

---

# List-decodeable Linear Regression

---

Anonymous Author(s)

Affiliation

Address

email

## Abstract

We give the first polynomial-time algorithm for robust regression in the list-decodable setting where an adversary can corrupt a greater than  $1/2$  fraction of examples. For any  $\alpha < 1$ , our algorithm takes as input a sample  $\{(x_i, y_i)\}_{i \leq n}$  of  $n$  linear equations where  $\alpha n$  of the equations satisfy  $y_i = \langle x_i, \ell^* \rangle + \zeta$  for some small noise  $\zeta$  and  $(1 - \alpha)n$  of the equations are *arbitrarily* chosen. It outputs a list  $L$  of size  $O(1/\alpha)$  - a fixed constant - that contains an  $\ell$  that is close to  $\ell^*$ .

Our algorithm succeeds whenever the inliers are chosen from a *certifiably* anti-concentrated distribution  $D$ . To complement our result, we prove that the anti-concentration assumption on the inliers is information-theoretically necessary. As a corollary of our algorithmic result, we obtain a  $(d/\alpha)^{O(1/\alpha^8)}$  time algorithm to find a  $O(1/\alpha)$  size list when the inlier distribution is standard Gaussian. For discrete product distributions that are anti-concentrated only in *regular* directions, we give an algorithm that achieves similar guarantee under the promise that  $\ell^*$  has all coordinates of same magnitude.

To solve the problem we introduce a new framework for list-decodable learning that strengthens the “identifiability to algorithms” paradigm based on the sum-of-squares method.

## 1 Introduction

In this work, we design algorithms for the problem of linear regression that are robust to training sets with an overwhelming ( $\gg 1/2$ ) fraction of adversarially chosen outliers.

Outlier-robust learning algorithms have been extensively studied (under the name *robust statistics*) in mathematical statistics [40, 34, 22, 20]. However, the algorithms resulting from this line of work usually run in time exponential in the dimension of the data [6]. An influential line of recent work [26, 1, 14, 30, 8, 27, 28, 21, 13, 15, 25] has focused on designing *efficient* algorithms for outlier-robust learning.

Our work extends this line of research. Our algorithms work in the *list-decodable learning* framework. In this model, the majority of the training data (a  $1 - \alpha$  fraction) can be adversarially corrupted leaving only an  $\alpha \ll 1/2$  fraction of *inliers*. Since uniquely recovering the underlying parameters is information-theoretically *impossible* in such a setting, the goal is to output a list (with an absolute constant size) of parameters, one of which matches the ground truth. This model was introduced in [3] to give a discriminative framework for clustering. More recently, beginning with [8], various works [16, 27] have considered this as a model of *untrusted* data.

There has been a phenomenal progress in developing techniques for outlier-robust learning with a *small* ( $\ll 1/2$ )-fraction of outliers (e.g. outlier *filters* [12, 13], separation oracles for inliers [12] or the *sum-of-squares* method [28, 21, 27, 25]). In contrast, progress on algorithms that tolerate the significantly harsher conditions in the list-decodable setting has been slower. The only prior

works [8, 16, 27] in this direction designed list-decodable algorithms for mean estimation via somewhat *ad hoc*, problem-specific methods.

In this paper, we develop a principled technique to give the first efficient list-decodable learning algorithm for the fundamental problem of *linear regression*. Our algorithm takes a corrupted set of linear equations with an  $\alpha \ll 1/2$  fraction of inliers and outputs a  $O(1/\alpha)$ -size list of linear functions, one of which is guaranteed to be close to the ground truth (i.e., the linear function that correctly labels the inliers). A key conceptual insight in this result is the observation that list-decodable regression information-theoretically requires the inlier-distribution to be *anti-concentrated*. Our algorithm succeeds whenever the distribution satisfies a stronger “algorithmically usable” *certifiable anti-concentration* condition. This class includes the standard gaussian distribution and more generally, any spherically symmetric distribution with strictly sub-exponential tails.

Prior to our work<sup>1</sup>, the state-of-the-art outlier-robust algorithms for linear regression [25, 17, 11, 36] could handle only a small ( $< 0.1$ )-fraction of outliers even under strong assumptions on the underlying distributions.

List-decodable regression generalizes the well-studied [10, 23, 19, 41, 2, 9, 42, 38, 31] and *easier* problem of *mixed linear regression*: given  $k$  “clusters” of examples that are labeled by one out of  $k$  distinct unknown linear functions, find the unknown set of linear functions. All known techniques for the problem rely on faithfully estimating *moment tensors* from samples and thus, cannot tolerate the overwhelming fraction of outliers in the list-decodable setting. On the other hand, since we can take any cluster as inliers and treat rest as outliers, our algorithm immediately yields new efficient algorithms for mixed linear regression. Unlike all prior works, our algorithms work without any pairwise separation or bounded condition-number assumptions on the  $k$  linear functions.

**List-Decodable Learning via the Sum-of-Squares Method** Our algorithm relies on a strengthening of the robust-estimation framework based on the sum-of-squares (SoS) method. This paradigm has been recently used for clustering mixture models [21, 27] and obtaining algorithms for moment estimation [28] and linear regression [25] relies on a strengthening of robust-estimation framework based on the sum-of-squares (SoS) method. This paradigm has been recently used for clustering mixture models [21, 27] and obtaining algorithms for moment estimation [28] and linear regression [25] that are resilient to a small ( $\ll 1/2$ ) fraction of outliers under the mildest known assumptions on the underlying distributions. This method reduces outlier-robust algorithm design to finding “simple” proofs of unique *identifiability* of the unknown parameter of the original distribution from a corrupted sample. However, this principled method works only in the setting with a small ( $\ll 1/2$ ) fraction of outliers. As a consequence, the work of [27] for mean estimation in the list-decodable setting relied on “supplementing” the SoS method with a somewhat *ad hoc*, problem-dependent technique.

