P-4

$f(x, y, z) = 3x^2 y z$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} 6xyz \\ 3x^2 z \\ 3x^2 y \end{bmatrix}$$

$$\nabla^2 f = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial x \partial z} & \frac{\partial^2 f}{\partial y \partial z} & \frac{\partial^2 f}{\partial z^2} \end{bmatrix} = \begin{bmatrix} 6yz & 6xz & 6xy \\ 6xz & 0 & 3x^2 \\ 6xy & 3x^2 & 0 \end{bmatrix}$$

$\frac{\partial^2 f}{\partial x^2} = 6yz$ $\qquad$ $\frac{\partial^2 f}{\partial x \partial y} = 6xz$

$\frac{\partial^2 f}{\partial y^2} = 0$ $\qquad$ $\frac{\partial^2 f}{\partial y \partial z} = 3x^2$

$\frac{\partial^2 f}{\partial z^2} = 0$ $\qquad$ $\frac{\partial^2 f}{\partial x \partial z} = 6xy$

As we can see $\nabla^2 f$ (Hessian) is a symmetric matrix.

P-2

$$B = \begin{pmatrix} 2 & 0 & 2 \\ 2 & 2 & 0 \\ 0 & 2 & 2 \end{pmatrix}$$

$$r_{11} = \left\| \begin{matrix} 2 \\ 2 \\ 0 \end{matrix} \right\| = 2\sqrt{2}$$

$$q_1 = \frac{1}{2\sqrt{2}} \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

$$r_{12} = q_1^T a_2$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} = \sqrt{2}$$

$$q_2' = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} - \sqrt{2} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

4+1+1

6

$$r_{22} = \|q_2'\| = \sqrt{6}$$

$$q_2 = \frac{q_2'}{r_{22}} = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{pmatrix}$$

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$r_{13} = q_1^T a_3 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} = \sqrt{2}.$

$r_{23} = q_2^T a_3 = \begin{pmatrix} -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} = -\frac{2}{\sqrt{6}} + \frac{4}{\sqrt{6}} = \frac{2}{\sqrt{6}}$

$\cancel{q_3' = a_3 - r_{13} q_1 - r_{23} q_2}$

$\cancel{= \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} - \sqrt{2} \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} - \frac{2}{\sqrt{6}} \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}}$

$\cancel{= \begin{pmatrix} 2(1-\sqrt{2}) \\ -2\sqrt{2} - \frac{4}{\sqrt{6}} \\ 2 - \frac{4}{\sqrt{6}} \end{pmatrix}}$

$\cancel{r_{33} = \|q_3'\| =}$

$q_3' = a_3 - r_{13} q_1 - r_{23} q_2$

$= \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} - \sqrt{2} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} - \frac{2}{\sqrt{6}} \begin{pmatrix} -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{pmatrix}$ $\quad -\frac{2}{6}$

$= \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$

$= \begin{pmatrix} 2 - 1 + \frac{1}{3} \\ 0 - 1 - \frac{1}{3} \\ 2 - 0 - \frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{4}{3} \\ -\frac{4}{3} \\ \frac{4}{3} \end{pmatrix}$

$r_{33} = \|q_3'\| = \frac{4}{3}\sqrt{3} = \frac{4}{\sqrt{3}}$

$q_3 = \frac{q_3'}{r_{33}} = \frac{\sqrt{3}}{4} \begin{pmatrix} \frac{4}{3} \\ -\frac{4}{3} \\ \frac{4}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$

$Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}$

$R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix}$

$= \begin{pmatrix} 2\sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{6} & \frac{2}{\sqrt{6}} \\ 0 & 0 & \frac{4}{\sqrt{3}} \end{pmatrix}$

P.1.

$$A = \begin{pmatrix} 2 & 2 & 2 & 2 \\ \frac{17}{10} & \frac{1}{10} & -\frac{17}{10} & -\frac{1}{10} \\ \frac{3}{5} & \frac{9}{5} & -\frac{3}{5} & -\frac{9}{5} \end{pmatrix}_{3\times 4}$$

$A = U \Sigma V^T$

Columns of U are eigenvector of AA^T

" V " " A^TA

$$AA^T = \begin{pmatrix} 2 & 2 & 2 & 2 \\ \frac{17}{10} & \frac{1}{10} & -\frac{17}{10} & -\frac{1}{10} \\ \frac{3}{5} & \frac{9}{5} & -\frac{3}{5} & -\frac{9}{5} \end{pmatrix}_{3\times 4} \begin{pmatrix} 2 & \frac{17}{10} & \frac{3}{5} \\ 2 & \frac{1}{10} & \frac{9}{5} \\ 2 & -\frac{17}{10} & -\frac{3}{5} \\ 2 & -\frac{1}{10} & -\frac{9}{5} \end{pmatrix}_{4\times 3}$$

$$= \begin{pmatrix} 16 & 0 & 0 \\ 0 & \frac{29}{5} & \frac{12}{5} \\ 0 & \frac{12}{5} & \frac{36}{5} \end{pmatrix}$$

