Chapter 2

Universal Constructions and the Algebra of Types

Universal Constructions

Def: **Terminal object**: Let $\mathcal{C}$ be a category. An object 1 in $\mathcal{C}$ is called as a terminal object if for every object c there is a unique morphism $!_c : c \to 1$.

(*) It's universal in the sense that we're talking about ALL objects.

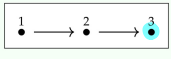
(*) The maps of the form "!" are also referred to as "bang"

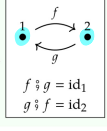
Proposition Any two terminal objects in the same category are isomorphic.

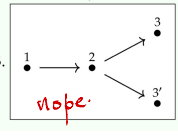
proof: Given two terminal objects, there exist unique maps for when going from one to the other. Further, the composition of these two must give you $\mathbb{1}$, $\exists!$ map $1 \circlearrowleft$, which has to be $\mathbb{1}$. □

Proposition $\mathbb{1} : 1 \circlearrowleft$ is the only endomorphism on 1.

Exercise 2.3. Which of the following categories has a terminal object?

1. 

2. 

3. 

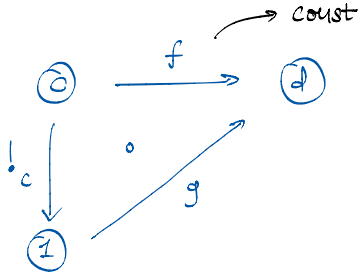
◊

eg: A singleton set is a terminal object in the category of sets.

Def: **Constant morphism**: We say that a morphism $f: c \to d$ is constant if there exists a morphism $g: 1 \to d$ such that $f = g \circ !_c$.

i.e., a morphism is constant if it factors through the terminal object.

(*) A morphism is constant if it sends every element and "1" to the same element.



(*) In a poset, a terminal object is the greatest element.

↳ In a poset we interpret morphisms from a to b as $a \leq b$.

Thus, $a \leq 1$, $\forall\, a$.

Def: **Global element**: Let 1 be a terminal object in $\mathcal{C}$. A global element of an object c is a morphism $1 \to c$.

Proposition TFAE: — check once.

1. The object c is terminal.
2. The object c has a unique global element.
3. $\exists$ object d for which there is a unique generalised element of shape d in c.
4. Property 3 happens for all objects d.

(*) In the category of categories, the category **1** is the terminal category.

Def: Initial object: An object 0 in $\mathcal{C}$ is called as an initial object if for every object c, there is a unique morphism

$$!_c : 0 \longrightarrow c .$$

eg:- In a poset, an initial object is a least element.

eg:- In **Set**, $\emptyset$ is an initial object.

(*) An initial object in $\mathcal{C}$ is a terminal object in $\mathcal{C}^{op}$.

eg:- The trivial category **0** with no objects or morphisms is an initial object in **Cat**.

(*) The initial object is defined by the mapping out property.

↳ As a shape, an initial object is a shape with no shape.

Def: Zero object: When its both initial and terminal.

(*) If $\exists$ a morphism from the terminal obj to the initial object, then the two objects are isomorphic.

[i.e., if the initial object has a global element]

Start with 2.2.3.