

Automaticity of Mapping Class Groups

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Outline of the presentation

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Background

Automatic Groups

The Word Problem for Groups

- **The word problem (Dehn 1911¹)**. Let $G = \langle S | R \rangle$ be a finitely presented group. Given a string w over S , can it be determined in **finite** time if w represents the identity in G ?

¹Dehn, M. “*Über unendliche diskontinuierliche Gruppen*”. In: *Mathematische Annalen* 71.1 (1911), pp. 116-144. doi:10.1007/BF01456932.

- **Definition.** An FSA \mathcal{M} over an alphabet S is a **finite directed graph**, such that:
 1. Vertices = '**states**', directed labeled edges = '**arrows**'.
 2. One state is specified as the **start state**.
 3. A subset of states is specified as the **accept states**.
 - FSA \mathcal{M} reads a string w one letter at a time, and changes its state as per the arrows.
 - A set L of strings over S is called as a **regular language** if there exists an FSA \mathcal{M} which recognises L .

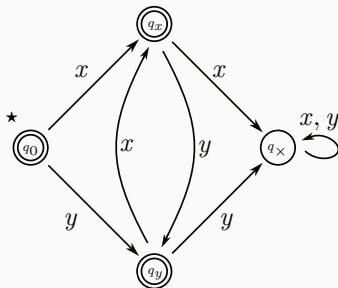
An Automatic Structure for Groups

- **Definition.** A **synchronous automatic structure** for G consists of a generating set S , a regular language L over S , and regular languages K_s over (S, S) for each $s \in S_\varepsilon$, such that:
 1. The canonical map $L \rightarrow G$ is surjective.
 2. $(w, v) \in K_s$ iff $w, v \in L$ and $ws = v$ in G .

The elements of L are called **normal forms**.

- **Definition.** If the FSA's accepting the languages K_s are asynchronous FSA's, then the structure is called as an **asynchronous automatic structure**.

Example: The FSA for $\mathbb{Z}_2 * \mathbb{Z}_2$



The FSA accepts a string *iff* it does not have xx and yy as a substring.

$\star \bigcirc$ = the start state \odot = an accept state \bigcirc = a reject state

- $w = xyx \Rightarrow q_0 \xrightarrow{x} q_x \xrightarrow{y} q_y \xrightarrow{x} q_x$
- $w = yxyy \Rightarrow q_0 \xrightarrow{y} q_y \xrightarrow{x} q_x \xrightarrow{y} q_y \xrightarrow{y} q_x$

Automatic Groupoids

Let Γ be a groupoid.

- Objects = ‘**vertices**’, morphisms = ‘**elements**’
- We assume connectedness: there is at least one element between any two vertices.
- Let Γ_x be the set of all elements with the same initial and final vertex x . Then, Γ_x is a group.
- The automatic structure is defined in a similar way, but a string denotes only a composable sequence of elements.

Theorem (Epstein 1992²)

A groupoid Γ is automatic iff Γ_x is an automatic group, for some vertex x .

²Epstein, David B. A. et al. *Word processing in groups*. Jones and Bartlett Publishers, Boston, MA, 1992.

Automatic Structures are Efficient

Theorem (Epstein 1992³)

If G admits an automatic structure, then the word problem for G can be solved in quadratic time.

Proof:

- Given any word w over S , the automatic structure can be used to construct an element $w_L \in L$ in quadratic time such that $w = w_L$ in G .
- Then, as both $w_L, e_G \in L$, \mathcal{M}_{e_G} can be used to solve the word problem. (\mathcal{M}_{e_G} corresponds to the FSA for regular language K_{e_G} .) □

³Epstein, David B. A. et al. *Word processing in groups*. Jones and Bartlett Publishers, Boston, MA, 1992.

Background

Mapping Class Groups

Mapping Class Groups

Let S be a connected, oriented surface. S is called a surface of **finite type** if $\pi_1(S)$ is finitely generated.

