On Convergence of Riemannian Manifolds

Departmental Seminar by Dr Atreyee Bhattacharya, 24 Oct 2024

⊛ "Sequence of metric spaces converging to a metric space" ⊛

$(M_i, d_i) \longrightarrow (M, d)$ — How do we interprete this?

⊛ Outline of the talk

- Hausdorff distance
- Gromov-Hausdorff distance
- Convergence of Riemann manifolds.

⊛ Hausdorff distance:

↳ dist betⁿ two subspaces of metric space.

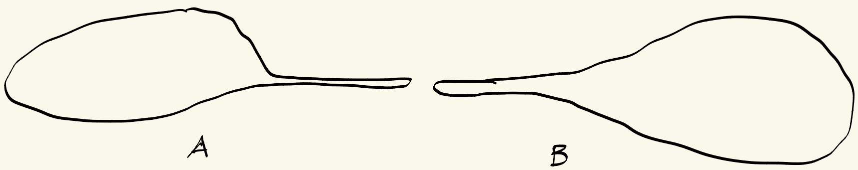
Let (X, d) be a metric space w/ bounded diameter

Let $A, B \subseteq X$ be closed

<u>Question</u>: How close are A and B?

↳ <u>one way</u>: $d(A, B) = \inf\limits_{\substack{a \in A \\ b \in B}} d(a, b)$

However, this is misleading, as something like this can happen:



i.e., some pts of A and B are close but most are far apart. $\Rightarrow$ the two sets are <u>NOT</u> close.

<u>Def</u>: $d(x, A) := \inf_{a \in A} d(x, a)$. — distance between a point and a set

$B(A, \varepsilon) := \{ x \in X \mid d(x, A) < \varepsilon \}$ — All points ε-away from the set.

<u>Def</u>: Hausdorff distance: Let $Y, Z \subseteq X$

$$d_H(Y, Z) := \inf_{\varepsilon} \{ \varepsilon \mid Y \subseteq B(Z, \varepsilon) \text{ and } Z \subseteq B(Y, \varepsilon) \}$$

✻ $\mathcal{C}(X) := \{ \text{all non-empty closed subsets of } X \}$

<u>Then</u> $(\mathcal{C}(X), d_H)$ forms a metric space.

<u>Theorem</u>: $(\mathcal{C}(X), d_H)$ forms a metric space. Further, it's (locally) compact <u>iff</u> (X, d) is (locally) compact.

✻ We investigate $X = \mathbb{R}$.

Consider these subsets: (Inductive definition)

$$A_0 := [0, \infty)$$

$$A_i := A_{i-1} \setminus \left(i, i + \tfrac{r}{2}\right) \qquad \text{for } r < 1/2.$$

<u>Then</u>, $d_H(A_i, A_j) = \frac{r}{2}$. for any i, j.

✻ The notion of Hausdorff dist makes sense even without the "closed" condition as above.

Closed subsets are required for realising $(\mathcal{C}(X), d_H)$ as a metric space.

⊛ Gromov-Hausdorff distance:

If you have two compact spaces, embed them both in a bigger space isometrically. Then use d_H on the image sets.

Def: Gromov-Hausdorff distance: Given compact spaces A, B:

$$d_{GH}(A,B) := \inf_{\varphi, \psi, X} \left\{ d_H(\varphi(A), \psi(B)) \;\middle|\; \begin{array}{l} \varphi : A \hookrightarrow X \\ \psi : B \hookrightarrow X \end{array} \text{ isometries} \right\}$$

⊛ If A and B are isometric, we see that $d_{GH}(A,B) = 0$.

⊛ Another interpretation of d_{GH}: Let $X = A \sqcup B$.

↳ In this case, you need to define the dist^n betw pts of A and pts of B — internally, you can use their own metrics, say d_A and d_B.

$$d_{GH}(A,B) = \inf_{(X,d)} \left\{ d_H(A,B) \;\middle|\; X = A \sqcup B \text{ with an admissible metric on } X \right\}$$

Exercise: Prove that the above two interpretations / defs are the same.

eg: Let A, B as above, st $\operatorname{diam}(A), \operatorname{diam}(B) \leq D$.

$X = A \sqcup B$. Define $d(a,b) = \frac{D}{2}$. — this creates a valid metric on $A \sqcup B$.

$$\Rightarrow d_{GH}(A,B) \leq \frac{D}{2}. \qquad \square$$

eg. $B = \{b\}$. (everything else as before)

$$\Rightarrow d_{GH}(A,B) = \frac{\operatorname{diam}(A)}{2}. \qquad \square$$

Theorem: (Gromov) Let $M = \{$isometry classes of compact metric spaces$\}$. Then, (M, d_{GH}) is a complete metric space.

↳ M = all equivalence classes of compact metric spaces under isometry.

(*) Alternate interpret^n of GH distance based on the above thm:

Y, Z — compact metric spaces

$f : Y \to Z$ is an ε – Gromov – Hausdorff Approximation (GHA) if:

(i) $B(f(Y), \varepsilon) = Z$

(ii) $\left| d(f(x_1), f(x_2)) - d(x_1, x_2) \right| < \varepsilon, \quad \forall x_1, x_2 \in Y$.

(*) $\hat{d}_{GH}(A,B) := \inf_{\varepsilon} \{ \exists\ \varepsilon\text{-GHA's } f : A \to B \text{ and } g : B \to A \}$

↳ However, this is not the genuine GH metric

↳ $\hat{d}_{GH}(A,B) = \operatorname{diam}(A)$ — this contradicts w/ the above result.
$\neq \operatorname{diam}(A)/2$.

However, this is equivalent to the genuine GH-distance.

Exercise: $\frac{2}{3} d_{GH} \le \hat{d}_{GH} \le 2 d_{GH}$.

relatively compact = pre-compact = the closure is compact

Lemma : (Gromov) $\mathcal{C} \subseteq M$ is pre-compact if for all $X \in \mathcal{C}$, $\text{diam}(X) \leq d$, and for every $\varepsilon > 0$, $\exists$ finite ε-net of size $\leq \ell(\varepsilon)$.

↳ ε-net : A subset "A" of a metric space M is called as an ε-net if :

(a) ε-dense : the nbhd of A is the entire space M.

(b) ε-separate : $\forall\, x, y \in A$, $d(x,y) > \varepsilon$.

(*) RANDOM FACT : For any $\varepsilon > 0$, there is a finite ε-net.

↳ What is $\ell(\varepsilon)$?

Some function of ε. All we mean is that the size of the ε-net is bounded by some function of ε, which is independent of the choice of $X \in \mathcal{C}$.

Thus, $\forall\, \varepsilon > 0$, $\ell(\varepsilon)$ denotes "the universal constant", and it's the same for all choices of $X \in \mathcal{C}$.

(*) Gromov Compactness Theorem : I did not understand :)

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