing, with $arccos(0) = \pi/2, arccos(1) = 0$ we note from (28) that

$$(190/81)^2(\log)^2(1/|r_0|) \times (\frac{H^2}{1 - H^2}) < arccos^2(|r_0|^{9/5}), \quad e^{-\pi^2/\sigma} < |r_0| < 1$$

Thus we need to set $\frac{H^2}{1-H^2} < A\sigma^2$ for some constant $A < 1$. Recall that $H = \frac{e^{2c\hat{\rho}}-1}{e^{2c\hat{\rho}}+1} \leq e^{2c\hat{\rho}} - 1$.
Thus

$$(e^{2c\hat{\rho}} - 1)^2 \leq \frac{A\sigma^2}{A\sigma^2 + 1} \implies H^2 \leq \frac{A\sigma^2}{A\sigma^2 + 1} \iff \frac{H^2}{1 - H^2} \leq A\sigma^2$$

This may be clearly achieved with $\hat{\rho} = \Theta_l(\sigma)$, for example (recall $0 < \sigma < 1$).

We thus have every point on the line joining $r, 1$ is in $A(r, 1)$.

We notice that $\frac{|r|}{|1-r|}$, has a lower bound given by $B = \frac{|r_0|^{10/9}}{1-|r_0|^{9/10}}$. It therefore follows that a disc of radius $B(1 - r)\sigma/k, (k > 1$ fixed constant) centered at the point r, admits a $\{\sigma\}$ extension that lies inside $A^{\{1\}}$. This is a direct conclusion from [1, Lemma 50]: A point α with $|\alpha| > |r|$ over

$\mathcal{C}^{\{\sigma\}}$ admits a disc of radius $|\alpha|\sigma/k, k>1$, centered around α, such that its σ extension lies inside $A^{\{1\}}$.

We also note that $1/B$ is bounded by an exponential in $1/\sigma$, and therefore, the same trick as in case 1 does not work. We need to place bigger discs as we move towards 1, to account for this.

We instead do the following:

The bound on $cos\theta$ forces the line joining $1, r$ and the line joining $0, r$ intersect at an angle $\leq 90^0$. Thus

$$|r+B(1-r)\sigma/k| \geq \sqrt{|r^2|+|1-r|^2B^2\sigma^2/k^2}.$$

Thus at the point $r+B\sigma(1-r)/k$, we may place a disc of radius:

$$(1-r)\sqrt{\frac{|r|^2}{|1-r|^2}+B^2\sigma^2/k^2} \geq (1-r)B\sqrt{1+\sigma^2/k^2}$$

(recall $B^2 \leq \frac{|r|^2}{|1-r|^2}$).

Thus we see that by placing a disc of radius $B(1-r)\sigma/k$ at the point r, a disc of radius $B\sqrt{1+\sigma^2/k^2}(1-r)$ at the point $r+(1-r)B\sigma/k$, a fibre of type $*$ at $r+(1-r)B\sigma/k$, and so on, we obtain a disjoint real cover.

In this process, we need $< N$ discs, where N is the smallest integer such that $B((1+\frac{\sigma^2}{k^2})^{\frac{N}{2}+1}-1) \geq \sqrt{1+k^2/\sigma^2}-k/\sigma$.

Thus N=poly$_l(1/\sigma)$, as required.

This finishes the proof of the claim for $\sigma<1, \rho=1$

$\square$

7 Appendix

In this section, we address the motivation behind the main constructions.

7.1 Recursive constructions

We provide a slight modification to [1, proposition 56]. This is particularly useful in the recursion step 5.2.6. We note that we added the boundaries as new sections in this step. However, it is not immediately clear if the clustering process from [1] with this additional section satisfies an analogous result as in [1].

As usual, we shall show the conclusions assuming all the sections are real, as the complex case can be dealt with in a similar way.