$\left(\frac{17}{10}\right)^2 \times 2 + \left(\frac{1}{10}\right)^2 \times 2$

$\frac{289}{100} \times 2 + \frac{1}{100} \times 2$

$\frac{1}{100}(578+2)$

$\frac{580}{100} = \frac{29}{5}$

$\left(\frac{3}{5}\right)^2 \times 2 + \left(\frac{9}{5}\right)^2 \times 2$

$= \frac{9}{25} \times 2 + \frac{81}{25} \times 2$

$= \frac{1}{25}(18+162)$

$= \frac{180}{25} = \frac{36}{5}$

$\frac{17}{10} \times \frac{3}{5} \times 2 + \frac{1}{10} \times \frac{9}{5} \times 2$

$\frac{51}{25} + \frac{9}{25}$

$\frac{60}{25} = \frac{12}{5}$

Eigenvalues of AA^T:

$$\begin{vmatrix} 16-\lambda & 0 & 0 \\ 0 & \frac{29}{5}-\lambda & \frac{12}{5} \\ 0 & \frac{12}{5} & \frac{36}{5}-\lambda \end{vmatrix} = 0 .$$

$$(16-\lambda)\left[\left(\frac{29}{5}-\lambda\right)\left(\frac{36}{5}-\lambda\right) - \frac{144}{25}\right] = 0 .$$

$\lambda = 16$, or

$$\left(\frac{29}{5}-\lambda\right)\left(\frac{36}{5}-\lambda\right) - \frac{144}{25} = 0 .$$

or $\lambda^2 - 13\lambda + \frac{29\times 36}{25} - \frac{144}{25} = 0$

$\lambda^2 - 13\lambda + \frac{\cancel{900}\,36}{\cancel{25}} = 0 .$

$\lambda^2 - 4\lambda - 9\lambda + 36 = 0$

$(\lambda-4)(\lambda-9) = 0 .$ $\quad \lambda = 4, \lambda = 9$

Singular values of A: 2, 3, 4.

$$\Sigma = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}_{3\times 4} .$$

Eigenvector for AA^T for $\lambda = 4$:

$$\begin{pmatrix} 16 & 0 & 0 \\ 0 & \frac{29}{5} & \frac{12}{5} \\ 0 & \frac{12}{5} & \frac{36}{5} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 4 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$\Rightarrow$ 1) $16x_1 = 4x_1$
$x_1 = 0$.

2) $\frac{29}{5}x_2 + \frac{12}{5}x_3 = 4x_2$

$\frac{9x_2}{5} + \frac{12x_3}{5} = 0$

$3x_2 + 4x_3 = 0$

Eigenvector: $\begin{pmatrix} 0 \\ -4 \\ 3 \end{pmatrix}$

Normalised eigenvector: $\frac{1}{5}\begin{pmatrix} 0 \\ -4 \\ 3 \end{pmatrix}$

Eigenvector for $\lambda = 9$, for AA^T

$$\begin{pmatrix} 16 & 0 & 0 \\ 0 & \frac{29}{5} & \frac{12}{5} \\ 0 & \frac{12}{5} & \frac{36}{5} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 9\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$16x_1 = 9x_1 \Rightarrow x_1 = 0$.

$\frac{29}{5}x_2 + \frac{12}{5}x_3 = 9x_2$

$\frac{12x_3}{5} = \frac{16x_2}{5}$

$3x_3 = 4x_2$

Eigenvector: $\begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix}$

Normalised eigenvector: $\frac{1}{5}\begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix}$

Eigenvector for $\lambda = 16$, for AA^T

$$\begin{pmatrix} 16 & 0 & 0 \\ 0 & \frac{29}{5} & \frac{12}{5} \\ 0 & \frac{12}{5} & \frac{36}{5} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 16\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$16x_1 = 16x_1$

$x_1 = 1$

$\frac{29}{5}x_2 + \frac{12x_3}{5} = 16x_2$

$\frac{12x_3}{5} = \frac{51x_2}{5}$ $\quad 4x_3 = 17x_2$

$\frac{12x_2}{5} + \frac{36x_3}{5} = 16x_3$

$\frac{12x_2}{5} = \frac{44x_3}{5}$

$3x_2 = 11x_3$

$\Rightarrow x_2 = x_3 = 0$.

