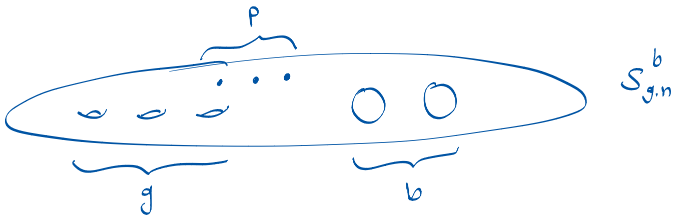
Liftable Mapping Class Groups - Lecture 1

Geometry-Topology seminar by Prof. Kashyap, 24 Jan 2025

GRAND GOAL: BIRMAN-HILDEN THEORY.

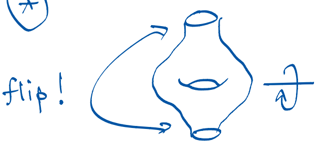
$S = S_{g,n}^{b}$. — Closed orientable / oriented surface of genus g with n punctures and b boundary components.



(*) Topologically, punctures can be seen as marked points.

→ $Homeo^{+}(S, \partial S)$ = the group of orientⁿ preserving homeos of S, restricting to identity on ∂S and preserving the set of punctures.

(*)



This is a non-example.
The boundary comps get switched.

→ MCG of S = $\pi_0(Homeo^{+}(S, \partial S))$ —— # path components.

(*) What's the topology on S?

⟹ Compact-open topology. →

Cpt open top — defined on a set of cts maps.

Let $C(X,Y)$ = all cts fⁿ $X \to Y$..
Let $K \subseteq X$ cpt, $U \subseteq Y$ open, and
$V(K,U) :=$ all $f \in C(X,Y)$ st $f(K) \subseteq U$ → cptness is preserved by cts maps.
Then, ALL such $V(K,U)$ form a subbasis for the cpt open top.

Does not always form a basis of top on $C(X,Y)$.

(*) π_0 does not always form a group.
Here, it does.

(*) MCG is a topological group.

(*) Elements of MCG are ori-preserving maps
(Points)

$\Rightarrow$ A path in MCG is a homotopy.

[path betⁿ maps].

Isotopy — When you take a path from a point f to point g inside MCG via homeomorphisms.

(*) Further,

$\mathrm{Homeo}_0^+(S, \partial S)$ = path component of the identity.

↳ it's a normal subgroup in a topological group.

$\rightarrow$ $MCG(S) \cong \mathrm{Homeo}^+(S, \partial S) / \mathrm{Homeo}_0^+(S, \partial S)$

$\rightarrow$ $\mathrm{Mod}(S) :=$

IMPORTANCE OF $\mathrm{Mod}(S_g)$

In the case $S = S_g$ $(n = b = 0)$:

(a) Describes the orbifold fundamental group of
$$M_g = \mathrm{Teich}(S_g) / \mathrm{Mod}(S_g)$$

(b) Classifies surface bundles with fibers S_g.

(c) $\mathrm{Mod}(S_g) \cong \mathrm{Out}(\pi_1(S_g))$ — Dehn-Nielsen-Baer theorem

(d) It generalises the classical <u>modular group</u>

as
$$\mathrm{Mod}(T^2) \cong SL(2,\mathbb{Z})$$

↳ From here we get the Mod notation for MCG

(e) In the case $b=1$, $g=0$:

$$\mathrm{Mod}(S_{0,n}) \cong B_n$$ — Braid group on n strands.

BRAID GROUP For a topological space X, define the configuration space as:

$$\mathrm{Conf}_n(X) = X^n \setminus \{\text{all those points st at least two coordinates are equal}\}$$

Now, S_n can act on this set. — $\Sigma_n := S_n$.

$$\mathrm{UConf}_n(X) = \mathrm{Conf}_n(X)/\Sigma_n$$ — n-fold conf. space of X.

Further, if X is connected, then

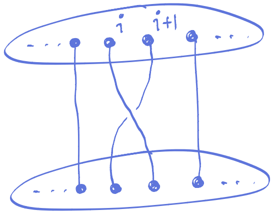
$$B_n(X) := \pi_1(\mathrm{UConf}_n(X))$$ — Braid group

$$P_n(X) := \pi_1(\mathrm{Conf}_n(X))$$??