As an important conceptual contribution, our work yields a framework for list-decodable learning that recovers some of the simplicity of the general blueprint. To do this, we give a general method for *rounding pseudo-distributions* in the setting with  $\gg 1/2$  fraction outliers. A key step in our rounding builds on the work of [29] who developed such a method to give a simpler proof of the list-decodable mean estimation result of [27]. In Section 2, we explain our ideas in detail.

The results in all the works above hold whenever the underlying distribution satisfies a certain *certified concentration* condition formulated within the SoS system via higher moment bounds. An important contribution of this work is formalizing an *anti-concentration* condition within the SoS system. Unlike the bounded moment condition, there is no canonical phrasing within SoS for such statements. We choose a form that allows proving “certified anti-concentration” for a distribution by showing the existence of a certain approximating polynomial. This allows showing certified anti-concentration of natural distributions via a completely modular approach that relies on a beautiful line of works that construct “weighted” polynomial approximators [32].

We believe that our framework for list-decodable estimation and our formulation of certified anti-concentration condition will likely have further applications in outlier-robust learning.

---

<sup>1</sup>There’s a long line of work on robust regression algorithms (see for e.g. [7, 24]) that can tolerate corruptions only in the *labels*. We are interested in algorithms robust against corruptions in both examples and labels.

## 1.1 Our Results

We first define our model for generating samples for list-decodable regression.

**Model 1.1** (Robust Linear Regression). For  $0 < \alpha < 1$  and  $\ell^* \in \mathbb{R}^d$  with  $\|\ell^*\|_2 \leq 1$ , let  $\text{Lin}_D(\alpha, \ell^*)$  denote the following probabilistic process to generate  $n$  noisy linear equations  $\mathcal{S} = \{\langle x_i, a \rangle = y_i \mid 1 \leq i \leq n\}$  in variable  $a \in \mathbb{R}^d$  with  $\alpha n$  inliers  $\mathcal{I}$  and  $(1 - \alpha)n$  outliers  $\mathcal{O}$ :

1. Construct  $\mathcal{I}$  by choosing  $\alpha n$  i.i.d. samples  $x_i \sim D$  and set  $y_i = \langle x_i, \ell^* \rangle + \zeta$  for additive noise  $\zeta$ ,
2. Construct  $\mathcal{O}$  by choosing the remaining  $(1 - \alpha)n$  equations arbitrarily and potentially adversarially w.r.t the inliers  $\mathcal{I}$ .

Note that  $\alpha$  measures the “signal” (fraction of inliers) and can be  $\ll 1/2$ . The bound on the norm of  $\ell^*$  is without any loss of generality. For the sake of exposition, we will restrict to  $\zeta = 0$  for most of this paper and discuss (see Remarks 1.6 and 4.4) how our algorithms can tolerate additive noise.

An  $\eta$ -approximate algorithm for list-decodable regression takes input a sample from  $\text{Lin}_D(\alpha, \ell^*)$  and outputs a *constant* (depending only on  $\alpha$ ) size list  $L$  of linear functions such that there is some  $\ell \in L$  that is  $\eta$ -close to  $\ell^*$ .

One of our key conceptual contributions is to identify the strong relationship between *anti-concentration inequalities* and list-decodable regression. Anti-concentration inequalities are well-studied [18, 39, 37] in probability theory and combinatorics. The simplest of these inequalities upper bound the probability that a high-dimensional random variable has zero projections in any direction.

**Definition 1.2** (Anti-Concentration). A  $\mathbb{R}^d$ -valued zero-mean random variable  $Y$  has a  $\delta$ -*anti-concentrated* distribution if  $\Pr[\langle Y, v \rangle = 0] < \delta$ .

In Proposition 2.4, we provide a simple but conceptually illuminating proof that anti-concentration is *sufficient* for list-decodable regression. In Theorem 6.1, we prove a sharp converse and show that anti-concentration is information-theoretically *necessary* for even noiseless list-decodable regression. This lower bound surprisingly holds for a natural distribution: uniform distribution on  $\{0, 1\}^d$  and more generally, uniform distribution on  $[q]^d$  for  $q = \{0, 1, 2, \dots, q\}$ .

**Theorem 1.3** (See Proposition 2.4 and Theorem 6.1). *There is a (inefficient) list-decodable regression algorithm for  $\text{Lin}_D(\alpha, \ell^*)$  with list size  $O(\frac{1}{\alpha})$  whenever  $D$  is  $\alpha$ -anti-concentrated. Further, there exists a distribution  $D$  on  $\mathbb{R}^d$  that is  $(\alpha + \epsilon)$ -anti-concentrated for every  $\epsilon > 0$  but there is no algorithm for  $\frac{\alpha}{2}$ -approximate list-decodable regression for  $\text{Lin}_D(\alpha, \ell^*)$  that returns a list of size  $< d$ .*

For our efficient algorithms, we need a *certified* version of the anti-concentration condition. To handle additive noise of variance  $\zeta^2$ , we need a control of  $\Pr[|\langle x, v \rangle| \leq \zeta]$ . Thus, we extend our notion of anti-concentration and then define a *certified* analog of it:

**Definition 1.4** (Certifiable Anti-Concentration). A random variable  $Y$  has a  $k$ -*certifiably*  $(C, \delta)$ -anti-concentrated distribution if there is a univariate polynomial  $p$  satisfying  $p(0) = 1$  such that there is a degree  $k$  sum-of-squares proof of the following two inequalities:

1.  $\forall v, \langle Y, v \rangle^2 \leq \delta^2 \mathbb{E} \langle Y, v \rangle^2$  implies  $(p(\langle Y, v \rangle) - 1)^2 \leq \delta^2$ .
2.  $\forall v, \|v\|_2^2 \leq 1$  implies  $\mathbb{E} p^2(\langle Y, v \rangle) \leq C\delta$ .