We first recall the proposition and then suggest the modifications:

Theorem 7.1.1. *[1, Proposition 56] Suppose $1/\rho > poly_l(v, |\log\gamma|)$. Then the following hold uniformly over $\hat{\mathcal{C}}$:*

1. *The fibers $\mathcal{F}_{i,q}, \mathcal{F}_{i,q+}$ are well-defined and cover $\mathbb{C} \setminus \{0\}$.*
2. *The domains $\mathcal{F}^{\gamma}_{i,q} \setminus \mathcal{F}_{i,q}$ do not contain any of the points y_j for $q = 1, \dots, m_i$.*
3. *The domains $\mathcal{F}^{\gamma}_{i,q+}$ do not contain any of the points y_j for $q = 0, \dots, m_i$.*

Where m_i denotes the number of clusters around y_i. In the above proposition, we note that there is no grouping as in our construction. Moreover, as noted before, the $\mathcal{F}_{i,q}, \mathcal{F}_{i,q^+}$ defined above are a δ extension of our $\mathcal{F}_{i,q}, \mathcal{F}_{i,q^+}$. We also recall that Binyamini-Novikov define the $\hat{y}_{,q}$ to be the **center of the cluster**. This is a different notion from the center of the groups that we define, however, they are important in this context, as we shall work with them.

Since our clustering process is recursive and we add an extra section in each recursive clustering, we need to account for it in our modification of the above theorem.

Say we are in the simple case 5.2, where all sections are real. As seen in 7, we have clusters of sections inside the annuli $\mathcal{F}_{0,q}$. Inside a specific $\mathcal{F}_{0,q}$, we have the discs $D_{k,p}$, as in 5.2.3 at p. Let us focus on this specific $D_{k,p}$. The center is $c_k(p)$. We now follow the steps followed in [1] to form clusters (Not groups!!) around c_k, but we also include the boundary of $D_{k,p}$ as a point in the consideration. We now proceed to construct annuli $\mathcal{F}_{k,q}, \mathcal{F}_{k,q+}$ as usual centered around c_k, with the additional point $b_k(p)$.

We now follow the recursive step 5.2.6. Notice that $c_k(p)$ inside $D_{k,p}$ are never the centers for the $D^2_{s,p}$ due to the nature of clustering. We need to show the constructions made over a point $p \in \mathbb{R}_+\hat{\mathcal{C}}$ can be made uniform and well defined over $\hat{\mathcal{C}}$. We now assume there is only a single disc $D_{k,p}$ and make the following claims, which are analogues of Lemma 55 and Proposition 56 in [1].

Proposition 7.1.2.

Suppose $1/\rho > poly_l(v, |\log \gamma|)$. Then, the following hold uniformly over $\hat{\mathcal{C}}$:

1. *The fibers $\mathcal{F}_{0,q}, \mathcal{F}_{0,q+}$ are well-defined and cover $\mathbb{C} \setminus (\{0\} \cup \partial\mathcal{F}_{0,q} \cup \partial\mathcal{F}_{0,q^+})$.*
2. *The domains $\mathcal{F}^{\gamma}_{0,q} \setminus \mathcal{F}_{0,q}$ do not contain any of the points y_j*
3. *The domains $\mathcal{F}^{\gamma}_{0,q+}$ do not contain any of the points y_j.*
4. *The boundary described by b_k does not meet any of the sections y_n, over the base cell $\hat{\mathcal{C}}$.*
5. *The fibers $\mathcal{F}_{k,q}, \mathcal{F}_{k,q+}, i \neq 0$ are well-defined and cover $\mathbb{C} \setminus (\{0\} \cup \partial\mathcal{F}_{k,q} \cup \partial\mathcal{F}_{k,q^+})$.*
6. *The domains $\mathcal{F}^{\gamma}_{k,q} \setminus \mathcal{F}_{k,q}$ do not contain any of the points y_j.*
7. *The domains $\mathcal{F}^{\gamma}_{k,q+}$ do not contain any of the points y_j.*

Proof.