- Let $\text{Homeo}^+(S)$ be the group of all self-homeomorphisms of S that fix ∂S pointwise, and preserve the set of punctures.
- Let $\text{Homeo}_0(S)$ be the normal subgroup of all self-homeomorphisms of S which are isotopic to identity.
- Then, the mapping class group of S is defined as

$$\mathcal{MCG}(S) := \text{Homeo}^+(S) / \text{Homeo}_0(S)$$

The Main Theorem

Main Theorem (Mosher 1995⁴)

For a surface of finite type, the mapping class group is automatic.

⁴Mosher, Lee. “*Mapping class groups are automatic*”. In: *Annals of Math. Second Series* 142.2 (1995), pp. 303–384. doi: 10.2307/2118637.

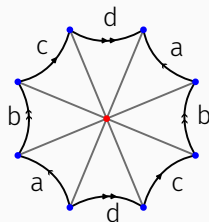
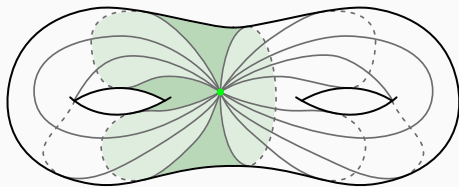
Asynchronous Structure of Oriented, Punctured Surfaces

Ideal Triangulations

- **Convention.** Each component of ∂S contains at least one puncture.
- **Definition.** Let $I := [0, 1]$. An **ideal arc** on S is the image of the map $f : (I, \partial I, \text{int}(I)) \rightarrow (S, P, S - P)$ such that:
 1. f is injective on $\text{int}(I)$, and if $f(I)$ is a loop, it is nontrivial.
 2. Either $f(I) \subseteq \partial S$, or $f(\text{int } I) \cap \partial S = \emptyset$ and $f(I)$ is not homotopic into ∂S .
- **Definition.** An **ideal arc system** δ is a set of ideal arcs such that:
 1. δ contains every boundary ideal arc.
 2. If $g \neq h \in \delta$, then g and h are not isotopic, and their interiors do not intersect.

Ideal Triangulations

- Definition.** An **ideal triangulation** is a maximal ideal arc system.



Examples of ideal triangulations on $S_{2,1}$ and $S_{2,2}$ respectively.

Each complementary component of an ideal triangulation is a triangle.

A General Method to Construct Groupoids

Let the action of G on \mathcal{D} be free. Define a groupoid Γ such that

- objects are the orbits of the action $G \curvearrowright \mathcal{D}$
- morphisms are the orbits of the diagonal action $G \curvearrowright \mathcal{D} \times \mathcal{D}$

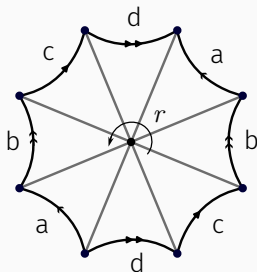
Denote the orbits as $\{x\}$ and $\{x, y\}$ respectively.

The composition rule is defined as $\{x, y\} \circ \{y, z\} := \{x, z\}$.

A free action is required to get an isomorphism $\theta : G \rightarrow \Gamma_v$ defined as $g \mapsto \{x, gx\}$, where v is a vertex of Γ .

Using Free Actions to Construct \mathcal{MCGD}

Currently, the action of \mathcal{MCG} on the isotopy classes of ideal triangulations is not free: a nontrivial mapping class can fix an isotopy class of a triangulation.



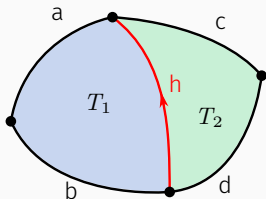
The $\mathbb{Z}_8 = \langle r | r^8 \rangle$ action induces a homeomorphism on S_2 which fixes the triangulation.

Labeled Ideal Triangulations Break Symmetries

- **Definition.** A labeled ideal triangulation is a pair (δ, h) , where δ is a triangulation, and h is an oriented ideal arc of δ .

Theorem

\mathcal{MCG} acts freely on the set of isotopy classes of labeled ideal triangulations.