Intuitively, certified anti-concentration asks for a *certificate* of the anti-concentration property of  $Y$  in the “sum-of-squares” proof system (see Section 3 for precise definitions). SoS is a proof system that reasons about polynomial inequalities. Since the “core indicator”  $\mathbf{1}(|\langle x, v \rangle| \leq \delta)$  is not a polynomial, we phrase the condition in terms of an approximating polynomial  $p$ . We are now ready to state our main result.

**Theorem 1.5** (List-Decodable Regression). *For every  $\alpha, \eta > 0$  and a  $k$ -certifiably  $(C, \alpha^2 \eta^2 / 10C)$ -anti-concentrated distribution  $D$  on  $\mathbb{R}^d$ , there exists an algorithm that takes input a sample generated according to  $\text{Lin}_D(\alpha, \ell^*)$  and outputs a list  $L$  of size  $O(1/\alpha)$  such that there is an  $\ell \in L$  satisfying  $\|\ell - \ell^*\|_2 < \eta$  with probability at least 0.99 over the draw of the sample. The algorithm needs a sample of size  $n = (kd)^{O(k)}$  and runs in time  $n^{O(k)} = (kd)^{O(k^2)}$ .*

Please note that sections 3-6 are in the supplementary material.

134 *Remark 1.6* (Tolerating Additive Noise). For additive noise (not necessarily independent) of variance  
 135  $\zeta^2$  in the inlier labels, our algorithm, in the same running time and sample complexity, outputs a list  
 136 of size  $O(1/\alpha)$  that contains an  $\ell$  satisfying  $\|\ell - \ell^*\|_2 \leq \frac{\zeta}{\alpha} + \eta$ . Since we normalize  $\ell^*$  to have unit  
 137 norm, this guarantee is meaningful only when  $\zeta \ll \alpha$ .

138 *Remark 1.7* (Exponential Dependence on  $1/\alpha$ ). List-decodable regression algorithms immediately  
 139 yield algorithms for mixed linear regression (MLR) without any assumptions on the components. The  
 140 state-of-the-art algorithm for MLR with gaussian components [31] has an exponential dependence on  
 141  $k = 1/\alpha$  in the running time in the absence of strong pairwise separation or small condition number  
 142 of the components. Liang and Liu [31] (see Page 10) use the relationship to learning mixtures of  $k$   
 143 gaussians (with an  $\exp(k)$  lower bound [35]) to note that algorithms with polynomial dependence on  
 144  $1/\alpha$  for MLR and thus, also for list-decodable regression might not exist.

145 **Certifiably anti-concentrated distributions** In Section 5, we show certifiable anti-concentration  
 146 of some well-studied families of distributions. This includes the standard gaussian distribution and  
 147 more generally any anti-concentrated spherically symmetric distribution with strictly sub-exponential  
 148 tails. We also show that simple operations such as scaling, applying well-conditioned linear transfor-  
 149 mations and sampling preserve certifiable anti-concentration. This yields:

150 **Corollary 1.8** (List-Decodable Regression for Gaussian Inliers). *For every  $\alpha, \eta > 0$  there's*  
 151 *an algorithm for list-decodable regression for the model  $\text{Lin}_D(\alpha, \ell^*)$  with  $D = \mathcal{N}(0, \Sigma)$  with*  
 152  $\lambda_{\max}(\Sigma)/\lambda_{\min}(\Sigma) = O(1)$  *that needs  $n = (d/\alpha\eta)^{O(\frac{1}{\alpha^4\eta^4})}$  samples and runs in time  $n^{O(\frac{1}{\alpha^4\eta^4})} =$*   
 153  $(d/\alpha\eta)^{O(\frac{1}{\alpha^8\eta^8})}$ .

154 We note that certifiably anti-concentrated distributions are more restrictive compared to the families of  
 155 distributions for which the most general robust estimation algorithms work [28, 27, 25]. To a certain  
 156 extent, this is inherent. The families of distributions considered in these prior works do not satisfy  
 157 anti-concentration in general. And as we discuss in more detail in Section 2, anti-concentration is  
 158 information-theoretically *necessary* (see Theorem 6.1) for list-decodable regression. This surprisingly  
 159 rules out families of distributions that might appear natural and “easy”, for example, the uniform  
 160 distribution on  $\{0, 1\}^n$ . In fact, our lower bound shows the impossibility of even the “easier” problem  
 161 of mixed linear regression on this distribution.

162 We rescue this to an extent for the special case when  $\ell^*$  in the model  $\text{Lin}(\alpha, \ell^*)$  is a “Boolean  
 163 vector”, i.e., has all coordinates of equal magnitude. Intuitively, this helps because while the the  
 164 uniform distribution on  $\{0, 1\}^n$  (and more generally, any discrete product distribution) is badly  
 165 anti-concentrated in sparse directions, they are well anti-concentrated [18] in the directions that are  
 166 far from any sparse vectors.

167 As before, for obtaining efficient algorithms, we need to work with a *certified* version (see Defini-  
 168 tion 4.5) of such a restricted anti-concentration condition. As a specific Corollary (see Theorem 4.6  
 169 for a more general statement), this allows us to show:

170 **Theorem 1.9** (List-Decodable Regression for Hypercube Inliers). *For every  $\alpha, \eta > 0$  there's an*  
 171  *$\eta$ -approximate algorithm for list-decodable regression for the model  $\text{Lin}_D(\alpha, \ell^*)$  with  $D$  is uniform*  
 172 *on  $\{0, 1\}^d$  that needs  $n = (d/\alpha\eta)^{O(\frac{1}{\alpha^4\eta^4})}$  samples and runs in time  $n^{O(\frac{1}{\alpha^4\eta^4})} = (d/\alpha\eta)^{O(\frac{1}{\alpha^8\eta^8})}$ .*

173 In Section 4.1, we obtain similar results for general product distributions. It is an important open  
 174 problem to prove certified anti-concentration for a broader family of distributions.

## 175 2 Overview of our Technique

176 In this section, we illustrate the important ideas in our algorithm for list-decodable regression. Thus,  
 177 given a sample  $\mathcal{S} = \{(x_i, y_i)\}_{i=1}^n$  from  $\text{Lin}_D(\alpha, \ell^*)$ , we must construct a constant-size list  $L$  of  
 178 linear functions containing an  $\ell$  close to  $\ell^*$ .

