# The Word Problem for Groups and Automaticity


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## The word problem for groups

**The word problem.** Throughout, let $G = \langle S|R\rangle$ be a fixed presentation, such that $R$ is finite. If it can be determined whether a string $w$ over $S$ represents $e_G$ in *finite time*, then the word problem for $G$ is said to be solvable.

**Observation.** $\tilde{e} = e_G \iff \tilde{e} = \prod_{i=1}^{n} v_i r_i v_i^{-1}$, for $r_i \in R$ and $v_i \in F(S)$.

**Novikov**, **Boone**, and **Britton** independently showed that there exists a finitely presented group with unsolvable word problem.

## Trivia on the word problem

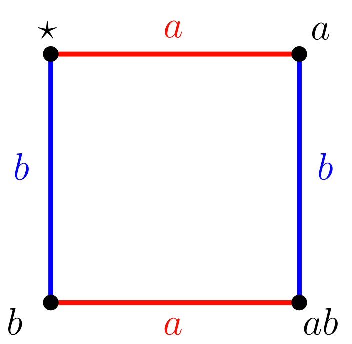
- Most commonly used group families have solvable word problem: finite groups, braid groups, hyperbolic groups, mapping class groups, Baumslag-Solitar groups, etc.
- All one-relator groups have solvable word problem. (Magnus, Karrass, and Solitar)
- A group has solvable word problem *iff* it can be embedded in *every* algebraically closed group. (Neumann and Macintyre)
- If $G$ is a finitely presented group that contains a copy of every group with solvable word problem, then $G$ itself must have *unsolvable* word problem. (Boone and Rogers)

## Cayley graphs for groups

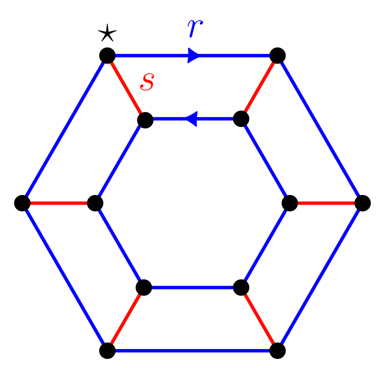
The **Cayley graph** for a finitely presented group $G = \langle S|R\rangle$ is a directed, labeled graph $\Gamma(G,S)$ whose vertex set is $G$, and there is a directed edge $g \xrightarrow{s} gs$, for each $g \in G, s \in S$.

The word length $|g|$ of $g \in G$ is the least number of edges between $e_G$ and $g$ in $\Gamma(G,S)$. The Cayley graph can then be metrized as $d(g,h) := |g^{-1}h|$.

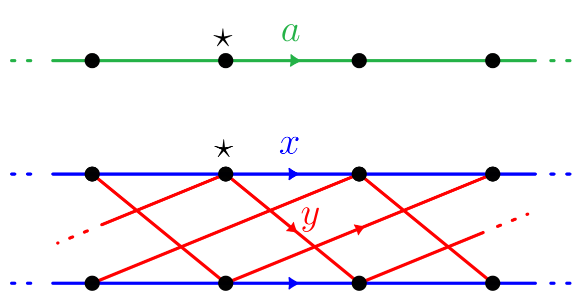
**Observation.** $\{\text{strings } w\} \longleftrightarrow \{\text{paths in } \Gamma\}$ and $\{\tilde{e}\} \longleftrightarrow \{\text{loops in } \Gamma\}$.




$K_4 = \langle a, b \mid a^2, b^2, (ab)^2\rangle$




$D_{12} = \langle r, s \mid r^6, s^2, srsr\rangle$




$\mathbb{Z} = \langle a\rangle$ and $\mathbb{Z} = \langle x, y \mid [x,y], x^3y^{-2}\rangle$