Proof:

- Let $f \in \text{Homeo}^+(S)$ fix both δ and an oriented arc $h \in \delta$.
- Then, $T_1 \mapsto T_1$.
- Thus, $a \mapsto a$ and $b \mapsto b$.
- By Alexander method, f is trivial. □

Asynchronous Structure of Oriented, Punctured Surfaces

The Mapping Class Groupoid \mathcal{MCGD}

The Mapping Class Groupoid

Let \mathcal{D} be the set of isotopy classes of labeled ideal triangulations. We construct $\mathcal{MCG}\mathcal{D}$ using the free action $\mathcal{MCG} \curvearrowright \mathcal{D}$.

- Vertices = set of combinatorial types of labeled triangulations
- Elements = combinatorial types of pairs of labeled triangulations

A **combinatorial type** of δ is an orbit of δ under the action of $\text{Homeo}^+(S)$.

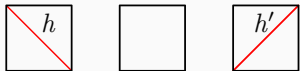
That is, existence of orientation preserving homeomorphisms is an equivalence relation. The equivalence classes are called the combinatorial types.

Asynchronous Structure of Oriented, Punctured Surfaces

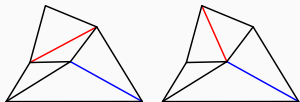
Elementary Moves

Elementary Moves

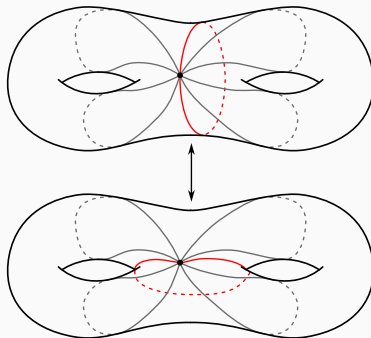
- Idea.** Removing an arc $h \in \delta$ and putting back its opposite diagonal arc h' gives a new triangulation δ' which is not isotopic to the original one. Such a move is denoted as $\delta \xrightarrow{h} \delta'$.



(a) The elementary move about h



(b) Elementary moves do not disturb other arcs



(c) Elementary moves are symmetric

Elementary Move Connectivity

Any ideal triangulation can be reached from any other via elementary moves.

Lemma

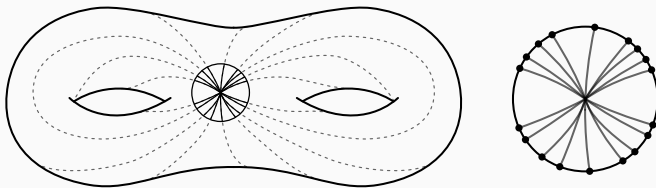
Given ideal triangulations δ and δ' , there exists a finite sequence of elementary moves

$$\delta = \delta_0 \rightarrow \delta_1 \rightarrow \delta_2 \rightarrow \cdots \rightarrow \delta_n$$

such that δ_n is isotopic to δ' .

Labeling Elementary Moves

- **Definition.** A **labeled** elementary move is a move between two labeled ideal triangulations, written as $(\delta, g) \xrightarrow{h} (\delta', g')$. Two cases arise depending on whether the move is done about the label.
 - The label is retained if $h \neq g$.
 - If $h = g$, then g' is chosen such that $\text{Tail}(g') = \text{Pred}(\text{Tail } g)$
- **Note.** This rule is combinatorial!



The orientation on S induces an order on $\mathcal{E}(\delta)$

- **Relabeling generators.** If (δ, h) and (δ, g) have the same underlying unlabeled ideal triangulation δ , then the element

$$\{(\delta, h), (\delta, g)\} \in \mathcal{MCGD}(S)$$

is called as a **relabeling generator**.

- **Elementary move generators.** The element

$$\{(\delta, g) \xrightarrow{h} (\delta', g')\} \in \mathcal{MCGD}(S)$$

is called as an **elementary move generator**.

Theorem

Any element of $MCGD(S)$ can be written as a finite sequence of elementary move generators followed by at most one relabeling generator.

Proof: We can factor any $m = \{(\delta, g), (\delta', g')\}$ in terms of labeled elementary moves:

- Sans the labels, get a finite sequence $\delta \rightarrow \delta_1 \rightarrow \dots \rightarrow \delta_N = \delta'$.
- Apply the combinatorial rule to generate the label g_i of δ_i .
- Change the label from g_N to g' .