179 Our algorithm is based on the sum-of-squares method. We build on the “identifiability to algorithms”  
 180 paradigm developed in several prior works [5, 4, 33, 28, 21, 27, 25] with some important conceptual  
 181 differences.

---

Please note that sections 3-6 are in the supplementary material.

182 **An inefficient algorithm** Let's start by designing an inefficient algorithm for the problem. This  
 183 may seem simple at the outset. But as we'll see, solving this relaxed problem will rely on some  
 184 important conceptual ideas that will serve as a starting point for our efficient algorithm.

185 Without computational constraints, it is natural to just return the list  $L$  of all linear functions  $\ell$  that  
 186 correctly labels all examples in some  $S \subseteq \mathcal{S}$  of size  $\alpha n$ . We call such an  $S$ , a large, *soluble* set. True  
 187 inliers  $\mathcal{I}$  satisfy our search criteria so  $\ell^* \in L$ . However, it's not hard to show (Proposition ?? ) that  
 188 one can choose outliers so that the list so generated has size  $\exp(d)$  (far from a fixed constant!).

189 A potential fix is to search instead for a *coarse soluble partition* of  $\mathcal{S}$ , if it exists, into disjoint  
 190  $S_1, S_2, \dots, S_k$  and linear functions  $\ell_1, \ell_2, \dots, \ell_k$  so that every  $|S_i| \geq \alpha n$  and  $\ell_i$  correctly computes  
 191 the labels in  $S_i$ . In this setting, our list is small ( $k \leq 1/\alpha$ ). But it is easy to construct samples  $\mathcal{S}$  for  
 192 which this fails because there are coarse soluble partitions of  $\mathcal{S}$  where every  $\ell_i$  is far from  $\ell^*$ .

193 **Anti-Concentration** It turns out that any (even inefficient) algorithm for list-decodable regression  
 194 provably (see Theorem 6.1) *requires* that the distribution of inliers<sup>2</sup> be sufficiently *anti-concentrated*:

195 **Definition 2.1** (Anti-Concentration). A  $\mathbb{R}^d$ -valued random variable  $Y$  with mean 0 is  $\delta$ -anti-  
 196 concentrated<sup>3</sup> if for all non-zero  $v$ ,  $\Pr[\langle Y, v \rangle = 0] < \delta$ . A set  $T \subseteq \mathbb{R}^d$  is  $\delta$ -anti-concentrated  
 197 if the uniform distribution on  $T$  is  $\delta$ -anti-concentrated.

198 As we discuss next, anti-concentration is also *sufficient* for list-decodable regression. Intuitively,  
 199 this is because anti-concentration of the inliers prevents the existence of a soluble set that intersects  
 200 significantly with  $\mathcal{I}$  and yet can be labeled correctly by  $\ell \neq \ell^*$ . This is simple to prove in the special  
 201 case when  $\mathcal{S}$  admits a coarse soluble partition.

202 **Proposition 2.2.** *Suppose  $\mathcal{I}$  is  $\alpha$ -anti-concentrated. Suppose there exists a partition*  
 203  *$S_1, S_2, \dots, S_k \subseteq \mathcal{S}$  such that each  $|S_i| \geq \alpha n$  and there exist  $\ell_1, \ell_2, \dots, \ell_k$  such that  $y_j = \langle \ell_i, x_j \rangle$*   
 204 *for every  $j \in S_i$ . Then, there is an  $i$  such that  $\ell_i = \ell^*$ .*

205 *Proof.* Since  $k \leq 1/\alpha$ , there is a  $j$  such that  $|\mathcal{I} \cap S_j| \geq \alpha |\mathcal{I}|$ . Then,  $\langle x_i, \ell_j \rangle = \langle x_i, \ell^* \rangle$  for every  
 206  $i \in \mathcal{I} \cap S_j$ . Thus,  $\Pr_{i \sim \mathcal{I}}[\langle x_i, \ell_j - \ell^* \rangle = 0] \geq \alpha$ . This contradicts anti-concentration of  $\mathcal{I}$  unless  
 207  $\ell_j - \ell^* = 0$ .  $\square$

208 The above proposition allows us to use *any* soluble partition as a *certificate* of correctness for the  
 209 associated list  $L$ . Two aspects of this certificate were crucial in the above argument: 1) *largeness*:  
 210 each  $S_i$  is of size  $\alpha n$  - so the generated list is small, and, 2) *uniformity*: every sample is used in  
 211 exactly one of the sets so  $\mathcal{I}$  must intersect one of the  $S_i$ s in at least  $\alpha$ -fraction of the points.

212 **Identifiability via anti-concentration** For arbitrary  $\mathcal{S}$ , a coarse soluble partition might not exist.  
 213 So we will generalize coarse soluble partitions to obtain certificates that exist for every sample  $\mathcal{S}$   
 214 and guarantee largeness and a relaxation of uniformity (formalized below). For this purpose, it is  
 215 convenient to view such certificates as distributions  $\mu$  on  $\geq \alpha n$  size soluble subsets of  $\mathcal{S}$  so any  
 216 collection  $\mathcal{C} \subseteq 2^{\mathcal{S}}$  of  $\alpha n$  size sets corresponds to the uniform distribution  $\mu$  on  $\mathcal{C}$ .

