Programming With Categories

Chapter 1

Functions, Categories, and Types

Categories and functions

The basics:

(*) Total : Every element in the domain gets at least one image.

Deterministic : Every element in the domain gets at most one image.

(*) A function has to be both.

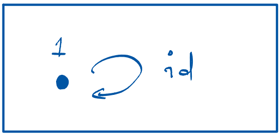
Def : Category : A category $\mathcal{C}$ consists of four things :

1. A set $\mathrm{Obj}\,\mathcal{C}$, elements of which are called as objects of $\mathcal{C}$.
2. For every ordered pair $A, B \in \mathrm{Obj}\,\mathcal{C}$, a set of morphisms, $\mathrm{Hom}(A, B)$, called as the homset.
3. For each $A \in \mathrm{Obj}\,\mathcal{C}$, a specified morphism $\mathbb{1}_A \in \mathrm{Hom}(A, A)$, called as the identity morphism.
4. For any three $A, B, C \in \mathrm{Obj}\,\mathcal{C}$ and morphisms $f : A \to B$ and $g : B \to C$, a chosen morphism $(g \circ f) : A \to C$ obtained by the composition.

Such that :

1. Left unital : for any $f : A \to B$, $\mathbb{1}_A \circ f = f$.
2. Right unital : for any $f : A \to B$, $f \circ \mathbb{1}_B = f$.
3. Associative : All compositions should be associative.

Examples of categories



① $\mathbf{1}$: One object, one morphism.

Also, $id \circ id = id$.

② **Discrete categories**:

In general, for every set S, we define $\mathbf{Disc}(S)$, called as the discrete category on S.

$\mathbf{Disc}(2)$

$$\begin{tikzcd} 1 \arrow[loop above, "id_1"] & 2 \arrow[loop above, "id_2"] \end{tikzcd}$$

$$Obj(\mathbf{Disc}(S)) = S$$

$$Hom(S)(A,B) = \begin{cases} \{id_A\} & \text{if } A = B \\ \emptyset & \text{else} \end{cases}$$

③ Empty category : $\mathbf{0} := \mathbf{Disc}(\emptyset)$.

④ Walking arrow category $= \mathbf{2} =$

$$\begin{tikzcd} 1 \arrow[r, "f"] \arrow[loop below, "id_1"] & 2 \arrow[loop below, "id_2"] \end{tikzcd}$$

Composition and associative laws are defined in the obvious way.

Note: We omit the obvious arrows.

Identity and composition laws are omitted.

⑤ **Ordinal categories**: (Generalising the walking arrow category)

$$\begin{tikzcd} \end{tikzcd}$$

$\mathbf{0}$

$$\begin{tikzcd} 1 \end{tikzcd}$$

$\mathbf{1}$

$$\begin{tikzcd} 1 \arrow[r] & 2 \end{tikzcd}$$

$\mathbf{2}$

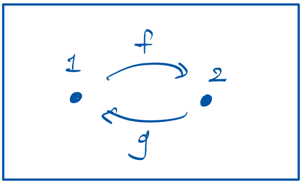
$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \end{tikzcd}$$

$\mathbf{3}$

⑥ The walking isomorphism.

Note that $f = g^{-1}$.

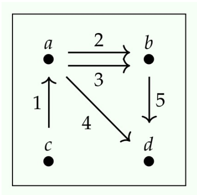
$\mathbb{I}$



Conversely, if a category contains only two objects and only two non-identity morphisms, then it has to be precisely the above category.

Free Categories

Def: Free categories: For any directed multigraph G, called the free category on G, denoted as $\mathrm{Free}(G)$.



$\mathrm{Obj}(\mathrm{Free}\ G) = \mathrm{Vertices}(G)$

$\mathrm{Morphisms} = \mathrm{Path\ in\ } G$

Monoids

Def: Monoid: A monoid $(M, e, \circ)$ consists of:

① A set M, the "carrier" set

② An element $e \in M$, the unit/identity.

③ A function $\circ : M \times M \to M$, the operation.

Such that the following holds:

① Unitality: $e \circ m = m \circ e = m$

② Associativity.

Equivalently, a monoid is a semigroup with 1.

Equivalently, a monoid is a category with only one object.

Preorders

A preorder is a category such that for every two objects (a, b), there is at most one morphism $a \to b$.

Def: Preorder: A preorder $(P, \leq)$ consists of:

1. a set P
2. a subset "$\leq$" of $P \times P$, called as an order.

Given $(p, q) \in P \times P$, we write "$p \leq q$" if $(p, q) \in \leq$.

Such that the following two hold.

1. Reflexivity.
2. Transitivity.

Examples of preorders:

1. $\mathbb{N}, \mathbb{Z}, \mathbb{R}$.
2. There is a preorder $\mathcal{P}$, whose objects are $Obj(\mathcal{P}) = \mathbb{N}_{\geq 1}$. and where

$$\mathrm{Hom}(a, b) := \{ x \in \mathbb{N} \mid x * a = b \}$$

↳ This is a preorder because either $a \mid b$ or not, so $\mathrm{Hom}(a, b)$ is either singleton or empty.

Building new categories from old.

(*) **Opposite category:** For any category $\mathcal{C}$, there is a category $\mathcal{C}^{op}$ defined by turning all the arrows around.

i.e., $Obj(\mathcal{C}) := Obj(\mathcal{C}^{op})$ and $\mathrm{Hom}_{\mathcal{C}^{op}}(a, b) := \mathrm{Hom}_{\mathcal{C}}(b, a)$.

(*) Product categories: Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

$$\text{obj}(\mathcal{C} \times \mathcal{D}) := \text{obj}(\mathcal{C}) \times \text{obj}(\mathcal{D})$$

$$\text{Hom}_{\mathcal{C}\times\mathcal{D}}((c,d),(c',d')) := \text{Hom}_{\mathcal{C}}(c,c') \times \text{Hom}_{\mathcal{D}}(d,d').$$

(*) Full Subcategories: Let $\mathcal{C}$ be a category. Let $D \subseteq \text{obj}(\mathcal{C})$.

The "full subcategories of $\mathcal{C}$ spanned by D"

$$\text{obj}(\mathcal{C}_{\text{obj}=D}) := D \quad \text{and} \quad \text{Hom}_{\mathcal{C}_{\text{obj}=D}}(d_1, d_2) = \mathcal{C}(d_1, d_2)$$

↳ You want only some objects but all morphisms between them.

(*) The category of finite sets is a full subcategory of the category Set, spanned by the finite sets.

The Yoneda Perspective

"How does x look from the perspective of y?"

(*) We are interested in the notion of "generalised elements"

↳ Let $a \in \text{obj}\,\mathcal{C}$. For any $c \in \text{obj}\,\mathcal{C}$, we are interested in "how does "a" look like from the point of view of "c"?"

The answer, obviously lies in the homset $\text{Hom}_{\mathcal{C}}(c,a)$.

(*) Note that elements in a set X are in bijection with the functions $1 \to X$.

We essentially think of elements as the pointer functions.

i.e., $\forall\, x \in X,\ \exists!$ function $f_x : 1 \to X$ by $1 \longmapsto x$.