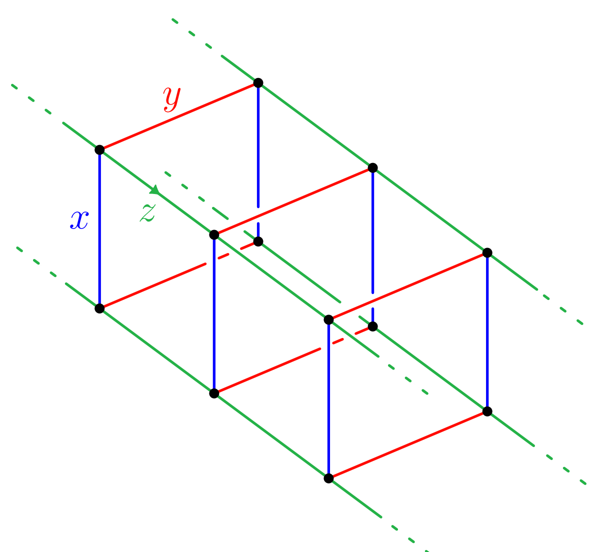
## The area and the perimeter

In the Cayley graph, $\tilde{e} = \prod_{i=1}^{n} v_i r_i v_i^{-1}$ corresponds to decomposing $\tilde{e}$ into smaller relator loops $r_i$'s, translated by the paths $v_i$.

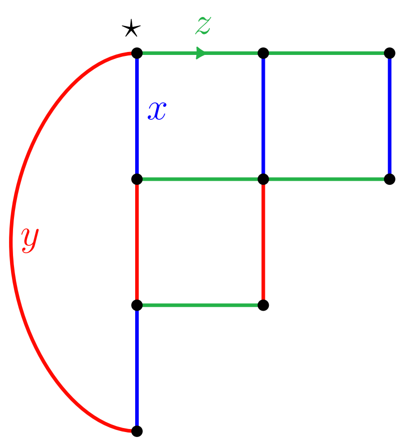
The **combinatorial area** of $\tilde{e}$ is the minimum such $n$ in any decomposition.

An **isoperimetric function** for $G$ is a non-decreasing function $\phi : \mathbb{N} \to [0,\infty)$ which measures the maximum area any loop can have given the maximum perimeter $k$.

$$\phi(k) := \max\{\text{area}(w) \mid |w| \leqslant k, w = e_G, w \text{ freely reduced}\}$$




$\mathbb{Z} \times K_4 \cong \langle x, y, z \mid x^2, y^2, (xy)^2, [y,z], [x,z]\rangle$




$w = z^2xz^{-1}yz^{-1}xy$ with combinatorial area 4

A function $f$ on $\mathbb{N}$ is called **recursive** if there is a finite procedure to calculate $f(n)$ given $n$.

**A bound on the area.** *$G$ has solvable word problem iff $\phi$ is a recursive function.*

**A bound on the paths.** *If $w = \prod_{i=1}^{n} v_i r_i v_i^{-1}$, then $w$ can be rewritten such that each $v_i$ has length at most $(|w| + 2k) \cdot 2^n$, where $k$ is the maximum length of the relators $r_i$.*

## Automatic groups

A **finite state automaton** is an abstract machine which can be modeled as a finite directed graph. These automata process strings and either accept or reject them.

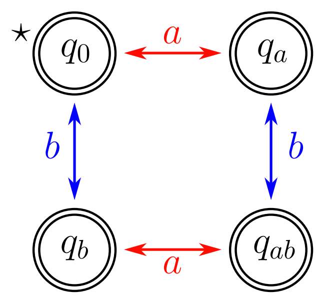
An **automatic structure** on $G$ is a finite generating set $S$ along with finite state automata $\mathcal{W}$ and $\mathcal{M}$ such that:

1. $\mathcal{W}$ accepts all elements of $G$.
2. $\mathcal{M}$ accepts the pair $(w,u)$ *iff* $w$ and $u$ differ by a generator.