Eigenvector $= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\because AA^T$ is symmetric its eigenvectors are orthonormal.

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.6 & -0.8 \\ 0 & 0.8 & 0.6 \end{pmatrix}$$

$$U\Sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.6 & -0.8 \\ 0 & 0.8 & 0.6 \end{pmatrix}_{3\times3}\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}_{3\times4}$$

$$= \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 1.8 & -1.6 & 0 \\ 0 & 2.4 & 1.2 & 0 \end{pmatrix}_{3\times4}$$

$$A^TA = \begin{pmatrix} 2 & \frac{17}{10} & \frac{3}{5} \\ 2 & \frac{1}{10} & \frac{9}{5} \\ 2 & -\frac{17}{10} & -\frac{3}{5} \\ 2 & -\frac{1}{10} & -\frac{9}{5} \end{pmatrix}_{4\times 3} \begin{pmatrix} 2 & 2 & 2 & 2 \\ \frac{17}{10} & \frac{1}{10} & -\frac{17}{10} & -\frac{1}{10} \\ \frac{3}{5} & \frac{9}{5} & -\frac{3}{5} & -\frac{9}{5} \end{pmatrix}_{3\times 4}$$

$$= \begin{pmatrix} \frac{29}{4} & \frac{21}{4} & \frac{3}{4} & \frac{11}{4} \\ \frac{21}{4} & \frac{29}{4} & \frac{11}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{11}{4} & \frac{29}{4} & \frac{21}{4} \\ \frac{11}{4} & \frac{3}{4} & \frac{21}{4} & \frac{29}{4} \end{pmatrix}$$

Eigenvalues of A^TA = 4, 9, 16, 0.

Eigenvectors will be orthogonal since A^TA is symmetric.

For $\lambda = 0$ eigenvalue,
eigenvector :

$$\begin{bmatrix} \frac{29}{4} & \frac{21}{4} & \frac{3}{4} & \frac{11}{4} \\ \frac{21}{4} & \frac{29}{4} & \frac{11}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{11}{4} & \frac{29}{4} & \frac{21}{4} \\ \frac{11}{4} & \frac{3}{4} & \frac{21}{4} & \frac{29}{4} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0$$

By row reduction we get,

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0 .$$

$x_1 + x_4 = 0$ $\qquad x_4 = -x_1$

$x_2 - x_4 = 0$ $\qquad x_4 = x_2$

$x_3 + x_4 = 0$ $\qquad x_4 = -x_3$

Eigenvector $\begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$ $\qquad$ Normalised form :

$$\frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

For $\lambda = 4$ eigenvalue,
eigenvector :

$$\begin{bmatrix} \frac{13}{4} & \frac{21}{4} & \frac{3}{4} & \frac{11}{4} \\ \frac{21}{4} & \frac{13}{4} & \frac{11}{4} & \frac{3}{4} \\ 3/4 & \frac{11}{4} & \frac{13}{4} & \frac{21}{4} \\ \frac{11}{4} & \frac{3}{4} & \frac{21}{4} & \frac{13}{4} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0$$

By row reduction we get

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0.$$

$$x_1 - x_4 = 0 \qquad x_4 = x_1$$

$$x_2 + x_4 = 0 \qquad x_4 = -x_2$$

$$x_3 + x_4 = 0 \qquad x_4 = -x_3.$$

Eigenvector: $\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$ Normalised eigenvector: $\frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$

For eigenvalue $\lambda = 9$,

$$\begin{bmatrix} -\frac{7}{4} & \frac{21}{4} & \frac{3}{4} & \frac{11}{4} \\ \frac{21}{4} & -\frac{7}{4} & \frac{11}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{11}{4} & -\frac{7}{4} & \frac{21}{4} \\ \frac{11}{4} & \frac{3}{4} & \frac{21}{4} & -\frac{7}{4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0.$$

By row reduction we get,

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

$$x_1 + x_4 = 0 \qquad x_4 = -x_1$$

$$x_2 + x_4 = 0 \qquad x_4 = -x_2$$

$$x_3 - x_4 = 0 \qquad x_4 = x_3$$

Eigenvector: $\begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$ Normalised eigenvector: $\frac{1}{2}\begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$

For eigenvalue $\lambda = 16$, eigenvector.

$$\begin{bmatrix} -\frac{35}{4} & \frac{21}{4} & \frac{3}{4} & \frac{11}{4} \\ \frac{21}{4} & -\frac{35}{4} & \frac{11}{4} & 3/4 \\ 3/4 & \frac{11}{4} & -\frac{35}{4} & \frac{21}{4} \\ \frac{11}{4} & \frac{3}{4} & \frac{21}{4} & -\frac{35}{4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

By row. reduction we get,

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

$x_1 - x_4 = 0$

$x_2 - x_4 = 0$

$x_3 - x_4 = 0$

$x_1 = x_4$

$x_2 = x_4$.