217 To precisely define uniformity, let  $W_i(\mu) = \mathbb{E}_{S \sim \mu}[\mathbf{1}(i \in S)]$  be the “frequency of  $i$ ”, that is,  
 218 probability that the  $i$ th sample is chosen to be in a set drawn according to  $\mu$ . Then, the uniform  
 219 distribution  $\mu$  on any coarse soluble  $k$ -partition satisfies  $W_i = \frac{1}{k}$  for every  $i$ . That is, all samples  
 220  $i \in \mathcal{S}$  are *uniformly* used in such a  $\mu$ . To generalize this idea, we define  $\sum_i W_i(\mu)^2$  as the *distance*  
 221 *to uniformity* of  $\mu$ . Up to a shift, this is simply the variance in the frequencies of the points in  $\mathcal{S}$   
 222 used in draws from  $\mu$ . Our generalization of a coarse soluble partition of  $\mathcal{S}$  is any  $\mu$  that minimizes  
 223  $\sum_i W_i(\mu)^2$ , the distance to uniformity, and is thus *maximally uniform* among all distributions  
 224 supported on large soluble sets. Such a  $\mu$  can be found by convex programming.

225 The following claim generalizes Proposition 2.2 to derive the same conclusion starting from any  
 226 maximally uniform distribution supported on large soluble sets.

227 **Proposition 2.3.** *For a maximally uniform  $\mu$  on  $\alpha n$  size soluble subsets of  $\mathcal{S}$ ,*  
 228  $\sum_{i \in \mathcal{I}} \mathbb{E}_{S \sim \mu}[\mathbf{1}(i \in S)] \geq \alpha |\mathcal{I}|$ .

**Please note that sections 3-6 are in the supplementary material.**

<sup>2</sup>As in the standard robust estimation setting, the outliers are arbitrary and potentially adversarially chosen.

<sup>3</sup>Definition 1.4 differs slightly to handle list-decodable regression with additive noise in the inliers.



229 The proof proceeds by contradiction (see Lemma 4.3). We show that if  $\sum_{i \in \mathcal{I}} W_i(\mu) \leq \alpha|\mathcal{I}|$ , then we  
 230 can strictly reduce the distance to uniformity by taking a mixture of  $\mu$  with the distribution that places  
 231 all its probability mass on  $\mathcal{I}$ . This allow us to obtain an (inefficient) algorithm for list-decodable  
 232 regression establishing identifiability.

233 **Proposition 2.4** (Identifiability for List-Decodable Regression). *Let  $\mathcal{S}$  be sample from  $\text{Lin}(\alpha, \ell^*)$   
 234 such that  $\mathcal{I}$  is  $\delta$ -anti-concentrated for  $\delta < \alpha$ . Then, there's an (inefficient) algorithm that finds a list  
 235  $L$  of size  $\frac{20}{\alpha-\delta}$  such that  $\ell^* \in L$  with probability at least 0.99.*

236 *Proof.* Let  $\mu$  be any maximally uniform distribution over  $\alpha n$  size soluble subsets of  $\mathcal{S}$ . For  $k = \frac{20}{\alpha-\delta}$ ,  
 237 let  $S_1, S_2, \dots, S_k$  be independent samples from  $\mu$ . Output the list  $L$  of  $k$  linear functions that  
 238 correctly compute the labels in each  $S_i$ .

239 To see why  $\ell^* \in L$ , observe that  $\mathbb{E}|S_j \cap \mathcal{I}| = \sum_{i \in \mathcal{I}} \mathbb{E}\mathbf{1}(i \in S_j) \geq \alpha|\mathcal{I}|$ . By averaging,  $\Pr[|S_j \cap \mathcal{I}| \geq$   
 240  $\frac{\alpha+\delta}{2}|\mathcal{I}|] \geq \frac{\alpha-\delta}{2}$ . Thus, there's a  $j \leq k$  so that  $|S_j \cap \mathcal{I}| \geq \frac{\alpha+\delta}{2}|\mathcal{I}|$  with probability at least  
 241  $1 - (1 - \frac{\alpha-\delta}{2})^{\frac{20}{\alpha-\delta}} \geq 0.99$ . We can now repeat the argument in the proof of Proposition 2.2 to  
 242 conclude that any linear function that correctly labels  $S_j$  must equal  $\ell^*$ .  $\square$

243 **An efficient algorithm** Our identifiability proof suggests the following simple algorithm: 1) find  
 244 any maximally uniform distribution  $\mu$  on soluble subsets of size  $\alpha n$  of  $\mathcal{S}$ , 2) take  $O(1/\alpha)$  samples  
 245  $S_i$  from  $\mu$  and 3) return the list of linear functions that correctly label the equations in  $S_i$ s. This is  
 246 inefficient because searching over distributions is NP-hard in general.

247 To make this into an efficient algorithm, we start by observing that soluble subsets  $S \subseteq \mathcal{S}$  of size  $\alpha n$   
 248 can be described by the following set of quadratic equations where  $w$  stands for the indicator of  $S$   
 249 and  $\ell$ , the linear function that correctly labels the examples in  $S$ .

$$\mathcal{A}_{w, \ell} : \left\{ \begin{array}{l} \sum_{i=1}^n w_i = \alpha n \\ \forall i \in [n]. \quad w_i^2 = w_i \\ \forall i \in [n]. \quad w_i \cdot (y_i - \langle x_i, \ell \rangle) = 0 \\ \|\ell\|^2 \leq 1 \end{array} \right\} \quad (2.1)$$

250 Our efficient algorithm searches for a maximally uniform *pseudo-distribution* on  $w$  satisfying (2.1).  
 251 Degree  $k$  pseudo-distributions (see Section 3 for precise definitions) are generalization of distributions  
 252 that nevertheless “behave” just as distributions whenever we take (pseudo)-expectations (denoted  
 253 by  $\tilde{\mathbb{E}}$ ) of a class of degree  $k$  polynomials. And unlike distributions, degree  $k$  pseudo-distributions  
 254 satisfying<sup>4</sup> polynomial constraints (such as (2.1)) can be computed in time  $n^{O(k)}$ .

255 For the sake of intuition, it might be helpful to (falsely) think of pseudo-distributions  $\tilde{\mu}$  as simply  
 256 distributions where we only get access to moments of degree  $\leq k$ . Thus, we are allowed to compute  
 257 expectations of all degree  $\leq k$  polynomials with respect to  $\tilde{\mu}$ . Since  $W_i(\tilde{\mu}) = \tilde{\mathbb{E}}_{\tilde{\mu}} w_i$  are just  
 258 first moments of  $\tilde{\mu}$ , our notion of maximally uniform distributions extends naturally to pseudo-  
 259 distributions. This allows us to prove an analog of Proposition 2.3 for pseudo-distributions and gives  
 260 us an efficient replacement for Step 1.