□

Theorem

\mathcal{MCGD} has finitely many elementary moves and relabeling generators.

Proof:

- The combinatorial type of δ is completely determined by $(\mathcal{E}(\delta), \text{Opp}, \text{Succ})$.
- The combinatorial type of (δ, g) is determined by $(\mathcal{E}(\delta), \text{Opp}, \text{Succ}, \text{Tail } g)$. (\Rightarrow finitely many vertices.)
- A move $(\delta, g) \xrightarrow{h} (\delta', g')$ is determined by $(\mathcal{E}(\delta), \text{Opp}, \text{Succ}, \text{Tail } g, c_h)$, where c_h is an involution of Opp . (\Rightarrow finitely many labeled moves.)
- A relabeling $\{(\delta, g), (\delta, g')\}$ is determined by $(\mathcal{E}(\delta), \text{Opp}, \text{Succ}, \text{Tail } g, \text{Tail } g')$. (\Rightarrow finitely many relabelings.) \square

Asynchronous Structure of Oriented, Punctured Surfaces

Normal Forms for \mathcal{MCGD}

Combing a Triangulation Along Another

Combings find an explicit path between two ideal triangulations.

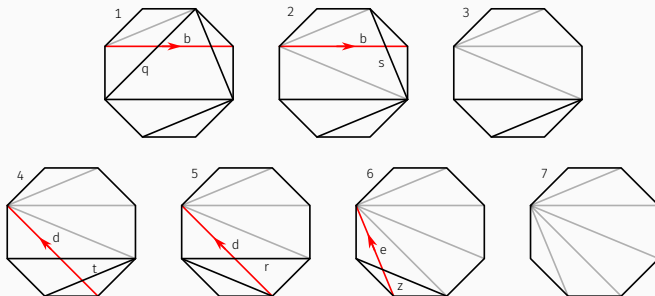
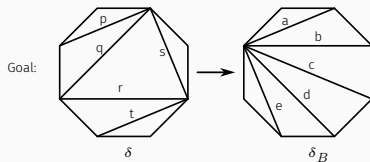
- We fix a base ideal triangulation δ_B .
- Given any δ , we comb δ along δ_B , one arc at a time.
- **Combing along an arc.** Let δ be a triangulation, and let g be any oriented ideal arc.

If $g \notin \delta$, then δ is said to be **uncombed** along g .

If uncombed, the **combing sequence** of δ relative to g is defined as $\delta = \delta_0 \rightarrow \delta_1 \rightarrow \cdots$, constructed inductively as:

1. There is some $h \in \delta$ such that h is the first arc crossed by g on S .
2. Do a move $\delta \xrightarrow{h} \delta_1$.
3. Choose the next arc $h_1 \in \delta_1$ such that h_1 is the first arc crossed by g . Do $\delta_1 \xrightarrow{h_1} \delta_2$, and so on.

Example of Combing



Combing δ along each arc of δ_B

The Combing Sequence Terminates I

Lemma

Given an ideal triangulation δ and an oriented ideal arc g , there exists some δ_N in the combing sequence of δ relative to g such that $g \in \delta_N$.

Proof:

- Suppose the move $\delta_m \xrightarrow{h_m} \delta_{m+1}$ is done. Let $\hat{\delta}_m := \delta_m - \{h_m\}$.
- Consider the (geometric) intersection number

$$\langle \hat{\delta}_m, g \rangle := \sum_{a \in \hat{\delta}_m} \langle a, g \rangle$$

- Note that $\langle \hat{\delta}_m, g \rangle = 0$ iff g is the opposite diagonal of h_m .
 - If so, then $g \in \delta_{m+1}$, and we are done.
 - If not, $\langle \hat{\delta}_{m+1}, g \rangle < \langle \hat{\delta}_m, g \rangle$. Thus, $\langle \hat{\delta}_m, g \rangle$ decreases monotonically to 0. □

The Combing Sequence Terminates II

Therefore, the combing sequence of δ relative to δ_B terminates.