$x_3 = x_4$.

Eigenvector $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

Normalised eigenvector $= \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

V =

$$V = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -1/2 \\ \frac{1}{2} & -\frac{1}{2} & -1/2 & 1/2 \\ \frac{1}{2} & 1/2 & -1/2 & -1/2 \\ \frac{1}{2} & 1/2 & 1/2 & 1/2 \end{bmatrix}$$

Final <u>SVD</u>

$$\Sigma = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \quad V = \begin{bmatrix} \frac{1}{2} & -1/2 & 1/2 & -1/2 \\ \frac{1}{2} & -1/2 & -1/2 & 1/2 \\ \frac{1}{2} & 1/2 & -1/2 & -1/2 \\ \frac{1}{2} & 1/2 & 1/2 & 1/2 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.6 & -0.8 \\ 0 & 0.8 & 0.6 \end{bmatrix}$$

b) 5 uses of <u>SVD</u>:

1) It is used to split a matrix into the product of a symmetric & a orthogonal matrix

$M = QS$

$= QV\Sigma V^T = U\Sigma V^T$

2) It can be used to compute pseudo inverse

of a matrix.

if $A = U \Sigma V^T$

then pseudoinverse of $A = V \cdot$ pseudoinv$(\Sigma) U^T$

for even non-square matrices.

3) It can be used to compute rowspace and range ~~nullspace~~ of a matrix.

$A = USV^T$

~~Therefore~~ If n is rank of A, colspace of A is the span of first n cols of U.

If n is rank of A, rowspace of A is the span of first n cols of V.

4) It can be used to compute nullspace of a matrix.

$A = USV^T$

The last cols of V (except the first n, n = rank of matrix A) ~~are~~ form the nullspace of matrix A.

5) It can be used to find the eigenvectors of a matrix.

$AU = US$ where U is a matrix of orthonormal eigenvectors

S is ~~a~~ the diagonal matrix of eigenvalues.

$A = USU^T$ → which satisfies the form of sym. matrices

if A is symmetric then ~~use~~ by SVD,

S is a diagonal matrix with its elements as eigenvalues and ~~U is orthonormal eigenvectors~~. cols of U are eigenvectors

Q-3.

$$[x_1 \; x_2 \; x_3][S]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 4(x_1 - 2x_2 + x_3)^2$$

a) RHS

$$4(x_1^2 + 4x_2^2 + x_3^2 - 4x_1x_2 - 4x_2x_3 + 2x_1x_3)$$

$$= 4x_1^2 + 16x_2^2 + 4x_3^2 - 16x_1x_2 - 16x_2x_3 + 8x_1x_3$$

$$[S] = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \quad : \sum_{i,j=1}^{3} S_{ij}\, x_i x_j$$

Clearly. $S_{11} = 4$ $\quad S_{22} = 16$ $\quad S_{33} = 4$

$S_{12} + S_{21} = -16$.

$$= \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$S = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}$$ where S is a symmetric matrix.

$S_{12} = S_{21}$, $\quad S_{13} = S_{31}$, $\quad S_{23} = S_{32}$

By comparing LHS and RHS we get,

$S_{11} = 4 \quad S_{22} = 16 \quad S_{33} = 4$

$S_{12} = S_{21} = -16 \qquad S_{23} = S_{32} = -16$

$S_{31} = S_{13} = 8$

$$S = \begin{bmatrix} 4 & -16 & 8 \\ -16 & 16 & -16 \\ 8 & -16 & 4 \end{bmatrix} = 4\begin{bmatrix} 1 & -4 & 2 \\ -4 & 4 & -4 \\ 2 & -4 & 1 \end{bmatrix}$$

b). Eigenvalues of S:

$$\begin{matrix} 1 & -4 & 2 \\ -1 & 1 & -1 \\ 2 & -4 & 1 \end{matrix}$$

$$\begin{vmatrix} 4-\lambda & -16 & 8 \\ -16 & 16-\lambda & -16 \\ 8 & -16 & 4-\lambda \end{vmatrix} = 0.$$

$$(4-\lambda)\left[(16-\lambda)(4-\lambda) - 256\right] - 16\left(-16\times 8 + 16(4-\lambda)\right)$$
$$+ 8\left(256 - 8(16-\lambda)\right) = 0$$

$$(4-\lambda)(64 - 20\lambda + \lambda^2 - 256) - 16(-128 + 64 - 16\lambda)$$
$$+ 8\times 256 - 64(16-\lambda) = 0$$