261 **Proposition 2.5.** *For any maximally uniform  $\tilde{\mu}$  of degree  $\geq 2$ ,  $\sum_{i \in \mathcal{I}} \tilde{\mathbb{E}}_{\tilde{\mu}}[w_i] \geq \alpha|\mathcal{I}| =$   
 262  $\alpha \sum_{i \in [n]} \tilde{\mathbb{E}}_{\tilde{\mu}}[w_i]$ .*

263 For Step 2, however, we hit a wall: it's not possible to obtain independent samples from  $\tilde{\mu}$  given only  
 264 low-degree moments. Our algorithm relies on an alternative strategy instead.

265 Consider the vector  $v_i = \frac{\tilde{\mathbb{E}}_{\tilde{\mu}}[w_i \ell]}{\tilde{\mathbb{E}}_{\tilde{\mu}}[w_i]}$  whenever  $\tilde{\mathbb{E}}_{\tilde{\mu}}[w_i] \neq 0$  (set  $v_i$  to zero, otherwise). This is simply the  
 266 (scaled) average, according to  $\tilde{\mu}$ , of all the linear functions  $\ell$  that are used to label the sets  $S$  of size  
 267  $\alpha n$  in the support of  $\tilde{\mu}$  whenever  $i \in S$ . Further,  $v_i$  depends only on the first two moments of  $\tilde{\mu}$ .

**Please note that sections 3-6 are in the supplementary material.**

<sup>4</sup>See Fact 3.3 for a precise statement.

268 We think of  $v_i$ s as “guesses” made by the  $i$ th sample for the unknown linear function. Let us focus  
 269 our attention on the guesses  $v_i$  of  $i \in \mathcal{I}$  - the inliers. We will show that according to the distribution  
 270 proportional to  $\tilde{\mathbb{E}}[w]$ , the average squared distance of  $v_i$  from  $\ell^*$  is at max  $\eta$ :

$$\frac{1}{\sum_{i \in \mathcal{I}} \tilde{\mathbb{E}}[w_i]} \sum_{i \in \mathcal{I}} \tilde{\mathbb{E}}[w_i] \|v_i - \ell^*\|_2 < \eta. \quad (\star)$$

271 Before diving into  $(\star)$ , let’s see how it gives us our efficient list-decodable regression algorithm:

- 272 1. Find a pseudo-distribution  $\tilde{\mu}$  satisfying (2.1) that minimizes distance to uniformity  
 273  $\sum_i \tilde{\mathbb{E}}_{\tilde{\mu}}[w_i]^2$ .
- 274 2. For  $O(\frac{1}{\alpha})$  times, independently choose a random index  $i \in [n]$  with probability proportional  
 275 to  $\tilde{\mathbb{E}}_{\tilde{\mu}}[w_i]$  and return the list of corresponding  $v_i$ s.

276 Step 1 above is a convex program and can be solved in polynomial time. Let’s analyze step 2 to see  
 277 why the algorithm works. Using  $(\star)$  and Markov’s inequality, conditioned on  $i \in \mathcal{I}$ ,  $\|v_i - \ell^*\|_2 \leq 2\eta$   
 278 with probability  $\geq 1/2$ . By Proposition 2.5,  $\frac{\sum_{i \in \mathcal{I}} \tilde{\mathbb{E}}[w_i]}{\sum_{i \in [n]} \tilde{\mathbb{E}}[w_i]} \geq \alpha$  so  $i \in \mathcal{I}$  with probability at least  $\alpha$ .  
 279 Thus in each iteration of step 2, with probability at least  $\alpha/2$ , we choose an  $i$  such that  $v_i$  is  $2\eta$ -close  
 280 to  $\ell^*$ . Repeating  $O(1/\alpha)$  times gives us the 0.99 chance of success.

281  **$(\star)$  via anti-concentration** As in the information-theoretic argument,  $(\star)$  relies on the anti-  
 282 concentration of  $\mathcal{I}$ . Let’s do a quick proof for the case when  $\tilde{\mu}$  is an actual distribution  $\mu$ .

283 *Proof of  $(\star)$  for actual distributions  $\mu$ .* Observe that  $\mu$  is a distribution over  $(w, \ell)$  satisfying (2.1).  
 284 Recall that  $w$  indicates a subset  $S \subseteq \mathcal{S}$  of size  $\alpha n$  and  $w_i = 1$  iff  $i \in S$ . And  $\ell \in \mathbb{R}^d$  satisfies all the  
 285 equations in  $S$ .

286 By Cauchy-Schwarz,  $\sum_i \|\mathbb{E}[w_i \ell] - \mathbb{E}[w_i] \ell^*\| \leq \mathbb{E}_\mu[\sum_{i \in \mathcal{I}} w_i \|\ell - \ell^*\|]$ . Next, as in Proposition 2.2,  
 287 since  $\mathcal{I}$  is  $\eta$ -anti-concentrated, and for all  $S$  such that  $|\mathcal{I} \cap S| \geq \eta |\mathcal{I}|$ ,  $\ell - \ell^* = 0$ . Thus, any such  $S$   
 288 in the support of  $\mu$  contributes 0 to the expectation above. We will now show that the contribution  
 289 from the remaining terms is upper bounded by  $\eta$ . Observe that since  $\|\ell - \ell^*\| \leq 2$ ,  
 290  $\mathbb{E}[\sum_{i \in \mathcal{I}} w_i \|\ell - \ell^*\|] = \mathbb{E}[\mathbf{1}(|S \cap \mathcal{I}| < \eta |\mathcal{I}|) \sum_{i \in S \cap \mathcal{I}} \|\ell - \ell^*\|] \leq 2\eta |\mathcal{I}|$ .  $\square$

291 **SoSizing Anti-Concentration** The key to proving  $(\star)$  for pseudo-distributions is a *sum-of-squares*  
 292 (SoS) proof of anti-concentration inequality:  $\Pr_{x \sim \mathcal{I}}[\langle x, v \rangle = 0] \leq \eta$  in variable  $v$ . SoS is a restricted  
 293 system for proving polynomial inequalities subject to polynomial inequality constraints. Thus, to  
 294 even ask for a SoS proof we must phrase anti-concentration as a polynomial inequality.

