Automaticity of Mapping Class Groups

Master's Thesis Synopsis

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1 Introduction

For a surface S of finite type, the mapping class group $\mathcal{MCG}(S)$ is the group of all self homeomorphisms of S up to isotopy. An important problem which has remained central to combinatorial group theory is *the word problem*, first posed by Max Dehn in 1911. Penner and Harer [1] proved that the word problem is solvable for $\mathcal{MCG}(S)$. More recently, due to the work of Epstein et al. [2], another important question to ask about finitely presented groups is whether they are *automatic*. Automatic groups come equipped with an *automatic structure*, which allows the word problem to be solved in *quadratic time*. As developed in [2], the theory of automatic groups has its roots in the idea of seeing groups as *regular languages* and their Cayley graphs as the naturally associated *deterministic finite automaton* (DFA). Studying automatic groups with the additional semantics of being mapping class groups provides insight into several aspects of low-dimensional topology, combinatorial group theory, and theory of computation.

2 Objectives

The objective of this thesis is to understand and implement the main result by Lee Mosher as obtained in [3], which asserts that:

Main Theorem. *If S is a surface of finite type, then $\mathcal{MCG}(S)$ is automatic.*

Mosher describes a complete framework in [3] to prove the automaticity of the mapping class group of a general finite type surface. It turns out that such a structure is completely constructive in some cases. This is the premise of Mosher's other paper [4], where he gives an explicit pen-paper construction of the automatic structure for oriented, closed, once-punctured surfaces using *chord diagrams*. Chord diagrams are well-behaved finite objects which are used both for pen-paper and computer calculations. Writing codes for the given algorithms and verifying the obtained results with the given data is a secondary objective of the thesis.
The main theorem also acts as an additional proof that the word problem is solvable in $\mathcal{MCG}(S)$. This route of proving the solvability has less prerequisites, and is overall easier than [1].

3 Plan of Study

The proof of the main theorem uses several techniques in mapping class groups as well as theory of computation. The background material on mapping class groups will be drawn from Farb and Margalit's book [5, Chapters 1–4], while that on theory of computation will be studied from Michael Sipser's book [6, Chapters 1, 3, and 7].

Here is an outline of the proof of the main theorem.

Generally, mapping class groups are infinite, and thus are the isotopy classes of curves under their action. This makes $\mathcal{MCG}$s incompatible with DFAs. The main idea of Mosher to overcome this issue is that the orbits of the action can be finite. Another auxiliary construction is of the *mapping class groupoid* $\mathcal{MCGD}$, which is the edge path groupoid of some finite cell complex (the complex X as below) with fundamental group $\mathcal{MCG}$.

For a general surface S, the orbits of the isotopy classes of triangulations are first shown to be finite. The automaticity is then characterized in terms of the paths in the Cayley graph satisfying the *synchronous* k-fellow traveller property. $\mathcal{MCGD}$ is then endowed with an *asynchronous* automatic structure, which is later converted into a synchronous one. Finally, the assertion is proved by showing that $\mathcal{MCGD}$ is an automatic groupoid *iff* $\mathcal{MCG}$ is an automatic group.

In case of once-punctured surfaces, Mosher takes a more hands-on approach in [4]. First, a contractible complex Y is defined using isotopy classes of ideal triangulations of the surface, on which $\mathcal{MCG}$ acts with finite cell stabilizers and with finitely many orbits. Another complex $\tilde{X}$ is defined using a cellular map $q : \tilde{X} \to Y$, such that $\mathcal{MCG}$ acts on $\tilde{X}$ freely with finitely many orbits. The quotient $X := \tilde{X}/\mathcal{MCG}$ thus obtained is such that the fundamental group of X is $\mathcal{MCG}$. As above, $\mathcal{MCGD}$ is constructed using the free action of $\mathcal{MCG}$ on $\tilde{X}^{(0)}$. Finally, an automaton $\mathcal{M}_0$ is defined using $\mathcal{MCGD}$ and a cellular map $p : \mathcal{M}_0 \to X$, which yields the desired automatic structure.

References

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