Determinant of S:

$4(16\times4-256)-16(-16\times8+64)+8(256-128)$

$= 1280$

$$\begin{vmatrix} 1-\lambda & -4 & 2 \\ -4 & 4-\lambda & -4 \\ 2 & -4 & 1-\lambda \end{vmatrix} = 0$$

$(1-\lambda)\ [(4-\lambda)(1-\lambda)-16]-4(-8+4(1-\lambda))+2(16-2(4-\lambda))=0$

$(1-\lambda)(4-5\lambda+\lambda^2-16)-4(-4-4\lambda)+32-4(4-\lambda)=0$

$(1-\lambda)(\lambda^2-5\lambda-12)+\cancel{16}+16\lambda+32-\cancel{16}+4\lambda=0$

$-\lambda^3+\lambda^2-5\lambda+5\lambda^2-12+12\lambda$
$+20\lambda+32=0.$

$-\lambda^3+6\lambda^2+27\lambda+20=0.$

$\lambda^3-6\lambda^2-27\lambda-20=0$

$\lambda^2(\lambda+1)-7\lambda(\lambda+1)-20(\lambda+1)=0$

$(\lambda+1)(\lambda^2-7\lambda-20)=0.$

$\lambda=-1.$

$\lambda=\dfrac{7\pm\sqrt{49+80}}{2}$

$=\dfrac{7\pm\sqrt{129}}{2}.$

Eigenvalues are

$\lambda=-4,\ \lambda=2(7-\sqrt{129}),\ \lambda=2(7+\sqrt{129})$

Rank of S: We see that S has 3 linearly independent rows & columns.

So rank = 3.

c). For S to be +ve definite,

$x^TSx>0 \quad \forall x\in\mathbb{R}^n \quad x\neq0$

$\because (x_1-2x_2+x_3)^2\geq0$ for $x=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

S is not +ve definite,

rather S is positive semi-definite.

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Q-5.

At pts of minima or maxima,

$$\nabla f(x,y) = \lambda \nabla g(x,y) \quad \text{subject to } 4x^2+2y^2=25$$

$f(x,y) = 3x-6y$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \end{bmatrix}$$

$$\nabla g = \begin{bmatrix} \frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} 8x \\ 4y \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ -6 \end{bmatrix} = \lambda \begin{bmatrix} 8x \\ 4y \end{bmatrix}$$

$$x = \frac{3}{8\lambda} \qquad y = \frac{-6}{4\lambda} = -\frac{3}{2\lambda}$$

$$4x^2+2y^2=25$$

$$\cancel{4} \times \frac{9}{\cancel{64}_{16}\lambda^2} + \cancel{2} \times \frac{9}{\cancel{4}_{2}\lambda^2} = 25$$

$$\frac{9}{16\lambda^2} + \frac{9}{2\lambda^2} = 25$$

$$\frac{9+72}{16\lambda^2} = 25$$

$$\frac{81}{16\lambda^2} = 25$$

$$16\lambda^2 \times 25 = 81$$

$$\lambda^2 = \frac{81}{16\times 25}$$

$$\lambda = \pm \frac{9}{20}$$

From Extreme Value Th. at $\lambda = \pm\frac{9}{20}$ $f(x,y)$ should have max/min.

$$\lambda = \frac{9}{20}, \quad x = \frac{3\times 20}{8\times 9} = \frac{5}{6}$$

$$y = -\frac{6\times 20}{4\times 9} = -\frac{10}{3}$$

Scanned by CamScanner

$3x - 6y$

$= 3 \times \frac{5}{\cancel{6}_2} - 6^2 \times -\frac{10}{\cancel{3}}$

$= \frac{5}{2} + 20 = 22.5$

$\lambda = -\frac{9}{20}$, $x = \frac{\cancel{3} \times \cancel{20}^5}{\cancel{8} \times -\cancel{9}_3} = -\frac{5}{6}$

$y = \frac{\cancel{3} \times \cancel{20}^{10}}{\cancel{2} \times -\cancel{9}_3} = \frac{10}{3}$

$3x - 6y$

$= 3 \times -\frac{5}{\cancel{6}_2} - 6^2 \times \frac{10}{\cancel{3}} = -\frac{5}{2} - 20 = -22.5$

Max. value of $f(x, y) = 22.5$
at $x = 5/6, y = -10/3$

Min " $= -22.5$
at $x = -5/6, y = 10/3$