295 To do this, let  $p(z)$  be a low-degree polynomial approximator for the function  $\mathbf{1}(z = 0)$ . Then, we  
 296 can hope to “replace” the use of the inequality  $\Pr_{x \sim \mathcal{I}}[\langle x, v \rangle = 0] \leq \eta \equiv \mathbb{E}_{x \sim \mathcal{I}}[\mathbf{1}(\langle x, v \rangle = 0)] \leq \eta$   
 297 in the argument above by  $\mathbb{E}_{x \sim \mathcal{I}}[p(\langle x, v \rangle)^2] \leq \eta$ . Since polynomials grow unboundedly for large  
 298 enough inputs, it is *necessary* for the uniform distribution on  $\mathcal{I}$  to have sufficiently light-tails to  
 299 ensure that  $\mathbb{E}_{x \sim \mathcal{I}} p(\langle x, v \rangle)^2$  is small. In Lemma ??, we show that anti-concentration and light-tails  
 300 are *sufficient* to construct such a polynomial.

301 We can finally ask for a SoS proof for  $\mathbb{E}_{x \sim \mathcal{I}} p(\langle x, v \rangle) \leq \eta$  in variable  $v$ . We prove such *certified*  
 302 anti-concentration inequalities for broad families of inlier distributions in Section 5.

## 303 References

- 304 [1] Pranjali Awasthi, Maria-Florina Balcan, and Philip M. Long. The power of localization for  
 305 efficiently learning linear separators with malicious noise. *CoRR*, abs/1307.8371, 2013. 1
- 306 [2] Sivaraman Balakrishnan, Martin J. Wainwright, and Bin Yu. Statistical guarantees for the EM  
 307 algorithm: From population to sample-based analysis. *CoRR*, abs/1408.2156, 2014. 2

---

Please note that sections 3-6 are in the supplementary material.

- 308 [3] Maria-Florina Balcan, Avrim Blum, and Santosh Vempala. A discriminative framework for  
309 clustering via similarity functions. In *STOC*, pages 671–680. ACM, 2008. [1](#)
- 310 [4] Boaz Barak, Jonathan A. Kelner, and David Steurer. Dictionary learning and tensor decomposi-  
311 tion via the sum-of-squares method [extended abstract]. In *STOC’15—Proceedings of the 2015*  
312 *ACM Symposium on Theory of Computing*, pages 143–151. ACM, New York, 2015. [4](#)
- 313 [5] Boaz Barak and Ankur Moitra. Noisy tensor completion via the sum-of-squares hierarchy. In  
314 *COLT*, volume 49 of *JMLR Workshop and Conference Proceedings*, pages 417–445. JMLR.org,  
315 2016. [4](#)
- 316 [6] Thorsten Bernholt. Robust estimators are hard to compute. Technical report, Technical  
317 Report/Universität Dortmund, SFB 475 Komplexitätsreduktion in Multivariaten Datenstrukturen,  
318 2006. [1](#)
- 319 [7] Kush Bhatia, Prateek Jain, Parameswaran Kamalaruban, and Purushottam Kar. Consistent robust  
320 regression. In *Advances in Neural Information Processing Systems 30: Annual Conference*  
321 *on Neural Information Processing Systems 2017, 4-9 December 2017, Long Beach, CA, USA*,  
322 pages 2107–2116, 2017. [2](#)
- 323 [8] Moses Charikar, Jacob Steinhardt, and Gregory Valiant. Learning from untrusted data. In *STOC*,  
324 pages 47–60. ACM, 2017. [1](#), [2](#)
- 325 [9] Yudong Chen, Xinyang Yi, and Constantine Caramanis. A convex formulation for mixed  
326 regression with two components: Minimax optimal rates. In *Proceedings of The 27th Conference*  
327 *on Learning Theory, COLT 2014, Barcelona, Spain, June 13-15, 2014*, pages 560–604, 2014. [2](#)
- 328 [10] Richard D. De Veaux. Mixtures of linear regressions. *Comput. Statist. Data Anal.*, 8(3):227–245,  
329 1989. [2](#)
- 330 [11] Ilias Diakonikolas, Gautam Kamath, Daniel M. Kane, Jerry Li 0001, Jacob Steinhardt, and Alis-  
331 tair Stewart. Sever: A robust meta-algorithm for stochastic optimization. *CoRR*, abs/1803.02815,  
332 2018. [2](#)
- 333 [12] Ilias Diakonikolas, Gautam Kamath, Daniel M. Kane, Jerry Li, Ankur Moitra, and Alistair  
334 Stewart. Robust estimators in high dimensions without the computational intractability. In  
335 *FOCS*, pages 655–664. IEEE Computer Society, 2016. [1](#)
- 336 [13] Ilias Diakonikolas, Gautam Kamath, Daniel M. Kane, Jerry Li, Ankur Moitra, and Alistair  
337 Stewart. Robustly learning a gaussian: Getting optimal error, efficiently. *CoRR*, abs/1704.03866,  
338 2017. [1](#)
- 339 [14] Ilias Diakonikolas, Gautam Kamath, Daniel M. Kane, Jerry Zheng Li, Ankur Moitra, and  
340 Alistair Stewart. Robust estimators in high dimensions without the computational intractability.  
341 *CoRR*, abs/1604.06443, 2016. [1](#)
- 342 [15] Ilias Diakonikolas, Daniel M. Kane, and Alistair Stewart. Learning geometric concepts with  
343 nasty noise. *CoRR*, abs/1707.01242, 2017. [1](#)
- 344 [16] Ilias Diakonikolas, Daniel M. Kane, and Alistair Stewart. List-decodable robust mean estimation  
345 and learning mixtures of spherical gaussians. In *Proceedings of the 50th Annual ACM SIGACT*  
346 *Symposium on Theory of Computing, STOC 2018, Los Angeles, CA, USA, June 25-29, 2018*,  
347 pages 1047–1060, 2018. [1](#), [2](#)
- 348 [17] Ilias Diakonikolas, Weihao Kong, and Alistair Stewart. Efficient algorithms and lower bounds  
349 for robust linear regression. In Timothy M. Chan, editor, *Proceedings of the Thirtieth Annual*  
350 *ACM-SIAM Symposium on Discrete Algorithms, SODA 2019, San Diego, California, USA,*  
351 *January 6-9, 2019*, pages 2745–2754. SIAM, 2019. [2](#)
- 352 [18] P. Erdős. On a lemma of littlewood and offord. *Bull. Amer. Math. Soc.*, 51(12):898–902, 12  
353 1945. [3](#), [4](#)
- 354 [19] Susana Faria and Gilda Soromenho. Fitting mixtures of linear regressions. *J. Stat. Comput.*  
355 *Simul.*, 80(1-2):201–225, 2010. [2](#)



