Dynamical Invariants under Strong Orbit Equivalence

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Def : Topological dynamical systems : pair (X,T), where X is
(TDS) a cpt metric space, $T: X \circlearrowleft$ homeo.

$T^n :=$ n-fold composition of T.

For $x \in X$, $Orb_T(x) = \{T^n x \mid n \in \mathbb{Z}\}$

Def : Minimal TDS : (X,T) is minimal if all orbits are dense.

Assumption : X is a Cantor space.

$\downarrow$

totally disconnected cpt metric space with no isolated points.

eg : Middle-third Cantor set is a Cantor set.

Cantor : all Cantor spaces are homeo

(*) Examples of minimal Cantor systems

(1) Symbolic dynamics

A : finite set = "alphabet"

$A^{\mathbb{Z}}$ – biinfinite sequence with symbols from A.

Def : Shift map — $S: A^{\mathbb{Z}} \circlearrowleft$ by

$S(\dots x_{-1} x_0 x_1 \dots) = \dots x_{-1} x_0 x_1 \dots$

Shift everything to the left by one place.

$d(x,y) := 2^{-k}$ where $k = \min\{|n| \mid x_n \neq y_n\}$

② Subshift : $X \subseteq A^{\mathbb{Z}}$ is a subshift if it's closed

s-invariant subset — $s(X) \subseteq X$.

eg: Substitution subshift:

A = finite alphabet.

A^* := all finite words.

Let τ be some morphism $\tau : A^* \circlearrowleft$.

eg. Fibonacci morphism

$$A = \{a, b\}$$

$$\tau(a) = ab$$

$$\tau(b) = a$$

$$\tau(u_1 \dots u_k) = \tau(u_1) \dots \tau(u_k).$$

We define the subshift wrt morphism τ as:

$$X_\tau := \{ x \in A^{\mathbb{Z}} \mid \text{all finite words appearing in } x \text{ is a subset of } \tau^n(a) \text{ for some } a \in A \}$$

For Fibo subshift:

$$a \xmapsto{\tau} ab \xrightarrow{\tau} aba \xrightarrow{\tau} abaab \xrightarrow{\tau} \dots$$

We say τ is primitive if $\exists\, n > 0$ st $\forall\, a, b$, b appears in $\tau^n(a)$

$\Rightarrow$ when τ is primitive, X_τ is minimal.

② Dyadic odometers : $T : \{0,1\}^{\mathbb{N}} \circlearrowleft$

$T(x) := x + (1000 \dots)$ with allowing carrying over

eg: $1100 \dots + 1000 \dots = 00100 \dots$

Note: $T(111\cdots)$ is defined to be $000\cdots$

(*) **Equivalences on TDS:**

(*) Topological conjugacy:

① (X_1, T_1) and (X_2, T_2) are top conjugates if $\exists\ \phi : X_1 \to X_2$ homeo st

$$\begin{tikzcd} X_1 \arrow[r, "T_1"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\circ"] & X_1 \arrow[d, "\phi"] \\ X_2 \arrow[r, "T_2"'] & X_2 \end{tikzcd}$$

(*) Topological entropy:

(X, S) be a subshift. Then for a complexity function $P_X : \mathbb{N} \circlearrowleft$, $P_X(n) = \#$ words of length n that appear in some sequence in X

then topological entropy is

$$h(X) := \lim_{n\to\infty} \frac{1}{n} \ln(P_X(n))$$

$$X = A^{\mathbb{Z}} \;—\; P_X(n) = |A|^n, \quad h(X) = \ln(|A|).$$

$\rightarrow$ Collectn of Borel probability invariant measures

(*) Orbit equivalences

Note: When two systems are conjugates, say (X_1, T_1) and (X_2, T_2), then

$$\phi(\mathrm{Orb}_{T_1}(x)) = \mathrm{Orb}_{T_2}(\phi(x)) \quad \text{---} \quad (\phi)$$

We say that they are orbit equi if $\exists\ \phi : X_1 \to X_2$ st (ϕ) holds

This defines two maps

$\alpha, \beta : X_1 \to \mathbb{Z}$ st

$$\phi(T_1^{\alpha(x)}(x)) = T_2\, \phi(x) \quad \&$$

$$\phi(T_1(x)) = T_2^{\beta(x)}\, \phi(x).$$

(*) If α, β are cts, then either T_1 is conj to T_2 or T_1 is conj to T_2^{-1}.

(*) Strong orbit equivalence: (SOE)
(X_1, T_1) and (X_2, T_2) are SOE if α, β are discts at at-most one point.

Theorem: Two MDS systems are SOE iff they have the same dimension group.

$\boxed{\text{Theorem}}$

(*) Borel probability invariant measures are preserved under SOE.

Theorem:

(*) Within any SOE class, and $\alpha \geq 0$ given, $\exists$ subshift within this class having entropy α.

(*) Asymptotic group and automorphism groups:

(*) x & $y \in X$ are (left-) asymptotic if $\lim_{n \to \infty} d(T^{-n}x, T^{-n}y) = 0$

(*) When X is a subshift, $x, y \in X$ are asymptotic if $\exists\, k \in \mathbb{Z}$ st $x_{[-\infty,k]} = y_{[-\infty,k]}$.

↳ the left tails agree completely.

(*) Two orbits are asymptotic if you can find two elets from the orbits which are asymptotic.

$$\mathrm{Orb}_T(x) \sim \mathrm{Orb}_T(y) \text{ if } \exists\, \tilde{x} \in \mathrm{Orb}_T(x),\ \tilde{y} \in \mathrm{Orb}_T(y) \text{ st } \tilde{x} \sim \tilde{y}.$$

→ When the orbit asym class is not reduced to a single point, we call it to be an asym group.

→ Automorphism group:

$\phi : X \circlearrowleft$ homeo is an auto of (X, T) if

$$\phi \circ T = T \circ \phi.$$

⊛ If $\mathrm{Orb}_T(x) \sim \mathrm{Orb}_T(y)$ and $\mathrm{Orb}_T(\phi(x)) \sim \mathrm{Orb}_T(\phi(y))$

$$f : \mathrm{Aut}(X, T) \longrightarrow \mathrm{Symm}(\underline{AC})$$

?

↳ asym component

If $|AC| = 1$, then $\mathrm{Aut}(X, T) = \langle T \rangle$.

⊛ (A_n) ⇝ sequence of finite alphabets.

$$\tau_n : A_{n+1}^* \longrightarrow A_n^*$$

$\tau = (\tau_n)$.

$X_\tau = \{ x \in A_0^* \mid$ each word appearing in x is a subword of $\tau_0 \circ \cdots \circ \tau_n(a)$ for some $n \geq 0$, $a \in A_{n+1} \}$

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