# Cohomological Dimension of Hyperbolic Groups

Geometry-Topology seminar by Harsh Patil, 17 Jan 2025

**Theorem:** (Bestvina – Mess) Let $G$ be a torsion-free, hyperbolic group. Let $\partial G$ be its Gromov boundary.

**Then:**

① $H^i(G, \mathbb{Z}G) \cong \check{H}^{i-1}(\partial G, \mathbb{Z})$

② $cd(G) = \dim(\partial G) + 1$

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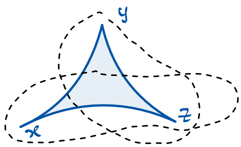



## ⊛ HYPERBOLIC GROUPS

⊛ Hyperbolic metric spaces:

A geodesic metric space $(X, d)$ is called as a $\delta$ – hyperbolic space if for any geodesic triangle $[x, y, z]$,

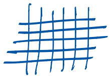
$$[x, y] \subseteq N_\delta([y, z] \cup [x, z]) \quad \text{for some } \delta.$$

(*) Let $G$ be a f.g. group. $S \subseteq G$. $s \in S$.

Cayley graph of $G \Rightarrow \Gamma(G, S)$

$\Gamma \Rightarrow V = G$

$E = \{(g, gs) \mid g \in G, s \in S\}$.

eg: $\mathbb{Z}^2 = \langle \{\pm(1,0), \pm(0,1)\} \rangle$ 

You can impose a metric on $\Gamma$. —— Shortest path

If that metric is hyperbolic, the group is called as a hyperbolic group.

Ex of hype groups:

1. Free groups.
2. Surface groups $\rightsquigarrow \pi_1$ of closed surfaces.
3. Small cancellation groups ??

(*) You can compactify a hyperbolic metric space in a nice way. —— This leads to Gromov boundary.

(*) Gromov bdry.

Let $X$ be a ________ ? Let $b \in X$.

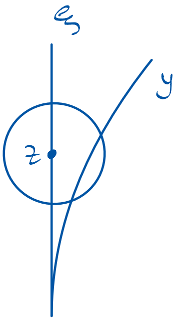
$$\partial_b X := \{ \gamma : [0,\infty) \,,\ \gamma \text{ is a geodesic},\ \gamma(0) = b \} \Big/ \sim$$

$P_1 \sim P_2$ if $\sup_t \{ d(P_1(t), P_2(t)) \} < \infty$.

(*) You can topologize $\partial_b X$ — Shadow topology !!

$\hookrightarrow$ $\xi \in \partial X$ $k > 3\delta$.

$$U(\xi, z) = \{ y \mid \forall\, xz,\ xy \cap B(z,k) \neq \emptyset \}$$




(*) Here, k is a fixed number. The topology entirely depends on k.

(*) Gromove bdry of the tetravalent tree is a Cantor set.

(*) Gromov bdry is independent of the initial pt — for two diff pts, the boundaries are homeo.

(*) Gromov bdry $(\mathbb{H}^n) \cong S^{n-1}$.

(*) For groups, $\partial G = \partial \Gamma(G, S)$.

(*) GROUP COHOMOLOGY:

$(G, M) \rightsquigarrow$ $G$ group

$M$ some $\mathbb{Z}G$ module → some ring

Then, $(G, M) \rightsquigarrow H^i(G, M)$

"PROJECTIVE RESOLUTIONS" are :

LES.

$$\cdots \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} \mathbb{Z} \longrightarrow 0$$

$P_i$'s $\rightsquigarrow$ projective $\mathbb{Z}G$ modules

$$\operatorname{Im} \partial_{i+1} = \ker \partial_i .$$

Given a projective resolutn. (or module?) you apply $\operatorname{Hom}_{\mathbb{Z}G}(\cdot, M)$

You get

$$\cdots \xleftarrow{\delta_2} \operatorname{Hom}(P_1, M) \xleftarrow{\delta_1} \operatorname{Hom}(P_0, M) \longleftarrow M \longleftarrow 0$$

↳ No longer an exact sequence.

Cohomology as:

Check the indices

$$\delta_i(f) = f \circ \partial_i , \qquad H^i(G, M) = \frac{\ker \partial_{i+1}}{\operatorname{Im} \partial_i}$$

(*) Looking at group action on CW complexes:

Suppose $G$ acts on contractible CW complex $X$ freely.
(takes 0 cells $\mapsto$ 0 cells, 1 cell $\mapsto$ 1 cell, etc)

Consider the chain complex

$$\cdots \longrightarrow C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} \mathbb{Z} \longrightarrow 0$$

(*) cd — cohomological dimension :

$$cd(G) = \sup\{ i \mid H^i(G, M) \neq 0 \text{ for some module} \}$$

eg: For $\mathbb{Z}^2 \curvearrowright \mathbb{R}^2$, the modules vanish after $i=2$ above.

$$\Rightarrow cd(\mathbb{Z}^2) \leq 2.$$

(*) Geometric dimension — $G$ be a group.

$$gd(G) := \inf\{ k \mid \exists \text{ a contractible CW complex } X \text{ st } G \curvearrowright X \text{ is free and } \dim X = k \}.$$

$\boxed{\text{Propn}}$ $cd(G) \leq gd(G)$

(*) Result: In most cases, the above inequality is actually an equality.

(*) Torsional group $\Rightarrow$ both $cd$ & $gd = \infty$.

(*) If $\nexists X$, a CW complex as above, $gd(G) = \infty$.

$\boxed{\text{Theorem}}$ (Eilenberg Ganea) Let $G$ be a group st $cd(G) < \infty$. If $cd(G) < 2$, then $cd(G) = gd(G)$

If $cd(G) \leq 2$, then $gd(G) \leq 3$.

Def: Type F: A group is said to be of type F if it acts on a finite-dim CW complex freely and cocompactly.

the quotient is cpt — in the quotient, there are only finitely many cells in each dim.

eg: $T^2$: 1 0-cell, 2 1-cells, 1 2-cell.

↳ " what's the resolution for $T^2$ " ?

[Vietoris Rips complex]

(*) Rips complex: — Specifically works for hyp grps

$G$ = torsion free hype grp.

$S \subseteq G$ — symmetric finite gen set.

Consider the metric on $\Gamma(G, S)$.

Consider the following simp complex:

$\lambda > 0$.

$\mathrm{Rips}_\lambda(\Gamma) = X \qquad X^{(0)} = G$

$X^{(1)} = \{ (g_1, g_2) \mid d(g_1, g_2) \le \lambda \}$.

Cayley metric?

and so on -

$\boxed{\text{Theorem}}$ (Rips) Let $\Gamma(G,S)$ be $\delta$-hyperbolic.

Let $\lambda > 4\delta + 2$. Then, $\mathrm{Rips}_\lambda(\Gamma)$ is contractible.

If $G$ is torsion free, the action of $G$ on $\mathrm{Rips}_\lambda(\Gamma)$.

is free. Also, $\mathrm{Rips}_\lambda(\Gamma)/G$ is cpt.

implies

$\hookrightarrow$ Any torsion-free hyperbolic group is of type-F.