Math 1512: Extra Practice for Exam 1 Key

Learning Goals for Exam 1

- Explain what $\lim_{x\to a} f(x) = L$ means. Illustrate with a sketch.
- Explain how left and right hand limits are related to the existence of a general limit.
- Give an example of a piecewise function $f(x)$ such that $\lim_{x\to 1} f(x) = 2$ but $\lim_{x\to 0} f(x)$ does not exist.
- Estimate $\lim_{x\to a} f(x)$ using a table.
- Use the slope of a secant line to determine the average velocity of a particle over a given interval.
- Use the limit of the secant slope to estimate the instantaneous velocity of a particle at a given time $t = a$.
- Use proper notation when applying limit laws to calculate a limit (pay particular attention to your use of equal signs). Even if your answer is correct, you will lose points for incorrect notation.
- Give an example of a limit that cannot be evaluated using direct substitution but can be computed after an algebraic simplification.
- Determine when a limit exists or does not exist, algebraically and/or graphically.
- There are two ways we defined the slope of the tangent line to the graph of a function $f(x)$ at $x = a$. Be able to state both definitions and sketch a graph that illustrates how we obtain the slope of the tangent line from the slope of the secant line.
- State the two equivalent definitions of the derivative of $f(x)$ at a number $x = a$. Provide a physical example of what $f'(a)$ represents.
- Use the graph of a function $f(x)$ to sketch the graph of $f'(x)$ (or vice versa).
- Give an example of a function that is not differentiable at $x = a$. Explain why.
- Know when and how to apply the power, sum/difference, product/quotient, and constant multiple rules.

Instructions These problems are meant to give you some more practice as you prepare for the exam. Please do not ONLY study these problems! Anything from or similar to the HW (both the hand-in and self-check problems) as well as anything covered in lecture are also fair game and are likely to appear on the exam. When you have questions please visit the calculus room in SMLC B-71, go to your instructor's office hours, or visit CAPS.

1. If a ball is thrown into the air at a velocity of 40 ft/s, its height in feet t seconds later is given by $h(t) = 40t - 16t^2$.

   (a) Find the average velocity for the time period beginning at $t = 2$ and lasting

      i. 0.5 second

      Answer: -32 ft/s

      ii. 0.1 second

      Answer: -25.6 ft/s

iii. 0.001 second

Answer: -24.16 ft/s

(b) What is the instantaneous velocity when $t = 2$ seconds?

Answer: -24 ft/s

2. Find the equation of the tangent line to the parabola $y = x^2$ at the point $(1, 1)$.

   Answer: $y = 2x - 1$

3. Estimate the instantaneous rate of change of $f(x) = 3\tan(2x)$ at $x = 0$ (x is in radians). Use this to compute the tangent line at the point $(0, 0)$.

   Answer: $y = 6x$

4. Use a table of values to evaluate each limit

   (a) $\lim_{x\to 0} \frac{\sqrt{x+4}-2}{x}$

   Answer: 0.25

   (b) $\lim_{x\to 0} \frac{\tan 3x}{\tan 5x}$

   Answer: 0.6

   (c) $\lim_{x\to 0} \frac{9^x - 5^x}{x}$

   Answer: About 0.59 (you'll see later the exact value is $\ln(9/5)$)

5. Sketch the graph of the function and use it to determine the values of a for which the limit $\lim_{x\to a} f(x)$ exists.

(a)

$$f(x) = \begin{cases} 1+x, & x < -1 \\ x^2, & -1 \le x < 1 \\ 2-x & x \ge 1 \end{cases}$$

Answer: For all a except $a = -1$

(b)

$$f(x) = \begin{cases} 1+\sin x, & x < 0 \\ \cos x, & 0 \le x < \pi \\ \sin x & x \ge \pi \end{cases}$$

Answer: For all a except $a = \pi$

6. Evaluate the following limits or explain why they do not exist.

(a)

$$\lim_{h\to 0} \frac{(2+h)^3 - 8}{h}$$

Answer: 12

(b)

$$\lim_{t\to 2} \frac{4-t^2}{t-2}$$

Answer: -4

(c)

$$\lim_{x\to 3} \frac{x+3}{x^2-9}$$

Answer: Does not exist

(d)

$$\lim_{h\to 0} \frac{\sqrt{25+h} - 5}{h}$$

Answer: 1/10

7. Determine the infinite limit.

   (a) $\lim_{x\to 1} \frac{2-x}{(x-1)^2}$

   Answer: ∞

   (b) $\lim_{x\to -2^+} \frac{x-1}{x^2(x+2)}$

   Answer: $-\infty$

   (c) $\lim_{x\to \pi^-} \cot x$

   Answer: $-\infty$

8. In the theory of relativity, the mass m of a particle with velocity v is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

   where m_0 is the mass of the particle at rest and c is the speed of light. What happens to the mass of the particle as $v \to c^-$?

   Answer: m approaches ∞!!!! Whoa...

9. Let

$$f(x) = \begin{cases} 1/(x+2), & x < -2 \\ x^2 - 5, & -2 < x \leq 3 \\ \sqrt{x+13}, & x > 3 \end{cases}$$

   Find

   (a) $\lim_{x\to -2} f(x)$

   Answer: DNE

   (b) $\lim_{x\to 0} f(x)$

   Answer: -5

   (c) $\lim_{x\to 3} f(x)$

   Answer: 4

   (d) Sketch a graph, labeling all intercepts clearly.