- [20] Frank R Hampel, Elvezio M Ronchetti, Peter J Rousseeuw, and Werner A Stahel. *Robust statistics: the approach based on influence functions*, volume 114. John Wiley & Sons, 2011. 1
- [21] Sam B. Hopkins and Jerry Li. Mixture models, robustness, and sum of squares proofs. 2017. 1, 2, 4
- [22] Peter J Huber. Robust statistics. In *International Encyclopedia of Statistical Science*, pages 1248–1251. Springer, 2011. 1
- [23] Michael I. Jordan and Robert A. Jacobs. Hierarchical mixtures of experts and the em algorithm. *Neural Computation*, 6(2):181–214, 1994. 2
- [24] Sushrut Karmalkar and Eric Price. Compressed sensing with adversarial sparse noise via l1 regression. In Jeremy T. Fineman and Michael Mitzenmacher, editors, *SOSA@SODA*, volume 69 of *OASICS*, pages 19:1–19:19. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2019. 2
- [25] Adam R. Klivans, Pravesh K. Kothari, and Raghu Meka. Efficient algorithms for outlier-robust regression. In *Conference On Learning Theory, COLT 2018, Stockholm, Sweden, 6-9 July 2018.*, pages 1420–1430, 2018. 1, 2, 4
- [26] Adam R. Klivans, Philip M. Long, and Rocco A. Servedio. Learning halfspaces with malicious noise. *Journal of Machine Learning Research*, 10:2715–2740, 2009. 1
- [27] Pravesh K. Kothari and Jacob Steinhardt. Better agnostic clustering via relaxed tensor norms. 2017. 1, 2, 4
- [28] Pravesh K. Kothari and David Steurer. Outlier-robust moment-estimation via sum-of-squares. *CoRR*, abs/1711.11581, 2017. 1, 2, 4
- [29] Pravesh K. Kothari and David Steurer. List-decodable mean estimation made simple. In *Manuscript*, 2019. 2
- [30] Kevin A. Lai, Anup B. Rao, and Santosh Vempala. Agnostic estimation of mean and covariance. In *FOCS*, pages 665–674. IEEE Computer Society, 2016. 1
- [31] Yuanzhi Li and Yingyu Liang. Learning mixtures of linear regressions with nearly optimal complexity. In *Conference On Learning Theory, COLT 2018, Stockholm, Sweden, 6-9 July 2018.*, pages 1125–1144, 2018. 2, 4
- [32] Doron S Lubinsky. A Survey of Weighted Approximation for Exponential Weights. *arXiv Mathematics e-prints*, page math/0701099, Jan 2007. 2
- [33] Tengyu Ma, Jonathan Shi, and David Steurer. Polynomial-time tensor decompositions with sum-of-squares. In *FOCS*, pages 438–446. IEEE Computer Society, 2016. 4
- [34] RARD Maronna, R Douglas Martin, and Victor Yohai. *Robust statistics*. John Wiley & Sons, Chichester. ISBN, 2006. 1
- [35] Ankur Moitra and Gregory Valiant. Settling the polynomial learnability of mixtures of gaussians. In *FOCS*, pages 93–102. IEEE Computer Society, 2010. 4
- [36] Adarsh Prasad, Arun Sai Suggala, Sivaraman Balakrishnan, and Pradeep Ravikumar. Robust estimation via robust gradient estimation. *CoRR*, abs/1802.06485, 2018. 2
- [37] Mark Rudelson and Roman Vershynin. The Littlewood-Offord problem and invertibility of random matrices. *Adv. Math.*, 218(2):600–633, 2008. 3
- [38] Hanie Sedghi, Majid Janzamin, and Anima Anandkumar. Provable tensor methods for learning mixtures of generalized linear models. In *AISTATS*, volume 51 of *JMLR Workshop and Conference Proceedings*, pages 1223–1231. JMLR.org, 2016. 2
- [39] Terence Tao and Van Vu. The Littlewood-Offord problem in high dimensions and a conjecture of Frankl and Füredi. *Combinatorica*, 32(3):363–372, 2012. 3
- [40] John W. Tukey. Mathematics and the picturing of data. pages 523–531, 1975. 1

- 401 [41] Xinyang Yi, Constantine Caramanis, and Sujay Sanghavi. Alternating Minimization for Mixed  
402 Linear Regression. *arXiv e-prints*, page arXiv:1310.3745, Oct 2013. [2](#)
- 403 [42] Kai Zhong, Prateek Jain, and Inderjit S. Dhillon. Mixed linear regression with multiple  
404 components. In *NIPS*, pages 2190–2198, 2016. [2](#)