<u>Braid relation</u>:

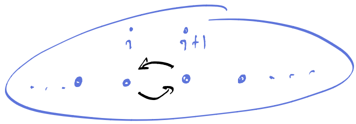
$$B_n = \left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; [\sigma_i, \sigma_j] = 1 \text{ if } |i-j|>1, \; \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \; 1 \le i \le n-2 \right\}$$

(*) An easier way of understanding B_n:



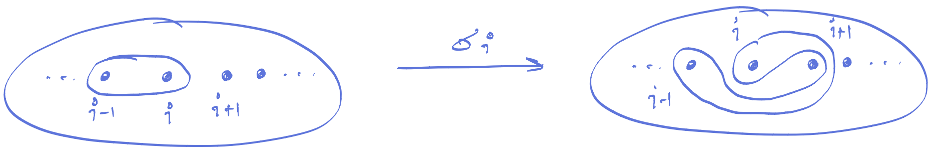
σ_i "braids" i^{th} and j^{th} marked points.

Represented this action, or the "braiding" in a single picture:



It switches only these two marked points.

How would that look for a curve nearby?

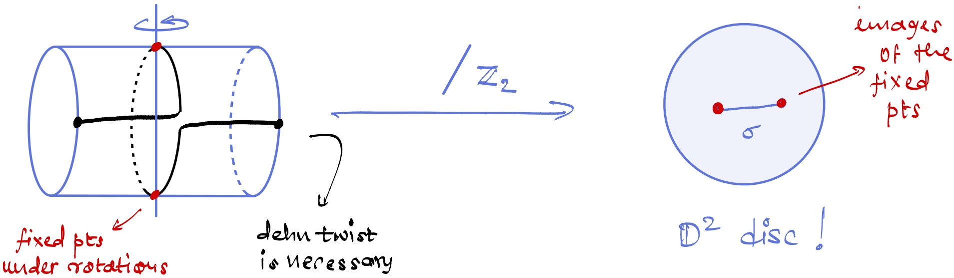


Question: Consider $S^1 \times I$ with a Dehn twist as required!

$$S^1 \times I \longrightarrow S^1 \times I \quad \text{by}$$

$$(x, t) \longmapsto (x e^{2\pi i t}, t)$$

$\mathbb{Z}_2$ action $\longleftarrow \pi$ rotation.

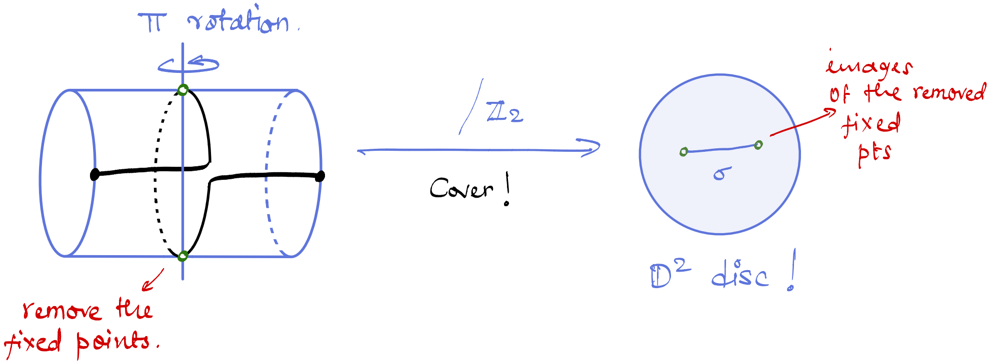


→ Branch covering: A map $p: \tilde{S} \to S$ is a branch covering if $\exists$ a finite set $\underbrace{B}_{\text{branch pts}} \subseteq S$ such that

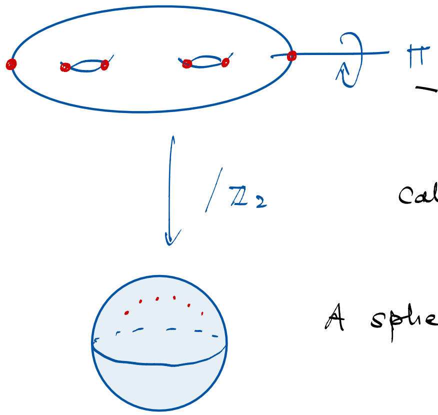
$$p\big|_{\tilde{S} \setminus p^{-1}(B)} : \tilde{S} \setminus p^{-1}(B) \longrightarrow S \setminus B$$

is a covering.

(*) In the example above, you remove the problematic fixed points, you actually get a cover!



Another example:



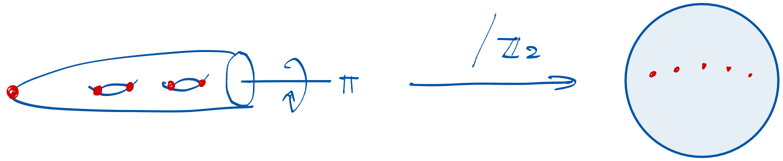
6 fixed points.

— Action of $\mathbb{Z}_2$

Called as the hyperelliptic cover.