10. Evaluate the limit, if it exists

   (a) $\lim_{x\to 4} \frac{x^2-4x}{x^2-3x-4}$

   Answer: 4/5

   (b) $\lim_{h\to 0} \frac{(-5+h)^2-25}{h}$

   Answer: -10

   (c) $\lim_{t\to 0} \left(\frac{1}{t\sqrt{1+t}} - \frac{1}{t} \right)$

Answer: -0.5

(d) $\lim_{x \to 1} \frac{2x+1}{x^2(x-1)}$

Answer: DNE

(e) $\lim_{h \to 0} \frac{(x+h)^4 - x^4}{h}$

Answer: $4x^3$

11. In the theory of relativity, the Lorentz contraction formula

$$L = L_0 \sqrt{1 - v^2/c^2}$$

expresses the length L of an object as a function of its velocity v with respect to an observer, where L_0 is the length of the object at rest and c is the speed of light. Find $\lim_{v \to c^-} L$ and interpret the result. Why is the left-hand limit necessary?

Answer: The length approaches zero. The left-hand limit is necessary since function is not defined for $v > c$.

12. Is there a number a such the limit

$$\lim_{x \to -2} \frac{3x^2 + ax + a + 3}{x^2 + x - 2}$$

exists?

Answer: $a = 15$

13. We have seen that the *derivative of* $f(x)$ *at* $x = a$ can be represented two ways:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

and

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

If $f(x) = \frac{1}{x+2}$, compute $f'(1)$ using both definitions.

Answer: -1/9

Do the same for $f(x) = \sqrt{x+3}$

Answer: 1/4

14. The displacement (in meters) of a particle moving in a straight line is given by the equation of motion $s(t) = \frac{1}{t^2}$, where t is measured in seconds. Find the velocity of the particle at time $t = 3$ seconds. Interpret the sign of your answer.

Answer: $-\frac{2}{27}$ m/s. The negative sign indicates that the particle is moving backward.

15. Sketch the graph of a function for which $f(0) = 0$, $f'(0) = 3$, $f'(1) = 0$, and $f'(2) = -1$

16. The number of bacteria after t hours in a controlled laboratory experiment is $n = f(t)$

(a) What is the meaning of the derivative $f'(5)$? What are its units?

Answer: $f'(5)$ is the rate of growth of the population when $t = 5$ hours. The units are bacteria/hour.

(b) Suppose there is an unlimited amount of space and nutrients for the bacteria. Which do you think is larger, $f'(5)$ or $f'(10)$? If the supply of nutrients is limited, does this affect your conclusion? Explain.

Answer: With unlimited space and nutrients, the rate of growth should increase at time increases, so $f'(5) < f'(10)$. If the supply of nutrients is limited, the rate of growth slows down (bacteria start dying off without food).

17. Find the derivative of the function using the definition. State the domain of the derivative.

(a) $f(x) = 5x - 9x^2$

Answer: $f'(x) = 5 - 18x$, Domain of f' is $\mathbb{R}$.

(b) $g(x) = \frac{1-2x}{3+x}$

Answer: $g'(x) = \frac{-7}{(3+x)^2}$, Domain of g' is $\{x|x \neq 3\}$.

18. Show that $g(x) = \sqrt{x}$ is not differentiable at $x = 0$.

Answer: $g'(0) = \lim_{x\to 0} \frac{g(x)-g(0)}{x-0} = \lim_{x\to 0} \frac{1}{2\sqrt{x}}$, which does not exist. There is a vertical tangent at $x = 0$.

19. Show that $g(x) = |x - 6|$ is not differentiable at $x = 6$. Sketch a graph of g, g'.

Answer:

$$g(x) = \begin{cases} x - 6, & x \geq 6 \\ -(x-6), & x < 6 \end{cases}$$

20. Differentiate the function.

(a) $f(x) = 2^{40}$

Answer: $f'(x) = 0$

(b) $R(a) = (3a + 1)^2$

Answer: $R'(a) = 18a + 6$

(c) $y = \frac{x^2+4x+3}{\sqrt{x}}$

Answer: $\frac{dy}{dx} = \frac{3}{2}\sqrt{x} + \frac{2}{\sqrt{x}} - \frac{3}{2x^{3/2}}$

(d) $g(t) = (t + t^{-1})^3$

Answer: $g'(t) = 3t^2 + 3 - 3t^{-2} - 3t^{-4}$

(e) $y = \frac{f(x)}{x^2}$, f is any differentiable function of x.

Answer: $\frac{dy}{dx} = \frac{xf'(x) - 2f(x)}{x^3}$

(f) $y = \sqrt{2} \cdot x + \sqrt{3x}$

Answer: $y' = \sqrt{2} + \frac{\sqrt{3}}{2\sqrt{x}}$

(g) $f(x) = \frac{x^2+2}{x^4 - 3x^2 + 1}$

Answer: $f'(x) = \frac{2x(-x^4 - 4x^2 + 7)}{(x^4 - 3x^2 + 1)^2}$

(h) $f(x) = \frac{x}{x + \frac{c}{x}}$

Answer: $f'(x) = \frac{2cx}{(x^2+c)^2}$

21. Find an equation of the tangent line to the curve $y = x\sqrt{x}$ that is parallel to $y = 1 + 3x$

Answer: $y = 3x - 4$

22. Find the points on the curve $y = 2x^3 + 3x^2 - 12x + 1$ where the tangent line is horizontal.

Answer: $(-2, 21)$ and $(1, -6)$