- (1),(2)&(3) This is immediate from Proposition 56 in [1].
- (4) This is clear from the construction of b_k. Say $b_k = y_n$ for some n at some point $z \in \hat{\mathcal{C}}$. Then it follows that $y_n(z) = b_k(z)$ in the normalization $y_0 = 0, \hat{y}_{0,q} = 1$. So y_n must be in the same fiber $\mathcal{F}_{0,q}$ as b_k. By construction, b_k (which is of the form $c_k + d_{\max,k} + 1/v^4$ in this normalization) is at least $1/v^4$ distance from y_n at $p \in \hat{\mathcal{C}}$, and thus if ρ is small enough in the above sense, $\mathrm{diam}(y_n, \mathbb{C} \setminus \{0,1\}) = O_l(v^3\rho)$, $\mathrm{diam}(c_k, \mathbb{C} \setminus \{0,1\}) = O_l(v^3\rho)$ and we get a contradiction. In fact, $\rho = 1/\mathrm{poly}_l(v)$ suffices as seen clearly for this.
- (5),(6)&(7) We know from above that b_k does not meet any of the other sections. Moreover, in the original fibre without any normalizations, b_k is of the form $c_k + d_{max,k}(\hat{y}_{0,q} - y_0)$, i.e b_k lifts to a univalued map to a cell of the form $\hat{C}_{a,b,c}$. Thus, we have by [1, fundamental Lemma]:

$$\begin{aligned} &\frac{b_k - y_\eta}{y_\lambda - y_\eta}(\hat{\mathcal{C}}_{a,b,c,d,e}) \subset B(\{0,1,\infty\}, e^{-\Omega_l(1/(v^5\rho))}; \mathbb{C}P^1) \quad \text{or} \\ &\mathrm{diam}(\frac{b_k - y_\eta}{y_\lambda - y_\eta}(\hat{\mathcal{C}}_{a,b,c,d,e}); \mathbb{C} \setminus \{0,1\}) = O_l(v^5\rho) \forall \text{ sections } y_\eta, y_\lambda. \end{aligned} \tag{29}$$

  We thus get an analogue of [1, lemma 55] as follows:

  Set c_k=0 Suppose $\rho = O_l(1/v^5)$. Let $n \neq k$ and write $R := \log |b_k/y_n|$. One of the following holds:

$$\begin{aligned} R|_{\hat{\mathcal{C}}} < -\Omega_l(1/v^5\rho) \quad &\text{or} \quad R|_{\hat{\mathcal{C}}} > \Omega_l(1/v^5\rho), \\ \mathrm{diam}(R(\hat{\mathcal{C}}), \mathbb{R}) &= O_l(v^5\rho) \\ \frac{\max_{z\in\hat{\mathcal{C}}} R(z)}{\min_{z\in\hat{\mathcal{C}}} R(z)} &< 1 + O_l(v^5\rho). \end{aligned} \tag{30}$$

We only require the above result with $c_k = 0$, for b_k only appears when clustering around c_k.

It is now clear that the claim follows from the above result. The proof is analogous to the proof in [1].

$\square$

7.1.2 clearly shows that we may indeed include b_k as a section when clusutering around c_k. While 7.1.2 only concerns the case where we only have a single disc D_k, we note that this may be analogously extended to say that there is a $1/\rho > \text{poly}_l(v, |\log\gamma|)$ such that the conditions hold for all such fibres around the respective centres.

We note the following point:

- Since we do not set $b_k = \hat{y}_{k,\alpha}$ when forming/coalescing groups inside D_k in 5.2, if at all we do need to form groups (Also recall that we set $\hat{y}_{k,\alpha} = 1$ when forming the simple groups), we see that the boundaries b_s^2 are of the form $d^2_{\max,s}(\hat{y}_{k,\alpha} - c_k)$, where the exponent 2 denotes the second level of recursion. As c_k is a section, we again get the same aymptotics as (29) and (30) for b_s^2 as well.

This ensures that we may indeed extend 7.1.2 to the case of multiple discs and multiple levels of recursions, with the aforementioned asymptotic on ρ.