A sphere with 6 marked points.

Example:



$S^1_{0,5}$

As you will get S^2 as in the prev. example, but with a disc removed — which is again a disc!

Central question in Birman-Hilden theory:

When does a MCG lift to a MCG in a cover?

→ Fiber preserving: Given a covering $p : \tilde{S} \to S$, we say a homeo $f : \tilde{S} \to \tilde{S}$ is fiber preserving if for each $x \in S$, $\exists\, y \in S$ such that

$$f(p^{-1}(x)) = p^{-1}(y)$$

→ Natural question: Given any pairs f_1, f_2 of fiber preserving homeo under p, are they homotopic through fiber preserving homeos?

The answer, in general, is <u>No</u>!

Covers in which each pairs of homeos satisfy this property are called as Birman-Hilden covers.

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Liftable Mapping Class Groups - Lecture 2

Geometry-Topology seminar by Prof. Kashyap, 31 Jan 2025

KNOWN RESULTS IN BIRMAN-HILDEN THEORY.

Recap: Given any pair f_1, f_2 of fiber-preserving homeos on $\tilde{S}$ (under $p: \tilde{S} \to S$ cover) (if homotopic) are they homotopic through fiber-preserving homeos?

$$\begin{tikzcd} \tilde{S} \arrow[r, "f_1, f_2"] \arrow[d] & \tilde{S} \arrow[d] \\ S \arrow[r, dashed, "f"] & S \end{tikzcd}$$

Covers satisfying above $\rightsquigarrow$ "Birman-Hilden covers", and they satisfy the Birman-Hilden prop.

Theorem (1972-73): (Birman-Hilden) Let $p: \tilde{S} \to S$ be a finite-sheeted regular branch covering, where $\tilde{S}$ is hyperbolic. If p is either unbranched or if $\mathrm{Deck}(p)$ is solvable (as a group), then p has the Birman-Hilden property.

→ What does the above theorem omit?

∞-sheeted covers.

→ What about the universal covers? Question!

(1973)

Theorem: (Maclachlan - Harvey) If $p: \tilde{S} \to S$ is a finite-sheeted regular branched covering map, where $\tilde{S}$ is hyperbolic, then p has B-H property.

→ Unramified: A preimage of a branch point (of a cover $p: \tilde{S} \to S$) is unramified if some nbhd is mapped injectively under p.

A p is said to be fully ramified if no branch pt has a ramified preimage.

→ Ramified $\overset{?}{=}$ fully ramified?

Theorem: (Winarski, 2015) Let $p: \tilde{S} \to S$ be a finite sheeted branch covering, where $\tilde{S}$ is hyperbolic, and p is fully ramified.

Then, p has the B-H property.

→ Remark: Regular and unbranched covers are all fully ramified.

MCG INTERPRETATION OF ALL THIS:

→ Let $p : \tilde{S} \to S$ be a branched cover. Treating each branch point as a marked pt on S and assuming all homeos on S preserve the set of branched points, we have the foll:

→ $\mathrm{LMod}_p(S)$ = Liftable MCG = subgroup of $\mathrm{Mod}(S)$ comprising mapping classes rep. by homeos that lift under p.

$\mathrm{SMod}_p(\tilde{S})$ be the subgroup of $\mathrm{Mod}(\tilde{S})$ rep. by homeos that are fiber-pres. under p

↳ Symmetric MCG.

<u>Prop^n</u>: Let $p : \tilde{S} \to S$ be a finite-sheeted branched cover, where $\tilde{S}$ is hyperbolic, without boundary.

<u>TFAE</u>: (1) p has the B-H property.

(2) The natural surjection $\mathrm{SMod}_p(\tilde{S}) \twoheadrightarrow \mathrm{LMod}_p(S)$ is well-defined.

(3) $$\mathrm{SMod}_p(\tilde{S}) / \mathrm{Deck}(p) \cong \mathrm{LMod}_p(S)$$

(4) $\exists$ a SES:

$$1 \to \mathrm{Deck}(p) \hookrightarrow \mathrm{SMod}_p(\tilde{S}) \twoheadrightarrow \mathrm{LMod}_p(S) \to 1.$$

Theorem (B-H) For a regular cyclic cover $p : \tilde{S} \to S$, we have :

$$SMod_p(\tilde{S}) = \text{Normalizer}_{Mod(\tilde{S})}(Deck(p))$$

$$[\text{Normalizer}_G(H)]$$

→ Also holds for non-cyclic covers.

due to this theorem.

