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Q.4 MAP estimate for Kumasswamy Distribution.

$$f(x|a,b) = a \cdot b \cdot x^{a-1} (1-x^a)^{b-1}$$

- b is sampled from a normal distribution with known mean $= \mu$, variance $= \sigma^2$.

$$b = N(\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(b-\mu)^2}{2\sigma^2}}$$

soln: using Baye's Theorem, we know

$$P(a,b|x) = \frac{P(x|a,b) \cdot P(a,b)}{P(x)}$$

from Question: $P(x|a,b) = a \cdot b \cdot x^{a-1} (1-x^a)^{b-1}$

$$P(a,b) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(b-\mu)^2}{2\sigma^2}}$$

$P(x) =$ Constant (can be ignored for simplicity

$$\therefore P(a,b|x) = \left[\left\{ a \cdot b \cdot x^{a-1} (1-x^a)^{b-1} \right\} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(b-\mu)^2}{2\sigma^2}} \right]$$

for n samples

$$P(a,b|x) = \prod_{k=1}^{n} \left[a \cdot b \cdot x_k^{a-1} (1-x_k^a)^{b-1} \right] \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(b-\mu)^2}{2\sigma^2}}$$

step 1: Taking log

$$\ln[P(a,b)|x] = \ln \left\{ \prod_{k=1}^{n} \left[a \cdot b \cdot x_k^{a-1} (1-x_k^a)^{b-1} \right] \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(b-\mu)^2}{2\sigma^2}} \right\}$$

$$= \sum_{k=1}^{n} \ln \left[a \cdot b \cdot x_k^{a-1} (1-x_k^a)^{b-1} \right] + \ln \left[\frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(b-\mu)^2}{2\sigma^2}} \right]$$

$$= \sum_{k=1}^{n} \left[\ln(a) + \ln(b) + (a-1)\ln(x) + (b-1)\ln(1-x^a) \right] + \left[-\frac{(b-\mu)^2}{2\sigma^2} - \ln(\sqrt{2\pi\sigma^2}) \right]$$

$$= \sum_{k=1}^{n} \left[\ln(a) + \ln(b) + (a-1)\ln(x) + (b-1)\ln(1-x^a) \right] + \left[-\frac{(b-\mu)^2}{2\sigma^2} - \frac{1}{2}\ln(2\pi\sigma^2) \right]$$

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$$= \sum_{k=1}^{n} \left[\ln(a) + \ln(b) + (a-1)\ln(x) + (b-1)\ln(1-x^a) \right]$$

$$+ \left[\frac{-(b-\mu)^2}{2\sigma^2} - \frac{1}{2}\ln(2\pi\sigma^2) \right]$$

step 2: Differentiating w.r.t. b

$$\frac{\partial f(x)}{\partial b} = \sum_{k=1}^{n} \left[0 + \frac{1}{b} + 0 + \ln(1-x_k^a)(1) \right] + \left[\frac{-1(\cancel{2})(b-\mu)}{\cancel{2}\sigma^2} \right]$$

$$= \sum_{k=1}^{n} \left[\frac{1}{b} + \ln(1-x_k^a) \right] + \left[\frac{-(b-\mu)}{\sigma^2} \right]$$

$$= \frac{n}{b} + \sum_{k=1}^{n} \ln(1-x_k^a) - \frac{(b-\mu)}{\sigma^2}$$

$$= \frac{n}{b} + \sum_{k=1}^{n} \ln(1-x_k^a) - \frac{b}{\sigma^2} + \frac{\mu}{\sigma^2}$$

step 3: Equating to zero.

$$\therefore \frac{n}{b} + \sum_{k=1}^{n} \ln(1-x_k^a) - \frac{b}{\sigma^2} + \frac{\mu}{\sigma^2} = 0$$

bringing b on same side

$$\frac{b}{\sigma^2} - \frac{n}{b} = \frac{\mu}{\sigma^2} + \sum_{k=1}^{n} \ln(1-x_k^a)$$

$$\frac{b^2 - n\sigma^2}{b\sigma^2} = \frac{\mu}{\sigma^2} + \sum_{k=1}^{n} \ln(1-x_k^a)$$

multiplying σ^2 throughout

$$\therefore \frac{b^2 - n\sigma^2}{b} = \mu + \sigma^2 \sum_{k=1}^{n} \ln(1-x_k^a)$$

$$\therefore b^2 - n\sigma^2 = b\left[\mu + \sigma^2 \sum_{k=1}^{n} \ln(1-x_k^a) \right]$$

$$\therefore b^2 - b\left[\mu + \sigma^2 \sum_{k=1}^{n} \ln(1-x_k^a) \right] - n\sigma^2 = 0 \quad \text{— (1)}$$

this is in the form of Quadratic equation.

$$ax^2 + bx + c = 0$$

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$\therefore$ eq ① becomes

$$b = \frac{\left(\mu + \sigma^2 \sum_{k=1}^{n} \ln(1 - x_k^a)\right) \pm \sqrt{\left[-\left(\mu + \sigma^2 \sum_{k=1}^{n} \ln(1 - x_k^a)\right)^2 - 4(1)(-n\sigma^2) \right]}}{2(1)}$$

$$\therefore \boxed{b = \frac{\left(\mu + \sigma^2 \sum_{k=1}^{n} \ln(1 - x_k^a)\right) \pm \sqrt{\left[-\left(\mu + \sigma^2 \sum_{k=1}^{n} \ln(1 - x_k^a)\right)^2 - 4(1)(-n\sigma^2) \right]}}{2}}$$