7.2 Groupings with $1/v^4$

We now show the motivation behind choosing to group with distances of order $1/v^4$ in 5.2.2. We noticed in 5.3 that the radius of D_k under the normalization $y_0 = 0, \hat{y}_{0,q} = 1$ is $\mathrm{O}_l(1/v^2)$. We also saw that inside D_k, under the normalization $c_k = 0\hat{y}_{k,\alpha} = 1$ ($y_{k,\alpha} \neq b_k$ unless there are no sections less than b_k in the same cluster), the D_s^2 lie inside D_k uniformly over $\hat{C}$.

It can be easily seen that if we had grouped with respect to a distance of $1/v^N$ instead of $1/v^4$, the radius of the D_k would be $O(1/v^{n-2})$ under the normalization $y_0 = 0, \hat{y}_{0,q} = 1$. Under the normalization $c_k = 0, \hat{y}_{k,\alpha} = 1$, it follows that the distance between b_k and the last section less than b_k inside $\mathcal{F}_{k,\alpha}$ is $\Omega(1/v^2)$, irrespective of N. And since in $c_k = 0, \hat{y}_{k,\alpha} = 1$ coordinates, the radius of D_s^2 is $O(1/v^{N-2})$, it follows that $1/v^4$ is in a sense that $1/v^4$ is a tight bound. We also need to subject this to the condition that $(\gamma^{-1} - 1)/2$ is bigger than $1/v$, and if not, we do of course find a power $1/v^S$ such that $(\gamma^{-1} - 1)/2$ is bigger than $1/v^S$, and group accordingly with $1/v^{S+4}$ instead.

7.3 The $\sim_\epsilon$ condition in Lion-Rolin Preparation theorem

Here we sketch the proof of how one derives equation (6), which implies the $\sim_\epsilon$ condition as stated in 4.

To begin with, we closely follow the construction procedure in 5.2. We note that in this construction, where we assumed the fibre $\mathcal{F}$ was an annulus, we first started by clustering around zero, then clustered around other real sections.

Moreover, the fibres we construct, if not centered around zero, lie in the half plane Re $z_l > 0$. We note that if the fibre $\mathcal{F}$ was a disc, then we also follow the same step. Indeed, in this case, one of the sections could be zero at some point in the base cell. If so, we simply cover a discriminant set; outside this set, the sections never meet zero. This argument is similar to how we showed that the sections do not meet each other outside a discriminant set.

In this way, we ensure that the fibres constructed are either centered around zero or are contained in the half plane Re$z_l > 0$. We therefore notice that, under the setup of 4:

$f_j's$ are prepared with $\tilde{G}$, where $\tilde{G}, f_j, \phi_l$ are as defined in 4.

Proposition 7.3.1. *Under the setup in 4, if ϕ_l is not identically zero and $\tilde{G} \circ f_j \neq 0$,then there is a constant $C > 1$, depending on σ, such that*

$$\begin{aligned} |x_l - \phi_l(z_{1,\dots l-1})| \leq \phi_l(z_{1,\dots l-1}) \times 1/C, \\ \forall z_{1\dots l-1} \in (\mathcal{C}_j)_{1,\dots l-1}. \end{aligned} \tag{31}$$

Proof. Notice that the arguments above ensure that if $\phi_l(z_{1\dots l-1})$ is not identically zero:

- Observation: $x_l = \pm z_l^{v_l} + \phi_l > 0$ over $\mathcal{C}_j^{\{\sigma\}}$.

Since z_l is bounded above by a function r (given by the absolute value of the radius of the disc/ outer radius of the annulus, depending on the type of $\mathcal{C}_j$ at l), we conclude from the above observation that $\phi_l(z_{1\dots l-1}) > Cr(z_{1\dots l-1})^{v_l}$ over $\mathcal{C}_j^{\{\sigma\}}$for some constant $C > 1$ depending only on σ. This gives us that over $\mathcal{C}_j$, $|x_l - \phi_l(z_{1,\dots l-1})| \leq r(z_{1\dots l-1})^{v_l} \leq \phi_l(z_{1,\dots l-1}) \times 1/C$. The claim follows. □

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