(1963)

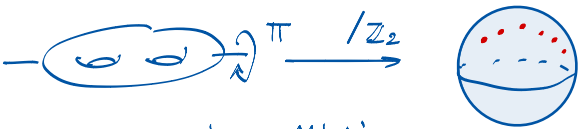
Theorem : (Nielsen-Kirchoff) Let S be a hyp surface without boundary. Every finite subgroup of $Mod(S)$ lifts under the canonical projection

$$Homeo^+(S) \longrightarrow Mod(S)$$

lifts to an isomorphic subgroup ?? of $Homeo^+(S)$.

RECENT DEVELOPMENTS

(Ghaswala - Winarski, 2017) Gave a presentation for $LMod_p(S_{0,n})$ under balanced hyperelliptic cover



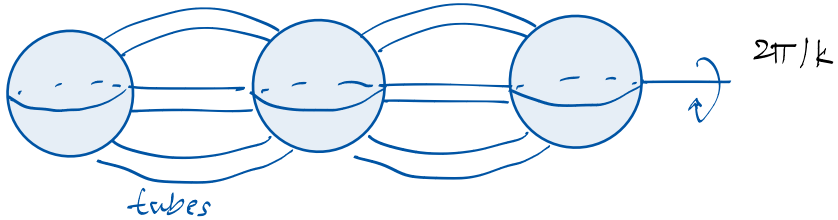
hyperelliptic cover.

<u>Theorem</u> : (Above guys) Proved $\exists$? of the follo :

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Mod}(S_2) \longrightarrow \mathrm{Mod}(S_{0,6}) \longrightarrow 1$$

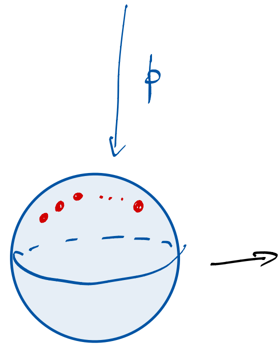
They also derived the presentation for $\mathrm{Mod}(S_2)$.

→ Take n-many spheres and attach k-many tubes betn each one of them.

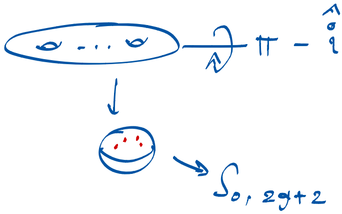


Characterized liftability of this cover.

i.e., presentation for $\mathrm{LMod}(S_{0,n})$.



→ $S_{0,k}$.



(*) Hyperelliptic involutions. $\hat{\iota}$:

$$\mathrm{LMod}_p(S_{0,2g+2}) = \mathrm{Mod}(S_{0,2g+2})$$

→ They also showed that not all mapping classes lift under branched covers of spheres.

⊛ OPEN PROBLEMS:

1. B-H prop for arbitrary covers.

2. When is LMod(S) equal to Mod(S)

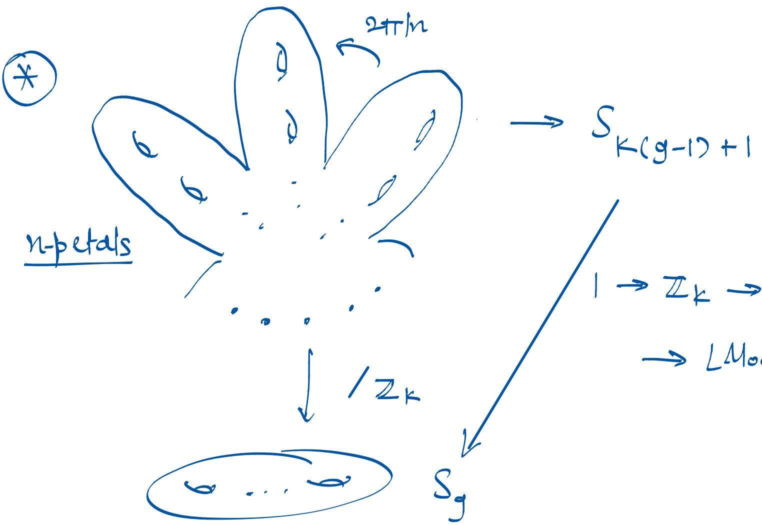
3. When is LMod(S) maximal ?

4. Maximal subgroups of $\mathrm{Mod}(S_g)$.

5. Presentation of LMod.

6. Normalizers and centralizers of finite subgroups of Mod(S).

⊛



$$1 \to \mathbb{Z}_k \to \mathrm{SMod}_{pk}(S_{k(g-1)+1}) \to \mathrm{LMod}_p(S_g) \to 1.$$

⊛ $/\mathbb{Z}_k \rightsquigarrow$ if k is prime $\Rightarrow \mathrm{LMod}_{pk}(S_g)$ is a maximal proper subgroup.

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