We get back the elementary move connectivity lemma as a corollary:

Lemma

Given ideal triangulations δ and δ' , there exists a finite sequence of elementary moves

$$\delta = \delta_0 \rightarrow \delta_1 \rightarrow \delta_2 \rightarrow \cdots \rightarrow \delta_n$$

such that δ_n is isotopic to δ' .

Therefore, combing sequences are sufficient to dictate a path between two triangulations.

Normal Forms for \mathcal{MCGD}

To construct a regular language L for \mathcal{MCGD} , we need to specify the alphabet and the normal forms.

- $R :=$ all relabeling generators.
- $E :=$ all elementary move generators and their inverses.
- **Alphabet.** $A_0 := R \cup E$

We fix the combinatorial type $\{(\delta_B, g_1)\}$ of the labeled triangulation (δ_B, g_1) as the base vertex in $\mathcal{MCGD}(S)$.
- **Normal forms.** Given (δ, h) , the corresponding normal form $w_0(\delta, h)$ is defined as the word over A_0 obtained by reversing the combing sequence $\delta \rightarrow \delta_0 \rightarrow \dots \rightarrow \delta_N = \delta_B$.
 $w_0(\delta, h)$ represents the element $\{(\delta_B, g_1), (\delta, h)\}$.
- **The language.** Let L_0 be the language over A_0 of all words $w_0(\delta, h)$.

Normal Forms are Regular I

Theorem

The language L_0 is a bijective regular language for $\mathcal{MCGD}(S)$ based at $\{\delta_B\}$.

Remarks.

- An FSA \mathcal{M}_0 can be explicitly constructed which accepts the language L_0 .
- The procedure to get an **uncombing** move yields a **unique** path. That is, $w_0(\delta, h)$ is the unique normal form representing the groupoid element $\{(\delta_B, g_1), (\delta, h)\}$.
 $\Rightarrow L_0$ bijects onto $\mathcal{MCGD}(S)$.
- We introduce a **marking** μ for δ which keeps track of all arcs $g_i \in \delta_B$ we have combed δ along, and the first uncombed arc.
- That is, markings track the **combing status** of δ relative to δ_B .

Normal Forms are Regular II

Lemma

Let (δ, μ) and (δ', μ') be marked ideal triangulations, and let $\delta \rightarrow \delta'$ be a labeled elementary move.

Then, $(\delta, \mu) \rightarrow (\delta', \mu')$ is a marked elementary move iff $\delta' \rightarrow \delta$ is the first step of combing δ' along some $g_j \in \delta$.




- The lemma forces uniqueness when choosing an uncombing sequence.
- From the lemma, the valid uncombings are precisely the marked elementary moves.
- The states in \mathcal{M}_0 are the combinatorial types of consistent marked ideal triangulations $\{\delta, \mu\}$.
- The arrows in \mathcal{M}_0 are marked elementary moves.
- However, it turns out that L_0 is an **asynchronous** structure. A *bad* elementary move can cause nearby paths in L_0 to diverge.

Plans for the Future

- **Recall.** An **automatic structure** for \mathcal{MCGD} consists of a generating set A_0 , a regular language L_0 over A_0 , and regular languages K_a over (A_0, A_0) for each $a \in A_0$, such that regular language K over (A_0, A_0) such that:
 1. The canonical map $L_0 \rightarrow \mathcal{MCGD}$ is ~~surjective~~ bijective. ✓
 2. $(w, v) \in K_a$ iff $w, v \in L_0$ and $wa = v$ in \mathcal{MCGD} .

We have obtained a language L_0 which is an asynchronous regular language for $\mathcal{MCGD}(S)$.

We need to make it synchronous, and construct the languages K_a 's.

-  Dehn, M. “Über unendliche diskontinuierliche Gruppen”. In: *Mathematische Annalen* 71.1 (1911), pp. 116–144. DOI: [10.1007/BF01456932](https://doi.org/10.1007/BF01456932).
-  Epstein, David B. A. et al. *Word processing in groups*. Jones and Bartlett Publishers, Boston, MA, 1992.
-  Mosher, Lee. “Mapping class groups are automatic”. In: *Annals of Math. Second Series* 142.2 (1995), pp. 303–384. DOI: [10.2307/2118637](https://doi.org/10.2307/2118637).

Thank You!