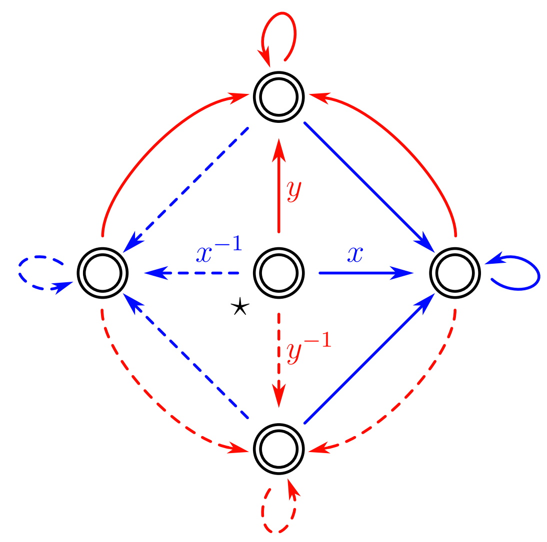
### Geometric characterization of automatic structure

Let $v_t$ be the prefix of $v$ of length $t$. Two words have the $k$-*fellow traveler property* if $d(w_t, v_t)$ is bounded by a constant $k$ independent of $w, v, t$, for all $t$.

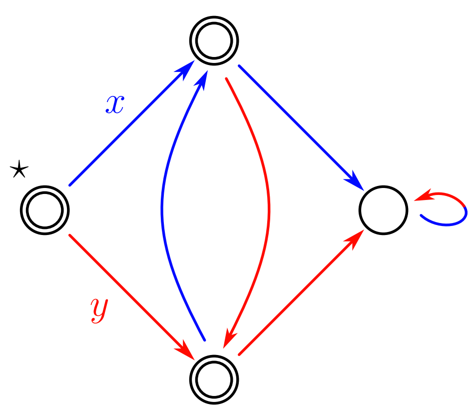
**Theorem.** *$G$ is automatic iff the finite state automaton $\mathcal{W}$ exists, and $G$ has the $k$-fellow traveler property for all $w, v$ with $d(w,v) \leqslant 1$.*




FSA for $K_4$




FSA for $F_2 = \langle x, y\rangle$




FSA for $\mathbb{Z}_2 * \mathbb{Z}_2$

## Why are automatic groups useful?

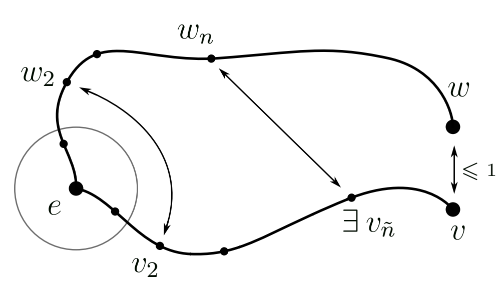
**Quadratic algorithm.** Let $(S, \mathcal{W})$ be an automatic structure for $G$. Then, given any word $w$, a string $\tilde{w}$ accepted by $\mathcal{W}$ can be found in quadratic time ($\propto |w|^2$) such that $w = \tilde{w}$ in $G$.

**Quadratic isoperimetric inequality.** An automatic group $G$ has to be finitely presented, and it satisfies quadratic isoperimetric inequality. That is, any string $w$ representing $e_G$ can be written as $w = \prod_{i=1}^{n} v_i r_i v_i^{-1}$ such that $n = \mathcal{O}(|w|^2)$.

**Invariance under generators.** If a group $G$ is automatic with respect to one generating set, then it is automatic with respect to any other finite generating set.

## Trivia on automatic groups

- Many common groups are automatic: finite groups, hyperbolic groups, Coxeter groups, mapping class groups, etc.
- While many are not: Baumslag–Solitar groups, non-Euclidean nilpotent groups, the Heisenberg group, etc.
- A non-abelian torsion-free nilpotent group is not automatic.
- The group $BS(1,2)$ is asynchronously automatic, but not synchronously.




Synchronous and asynchronous fellow travelers

## Some generalizations of the automatic structure

**Asynchronous fellow travelers.** If $u$ and $w$ differ by a generator, then there exists a constant $k$, independent of $u, w$, and their prefixes, such that any vertex on $w$ is within $k$ distance of some vertex on $u$, and vice versa.

**Synchronous and asynchronous fellows are equivalent.** If *geodesics* $u, w$ in $\Gamma(G,S)$ are asynchronous $k$-fellows, then they are synchronous $2k$-fellows.

**Automatic groupoids.** A groupoid $\tilde{G}$ is automatic iff the associated group $\tilde{G}_x$ is automatic.